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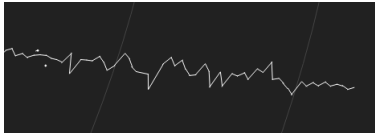
Régression sur un groupe de Lie avec une connexion quelconque

**55èmes Journées de Statistique de la Sfds
Université de Bordeaux**

Johan AUBRAY, Florence NICOL and Stéphane PUECHMOREL
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Monday May 27th, 2024

Our application

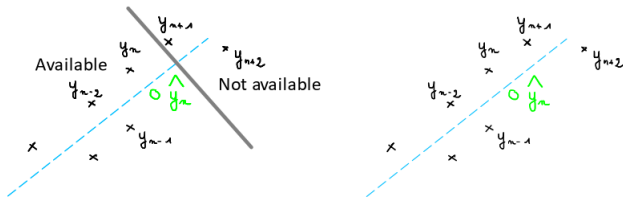
- Estimating the true position of a mobile object (aircraft, drone, ...) from measured (therefore noisy) raw positions



- This may be difficult with a highly maneuvering target

Our application

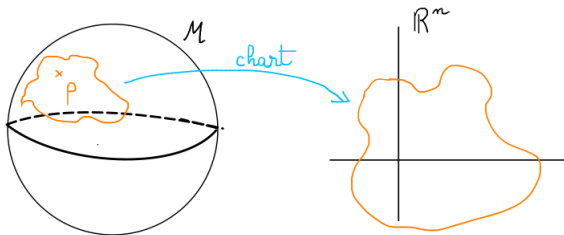
- Estimating the true position of a mobile object (aircraft, drone, ...) from measured (therefore noisy) raw positions
- Two cases :
 - From past positions: prediction
 - Frequently solved with linear / extended / unscented Kalman filters, or particle filters
 - From past and future positions: smoothing



- What comes next is based on the second case: smoothing

Our framework

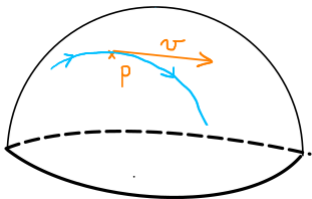
- Instead of considering the trajectory in the state space, the movement of the mobile object is considered as a sequence of displacements: rotations and translations in \mathbb{R}^3
- This set is called the Special Euclidean group $SE(3)$
- $SE(3)$ is a Lie group, which is both:
 - a differentiable manifold



- a group in which both the product and inverse maps are smooth

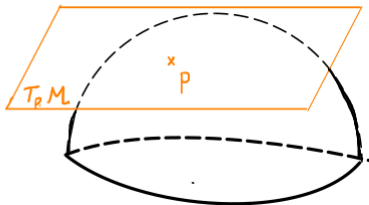
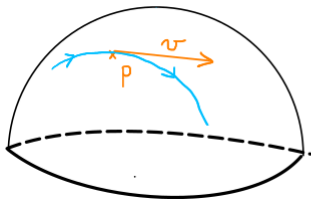
Going off on a tangent (space)

- At any point p of a manifold \mathcal{M} , one may define a tangent vector v as the derivative of a curve going through p :
 $v = c'(0)$ with $c:] - \epsilon, \epsilon[\rightarrow \mathcal{M}$, $c(0) = p$



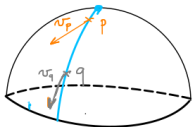
Going off on a tangent (space)

- At any point p of a manifold \mathcal{M} , one may define a tangent vector v as the derivative of a curve going through p :
 $v = c'(0)$ with $c:]-\epsilon, \epsilon[\rightarrow \mathcal{M}, c(0) = p$
- The set of all tangent vectors at p is a vector space called the tangent space at p denoted $T_p\mathcal{M}$



Manifolds and geodesics

- A geodesic in a manifold is the equivalent of a straight line
- A curve γ is a geodesic if it has a "constant" velocity vector
 - Problem: how can we compare a vector $v_p \in T_p\mathcal{M}$ with a vector $v_q \in T_q\mathcal{M}$?



- A connection ∇ is a rule for taking the directional derivative of a vector field along a tangent vector
- A curve γ is called a geodesic if

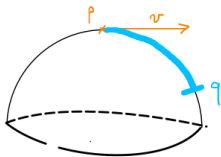
$$\nabla_{\gamma'} \gamma' = \frac{D}{dt} \gamma'(t) = 0$$

(we "parallel-transport" the $\gamma'(t)$ vector along the curve γ)

Exponential and logarithm on a manifold

- An initial point $p = \gamma(0)$ and an initial vector $v = \gamma'(0)$ are enough to define (at least locally) a unique geodesic
- This defines a map from the tangent bundle of \mathcal{M} $T\mathcal{M} = \cup_p T_p\mathcal{M}$ to the manifold \mathcal{M} called Exp:

$$\begin{aligned} \text{Exp}: T\mathcal{M} &\rightarrow \mathcal{M} \\ (p, v) &\mapsto q \end{aligned}$$

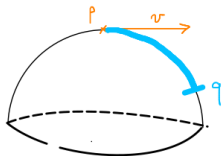


- This is the equivalent of $q = p + \vec{v}$ in \mathbb{R}^n

Exponential and logarithm on a manifold

- Likewise, one may define the inverse map called Log:

$$\begin{aligned} \text{Log}_p: \mathcal{M} &\rightarrow T_p\mathcal{M} \\ q &\mapsto v \end{aligned}$$



- This is the equivalent of $\vec{v} = \vec{pq} = q - p$ in \mathbb{R}^n

How about Lie groups?

- The group structure of a Lie group G makes it possible to bring most operations to the identity e
- The tangent space at e of G is called the Lie algebra of G , denoted \mathfrak{g}
 - it "linearises" the group and captures most of its properties

Lie group: $SE(3)$, Lie algebra: $\mathfrak{se}(3)$

- An element of $SE(3)$ can be represented by a 4-by-4 matrix

$$\begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix}, R \in SO(3), T \in \mathbb{R}^3 :$$

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{pmatrix} = \begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{pmatrix} = \begin{pmatrix} R \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + T \\ 1 \end{pmatrix}$$

- An element of $\mathfrak{se}(3)$ can be represented by a 4-by-4 matrix:

$$\begin{pmatrix} \Omega & U \\ 0 & 0 \end{pmatrix}, \Omega \text{ skew-symmetric}, U \in \mathbb{R}^3$$

But where's the regression in all of this?

In an Euclidean space

Let $X \in \mathbb{R}$ a non-random variable
and $Y \in \mathbb{R}^n$ a random variable

Regression model: $Y = \alpha + X\beta + \epsilon$

α is the intercept

β is the slope

$$(\hat{\alpha}, \hat{\beta}) = \arg \min \|Y - \alpha - X\beta\|^2$$

A closed-form solution exists

In a Riemannian manifold

Let $X \in \mathbb{R}$ a non-random variable
and $Y \in \mathcal{M}$ a random variable

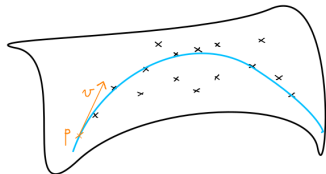
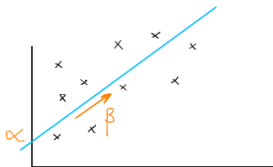
Regression model: $Y = \text{Exp}(\text{Exp}(p, Xv), \epsilon)$

p is the initial point

v is the initial velocity vector

$$(\hat{p}, \hat{v}) = \arg \min d(\text{Exp}(p, Xv), Y)^2$$

There is usually **NO** closed-form solution



But where's the regression in all of this?

In an Euclidean space

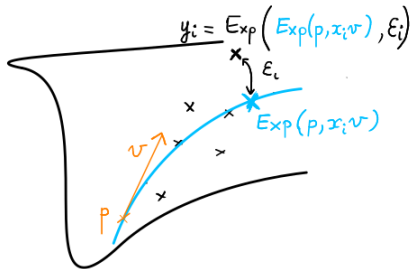
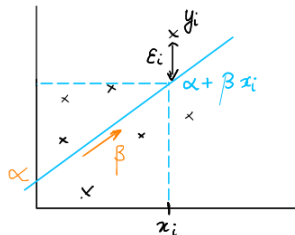
Regression model: $Y = \alpha + X\beta + \epsilon$

$$(\hat{\alpha}, \hat{\beta}) = \arg \min \|\mathbf{Y} - \alpha - X\beta\|^2$$

In a Riemannian manifold

Regression model: $Y = \text{Exp}(\text{Exp}(p, Xv), \epsilon)$

$$(\hat{p}, \hat{v}) = \arg \min d(\text{Exp}(p, Xv), Y)^2$$



"Optimal" curve

- The "optimal" curve of order k is obtained by computing the parameters that minimise the least squares criterion:

$$E(\gamma) = \frac{1}{N} \sum_{j=1}^N d(\gamma(t_j), y_j)^2 \quad (1)$$

$$\text{under the constraint } \nabla_{\gamma'}^{(k)} \gamma'(t) = \frac{D^{(k)}}{dt} \gamma'(t) = 0 \quad (2)$$

- 1 Cancelling the differential form dE ,
- 2 We take advantage of the Lie group structure by first solving in \mathfrak{g} , then "transporting" the solution in G .

"Any" connection?

- There is a unique metric-compatible, torsion-free connection called the **Levi-Civita** connection ∇^{LC}
- For another connection ∇ : the mapping

$$(t_1, \dots, t_n) \in [0, 1]^n \mapsto \text{Exp} \left(x, \sum_{i=1}^n t_i e_i \right)$$

with (e_1, \dots, e_n) be a basis of $T_x \mathcal{M}$, defines a local coordinate system in a neighborhood of x

Goodness of fit

We can use (P.T. Fletcher, 2011) a "Riemannian" coefficient of determination R^2 , defined as:

$$R^2 = 1 - \frac{SSE}{\text{Data Fréchet variance}} = 1 - \frac{\sum_j d(\gamma(t_j), y_j)^2}{\min_{y \in \mathcal{M}} \sum_j d(y, y_j)^2}$$

What's next?

- Local polynomials
- Description of the noise: the Gaussian hypothesis holds only if the noise is "not too large"