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Common structures of optimal solutions for a crop irrigation problem under various constraints and criteria*

Ruben Chenevat¹, Bruno Cheviron², Sébastien Roux¹ and Alain Rapaport¹

Abstract—We consider a simplified crop irrigation model written as a non-autonomous, non-smooth controlled system. Different operating contexts and objectives lead to the study of optimal control problems with various state constraints, criterion and targets. We look for feedback solutions and we derive the optimality necessary conditions for the unified formulation. We show that there are only two parametrized time-varying feedback strategies consisting of the best and the worst practices, independent of the chosen objective, and suitable for numerical analysis.

I. INTRODUCTION

In the present context where the world population keeps growing and climate tensions intensify, one has to wonder about natural resources management, especially in agriculture. In particular there is an increase of areas where irrigation needs to complement rainfalls for crop production, while the amount of available water per capita is expected to decrease in the near future. In order to match with the upcoming need to step up the world food production while adapting to environmental pressures, different approaches of improvement exist. It is possible to deal with saving water through agronomic or technological levers (choice of species, equipment, planning). The latter seems to have promising water saving potential in the crucial context of deficit irrigation (see [15], [20]).

Several mathematical models have been developed to study crop irrigation. A certain amount of them are more or less complex numerical models describing quite precisely the physiological processes of the crop evolution ([10], [8], [3], [12], [19]). On the other hand, some more parsimonious models are adapted to a simpler dynamical system model and decision making through a limited number of decision variables. These models have inspired several mathematical formulations of optimal control problems for crop irrigation in order to derive analytical results (see [1], [9], [13], [18], [21], [22]). They typically use a water balance with few hypotheses on the functions modeling the biological process, enabling a general study of their problem (maximizing a financial cost or minimizing the water used to keep the soil moisture above a threshold). Some of these models give

a more detailed description of the evapotranspiration and maximizes the production of biomass under the context of water scarcity.

The interest of analytical results is to eventually provide strategies describing optimal irrigation through time-varying feedback control suited to the limited number of observed variables and manipulated inputs on the field. Many agronomic objectives can be of interest : maximizing the biomass under water constraint, minimizing the water used for reaching a target of biomass, maximizing a financial cost, keeping the soil moisture above a stress threshold.

In this work, we aim at studying these problems. To this aim, we choose to use the formulation from [1] but with different criteria and different constraints. We extend its work to the aforementioned objectives, and we will not only look for the best strategies, but we will also investigate the worst strategies since it seems crucial to be able to quantify the potential gain margin that we might expect from an appropriate water management. Moreover we consider a constraint on lower values of water stress to preserve crops in good health guaranteeing the quality of the harvested biomass at the end of the season. These different criteria and constraint are gathered to define six optimization problems (three objectives, best and worst cases) that we seek to analyze them in a unified way. Through an analysis of the necessary conditions of optimality, we wonder if there exists a unified optimal structure for the optimal strategies, in terms of state feedback. These feedback strategies are then expected to be implemented and tested in situations closer to real world, and tested on more complex numerical models.

In the Section II, we describe the model with its general hypotheses, and we present the different problems (depending on their different objectives) that we investigate altogether through a unified formulation. After some general results on the trajectories of the solutions to our system, we are able to show some necessary properties on optimal solutions that we present in Section III. Then we apply the non-smooth Pontryagin Maximum Principle and we derive and exploit the necessary conditions of optimality in Section IV, to finally obtain optimal structures for the best and the worst strategies. We then illustrate the theoretical results with numerical simulations in Section V and we discuss about their application.

II. MODEL AND FRAMEWORK

A. Model Description

We consider the simple crop model inspired by previous existing literature ([1], [14], [17]), representing a soil mois-

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ture balance in terms of soil evaporation, plant transpiration and water contribution through irrigation, and biomass production being influenced by water stress.

$$\dot{S}(t) = k_1(-\varphi(t)K_S(S(t)) - (1 - \varphi(t))K_R(S(t)) + k_2u(t)) \quad (1)$$

$$\dot{B}(t) = \varphi(t)K_S(S(t)) \quad (2)$$

$$\dot{V}(t) = u(t) \quad (3)$$

The system evolves over a fixed time interval $[0, T]$ representing the total season from sowing to harvest, S denotes the relative soil water content, B denotes the biomass production, V denotes the water consumption from the beginning of the season, with initial conditions $B(0) = 0$ and $V(0) = 0$. The normalizing positive constants k_1 and k_2 come from agronomical properties (and we assume that $k_2 > 1$ to ensure the local controllability of the system). The renormalized controls are measurable functions defined on $[0, T]$ and taking values in $[0, 1]$. The function φ represents the evolution of the radiation interception efficiency related to the plant cover rate, and it is supposed to be C^1 , increasing, and taking values in $[0, 1]$. The functions K_S and K_R are continuous, piecewise affine, non-decreasing, and defined by the following expressions :

$$K_S(S) = \begin{cases} 0 & , S \in [0, S_w] \\ \frac{S - S_w}{S^* - S_w} & , S \in [S_w, S^*] \\ 1 & , S \geq S^* \end{cases} \quad (4)$$

$$K_R(S) = \begin{cases} 0 & , S \in [0, S_h] \\ \frac{S - S_h}{1 - S_h} & , S \geq S_h \end{cases} \quad (5)$$

where $0 < S_h < S_w < S^* < 1$ are soil water limits thresholds depending on soil and plant properties (hygroscopic point, wilting point, and hydric stress threshold). Thus defined, the functions K_S and K_R describe respectively the behaviour of the soil and the plant when subject to different moisture levels, especially when the plant suffers from water stress.

B. Targets and Constraints

The system will be subject to both biological and operating constraints. First the soil moisture has to belong to $[0, 1]$. Moreover the operator might want to prevent plants to undergo too much of a water stress, therefore we add a tolerance threshold $S_w < S_{tol} < S^*$ and the following constraint :

$$\forall t \in [0, T], S_{tol} \leq S(t) \leq 1 \quad (6)$$

As different objectives may be aimed at by the operator, we introduce different problems corresponding to these different targets.

- (B1) : Maximizing the biomass production using a limited quota of water. This means maximizing $B(T)$ under the terminal condition $V(T) \leq V_{init}$ where V_{init} is the initial quantity of available water.
- (B2) : Minimizing the water consumption to reach a certain biomass production. This means minimizing $V(T)$ under the terminal condition $B(T) \geq B_{target}$ where B_{target} is the objectif of biomass production.

- (B3) : Maximizing a financial balance between biomass sellings and water consumption cost. This means maximizing $g(B(T)) - c(V(T))$ where g and c are functions representing the earnings and spendings of the exploitation, and satisfy $g(0) = c(0) = 0$, $g' > 0$ and $c' > 0$.

For application considerations, we would like to assess the potential margin that an operator could obtain by solving these problems, then we introduce their mirror problems where we aim for the worst possible outcome according to the objective.

- (W1) : Minimizing the biomass production using at least a certain quota of water. This means minimizing $B(T)$ under the terminal condition $V(T) \geq V_{init}$.
- (W2) : Maximizing the water consumption to reach at most a certain biomass production. This means maximizing $V(T)$ under the terminal condition $B(T) \leq B_{target}$.
- (W3) : Minimizing the financial balance between biomass sellings and water consumption cost, i.e. minimizing $g(B(T)) - c(V(T))$.

C. Framework

Since the soil is conveniently refilled in water during winter, we consider $S_0 \in (S^*, 1]$ as the initial state of the soil moisture. In the following, we denote $S_{t_i, S_i, u(\cdot)}(\cdot)$ the trajectory of S induced by the control u with initial time t_i and initial condition S_i . We introduce the times $t^* = \sup\{t \in [0, T], S_{0, S_0, 0}(t) > S^*\}$ and $t_{tol} = \sup\{t \in [0, T], S_{0, S_0, 0}(t) > S_{tol}\}$, we assume that these times are lower than T , and we define the following controls :

$$U^*(t) = \begin{cases} 0 & , t \in [0, t^*] \\ u_{sing}(t, S^*) & , t \in [t^*, T] \end{cases} \quad (7)$$

$$U_{tol}(t) = \begin{cases} 0 & , t \in [0, t_{tol}] \\ u_{sing}(t, S_{tol}) & , t \in [t_{tol}, T] \end{cases} \quad (8)$$

where we used the expression of the control forcing the soil moisture to remain constant at a value $\tilde{S} \in [0, 1]$:

$$u_{sing}(t, \tilde{S}) = \frac{\varphi(t)K_S(\tilde{S}) + (1 - \varphi(t))K_R(\tilde{S})}{k_2} \in [0, 1] \quad (9)$$

Notice that this control is feasible under the assumption $k_2 > 1$. We then see that the control U^* represents a situation where the plant does not suffer from water stress, and the control U_{tol} corresponds to an extreme trajectory according to the state constraint (6). We define the following notations :

$$V^* = \int_0^T U^*(t) dt, V_{tol} = \int_0^T U_{tol}(t) dt \quad (10)$$

$$B_{min} = \int_0^T \varphi(t)K_S(S_{0, S_0, U_{tol}(\cdot)}(t)) dt, B_{max} = \int_0^T \varphi(t) dt \quad (11)$$

The terminal conditions for the different problems have to be reachable, and we place our work in the context of water scarcity (or deficit irrigation, as in [11], [16]), meaning that we are not able to maintain a sufficient soil

moisture to prevent any hydric stress (when we try to do the best possible). Consequently, we can precise the terminal conditions for our different problems :

- $(B1), (W1) : V_{tol} < V_{init} < V^*$
- $(B2), (W2) : B_{min} < B_{target} < B_{max}$

III. THE OPTIMIZATION PROBLEM

As we seek to apply the Pontryagin Maximum Principle, we have to deal with the state constraint (6) first. Following an idea in [7], we replace the constraint by additional state equations :

$$\dot{Y}(t) = \min(0, S(t) - S_{tol}) \quad , \quad \dot{Z}(t) = \max(0, S(t) - 1) \quad (12)$$

with complementary initial and final conditions :

$$Y(0) = Z(0) = 0 \quad , \quad Y(T) = Z(T) = 0 \quad (13)$$

We then rewrite our system of equations as follows :

$$\begin{cases} \dot{S}(t) = k_1(-\varphi(t)K_S(S(t)) - (1 - \varphi(t))K_R(S(t)) + k_2u(t)) \\ \dot{B}(t) = \varphi(t)K_S(S(t)) \\ \dot{V}(t) = u(t) \\ \dot{Y}(t) = \min(0, S(t) - S_{tol}) = f_Y(S(t)) \\ \dot{Z}(t) = \max(0, S(t) - 1) = f_Z(S(t)) \end{cases} \quad (14)$$

with initial conditions $S(0) = S_0$, $B(0) = V(0) = Y(0) = Z(0) = 0$ and terminal conditions according to the problem we investigate. Denoting $X = (S, B, V, Y, Z)$ and \mathcal{X}_{Bj} (resp. \mathcal{X}_{Wj}), for $j = 1, \dots, 3$, the target set for the problem (Bj) (resp. (Wj)), we say that a control $u(\cdot)$ is feasible for the problem (Bj) (resp. (Wj)) if the corresponding solution of (14) satisfies the terminal condition $X(T) \in \mathcal{X}_{Bj}$ (resp. \mathcal{X}_{Wj}). We denote \mathcal{F}_{Bj} (resp. \mathcal{F}_{Wj}) the set of the feasible controls for the problem (Bj) (resp. (Wj)).

A. General Formulation

We are now able to write the general formulation for our problems :

$$(P) \quad \begin{cases} \text{opti} \quad \Omega(X(T)) \\ u(\cdot) \in \mathcal{F} \\ X(T) \in \mathcal{X} \end{cases} \quad (15)$$

where X is a solution to the system (14), *opti* stands for *inf* or *sup* according to the problem, and \mathcal{F} , Ω and \mathcal{X} are the corresponding feasible set, terminal cost and target set. We can sum all this up as in Table I.

TABLE I
PROBLEMS SETTINGS

P	opti	\mathcal{F}	$\Omega(X)$	\mathcal{X}
$B1$	sup	\mathcal{F}_{B1}	B	$\mathbb{R} \times \mathbb{R} \times]-\infty, V_{init}] \times \{0, 0\}$
$B2$	inf	\mathcal{F}_{B2}	V	$\mathbb{R} \times [B_{target}, +\infty[\times \mathbb{R} \times \{0, 0\}$
$B3$	sup	\mathcal{F}_{B3}	$g(B) - c(V)$	$\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \{0, 0\}$
$W1$	inf	\mathcal{F}_{W1}	B	$\mathbb{R} \times \mathbb{R} \times [V_{init}, +\infty[\times \{0, 0\}$
$W2$	sup	\mathcal{F}_{W2}	V	$\mathbb{R} \times]-\infty, B_{target}] \times \mathbb{R} \times \{0, 0\}$
$W3$	inf	\mathcal{F}_{W3}	$g(B) - c(V)$	$\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \{0, 0\}$

Because of lack of space, we shall give only parts of the proofs. We refer the reader to the preprint [2] that provides detailed proofs.

B. Preliminary Results

We begin with a result of comparison between trajectories for the soil moisture and consequences on the water consumption.

Lemma 3.1: Let $(S_1, B_1, V_1, Y_1, Z_1)$ and $(S_2, B_2, V_2, Y_2, Z_2)$ be two solutions of the system (14) on $[t_1, t_2]$ with $0 \leq t_1 < t_2 \leq T$, and denote u_1, u_2 their respective associated control. Assume that $S_1(t_1) = S_2(t_1)$ and $S_1(t) \geq S_2(t)$ for all $t \in [t_1, t_2]$. Then :

$$\forall t \in [t_1, t_2], \int_{t_1}^t u_1(\tau) d\tau \geq \int_{t_1}^t u_2(\tau) d\tau \quad (16)$$

Moreover if the trajectories of S_1 and S_2 belong to $[S_h, 1]$, then the inequality (16) is strict unless S_1 and S_2 are identical.

Proof: Let $t \in [t_1, t_2]$. One has :

$$\begin{aligned} S_1(t) - S_2(t) &= k_1 \int_{t_1}^t \varphi(\tau) (K_S(S_2(\tau)) - K_S(S_1(\tau))) d\tau \\ &\quad + k_1 \int_{t_1}^t (1 - \varphi(\tau)) (K_R(S_2(\tau)) - K_R(S_1(\tau))) d\tau \\ &\quad + k_1 k_2 \int_{t_1}^t u_1(\tau) - u_2(\tau) d\tau \end{aligned}$$

Since K_S and K_R are non-decreasing in S , we get :

$$0 \leq S_1(t) - S_2(t) \leq k_1 k_2 \int_{t_1}^t u_1(\tau) - u_2(\tau) d\tau \quad (17)$$

Moreover for any fixed $t \in [0, T]$ the function $S \mapsto \varphi(t)K_S(S) + (1 - \varphi(t))K_R(S)$ is increasing over $[S_h, 1]$, then if $S_1(\cdot)$ and $S_2(\cdot)$ belong to $[S_h, 1]$ the inequality (17) is strict unless $S_1 = S_2$. ■

This result guarantees that for any $V_{init} \in (V_{tol}, V^*)$ there exists an admissible control such that the corresponding trajectory satisfies the constraint (6) and such that $V(T) = V_{init}$. Furthermore we easily have a similar property stating that $S_1 \geq S_2 \Rightarrow B_1(T) \geq B_2(T)$, then for any $B_{target} \in (B_{min}, B_{max})$ there exists an admissible control such that the corresponding trajectory verifies (6) and $B(T) = B_{target}$. This leads to the unavoidable classical result on the existence of an optimal solution to the optimization problems (P).

Lemma 3.2: Under the framework described in section II.C, for each problem (P) with its corresponding setting in Table I, there exists u_p^* an optimal solution.

C. Properties of Optimal Solutions

Before applying the PMP, we can already give some properties on the optimal solutions to our different problems only using arguments based on improving the cost by building well-chosen controls.

Proposition 3.3: Let u_{Bj}^* be the optimal controls of their respective problems (with $j = 1, \dots, 3$), and denote the corresponding solutions by $(S_{Bj}, B_{Bj}, V_{Bj}, Y_{Bj}, Z_{Bj})$. Then :

$$u_{Bj}^*(t) = 0, \quad \text{for a.e. } t \in [0, t^*] \quad (18)$$

$$\forall t \in [\hat{t}^*, T], S_{B_j}(t) \leq S^* \quad (19)$$

Proof: For any feasible control $v(\cdot)$, consider $S_v(\cdot), B_v(\cdot), V_v(\cdot)$ the corresponding solutions. Suppose that v is optimal for (Bj). Let $\mathcal{E} = \{t \in [0, T], S_v(t) \leq S^*\}$ and $\hat{t} = \inf \mathcal{E} \geq \hat{t}^*$. One has :

$$\int_0^{\hat{t}} U^*(t) dt \leq \int_0^{\hat{t}} v(t) dt \quad \text{and} \quad K_S(S_0, S_0, U^*(\cdot)(t)) = 1$$

The inequality $\int_{\hat{t}}^T v(t) dt \geq \int_{\hat{t}}^T U^*(t) dt$ is thus not possible for (B1) or (B2), and in the case (B3) we would have :

$$g(B_v(T)) - c(V_v(T)) \leq g(B_{max}) - c(V^*)$$

which means that v would not be optimal for (B3).

Then consider that $\int_{\hat{t}}^T v(t) dt < \int_{\hat{t}}^T U^*(t) dt$, one necessarily has $\hat{t} < T$. We set :

$$\mathcal{E}_1 = \{t \in [\hat{t}, T], v(t) < 1\}$$

The set \mathcal{E}_1 has non null measure, because $U^*(t) < 1$ on $[\hat{t}, T]$, and the intersection $\mathcal{E} \cap \mathcal{E}_1$ has also non null measure (otherwise we would have $v = 1$ a.e. in \mathcal{E} , which would imply S increasing over \mathcal{E} , that prevents S to reach the domain below S^*).

Assume that $\hat{t} > \hat{t}^*$. Then $\int_0^{\hat{t}} U^*(t) dt < \int_0^{\hat{t}} v(t) dt$. We consider the control $y(\cdot)$, and its corresponding solutions $S_y(\cdot), B_y(\cdot), V_y(\cdot)$, defined by :

$$\begin{cases} y(t) = U^*(t) & , t \in [0, \hat{t}) \\ y(t) = v(t) & , t \in [\hat{t}, T] \setminus (\mathcal{E} \cap \mathcal{E}_1) \text{ and } S_y(t) < 1 \\ y(t) = \min(v(t), 1/k_2) & , t \in [\hat{t}, T] \setminus (\mathcal{E} \cap \mathcal{E}_1) \text{ and } S_y(t) = 1 \\ y(t) \in [v(t), 1] & , t \in \mathcal{E} \cap \mathcal{E}_1 \end{cases}$$

with :

$$0 < \int_{\mathcal{E} \cap \mathcal{E}_1} y(t) - v(t) dt < \int_0^{\hat{t}} v(t) - U^*(t) dt$$

Then we have :

$$\begin{aligned} V_y(T) &= \int_0^{\hat{t}} U^*(t) dt + \int_{\mathcal{E} \cap \mathcal{E}_1} y(t) dt + \int_{[\hat{t}, T] \setminus (\mathcal{E} \cap \mathcal{E}_1)} y(t) dt \\ &< \int_0^{\hat{t}} v(t) dt + \int_{\mathcal{E} \cap \mathcal{E}_1} v(t) dt + \int_{[\hat{t}, T] \setminus (\mathcal{E} \cap \mathcal{E}_1)} v(t) dt \end{aligned}$$

The associated solution $S_y(\cdot)$ satisfies :

$$\forall t \in [0, T], S_y(t) \leq 1 \quad \text{and} \quad \forall t \in [\hat{t}, T], S_y(t) \geq S_v(t)$$

with :

$$\int_{\mathcal{E} \cap \mathcal{E}_1} S_y(t) dt > \int_{\mathcal{E} \cap \mathcal{E}_1} S_v(t) dt$$

Moreover one has $S_v(t) \leq S^*$ over $\mathcal{E} \cap \mathcal{E}_1$, and there exists $\mathcal{J} \subset \mathcal{E} \cap \mathcal{E}_1$ of non null measure such that $S_v(t) < S^*$ over \mathcal{J} . Then :

$$\int_{\mathcal{E} \cap \mathcal{E}_1} \varphi(t) K_S(S_y(t)) dt > \int_{\mathcal{E} \cap \mathcal{E}_1} \varphi(t) K_S(S_v(t)) dt$$

This leads to :

$$\begin{aligned} B_y(T) &> \int_0^{\hat{t}} \varphi(t) K_S(S_v(t)) dt + \int_{\mathcal{E} \cap \mathcal{E}_1} \varphi(t) K_S(S_v(t)) dt \\ &+ \int_{[\hat{t}, T] \setminus (\mathcal{E} \cap \mathcal{E}_1)} \varphi(t) K_S(S_v(t)) dt \end{aligned}$$

To conclude, we have :

$$V_y(T) < V_v(T) \quad \text{and} \quad B_y(T) > B_v(T)$$

Thus the control $y(\cdot)$ is feasible for (Bj), and the control $v(\cdot)$ is therefore non optimal for (Bj). Hence $\hat{t} = \hat{t}^*$ and (18) is proven.

Next, for any feasible control $v(\cdot)$ such that $v(t) = 0$ over $[0, \hat{t}^*]$, we denote $S_v(\cdot), B_v(\cdot), V_v(\cdot)$ the corresponding trajectories. Suppose that v is optimal for (Bj), and consider the sets \mathcal{E} and \mathcal{E}_1 as before. We set :

$$\mathcal{G} = \{t \in [\hat{t}^*, T], S_v(t) > S^*\}$$

Assume that $\mathcal{G} \neq \emptyset$. Then we have :

$$\int_{\mathcal{G}} u_{sing}(t, S^*) dt < \int_{\mathcal{G}} v(t) dt$$

Let us consider the following control $z(\cdot)$, and its associated solution $S_z(\cdot), B_z(\cdot), V_z(\cdot)$, defined by :

$$\begin{cases} z(t) = 0 & , t \in [0, \hat{t}^*] \\ z(t) = u_{sing}(t, S^*) & , t \in \mathcal{G} \\ z(t) = v(t) & , t \in \mathcal{E} \setminus \mathcal{E}_1 \text{ and } S_z(t) < 1 \\ z(t) = \min(v(t), 1/k_2) & , t \in \mathcal{E} \setminus \mathcal{E}_1 \text{ and } S_z(t) = 1 \\ z(t) \in [v(t), 1] & , t \in \mathcal{E} \cap \mathcal{E}_1 \end{cases}$$

with :

$$0 < \int_{\mathcal{E} \cap \mathcal{E}_1} z(t) - v(t) dt < \int_{\mathcal{G}} v(t) - u_{sing}(t, S^*) dt$$

By similar arguments as above, we get :

$$V_z(T) < V_v(T) \quad \text{and} \quad B_z(T) > B_v(T)$$

Thus the control $z(\cdot)$ is feasible for (Bj), and the control $v(\cdot)$ is therefore non optimal for (Bj). Hence $\mathcal{G} = \emptyset$ and (19) is proven. \blacksquare

The next result indicates that the terminal conditions of the different problems with a target are saturated (a detailed proof can be found in [2]).

Proposition 3.4: Let $u_{B1}^*, u_{B2}^*, u_{W1}^*, u_{W2}^*$ be the optimal controls of their respective problems, and denote the corresponding solutions as before. Then :

$$V_{B1}(T) = V_{W1}(T) = V_{init} \quad (20)$$

$$B_{B2}(T) = B_{W2}(T) = B_{target} \quad (21)$$

IV. OPTIMAL SYNTHESIS

We are going to apply the Pontryagin Maximum Principle for nonsmooth systems (see [5]). According to the dynamics (14) and the optimization problem (15), we can write a unified Hamiltonian for all our settings :

$$\begin{aligned} H_P &= \lambda_P^S k_1 (k_2 u - (\varphi(t) K_S(S) + (1 - \varphi(t)) K_R(S))) \\ &+ \lambda_P^B \varphi(t) K_S(S) + \lambda_P^V u + \lambda_P^Y f_Y(S) + \lambda_P^Z f_Z(S) \end{aligned}$$

We have its following adjoint equations :

$$\begin{cases} \dot{\lambda}_P^S(t) \in \varphi(t)(\lambda_P^S(t)k_1 - \lambda_P^B) \partial_C K_S(S(t)) \\ \quad + (1 - \varphi(t))\lambda_P^S(t)k_1 \partial_C K_R(S(t)) \\ \quad - \lambda_P^Y \partial_C f_Y(S(t)) - \lambda_P^Z \partial_C f_Z(S(t)) \\ \lambda_P^B = 0 \\ \lambda_P^V = 0 \\ \lambda_P^Y = 0 \\ \lambda_P^Z = 0 \end{cases} \quad (22)$$

where ∂_C denotes the Clarke generalized gradient (see [5]). Thus, for any optimal solution u_P^*, X_P^* of (15), there exists a vector $\Lambda_P = (\lambda_P^S, \lambda_P^B, \lambda_P^V, \lambda_P^Y, \lambda_P^Z)$ which is an absolutely continuous solution of (22) that maximizes the Hamiltonian H_P , and satisfies for all $t \in [0, T]$:

$$|\lambda_P^S(t)| + |\lambda_P^B(t)| + |\lambda_P^V(t)| + |\lambda_P^Y(t)| + |\lambda_P^Z(t)| \neq 0 \quad (23)$$

and the transversality condition :

$$\Lambda_P(T) \in -N_{\mathcal{X}}(X_P^*(T)) - \text{sgn}(m)\nabla\Omega(X_P^*(T)) \quad (24)$$

where $N_{\mathcal{X}}(X)$ is the normal cone to the set \mathcal{X} at the point X , and $\text{sgn} : \begin{cases} \min \mapsto 1 \\ \max \mapsto -1 \end{cases}$. More precisely, these conditions are detailed in Table II.

TABLE II
TRANSVERSALITY CONDITIONS

P	$\lambda_P^S(T)$	$\lambda_P^B(T)$	$\lambda_P^V(T)$	$\lambda_P^Y(T)$	$\lambda_P^Z(T)$
B1	0	1	≤ 0	$\in \mathbb{R}$	$\in \mathbb{R}$
B2	0	≥ 0	-1	$\in \mathbb{R}$	$\in \mathbb{R}$
B3	0	$g'(B_{\mathcal{P}}(T))$	$-c'(V_{\mathcal{P}}(T))$	$\in \mathbb{R}$	$\in \mathbb{R}$
W1	0	-1	≥ 0	$\in \mathbb{R}$	$\in \mathbb{R}$
W2	0	≤ 0	1	$\in \mathbb{R}$	$\in \mathbb{R}$
W3	0	$-g'(B_{\mathcal{P}}(T))$	$c'(V_{\mathcal{P}}(T))$	$\in \mathbb{R}$	$\in \mathbb{R}$

Defining the switching function :

$$\phi_P(t) = \lambda_P^S(t)k_1k_2 + \lambda_P^V \quad (25)$$

the maximization of H_P gives for a.e. $t \in [0, T]$:

$$\begin{cases} u_P^*(t) = 0 & \text{if } \phi_P(t) < 0 \\ u_P^*(t) \in [0, 1] & \text{if } \phi_P(t) = 0 \\ u_P^*(t) = 1 & \text{if } \phi_P(t) > 0 \end{cases} \quad (26)$$

A. Optimality Necessary Conditions

First we show that the signs of the adjoints λ_P^S do not change over time.

Proposition 4.1: We have :

$$\forall t \in [0, T], \lambda_{B_j}^S(t) \geq 0 \text{ and } \lambda_{W_j}^S(t) \leq 0 \quad (27)$$

Proof: Suppose that there exists $\hat{t} \in [0, T]$ such that $\lambda_{B_j}^S(\hat{t}) < 0$. Then there exists $\varepsilon > 0$ such that $\lambda_{B_j}^S(t) < 0$ for all $t \in (\hat{t} - \varepsilon, \hat{t} + \varepsilon)$. Therefore $\phi_{B_j}(t) < 0$ for all $t \in (\hat{t} - \varepsilon, \hat{t} + \varepsilon)$, and the function $S_{B_j}(\cdot)$ is decreasing over this interval. Since the trajectory of $S_{B_j}(\cdot)$ verifies the constraint (6), then

we deduce that $S_{B_j}(t) \in (S_{tol}, 1)$ for all $t \in (\hat{t} - \varepsilon, \hat{t} + \varepsilon)$. As a consequence, one has for $t \in (\hat{t} - \varepsilon, \hat{t} + \varepsilon)$:

$$\begin{aligned} \dot{\lambda}_{B_j}^S(t) \in \varphi(t)(\lambda_{B_j}^S(t)k_1 - \lambda_{B_j}^B) \partial_C K_S(S_{B_j}(t)) \\ + (1 - \varphi(t))\lambda_{B_j}^S(t)k_1 \partial_C K_R(S_{B_j}(t)) \end{aligned}$$

Since $0 \leq \varphi \leq 1$, $\lambda_{B_j}^B \geq 0$, $\partial_C K_S \subset \mathbb{R}^+$ and $\partial_C K_R \subset \mathbb{R}^+$, then one has :

$$\dot{\lambda}_S^{(B_j)}(t) \in \mathbb{R}^-, \quad \forall t \in (\hat{t} - \varepsilon, \hat{t} + \varepsilon)$$

Let $\mathcal{A} = \{t \in [0, T], \lambda_{B_j}^S(t) < 0\}$.

The function $\lambda_{B_j}^S(\cdot)$ is then non increasing over \mathcal{A} , which contradicts the necessary terminal condition $\lambda_{B_j}^S(T) = 0$. Hence $\lambda_{B_j}^S(t) \geq 0$ for all $t \in [0, T]$.

Next, suppose that there exists $\check{t} \in [0, T]$ such that $\lambda_{W_j}^S(\check{t}) > 0$. Then there exists $\delta > 0$ such that $\lambda_{W_j}^S(t) > 0$ for all $t \in (\check{t} - \delta, \check{t} + \delta)$. Therefore $\phi_{W_j}(t) > 0$ for all $t \in (\check{t} - \delta, \check{t} + \delta)$, and the function $S_{W_j}(\cdot)$ is increasing over this interval. Since the trajectory of $S_{W_j}(\cdot)$ verifies the constraint (6), then we deduce that $S_{W_j}(t) \in (S_{tol}, 1)$ for all $t \in (\check{t} - \delta, \check{t} + \delta)$. As a consequence, one has for $t \in (\check{t} - \delta, \check{t} + \delta)$:

$$\begin{aligned} \dot{\lambda}_{W_j}^S(t) \in \varphi(t)(\lambda_{W_j}^S(t)k_1 - \lambda_{W_j}^B) \partial_C K_S(S_{W_j}(t)) \\ + (1 - \varphi(t))\lambda_{W_j}^S(t)k_1 \partial_C K_R(S_{W_j}(t)) \end{aligned}$$

Since $0 \leq \varphi \leq 1$, $\lambda_{W_j}^B \leq 0$, $\partial_C K_S \subset \mathbb{R}^+$, and $\partial_C K_R \subset \mathbb{R}^+$, then one has :

$$\dot{\lambda}_{W_j}^S(t) \in \mathbb{R}^+, \quad \forall t \in (\check{t} - \delta, \check{t} + \delta)$$

Let $\mathcal{B} = \{t \in [0, T], \lambda_{W_j}^S(t) > 0\}$.

From above, we deduce that the function $\lambda_{W_j}^S(\cdot)$ is non decreasing over \mathcal{B} , which contradicts the necessary terminal condition $\lambda_{W_j}^S(T) = 0$. Hence $\lambda_{W_j}^S(t) \leq 0$ for all $t \in [0, T]$. ■

The other adjoints are constant according to the equations (21), and we can precise that there is no null adjoint amongst the λ_P^B and λ_P^V (a detailed proof can be found in [2]).

Proposition 4.2: We have :

$$\lambda_{B1}^V < 0, \lambda_{B2}^B > 0, \lambda_{W1}^V > 0, \lambda_{W2}^B < 0 \quad (28)$$

B. Singular Arcs

The next property is a crucial remark enabling us to state that singular arcs are associated to trajectories where $S(\cdot)$ remain constant. Moreover there is only some particular values of S where a singular arc can occur.

Proposition 4.3: A singular arc for a problem (P) satisfies $S_P(t) = \tilde{S}$ where \tilde{S} is a non-differentiable point of one of the functions K_S, K_R, f_Y, f_Z .

Proof: Consider a closed interval $I = [t_1, t_2]$ of non-null measure, such that $\phi_P(t) = 0$ on I . This implies to have λ_P^S being constant equal to $\tilde{\lambda}_P^S = -\frac{\lambda_P^V}{k_1k_2} \neq 0$ on this interval. If the functions K_S, K_R, f_Y and f_Z are differentiable at $S(\hat{t})$ where $\hat{t} \in I$, then by construction the quantities $\partial_C K_S(S_P(t))$, $\partial_C K_R(S_P(t))$, $\partial_C f_Y(S_P(t))$ and $\partial_C f_Z(S_P(t))$ are constant equal to $K'_S(S_P(\hat{t}))$, $K'_R(S_P(\hat{t}))$, $f'_Y(S_P(\hat{t}))$ and $f'_Z(S_P(\hat{t}))$ on a neighborhood $(\hat{t} - \varepsilon, \hat{t} + \varepsilon)$ of \hat{t} . More precisely, since we know that $S_{tol} \leq S_P(t) \leq 1$, one has $K'_R(S_P(\hat{t})) > 0$ and $f'_Y(S_P(\hat{t})) =$

$f'_Z(S_P(\hat{t})) = 0$. Thus, from the equation verified by λ_p^S , we get for a.e. $t \in I \cap (\hat{t} - \varepsilon, \hat{t} + \varepsilon)$:

$$\begin{aligned} 0 &\neq -\tilde{\lambda}_p^S k_1 K'_R(S_P(\hat{t})) \\ &= \varphi(t) \left((\tilde{\lambda}_p^S k_1 - \lambda_p^B) K'_S(S_P(\hat{t})) - \tilde{\lambda}_p^S k_1 K'_R(S_P(\hat{t})) \right) \end{aligned}$$

As the function φ is increasing, we deduce that this equality cannot be satisfied for a.e. $t \in I \cap (\hat{t} - \varepsilon, \hat{t} + \varepsilon)$. Consequently, a singular arc can only occur at a point $S_P = \tilde{S}$ where one of the functions K_S , K_R , f_Y or f_Z is not differentiable. ■

Remark 4.4: If a singular arc happens at the value $S = \tilde{S}$, we notice that the associated singular control is in fact known and its expression is $\tilde{u}(t) = u_{sing}(t, \tilde{S})$.

Furthermore, the set of possible singular points can even be improved thanks to the following result (a detailed proof can be found in [2]).

Proposition 4.5: For a problem (Bj) , a singular arc can only occur at $\tilde{S} = S_{tol}$ or $\tilde{S} = S^*$.

For a problem (Wj) , a singular arc can only occur at $\tilde{S} = S_{tol}$ or $\tilde{S} = 1$.

C. Structures of Optimal Solutions

The following proposition is the final key for describing precisely the optimal structures, since it will help to determine the amount and nature of the different bang arcs (a detailed proof can be found in [2]).

Proposition 4.6: For a problem (Bj) , the set $\chi_{Bj} = \{t \in [t^*, T], \phi_{Bj}(t) \geq 0\}$ is connected.

For a problem (Wj) , the set $\chi_{Wj} = \{t \in [0, T], \phi_{Wj}(t) \leq 0\}$ is connected.

We define now the Ψ^+ (resp. Ψ^-) strategy, and we prove that the optimal controls for the problems (Bj) (resp. (Wj)) belong to this class of time-varying feedbacks.

Definition 4.1: For t_d, V_c, t_c , we define the following time-varying feedback control :

$$\Psi_{t_d, V_c, t_c}^+(t, X) = \begin{cases} 0 & \text{if } (S \geq S_{tol} \wedge t \leq t_d) \vee V = V_c \vee t \geq t_c \\ u_{sing}(t, S_{tol}) & \text{if } S = S_{tol} \wedge t \leq t_d \\ u_{sing}(t, S^*) & \text{if } S = S^* \wedge V < V_c \wedge t < t_c \\ 1 & \text{otherwise} \end{cases}$$

Theorem 4.7: In the framework of the problem (Bj) , there exists $t_d \in [t^*, T)$, $V_c \in (0, V_{init}]$, $t_c \in (t_d, T)$ such that the feedback control Ψ_{t_d, V_c, t_c}^+ is optimal for (Bj) .

Proof: Let $u_{Bj}^*(\cdot)$ be an optimal control for the problem (Bj) . As seen above, the set χ_{Bj} is non-empty and connected, and since the function $\phi_{Bj}(\cdot)$ is continuous, we deduce that χ_{Bj} is an interval $[t_a, t_c]$ where $0 \leq t_a < t_c \leq T$, and the control $u_{Bj}^*(\cdot)$ is null outside this interval. From previous results, we know that $t_a \geq t^*$ and $t_c < T$. At any $t \in \chi_{Bj}$ the switching function ϕ_{Bj} is non-negative, then for a.e. $t \in \chi_{Bj}$ one has either $u_{Bj}^*(t) = 1$ or $u_{Bj}^*(t) = u_{sing}(t, \tilde{S})$ where $\tilde{S} \in \{S_{tol}, S^*\}$. As a consequence, the quantity $V_{Bj}(\cdot)$ is increasing on χ_{Bj} and $S_{Bj}(\cdot)$ is non-decreasing on χ_{Bj} . Furthermore, the trajectory $S_{Bj}(\cdot)$ is thus composed of an increasing part (with $u_{Bj}^* = 1$) and possible singular parts (with $u_{Bj}^* = u_{sing}$ and $S_{Bj} = S_{tol}$ or $S_{Bj} = S^*$). The increasing part is thus over an interval $[t_d, t_f]$ with $t_a \leq t_d < t_f \leq t_c$. At $t = t_c$, we know that

$S_{Bj}(t_c) \leq S^*$ and since V_{Bj} is increasing on χ_{Bj} we deduce that t_c and $V(t_c)$ are equivalently defined. We call t_c the "cut-off time" and we set $V_c = V(t_c)$. According to what was said above, we then have either $t_a = t_d$ or t_a fully determined by the trajectory $S_{0, S_{0,0}}(\cdot)$ such that $S_{0, S_{0,0}}(t_a) = S_{tol}$. Moreover, the time t_f is either equal to t_c or fully determined by the trajectory $S_{t_d, S(t_d), 1}(\cdot)$ such that $S_{t_d, S(t_d), 1}(t_f) = S^*$. Finally, the control $u_{Bj}^*(\cdot)$ is fully determined by the two parameters t_d and t_c (or equivalently t_d and V_c), and we easily check that it fulfills $u_{Bj}^*(t) = \Psi_{t_d, V_c, t_c}^+(t, S(t), B(t), V(t))$ for a.e. $t \in [0, T]$. ■

The corresponding structure and optimality for the problems (Wj) are obtained in a similar way.

Definition 4.2: For V_d, V_c, t_c , we define the following time-varying feedback control :

$$\Psi_{V_d, V_c, t_c}^-(t, X) = \begin{cases} 1 & \text{if } (V < V_d \wedge S < 1) \vee V_c - V = T - t \\ u_{sing}(t, 1) & \text{if } V < V_d \wedge S = 1 \\ u_{sing}(t, S_{tol}) & \text{if } S = S^* \wedge V < V_c \wedge t < t_c \\ 0 & \text{otherwise} \end{cases}$$

Theorem 4.8: In the framework of the problem (Wj) , there exists $V_d \in [0, V_{init})$, $V_c \in [V_d, V_{init})$, $t_c \in (0, T)$ such that the feedback control Ψ_{V_d, V_c, t_c}^- is optimal for (Wj) .

V. NUMERICAL ILLUSTRATIONS

Thanks to theoretical analysis we can easily find the optimal solutions by scouting the low dimensionned sets of Ψ^+ and Ψ^- strategies. We then numerically compute (and verify with BOCOP [23]) optimal solutions of examples to visualize their structure. We have considered a parametrized increasing function $\varphi = \varphi_{a,b}$ (inspired by the shape of the leaf area index in [4]) such that $\varphi_{a,b}(0) = 0$ and $\varphi_{a,b}(T) = 1$ by :

$$\forall t \in [0, T], \varphi_{a,b}(t) = \frac{\gamma_{a,b}(t) - \gamma_{a,b}(0)}{\gamma_{a,b}(1) - \gamma_{a,b}(0)}$$

where $\gamma_{a,b}(t) = 1 / (1 + \exp(a - 2a(t/T)^b))$ with $a > 0, b > 0$. Then we have chosen a plausible set of (eventually rescaled) values for the model parameters. In order to illustrate an example of trajectories and control for the problems $(B1)$ and $(W1)$, we made sure to select $V_{init} \in (V_{tol}, V^*)$, and the set of parameters can be found in Table III.

TABLE III
PARAMETERS FOR $(B1)$ AND $(W1)$

T	S_0	k_1	k_2	S_h	S_w	S^*	a	b	S_{tol}	V_{init}
150	0.9	0.02	3	0.2	0.4	0.7	3.8	1	0.5	30

By implementing the parametrized strategies Ψ_{t_d, V_c, t_c}^+ and Ψ_{V_d, V_c, t_c}^- , we numerically found the optimal trigger times $t_{B1} = 58.4$ and $t_{W1} = 28.5$ to obtain the corresponding maximal biomass $B_{B1} = 67.6$ and minimal biomass $B_{W1} = 33.6$. We present the trajectories and their associated controls in Fig. 1, Fig. 2, Fig. 3 and Fig. 4.

Moreover we can associate for each admissible initial condition V_{init} two values of biomass : $B_{B1}(T)$ corresponding

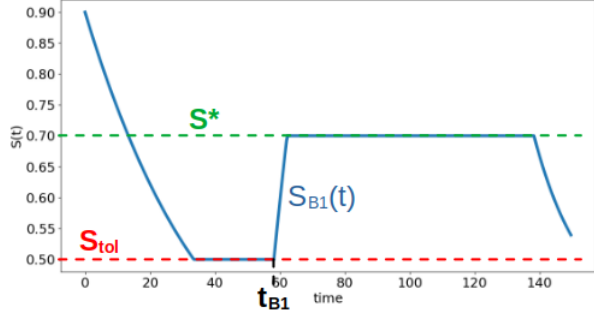


Fig. 1. Optimal trajectory of S_{B1} for the initial condition $V_{init} = 30$

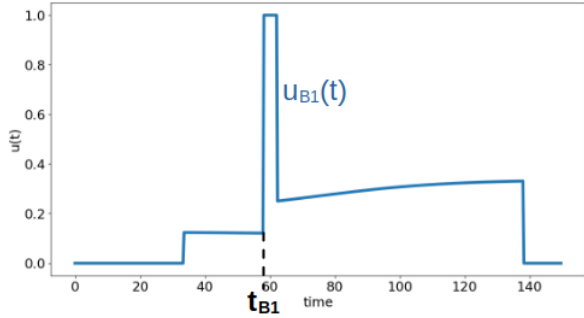


Fig. 2. Optimal control u_{B1}^* for the initial condition $V_{init} = 30$

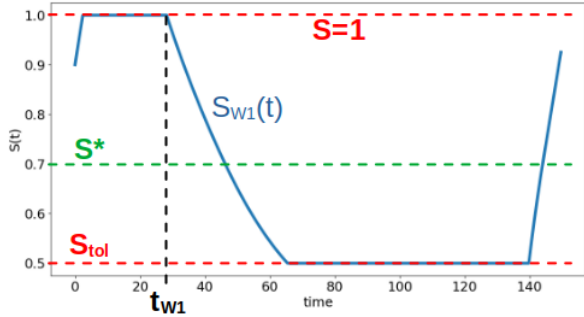


Fig. 3. Optimal trajectory of S_{W1} for the initial condition $V_{init} = 30$

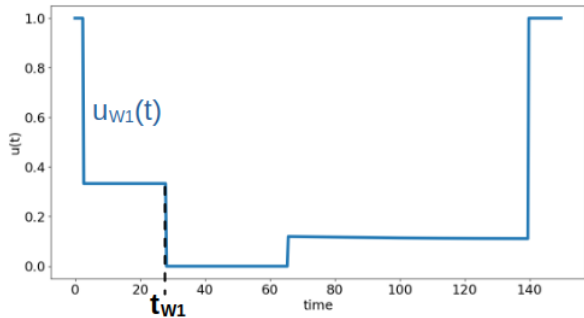


Fig. 4. Optimal control u_{W1}^* for the initial condition $V_{init} = 30$

to the best amount of biomass attainable using exactly the quota V_{init} , and $B_{W1}(T)$ corresponding to the worst amount of biomass attainable using exactly the quota V_{init} . This leads to draw two production curves as in Fig. 5 that we may compare in order to quantify the difference between the strategies.

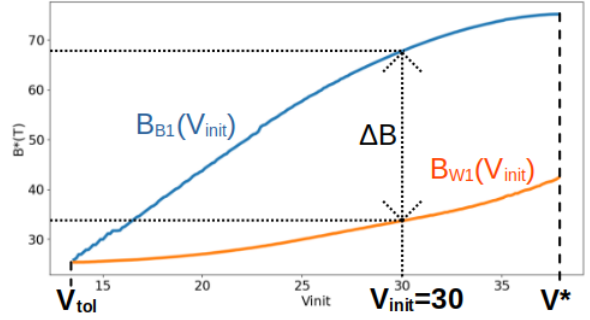


Fig. 5. Production curves for (B1) and (W1)

We can also illustrate the problem with a financial cost. Using the definition of the functions g and c , we notice that an optimal solution of (B3) has to verify $V_{B3}(T) \in [V_{tol}, v^*]$ and $B_{B3}(T) \in [B_{min}, B_{max}]$. Then, for any $\tilde{V} \in [V_{tol}, V^*]$ we can associate a quantity $B_{B1}(T|\tilde{V}) \in [B_{min}, B_{max}]$ corresponding to the biomass production with the optimal strategy from the problem (B1) where $V_{init} = \tilde{V}$. We can draw the graph of $\tilde{V} \mapsto g(B_{B1}(T|\tilde{V})) - c(\tilde{V})$, and the summit gives the value of the maximal cost for (B3). It is then possible to read the optimal coordinates of the problem (B3) on this graph, see an example in Fig. 6, computed with the parameters $(T, S_0, k_1, k_2, S_h, S_w, S^*, a, b, S_{tol})$ given in Table III and renormalized costs functions as follows (the sales of biomass is considered concave and the cost of water is considered convex, according to recent behaviour) :

$$g(B) = B^{0.8} \quad \text{and} \quad c(V) = 0.15 * V^{1.2}$$

The values of V_{B3} and $g(B_{B3}) - c(V_{B3})$ can also be directly computed by exploring the whole set of parametrized Ψ_{t_d, V_c, t_c}^+ strategies, and we numerically obtain $V_{B3} = 33.4$ and $g(B_{B3}) - c(V_{B3}) = 20.6$.

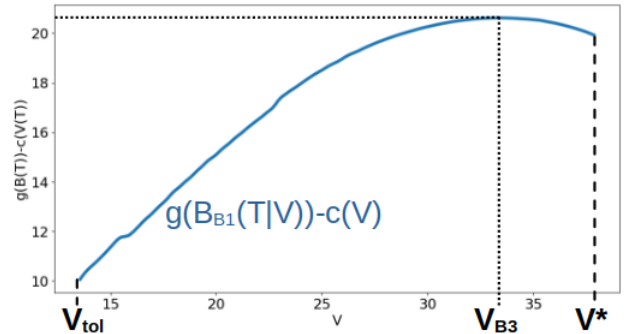


Fig. 6. Production curve for (B3)

VI. CONCLUSION

Considering six optimization problems based on the same set of dynamics of a simplified crop model, we have high-

lighted two structures for optimal strategies depending on the criterion and target. These strategies can be implemented as parametrized controls in small dimension, hence the optimal solutions can be computed numerically and efficiently. The perspectives are first to perform a numerical exploration with a large number of parameter sets in order to determine the respective influence of the different variables on the problem, and then to assess the potential gain margin between the best and the worst theoretical practices in order to bring decision support with quantitative results in a concrete application.

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