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## Regression on a Lie group with any connection

Johan Aubray, Florence Nicol, Stéphane Puechmorel

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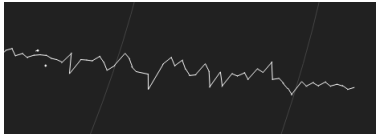
## **Régression sur un groupe de Lie avec une connexion quelconque**

**55èmes Journées de Statistique de la Sfds  
Université de Bordeaux**

**Johan AUBRAY, Florence NICOL and Stéphane PUECHMOREL**  
Ecole Nationale de l'Aviation Civile, Toulouse, France  
Monday May 27th, 2024

# Our application

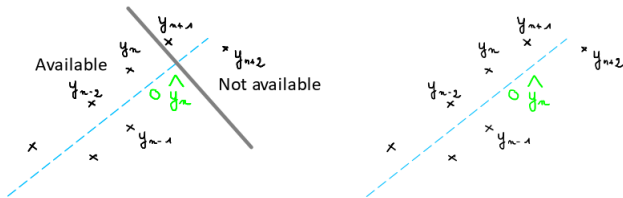
- Estimating the true position of a mobile object (aircraft, drone, ...) from measured (therefore noisy) raw positions



- This may be difficult with a highly maneuvering target

# Our application

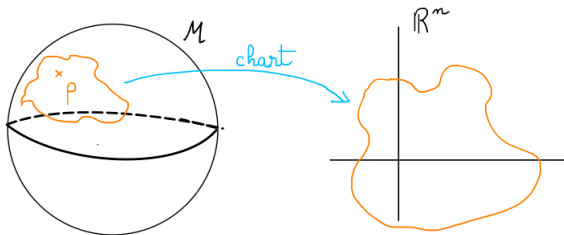
- Estimating the true position of a mobile object (aircraft, drone, ...) from measured (therefore noisy) raw positions
- Two cases :
  - From past positions: prediction
    - Frequently solved with linear / extended / unscented Kalman filters, or particle filters
  - From past and future positions: smoothing



- What comes next is based on the second case: smoothing

## Our framework

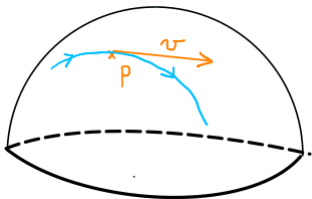
- Instead of considering the trajectory in the state space, the movement of the mobile object is considered as a sequence of displacements: rotations and translations in  $\mathbb{R}^3$
- This set is called the Special Euclidean group  $SE(3)$
- $SE(3)$  is a Lie group, which is both:
  - a differentiable manifold



- a group in which both the product and inverse maps are smooth

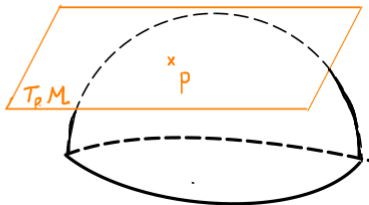
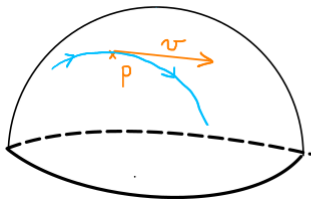
## Going off on a tangent (space)

- At any point  $p$  of a manifold  $\mathcal{M}$ , one may define a tangent vector  $v$  as the derivative of a curve going through  $p$ :  
 $v = c'(0)$  with  $c: ]-\epsilon, \epsilon[ \rightarrow \mathcal{M}$ ,  $c(0) = p$



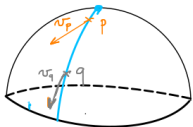
## Going off on a tangent (space)

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 $v = c'(0)$  with  $c: ]-\epsilon, \epsilon[ \rightarrow \mathcal{M}, c(0) = p$
- The set of all tangent vectors at  $p$  is a vector space called the tangent space at  $p$  denoted  $T_p\mathcal{M}$



## Manifolds and geodesics

- A geodesic in a manifold is the equivalent of a straight line
- A curve  $\gamma$  is a geodesic if it has a "constant" velocity vector
  - Problem: how can we compare a vector  $v_p \in T_p\mathcal{M}$  with a vector  $v_q \in T_q\mathcal{M}$ ?



- A connection  $\nabla$  is a rule for taking the directional derivative of a vector field along a tangent vector
- A curve  $\gamma$  is called a geodesic if

$$\nabla_{\gamma'} \gamma' = \frac{D}{dt} \gamma'(t) = 0$$

(we "parallel-transport" the  $\gamma'(t)$  vector along the curve  $\gamma$ )

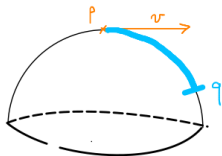


## Exponential and logarithm on a manifold

- An initial point  $p = \gamma(0)$  and an initial vector  $v = \gamma'(0)$  are enough to define (at least locally) a unique geodesic
- This defines a map from the tangent bundle of  $\mathcal{M}$   $T\mathcal{M} = \cup_p T_p\mathcal{M}$  to the manifold  $\mathcal{M}$  called Exp:

$$\text{Exp}: T\mathcal{M} \rightarrow \mathcal{M}$$

$$(p, v) \mapsto q$$

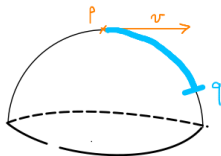


- This is the equivalent of  $q = p + \vec{v}$  in  $\mathbb{R}^n$

# Exponential and logarithm on a manifold

- Likewise, one may define the inverse map called Log:

$$\begin{aligned} \text{Log}_p: \mathcal{M} &\rightarrow T_p\mathcal{M} \\ q &\mapsto v \end{aligned}$$



- This is the equivalent of  $\vec{v} = \vec{pq} = q - p$  in  $\mathbb{R}^n$

# How about Lie groups?

- The group structure of a Lie group  $G$  makes it possible to bring most operations to the identity  $e$
- The tangent space at  $e$  of  $G$  is called the Lie algebra of  $G$ , denoted  $\mathfrak{g}$ 
  - it "linearises" the group and captures most of its properties

## Lie group: $SE(3)$ , Lie algebra: $\mathfrak{se}(3)$

- An element of  $SE(3)$  can be represented by a 4-by-4 matrix

$$\begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix}, R \in SO(3), T \in \mathbb{R}^3 :$$

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{pmatrix} = \begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{pmatrix} = \begin{pmatrix} R \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + T \\ 1 \end{pmatrix}$$

- An element of  $\mathfrak{se}(3)$  can be represented by a 4-by-4 matrix:

$$\begin{pmatrix} \Omega & U \\ 0 & 0 \end{pmatrix}, \Omega \text{ skew-symmetric}, U \in \mathbb{R}^3$$

# But where's the regression in all of this?

In an Euclidean space

Let  $X \in \mathbb{R}$  a non-random variable  
and  $Y \in \mathbb{R}^n$  a random variable

Regression model:  $Y = \alpha + X\beta + \epsilon$

$\alpha$  is the intercept

$\beta$  is the slope

$$(\hat{\alpha}, \hat{\beta}) = \arg \min \|Y - \alpha - X\beta\|^2$$

A closed-form solution exists

In a Riemannian manifold

Let  $X \in \mathbb{R}$  a non-random variable  
and  $Y \in \mathcal{M}$  a random variable

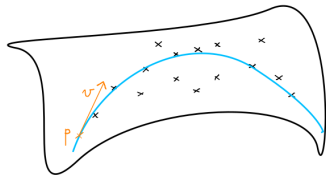
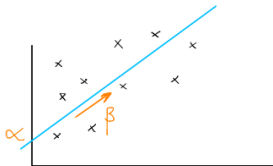
Regression model:  $Y = \text{Exp}(\text{Exp}(p, Xv), \epsilon)$

$p$  is the initial point

$v$  is the initial velocity vector

$$(\hat{p}, \hat{v}) = \arg \min d(\text{Exp}(p, Xv), Y)^2$$

There is usually **NO** closed-form solution



# But where's the regression in all of this?

In an Euclidean space

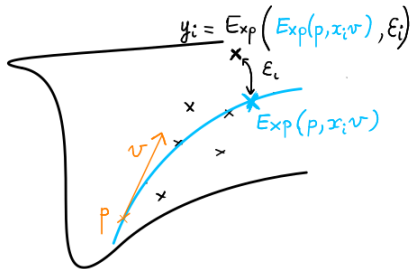
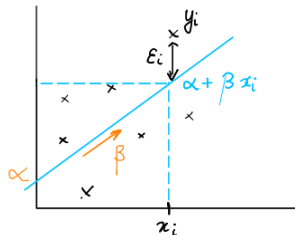
Regression model:  $Y = \alpha + X\beta + \epsilon$

$$(\hat{\alpha}, \hat{\beta}) = \arg \min \|\mathbf{Y} - \alpha - X\beta\|^2$$

In a Riemannian manifold

Regression model:  $Y = \text{Exp}(\text{Exp}(p, Xv), \epsilon)$

$$(\hat{p}, \hat{v}) = \arg \min d(\text{Exp}(p, Xv), Y)^2$$



# "Optimal" curve

- The "optimal" curve of order  $k$  is obtained by computing the parameters that minimise the least squares criterion:

$$E(\gamma) = \frac{1}{N} \sum_{j=1}^N d(\gamma(t_j), y_j)^2 \quad (1)$$

$$\text{under the constraint } \nabla_{\gamma'}^{(k)} \gamma'(t) = \frac{D^{(k)}}{dt} \gamma'(t) = 0 \quad (2)$$

- 1 Cancelling the differential form  $dE$ ,
- 2 We take advantage of the Lie group structure by first solving in  $\mathfrak{g}$ , then "transporting" the solution in  $G$ .

# "Any" connection?

- There is a unique metric-compatible, torsion-free connection called the **Levi-Civita** connection  $\nabla^{LC}$
- For another connection  $\nabla$ : the mapping

$$(t_1, \dots, t_n) \in [0, 1]^n \mapsto \text{Exp} \left( x, \sum_{i=1}^n t_i e_i \right)$$

with  $(e_1, \dots, e_n)$  be a basis of  $T_x \mathcal{M}$ , defines a local coordinate system in a neighborhood of  $x$



# Goodness of fit

We can use (P.T. Fletcher, 2011) a "Riemannian" coefficient of determination  $R^2$ , defined as:

$$R^2 = 1 - \frac{SSE}{\text{Data Fréchet variance}} = 1 - \frac{\sum_j d(\gamma(t_j), y_j)^2}{\min_{y \in \mathcal{M}} \sum_j d(y, y_j)^2}$$

# What's next?

- Local polynomials
- Description of the noise: the Gaussian hypothesis holds only if the noise is "not too large"