Regression on a Lie group with any connection

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Régression sur un groupe de Lie avec une connexion quelconque

55èmes Journées de Statistique de la Sfds
Université de Bordeaux

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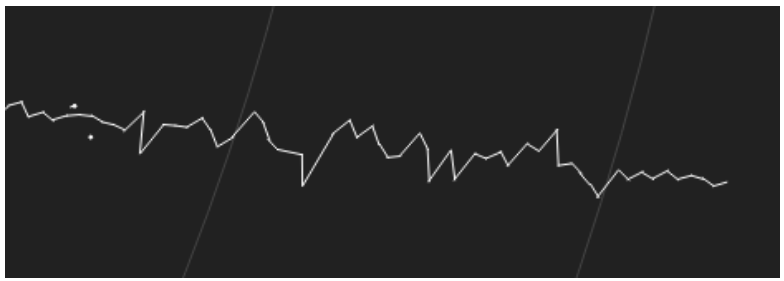
Our application

- Estimating the true position of a mobile object (aircraft, drone, ...) from measured (therefore noisy) raw positions



- This may be difficult with a highly maneuvering target

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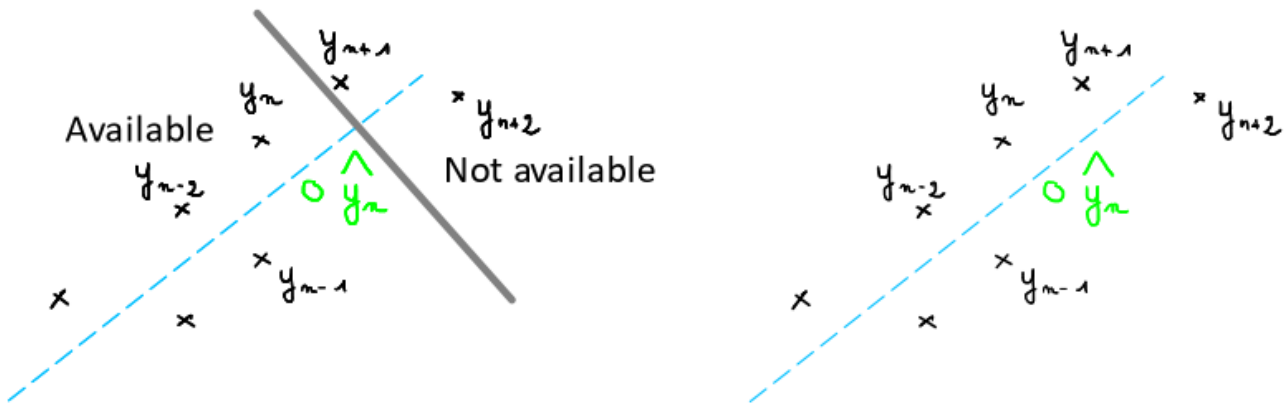
Our application

- Estimating the true position of a mobile object (aircraft, drone, ...) from measured (therefore noisy) raw positions
- Two cases :
  - From past positions: prediction
    - Frequently solved with linear / extended / unscented Kalman filters, or particle filters
  - From past and future positions: smoothing



- What comes next is based on the second case: smoothing

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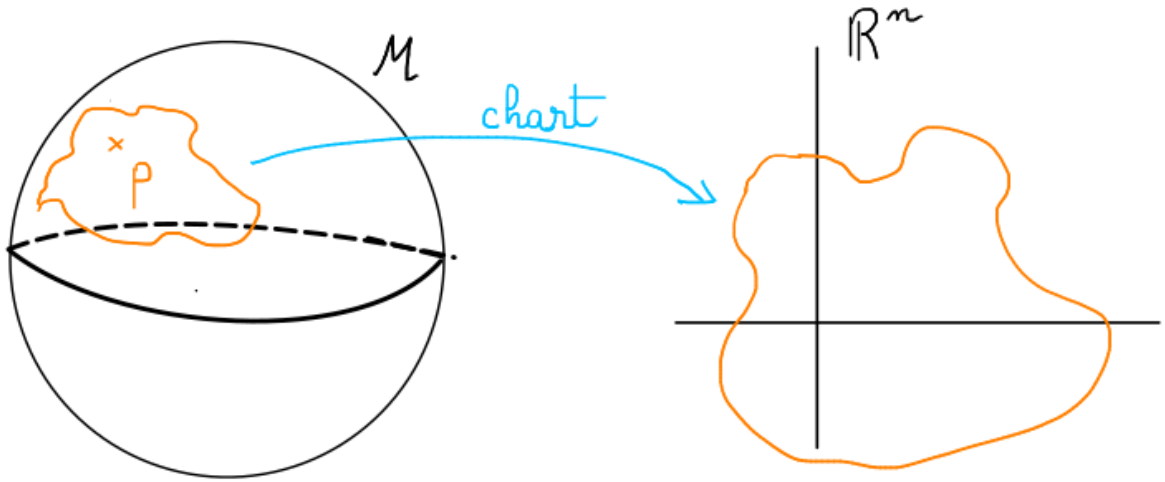
Our framework

- Instead of considering the trajectory in the state space, the movement of the mobile object is considered as a sequence of displacements: rotations and translations in $\mathbb{R}^3$
- This set is called the Special Euclidean group $SE(3)$
- $SE(3)$ is a Lie group, which is both:
  - a differentiable manifold



  - a group in which both the product and inverse maps are smooth

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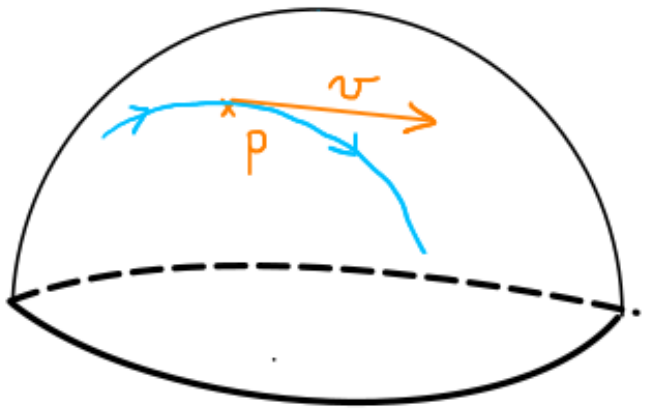
Going off on a tangent (space)

- At any point p of a manifold $\mathcal{M}$, one may define a tangent vector v as the derivative of a curve going through p: $v = c'(0)$ with $c\colon]-\epsilon, \epsilon[\to \mathcal{M}$, $c(0) = p$



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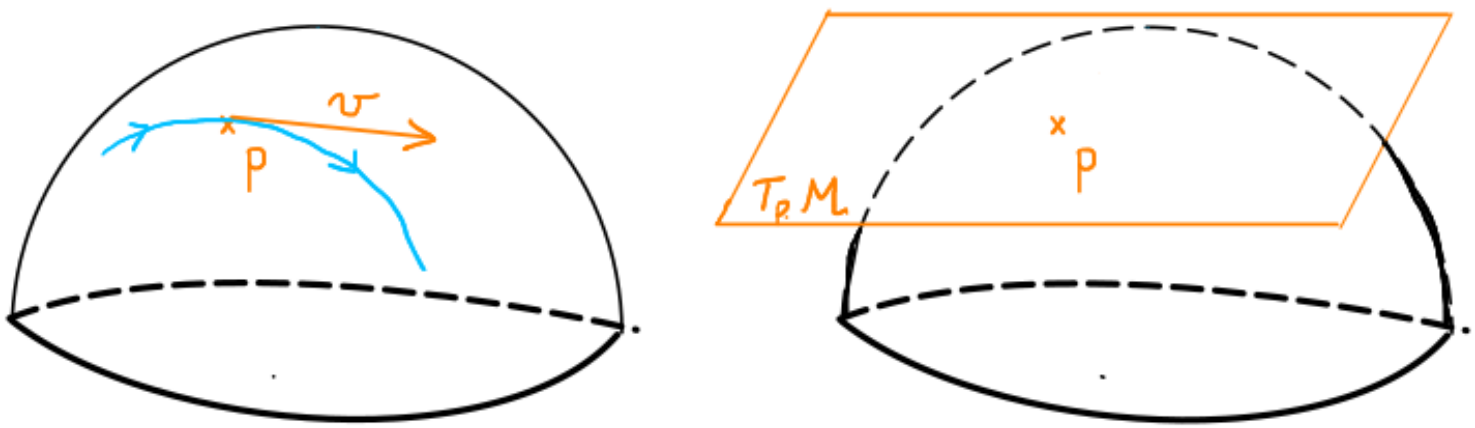
- The set of all tangent vectors at p is a vector space called the tangent space at p denoted $T_p\mathcal{M}$



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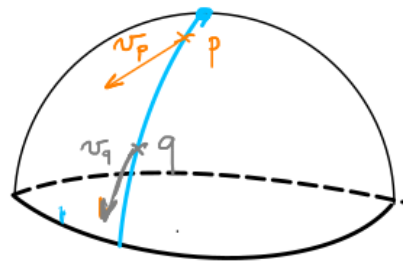
Manifolds and geodesics

- A geodesic in a manifold is the equivalent of a straight line
- A curve γ is a geodesic if it has a "constant" velocity vector
  - Problem: how can we compare a vector $v_p \in T_p\mathcal{M}$ with a vector $v_q \in T_q\mathcal{M}$?



  - A connection ∇ is a rule for taking the directional derivative of a vector field along a tangent vector
  - A curve γ is called a geodesic if

$$\nabla_{\gamma'}\gamma' = \frac{D}{dt}\gamma'(t) = 0$$

(we "parallel-transport" the $\gamma'(t)$ vector along the curve γ)

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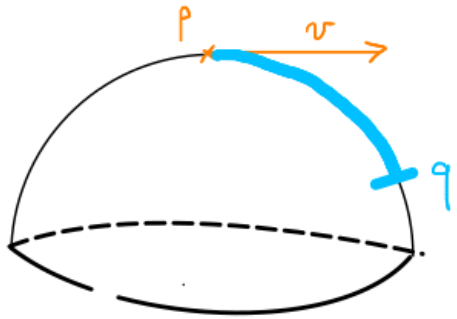
Exponential and logarithm on a manifold

- An initial point $p = \gamma(0)$ and an initial vector $v = \gamma'(0)$ are enough to define (at least locally) a unique geodesic
- This defines a map from the tangent bundle of $\mathcal{M}$ $T\mathcal{M} = \cup_p T_p\mathcal{M}$ to the manifold $\mathcal{M}$ called Exp:

$$\mathrm{Exp}: T\mathcal{M} \to \mathcal{M}$$
$$(p, v) \mapsto q$$



- This is the equivalent of $q = p + \overrightarrow{v}$ in $\mathbb{R}^n$

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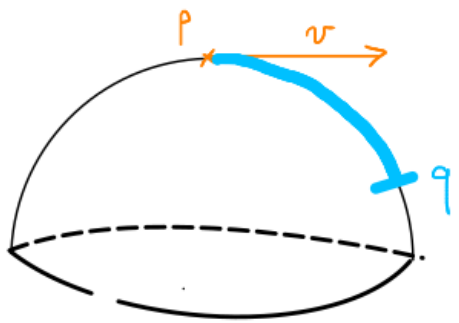
Exponential and logarithm on a manifold

- Likewise, one may define the inverse map called Log:

$$\mathrm{Log}_p : \mathcal{M} \to T_p\mathcal{M}$$
$$q \mapsto v$$



- This is the equivalent of $\overrightarrow{v} = \overrightarrow{pq} = q - p$ in $\mathbb{R}^n$

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How about Lie groups?

- The group structure of a Lie group G makes it possible to bring most operations to the identity e
- The tangent space at e of G is called the Lie algebra of G, denoted $\mathfrak{g}$
  - it "linearises" the group and captures most of its properties

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Lie group: SE(3), Lie algebra: $\mathfrak{se}(3)$

- An element of $SE(3)$ can be represented by a 4-by-4 matrix

$$\begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix}, R \in SO(3), T \in \mathbb{R}^3 :$$

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{pmatrix} = \begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{pmatrix} = \begin{pmatrix} R \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + T \\ 1 \end{pmatrix}$$

- An element of $\mathfrak{se}(3)$ can be represented by a 4-by-4 matrix:

$$\begin{pmatrix} \Omega & U \\ 0 & 0 \end{pmatrix}, \Omega \text{ skew-symmetric}, U \in \mathbb{R}^3$$

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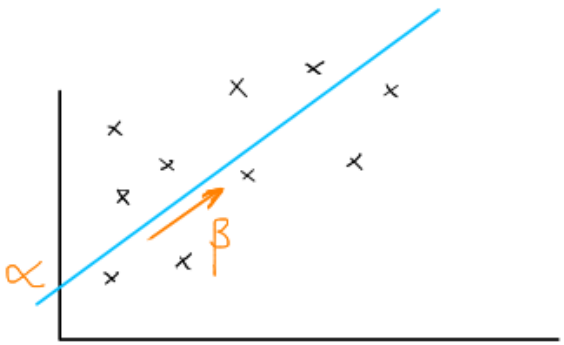
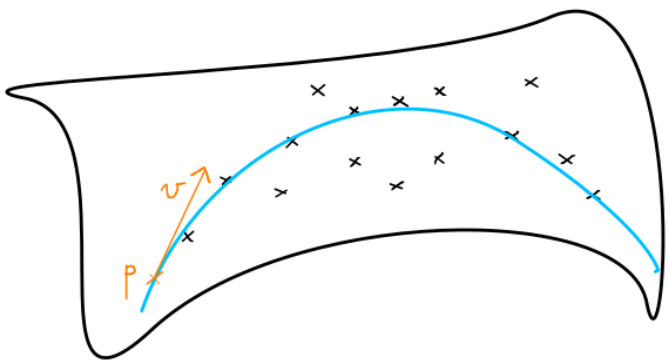
But where's the regression in all of this?

In an Euclidean space	In a Riemannian manifold
Let $X \in \mathbb{R}$ a non-random variable and $Y \in \mathbb{R}^n$ a random variable	Let $X \in \mathbb{R}$ a non-random variable and $Y \in \mathcal{M}$ a random variable
Regression model: $Y = \alpha + X\beta + \epsilon$	Regression model: $Y = \mathrm{Exp}\Big(\mathrm{Exp}(p, Xv), \epsilon\Big)$
α is the intercept β is the slope	p is the initial point v is the initial velocity vector
$(\hat{\alpha}, \hat{\beta}) = \arg\min \lvert\lvert Y - \alpha - X\beta \rvert\rvert^2$	$(\hat{p}, \hat{v}) = \arg\min d\Big(\mathrm{Exp}(p, Xv), Y\Big)^2$
A closed-form solution exists	There is usually **NO** closed-form solution





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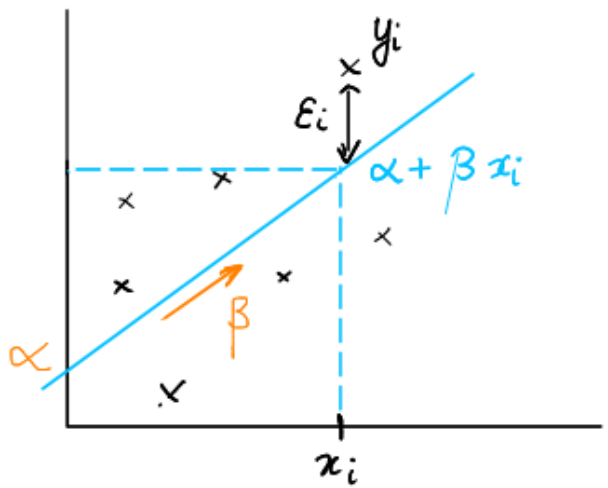

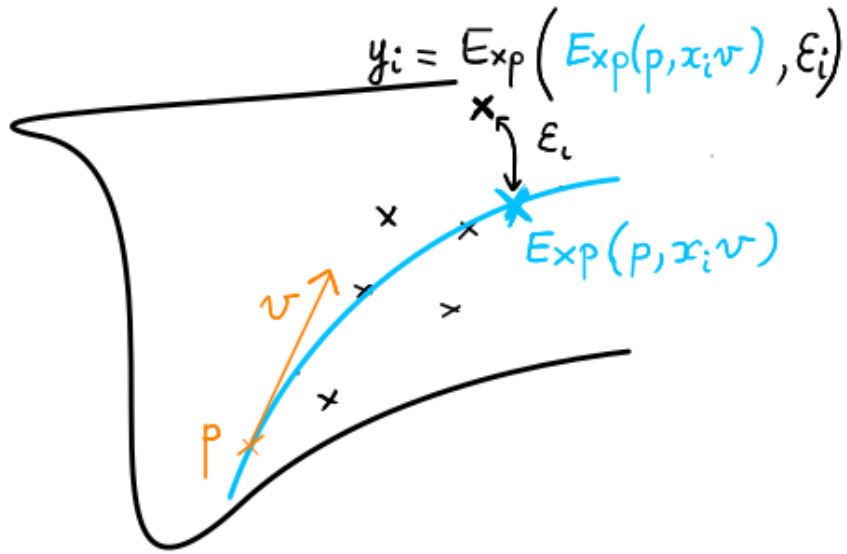

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"Optimal" curve

- The "optimal" curve of order k is obtained by computing the parameters that minimise the least squares criterion:

$$E(\gamma) = \frac{1}{N} \sum_{j=1}^{N} d(\gamma(t_j), y_j)^2 \tag{1}$$

$$\text{under the constraint } \nabla^{(k)}_{\gamma'} \gamma'(t) = \frac{D}{dt}^{(k)} \gamma'(t) = 0 \tag{2}$$

1. Cancelling the differential form dE,
2. We take advantage of the Lie group structure by first solving in $\mathfrak{g}$, then "transporting" the solution in G.

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"Any" connection?

- There is a unique metric-compatible, torsion-free connection called the **Levi-Civita** connection ∇^{LC}
- For another connection ∇: the mapping

$$(t_1, \ldots, t_n) \in [0,1]^n \mapsto \mathrm{Exp}\left(x, \sum_{i=1}^{n} t_i e_i\right)$$

with $(e_1, \ldots, e_n)$ be a basis of $T_x\mathcal{M}$, defines a local coordinate system in a neighborhood of x

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Goodness of fit

We can use (P.T. Fletcher, 2011) a "Riemannian" coefficient of determination R^2, defined as:

$$R^2 = 1 - \frac{SSE}{\text{Data Fréchet variance}} = 1 - \frac{\sum_j d(\gamma(t_j), y_j)^2}{\min_{y \in \mathcal{M}} \sum_j d(y, y_j)^2}$$

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What's next?

- Local polynomials
- Description of the noise: the Gaussian hypothesis holds only if the noise is "not too large"

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