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How to teach geometry in continuity along schooling?

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DISCUSSION AND CONCLUSION

We have analysed extracts from teachers' lesson planning, lessons and their reflections on introducing and defining similar triangles. We notice that the teachers paid attention to LRGT practices of word use (naming and elaborating meanings) and linking these to representations, perhaps attempting to apply what they learnt in PD about LRGT. Of interest are two dilemmas for the teachers in regard to moving between naming concepts (and so the formal register) and representations to be used to bring out the mathematical talk. The first dilemma is whether to ask learners to explore a concept before they define it or define it first then ask learners to explore it. This dilemma relates to moving between the everyday and formal registers of LRT (Hardman, 2021) and confirms the Vygotskian view that language plays an important role on linking everyday concepts to scientific concepts (Vygotsky, 1978). When the teachers started from asking for the everyday meaning of the concept 'similar' then show everyday representations of similar objects, the definition of 'similar objects' remained in the everyday and school mathematics registers. This implies that the idea of starting with everyday language and everyday visual representations only opened learning opportunities for understanding the concept 'similar' in everyday and school mathematics language but did not support linking these to formal language.

The second dilemma which was observed during teaching 1 reflection was about how to link representations and formal language; how/whether to use everyday representations to link to formal language or how/whether to use geometric representations to link to formal language. The teachers had opportunity to realise that the representations used in teaching 1 did not support linking to formal language, and decided to change the representations and how to introduce the concept 'similar'. Although the teachers only thought that the everyday objects were not suitable to start with because they lead to naming unnecessary aspects, we further add that the use of the three dimensional and circular objects complicated the teaching further because these did not help to solve the highlighted challenges of clarifying the critical aspects of proportionality of sides, equal corresponding angles and scale factor in similarity (Seago et al., 2014). This was evidenced in teaching 2, whereby when the teacher introduced 'similarity' with visual geometric representations, the aspect of equal angles emerged and the discussion moved between the formal and school mathematics with clear link between the visuals and the formal language. In later episodes, the geometric objects were also used to introduce the aspect of proportional sides and the teacher ended the lesson by bringing in the everyday representations and learners were able to talk about their similarity as same shape and different sizes. Agreeing with Hardman (2021) that for some concepts, linking the abstract and the everyday is effective when the teacher starts from the formal such as naming and defining similar triangles, later relating these to the everyday, for example showing or naming examples of similar objects. The findings also confirm with literature that the teaching and learning of mathematics involves mediating the complex relationships among linguistic, symbolic, visual forms of representation of mathematical knowledge and in geometry these are crucial for learners understanding (Mwadzaangati et al., 2022; Seago et al., 2014).

The opportunity for teachers' learning that the interaction between the representations and the registers is important in geometry teaching was through LS processes of planning, teaching and reflecting. The dilemmas that the teachers experienced and discussing how to work around these during LS opened the teachers' learning opportunities about LRGT. The teachers' long debates and

dilemmas in these LRGT practices of naming and representing imply that these practices are not easy to deal with in a single LS cycle and confirms the need for continuous PDs in LRT.

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MULTI-PERSPECTIVITY: A ‘RED THREAD’ THROUGH DISCUSSIONS ON GEOMETRY FOR TEACHING AND LEARNING

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What makes geometry so special in school mathematics? — This paper discusses a comprehensive and multi-perspective structure within one can speak about geometry in the context of teaching and learning. It starts with some fundamental aspects providing a general frame. Then, three dimensions organize the field: General views of geometry; approaches to geometry; geometrical activities. Finally, a few remarks are given how to probe the idea of multi-perspectivity.

The 9th ICMI Study of 1995 on “Perspectives on the Teaching of Geometry for the 21st Century” (Mammana & Villani, 1998) contains several chapters (esp. Chap. 6, Chap. 7) on evolutions, changes, trends of geometry such as changes in textbooks, curricula, technology, but also discusses deeper influences caused by the epistemology of mathematics, by the learning sciences, by social changes, etc. Constantly, the issue of the multi-perspectivity of geometry was highlighted, both, for geometry as a mathematical topic, and as a subject of school mathematics. Being so rich in perspectives seems to be characteristic for geometry (cf. Herbst et al., 2018; Graumann et al., 1996).

However, can we tap this *prima facie* vague idea of multi-perspectivity into a coherent, overarching, more or less systematic structure?

The “model” we propose sets a general frame for thinking about geometry in the context of teaching, learning, and education (first section), it states three dimensions to describe multi-perspectivity within a broad spectrum of geometry as a school topic (second section), and finally gives a few remarks to fields of probation of multi-perspectivity (third section).

A GENERAL FRAME FOR EDUCATIONAL THINKING ABOUT GEOMETRY: GUIDELINES AND FUNDAMENTAL ASPECTS

A still remarkable contribution is the British 2001-Report on teaching and learning geometry (The Royal Society & Joint Mathematical Council, 2001). This report does not only proceed – as to expect for a policy document – to a list of comprehensive and action related recommendations, but considers, for that reason, fundamental issues of geometry in schools, esp. for the 11-19 years old. A preface by the chairman, Adrian Oldknow, sets the tone: Being aware of the long history of the topic and of the ubiquity of geometric images, forms, models in daily live, “... geometry should be one of the easiest branches of mathematics to teach. But this is not the case ...” (RS/JMC 2001, p VII). The reasons lie in some “pitfalls”, and these root in the concept highlighted in this paper, multi-perspectivity. Thus, we must be careful. Oldknow points first to the danger of “abundance”:

[Geometry] suffers from an embarrassment of riches in terms of theories, results, techniques and applications. [...] We might refer to this, not unwelcome, problem as one of **abundance** [bold by author]. (RS/JMC 2001, p VIII).

Then, however, choices are necessary (in lessons, in the curriculum), and bring along new dangers:

At one extreme there is a danger of choosing eclectically from this abundance in a way that leads to the teaching of a lot of apparently unconnected ‘bits’. At the other extreme there is a danger of developing a tightly organised body of knowledge which addresses only a very small part of geometry. Our challenge has been to combine breadth with both educational and mathematical coherence – a problem we refer to as **coherence**. (RS/JMC 2001, p VIII)

For curriculum construction, a further problem appears. In their school career students encounter a lot of material in geometry. Besides the personal overload one seeks to avoid by establishing “coherence”, there is also the mission of the educational administration to create for a certain topic a unified content with relations between the lower and the higher Grades. Oldknow says:

We refer to this issue as one of **progression**. (RS/JMC 2001, p VIII).

The three terms abundance, coherence, and progression mark guidelines of educational thinking about geometry. They correspond to the three fundamental aspects described below: “Abundance” is an epistemological issue, since it states characteristics of geometry. “Coherence” appeals to pedagogy, since it is oriented to the personal learning, targets the construction of meaning, shows the longer chains of connections. “Progression” addresses the didactical intention of sensemaking curricula. These three aspects indicate the special focus when talking about a mathematical topic under educational viewpoints. It really makes a difference if we are speaking about geometry in the context of a sub-domain of mathematics, or about geometry as an essential part of mathematics in school, or even more general, about geometry for the purpose of building up a reflected view of “the world”. It therefore deserves attention to separate the aspects to speak about geometry.

The epistemological aspect

Speaking about geometry as a mathematical topic should consider the multi-perspectivity of geometry itself. Thus, studies about historical developments, on the logical foundations, on mathematical standards in geometrical work, etc. are all welcome and fruitful, but in the educational context they always have to take into the consideration the origin of geometry as a human creation and the roots of geometry in human activity, be it cognitive, even mystic, aesthetical, technical, or in relation to the wider environment.

The pedagogical aspect

Geometry as a school subject aims at education. In Germany we call that intention “Bildung” (Neubrand & Lengnink, 2023). “Bildung” does not address the content exhibited in the curriculum alone, but targets wider connections. For any content in school, one must admit the question in how far that content can contribute to human development. The German educator Heinz-Elmar Tenorth coined it as “Kultivierung der Lernfähigkeit” (Tenorth, 1994, S. 94 ff.), i.e. *cultivating the ability to learn*, and fostering the cognitive solution to any issue in our life. In that sense, “Bildung” is a target and a corrective for teaching and learning in school, even with respect to geometry.

The didactical aspect

In the Anglo-Saxon tradition the term Curriculum means more than just the creation of syllabuses in school; in the continental tradition we prefer speaking of “Didaktik” (Westbury et al., 2000; Blum et al., 2019). In both traditions, school geometry is not only a set of contents (as important this is); rather, implementations must contain the use of that specific knowledge in contexts whatsoever

(science, practice and customs in trade, craft, business, industry), a wide range of possible approaches, and the acknowledgment of the experiences of the student.

THREE DIMENSIONS TO ORGANIZE DISCOURSES OF GEOMETRY FOR TEACHING AND LEARNING

To come from these fundamental, but rather abstract aspects (as an underlying frame) to the issues of the daily agenda of geometry in school, and still not falling behind the multi-perspectivity strived for, we differentiate three dimensions to organize discourses on geometry. Each dimension should follow and explicate the three fundamental aspects, i.e. the epistemological, pedagogical, didactical foci. They are to structure multi-perspectivity.

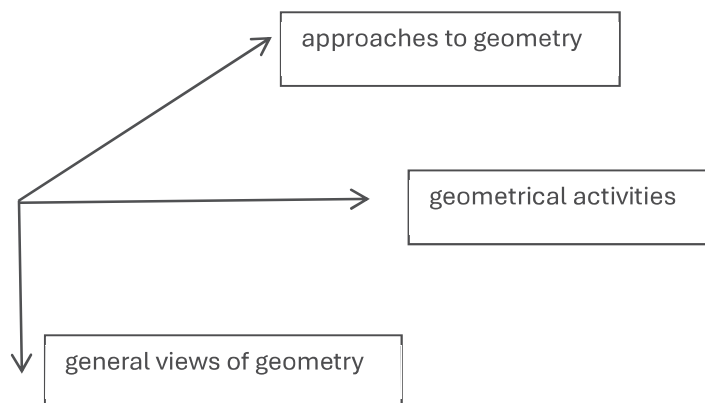


Figure 1: Three dimensions to organize discourses of geometry for teaching and learning

General views of geometry

Under the sub-domains of school mathematics geometry is unique in showing so many views. This opens a wide field of possibilities, still however, aware of the danger of abundance, and recalling that pursuing coherence is always on the agenda. What one can learn is that none of the so many views should be inferior or of less value, and equally, none of the views can stand alone. This stands behind the statement from above that the problem of abundance is “not unwelcome” (RS/JMC 2001, p VIII). The issue of multi-perspectivity of geometry itself can be traced back for long periods; an early source is Artmann (1979), a recent witness of the idea, albeit with another focus, is Kusniak (2018); cf. also the whole book of Herbst et al. (2018).

Here is a list, not claiming to be complete. Geometry can be viewed ...

- ... as a “ready to use” body of knowledge.

Of course, this view is, as a background, present whenever we discuss about geometry (as it would be for all other parts of school mathematics).

- ... as field which gives a blueprint of “doing mathematics”.

In no other field of school mathematics, experiences of doing mathematics seem as accessible as in geometry. There is the long history, but the central reason is that in geometry the way to the abstract theory is not as technically demanding, as in some other fields (say, e.g., calculus). Geometry has a wide range of theories; it could be formal, but there could be also, still strong and serious, theories keeping open the appeal to the practices (Bender & Schreiber, 1980). One even can claim that geometry is in itself a model for mathematics. Benno Artmann (1979) called it by the German word “Vorbild” (literally “preset picture”, meaning something like a “template”); we list later some

mathematical activities having authentic blueprints in geometry. Since one can hold the informal level for long, geometry is a good area for “speaking about mathematics” (Neubrand, 2000).

- ... as a rich source of problems of a big variety.

Geometry opens a lot of possibilities of problems of different characters and wide-ranging difficulties. It goes from puzzles to severe problems. However, geometry problems often claim not only for a local solution but for embedding the problems into wider connections. This, among other reasons, is the potential of the classical (Euclidean) triangle geometry.

- ... as a basis to describe, plan, construct, realize technical equipment.

This is a very specific aspect of geometry. In the real world lots of questions with geometrical background must be tackled: Streets, tunnels, ramps, bridges; buildings; gears; etc. Geometry is not only in drawing and construction but in the deeper questions like stability or directions of forces. Geometry aims at understanding, not just description. Hahn (2012) gives many examples.

- ... as a basis to understand the space we live in.

Going beyond of just constructing practical things, geometry enables us to conceptualize what we see around us. In school we should use the full range of this view, from local orientation (maps, schematic plans), to the geometry which guides us through the environment we live in (the neighborhood, the earth, including weather, spreading of pollution), up to the space.

- ... as a cultural achievement, as a product of the development of mankind.

This is not meant as a source of anecdotal stories alone. It strives what was said before under the pedagogical aspect (“Bildung”): Human development is in no field of school mathematics as clear as in geometry. Geometry is the origin of mathematics in the cultural history. The Latin term “more geometrico” is in the Western culture a metaphor for stringent thinking. But geometry is universal: I just point to Fukagawa & Rothman (2008; Japan) or to Gerdes (2010; Africa).

- ... as a rich supplier of forms for observation, interpretation, creation: visualization.

One can assume that this view of geometry is often unattended even neglected in schools, since there seem to be too less paths into the formal reign of mathematics. The opposite, however, is true. Geometry is a fundament, but also vice versa a product of visualization. Visualization as a concept strives aesthetics (Sinclair et al., 2007), it contains many far-reaching relations to arts (perspectivity, symmetries), it reaches out until the principles of the technical or even the evolutionary development (Hildebrandt & Tromba, 1996). Whatsoever, teaching and learning geometry should exploit that big stock of information, motivation, and launching platforms; see National Research Council (2006) for many suggestions.

Approaches to geometry

Multi-perspectivity in the general views opens multi-perspectivity in the approaches. No single approach fits it all or should be dominant; thus, teachers should be aware of the many possibilities. Each approach has its own dialectic. It can work for some students, or some topics; the didactical situation in the class calls for local decisions. And essentially, the term approach is too narrowly understood if we think about it only as a matter of motivation. The three dimensions influence each other. The approaches are so rich since geometry is so rich of views. The epistemological aspect triggers pedagogical orientations and didactical decisions.

Geometry can be approached (in the class, and equally by the individuals) ...

- ... by the relations to reality.

The condition, however, is to keep open that reality has many categories. It could start with the direct contact to things like the closer or wider environment, technical devices, facts from biology, architecture, etc. However, any approach needs goals going farther than motivation in the beginning. From the pedagogical aspect, an approach by reality should lead to deeper understanding. The German mathematics educator Heinrich Winter took the word “sublimation” to indicate that teaching geometry for “Bildung” should foster the ability to articulate the structural aspects of the world around us, it should make sensible for visual perception, transfer our observations conceptually, including a reflective attitude (Winter, 1997, p 29; my free translation).

- ... by the disclosure of the inner connections, by the wish to master a certain topic.

Too often approaches to geometry are thought of as to come from outside. But approaching can also come from inside. Wishing coherence is a universal human attitude. Thus, the drive to logical order, the wish to explain, in the sequel even to prove something, is not necessarily a sign for a “ready-to-use” geometry but can be guide and generate progress. It requires a metacognitive attitude in the class (Kaune, 2006), which can indeed be well realized by geometrical topics (e.g. Neubrand, 2000: the systematization of the set of quadrilaterals). Similarly, mastering something is human as well. But sheer repetition is not enough, practicing needs connection (see many papers of Erich Wittmann, 2021). It can be quite plain in geometry. Here is an “integrated exercise”: Look at all the various intersection this figure bears. What lengths of segments do appear?

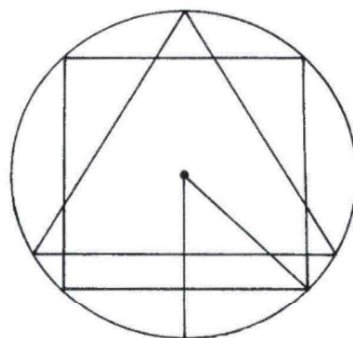


Figure 2. A circle, an equilateral triangle, a square – and many intersections

- ... by using materials, by handling geometrical devices, by measuring with instruments, by reflecting digital systems.

This approach is not a plea to return to the “old” ruler-and-compass times. The essence is that these approaches come via the material manifestation of geometrical concepts. A striking example is the phenomenon of “touching”, a concept reaching far into higher mathematics. Approaching geometry by devices and instruments is, furthermore, not meant as a subordination of digital approaches; rather, it claims for reflections about the differences, the advantages, and the pitfalls of each approach. The idea of “touching” is once more a good example: it calls for construction in the ruler-and-compass world, and often sticks to zoom-in / zoom-out strategies in the digital programs.

- ... through curiosity, exploration, investigation, by seeking for understanding.

Since geometry is as rich in views and contexts as described, individual approaches must be valued. Nothing is more personal than curiosity. Thus, geometry is greatly welcome as an extraordinary field

in which exploration and investigation can be developed within a huge variety of difficulty levels, contexts, interests, etc. This makes geometry special, and in this way one can find the strongest signals that understanding is the final aim in school.

Geometrical activities

The variety of approaches together with the pedagogical idea that learning is idiosyncratic in its nature produce the claim: Geometry becomes vivid if the students do it. There are many ways, but again no single activity marks the king's road. The more activities are done, the more options appear. One cannot do too much, rather it's dangerous to ignore or disregard possible activities.

Geometry allows activities like ...

- ... doing geometry by hand.

Sometimes one considers hands-on activities as the origin of thinking. At least in geometry we have a rich scale for that: folding, cutting, gluing, rolling, assembling, moving, etc. The impetus, then, to reflect these activities leads to understanding. It begins with noticing, it can continue with abbreviations, replacing an action symbolically, using a specific technique, etc., and on each stage with giving reasons for that what was done.

- ... drawing (with mechanical instruments, and within digital systems).

This is geometry's specialty. There should not be no verdict that the one is the more valuable than the other. Both represent geometry on the level of creating visible products and considering their manipulation.

- ... using numbers and calculation, realizing geometry with numbers.

Utilizing numbers, variables, formulas in the geometric context has many facets. It starts as early as with the geometric interpretation of numerical operations, patterns, and graphical manipulations; measurement is the next step. In this view, numbers and formulas express geometrical relations. However, there is also the other way round: Geometrical ideas contribute to the creation of analytic techniques: A "good model" of geometry requires realizations of concepts like location, distance, angles, volumes, and therefore one needs more than just the coordinates to build up Analytic Geometry (and, by that way, the door becomes open to generalize to higher dimensions). Anyway, geometry and numbers form a productive coexistence.

- ... visualization.

The central human activity of seeing, i.e. using the eyes, is specific for geometry. The activity of "look at and see" should be kept open in geometry lessons as long as possible. But then, two sides can be stated: Geometry calls for assuring oneself that the seen is what really happened; from there the road is open into argumentation at various levels. On the other side, geometry provides an arsenal for the active visualizing of facts, relations, operations. I called that double nature of visualization "contemplative" vs. "active" (Neubrand, 1987). In the era of digitalization, visualization becomes ubiquitous, and hence it increasingly plays its role in geometry education. Being aware of aesthetical categories forms a background for all that. (Sinclair et al., 2007).

- ... all the many typical mathematical working activities.

We already pointed to geometry as a "blueprint of mathematics". Here is a list of what can happen authentically when teaching and learning geometry: clarifying of phenomena; ordering; establishing