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Approximation numbers of composition operators on the Hardy and Bergman spaces of the ball and of the polydisk

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Abstract

We give general estimates for the approximation numbers of composition operators on the Hardy space on the ball B_d and the polydisk \mathbb{D}^d and of composition operators on the Bergman space on the polydisk.

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1. Introduction

This work is an attempt to investigate approximation numbers of composition operators on the Hardy space $H^2(\Omega)$ where Ω is an open subset of \mathbb{C}^d , i.e. when we work with d complex variables instead of one. In fact, we will essentially consider the two cases when $\Omega = B_d$ is the unit ball of \mathbb{C}^d endowed with its usual hermitian norm $\|z\| = (\sum_{j=1}^d |z_j|^2)^{1/2}$ and $\Omega = \mathbb{D}^d$ is the unit ball of \mathbb{C}^d endowed with the sup-norm $\|z\|_\infty = \sup_{j=1}^d |z_j|$, that is when Ω is the unit polydisk of \mathbb{C}^d . In order to treat these two cases jointly, we will work in the setting of bounded symmetric domains.

An interesting feature is that the rate of decay of approximation numbers *highly depends on d* , becoming slower and slower as d increases, which might lead to think that no compact composition operators exist for truly infinite-dimensional symbols. We will see in the forthcoming paper [27] that this is not the case.

2. Notation and background

A *bounded symmetric domain* of \mathbb{C}^d is a bounded open convex and circled subset Ω of \mathbb{C}^d such that for every point $a \in \Omega$, there is an involutive bi-holomorphic map $u : \Omega \rightarrow \Omega$ such that a is an isolated fixed point of u (equivalently, $u(a) = a$ and $u'(a) = -id$: see [32, proposition 3-1-1]). For this definition, see [10, definition 16, and theorem 17], or [11, definition 5 and theorem 4]. É. Cartan showed that every bounded symmetric domain of \mathbb{C}^d is homogeneous, i.e. the group of automorphisms of Ω acts transitively on Ω : for every $a, b \in \Omega$, there is an automorphism u of Ω such that $u(a) = b$ (see [32], p. 250). The unit ball B_d and the polydisk \mathbb{D}^d are examples of bounded symmetric domains.

The Shilov boundary S_Ω of such a domain Ω is the smallest closed set $F \subseteq \partial\Omega$ such that $\sup_{z \in \bar{\Omega}} |f(z)| = \sup_{z \in F} |f(z)|$ for every function f holomorphic in some neighbourhood of $\bar{\Omega}$ (see [10, section 4-1]). Since Ω is convex, it is also the Shilov boundary of the algebra of the continuous functions on $\bar{\Omega}$ which are holomorphic in Ω (because every such functions can easily be approximated by a sequence of the former ones: see [15, pp. 152–154]). For example, the Shilov boundary of the bidisk is $S_{\mathbb{D}^2} = \{(z_1, z_2) \in \mathbb{C}^2; |z_1| = |z_2| = 1\}$, whereas, its usual boundary $\partial\mathbb{D}^2$ is $\{(z_1, z_2) \in \mathbb{C}^2; |z_1|, |z_2| \leq 1 \text{ and } |z_1| = 1 \text{ or } |z_2| = 1\}$; for the unit ball B_d , the Shilov boundary is equal to the usual boundary \mathbb{S}^{d-1} [10, section 4-1]. Equivalently (see [10, theorem 33], or [11, theorem 10]), S_Ω is the set of the extreme points of the convex set $\bar{\Omega}$.

If σ is the unique probability measure on S_Ω invariant under the automorphisms u of Ω such that $u(0) = 0$, the *Hardy space* $H^2(\Omega)$ is the space of all complex-valued holomorphic functions f on Ω such that:

$$\|f\|_{H^2(\Omega)} := \left(\sup_{0 < r < 1} \int_{S_\Omega} |f(r\xi)|^2 d\sigma(\xi) \right)^{1/2}$$

(see [17]). It is a Hilbert space (see [16]).

A Schur map, associated with Ω , will be a *non-constant* analytic self-map of Ω into itself. It will be called *truly d -dimensional* if the differential $\varphi'(a) : \mathbb{C}^d \rightarrow \mathbb{C}^d$ is an invertible linear map for at least one point $a \in \Omega$. Then, by the implicit function theorem, and the rank theorem on local conjugation of φ to the map $(z_1, \dots, z_d) \mapsto (z_1, \dots, z_r, 0, \dots, 0)$, this is equivalent to saying that $\varphi(\Omega)$ has non-void interior. We say that the Schur map φ is a *symbol* if it defines a *bounded* composition operator $C_\varphi : H^2(\Omega) \rightarrow H^2(\Omega)$ by $C_\varphi(f) = f \circ \varphi$.

Let us recall that although any Schur function generates a bounded composition operator on $H^2(\mathbb{D}^d)$ when $d = 1$, this is no longer the case as soon as $d \geq 2$, as shown for example by the Schur map $\varphi(z_1, z_2) = (z_1, z_1)$. Indeed (see [2]), if say $d = 2$, taking $f(z) = \sum_{j=0}^n z_1^j z_2^{n-j}$, we see that

$$\|f\|_2 = \sqrt{n+1} \quad \text{while} \quad \|C_\varphi f\|_2 = \|(n+1)z_1^n\|_2 = n+1.$$

The same phenomenon occurs on $H^2(B_d)$ ([28]; see also [7, 8, 12]).

If H is a Hilbert space and $T : H \rightarrow H$ is a bounded linear operator, the *approximation numbers* of T are defined, for $n \geq 1$ by:

$$a_n(T) = \inf_{\text{rank } R < n} \|T - R\|. \tag{2.1}$$

One has $\|T\| = a_1(T) \geq a_2(T) \geq \dots \geq a_n(T) \geq a_{n+1}(T) \geq \dots$, and T is compact if and only if $a_n(T) \xrightarrow{n \rightarrow \infty} 0$.

The approximation numbers have (obviously) the following ideal property: for every bounded linear operators $S, U : H \rightarrow H$, one has:

$$a_n(STU) \leq \|S\| \|U\| a_n(T), \quad n = 1, 2, \dots$$

For an operator $T : H^2(\Omega) \rightarrow H^2(\Omega)$ with approximation numbers $a_n(T) = a_n$, we will introduce the non-negative numbers $0 \leq \gamma_d^-(T) \leq \gamma_d^+(T) \leq \infty$ defined by:

$$\gamma_d^-(T) = \liminf_{n \rightarrow \infty} \frac{\log 1/a_n}{n^{1/d}} \quad \text{and} \quad \gamma_d^+(T) = \limsup_{n \rightarrow \infty} \frac{\log 1/a_n}{n^{1/d}}. \tag{2.2}$$

The relevance of those parameters to the decay of approximation numbers is indicated by the following obvious facts, in which $0 < c \leq C < \infty$ denote constants independent of n :

$$\gamma_d^-(T) > 0 \iff a_n \leq C e^{-c n^{1/d}}, \quad n = 1, 2, \dots \tag{2.3}$$

$$\gamma_d^+(T) < \infty \iff a_n \geq c e^{-C n^{1/d}}, \quad n = 1, 2, \dots \tag{2.4}$$

So, the positivity of $\gamma_d^-(T)$ indicates that a_n is ‘‘small’’ and the finiteness of $\gamma_d^+(T)$ indicates that a_n is ‘‘big’’.

Equivalently, one may define:

$$\beta_d^-(T) = \liminf_{n \rightarrow \infty} (a_{n^d})^{1/n} \quad \text{and} \quad \beta_d^+(T) = \limsup_{n \rightarrow \infty} (a_{n^d})^{1/n}. \tag{2.5}$$

As usual, the notation $A \lesssim B$ means that there is a constant c such that $A \leq c B$ and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

3. Lower bound

The next theorem shows that the approximation numbers of composition operators cannot be very small. We have already seen that in the one-dimensional case in [23]. The important fact here is that this lower bound depends highly on the dimension.

THEOREM 3.1. *Let Ω be a bounded symmetric domain of \mathbb{C}^d and $\varphi : \Omega \rightarrow \Omega$ be a truly d -dimensional Schur map inducing a compact composition operator $C_\varphi : H^2(\Omega) \rightarrow H^2(\Omega)$. Then, for some constants $0 < c \leq C < \infty$, independent of n , we have:*

$$a_n(C_\varphi) \geq c e^{-C n^{1/d}}, \quad \forall n \geq 1,$$

that is $\gamma_d^+(C_\varphi) < \infty$, or $\beta_d^-(C_\varphi) > 0$.

For proving this result, we shall use the following results, the first of which is due to D. Clahane [9, theorem 2.1] (and to B. MacCluer [28] in the particular case of the unit ball B_d).

THEOREM 3.2 (D. Clahane). *Let Ω be a bounded symmetric domain of \mathbb{C}^d and $\varphi: \Omega \rightarrow \Omega$ be a holomorphic map inducing a compact composition operator $C_\varphi: H^2(\Omega) \rightarrow H^2(\Omega)$. Then φ has a unique fixed point $z_0 \in \Omega$ and the spectrum of C_φ consists of 0, and all possible products of eigenvalues of the derivative $\varphi'(z_0)$.*

When φ is truly d -dimensional, 0 cannot be an eigenvalue of C_φ since if $f \circ \varphi = 0$, then f vanishes on $\varphi(\Omega)$ which has a non-void interior, and hence $f \equiv 0$. Note that 1 is an eigenvalue, by taking an empty product of eigenvalues of $\varphi'(z_0)$.

In fact, in our case, we will not need the existence of z_0 , for we will force 0 to be a fixed point by a harmless change of the symbol φ . By the multi-dimensional version of Schwarz’s lemma [30, theorem 8.1.2], $\varphi'(0)$ has norm ≤ 1 if \mathbb{C}^d is normed by taking Ω as open unit ball; it follows that the eigenvalues of $\varphi'(0)$ have modulus ≤ 1 . In fact, when C_φ is compact, its eigenvalues tend to 0 and it follows from Theorem 3.2 that the eigenvalues of $\varphi'(0)$ have modulus < 1 .

One may also add that the spectral radius of $\varphi'(0)$ is always ≤ 1 (for Ω bounded and connected) and has at least an eigenvalue of modulus < 1 if and only if φ is not an automorphism (see [19, Cartan–Caratheodory–Kaup–Wu theorem, page 374]).

LEMMA 3.3. *Let H be a complex Hilbert space and $T: H \rightarrow H$ be a compact operator with eigenvalues $\lambda_1, \dots, \lambda_n, \dots$ arranged so that $(|\lambda_n|)_{n \geq 1}$ is non-increasing. Then, for any $n \geq 1$,*

$$|\lambda_{2n}|^2 \leq a_1 a_n. \tag{3.1}$$

Indeed, it suffices to apply an immediate consequence of Weyl’s inequalities (see [6, page 157]), namely $|\lambda_n| \leq (a_1 \cdots a_n)^{1/n}$, with n changed into $2n$, and square to get

$$|\lambda_{2n}|^2 \leq (a_1 \cdots a_{2n})^{1/n} \leq (a_1^n a_n^n)^{1/n} = a_1 a_n.$$

LEMMA 3.4. *Let N_p be the number of multi-indices $\alpha = (\alpha_1, \dots, \alpha_d)$ such that $|\alpha| = \alpha_1 + \cdots + \alpha_d \leq p$. Then, as p goes to infinity:*

$$N_p \sim \frac{p^d}{d!}. \tag{3.2}$$

Proof. Let n_k be the number of multi-indices $(\alpha_1, \dots, \alpha_d, \alpha_{d+1})$ such that $\alpha_1 + \cdots + \alpha_d + \alpha_{d+1} = k$. We have (see [22, page 498]), classically, for $|t| < 1$:

$$\sum_{p=0}^{\infty} n_p t^p = \left(\sum_{\alpha_1=0}^{\infty} t^{\alpha_1} \right) \cdots \left(\sum_{\alpha_{d+1}=0}^{\infty} t^{\alpha_{d+1}} \right) = \left(\sum_{k=0}^{\infty} t^k \right)^{d+1} = \frac{1}{(1-t)^{d+1}};$$

hence

$$n_p = \binom{d+p}{p}.$$

But $N_p = n_p$, and hence:

$$N_p = \frac{(p+d) \cdots (p+1)}{d!} \sim \frac{p^d}{d!},$$

as claimed.

CLAIM 3.5. We may assume that $\varphi(0) = 0$ and $\varphi'(0)$ is invertible.

Proof. Since φ is truly d -dimensional, there exists $a \in \Omega$ such that $\varphi'(a)$ is invertible. Since Ω is homogeneous, there exist two automorphisms Φ_a and $\Phi_{\varphi(a)}$ of Ω such that $\Phi_a(0) = a$ and $\Phi_{\varphi(a)}[\varphi(a)] = 0$. Set $\psi = \Phi_{\varphi(a)} \circ \varphi \circ \Phi_a$. Then $\psi(0) = 0$. Now, every analytic automorphism Φ of Ω induces a bounded composition operator on $H^2(\Omega)$ and $C_\Phi^{-1} = C_{\Phi^{-1}}$ [9, theorem 3.1]; hence we can write $C_\psi = C_{\Phi_a} \circ C_\varphi \circ C_{\Phi_{\varphi(a)}}$ and it follows that C_ψ , as well as C_φ , is compact. The ideal property of approximation numbers implies that, for $n = 1, 2, \dots$, one has:

$$\left(\|C_{\Phi_a}\| \|C_{\Phi_{\varphi(a)}}\|\right)^{-1} a_n(C_\varphi) \leq a_n(C_\psi) \leq \|C_{\Phi_a}\| \|C_{\Phi_{\varphi(a)}}\| a_n(C_\varphi),$$

so $\gamma_d^-(C_\psi) = \gamma_d^-(C_\varphi)$. Moreover, using the chain rule, we see that $\psi'(0)$ is invertible, since $\varphi'(a)$ is.

Proof of Theorem 3.1. Let μ_1, \dots, μ_d be the eigenvalues of $\varphi'(0)$, with $|\mu_1| \geq \dots \geq |\mu_d|$. As we said above, we have $|\mu_1| < 1$. Set $|\mu_d| = e^{-A}$, with $A > 0$. By Theorem 3.2, the eigenvalues $\lambda_1, \dots, \lambda_n$ of C_φ are the numbers $z_\alpha = \mu_1^{\alpha_1} \dots \mu_d^{\alpha_d}$ rearranged in (modulus) descending order. We claim that:

$$|\lambda_{N_p}| \geq |\mu_d|^p = e^{-Ap}.$$

Indeed, there are N_p numbers $|z_\alpha|$ with $|\alpha| \leq p$ (cf. Lemma 3.4), and all of them are $\geq |\mu_d|^p$ since $|z_\alpha| \geq |\mu_d|^{|\alpha|} \geq |\mu_d|^p$ (because $|\mu_d| \leq 1$). This proves the claim.

If $M_p = \lfloor N_p \rfloor / 2$ where $\lfloor \cdot \rfloor$ stands for the integer part, equation (3.1) gives:

$$e^{-2Ap} \leq |\lambda_{N_p}|^2 \leq |\lambda_{2M_p}|^2 \leq a_1 a_{M_p}.$$

Since $M_p \sim C_d p^d$ in view of Lemma 3.4, inverting this relation and using the monotonicity of the a_n 's clearly give the claimed result.

Remark 3.6. When φ is not supposed to be truly d -dimensional, we may modify the statement as follows: Let $d_\varphi = \max\{\text{rank } \varphi'(a); a \in \Omega\}$ be the dimension of φ . Then there exists $c, C > 0$ such that:

$$a_n(C_\varphi) \geq c e^{-Cn^{1/d_\varphi}}.$$

The proof remains almost identical.

4. An alternative approach for the polydisk and the unit ball

The previous proof of Theorem 3.1 is essentially a ‘‘functional analysis’’ one. It is interesting to give a proof using complex analysis tools instead of functional analysis ones. Moreover, this approach will be useful for the examples in Section 6.

In this section, we restrict ourselves to the polydisk. The same approach works for the unit ball, by using results of B. Berndtsson in [4]. We will make use of the following theorem of P. Beurling [14, p. 285], in which the word *interpolating sequence* refers to the space H^∞ of bounded analytic functions on Ω ($\Omega = \mathbb{D}$ or \mathbb{D}^d), the interpolation constant M_S of the sequence $S = (z_j)$ being the smallest number M such that, for any sequence (w_j) of data satisfying $\sup |w_j| \leq 1$, there exists $f \in H^\infty(\Omega)$ such that $f(z_j) = w_j$ and $\|f\|_\infty \leq M$.

THEOREM 4.1 (P. Beurling). *Let (z_j) be an interpolating sequence in the unit disk \mathbb{D} , with interpolation constant M . Then, there exist analytic functions f_j , $j \geq 1$, on \mathbb{D} such that:*

$$f_j(z_k) = \delta_{j,k} \quad \text{and} \quad \sum_{j=1}^{\infty} |f_j(z)| \leq M, \quad \forall z \in \mathbb{D}.$$

As a consequence, if $A = (a_j)$ and $B = (b_k)$ are interpolating sequences of \mathbb{D} with respective interpolation constants M_A and M_B , their “cartesian product” $(p_{j,k})_{j,k} = ((a_j, b_k))_{j,k}$ is an interpolating sequence, with respect to $H^\infty(\mathbb{D}^2)$, with interpolation constant $\leq M_A M_B$.

The consequence was observed in the paper [5]. Indeed, if (f_j) and (g_k) are P. Beurling’s functions associated to A and B respectively, any sequence $(w_{j,k})$ with $\sup_{j,k} |w_{j,k}| \leq 1$ can be interpolated by the bounded analytic function

$$f(z, w) = \sum_{j,k \geq 1} w_{j,k} f_j(z) g_k(w)$$

which satisfies $\|f\|_\infty \leq M_A M_B$.

Alternatively, in the sequel, we might use the result of [5] on the sufficiency of Carleson’s condition on products of Gleason distances in the case of several variables. But we will stick to the previous approach. We now make use of the following result of [24] which was enunciated in the one-dimensional case, but whose proof works word for word in our new setting; indeed, the space of multipliers of $H^2(\mathbb{D}^d)$ is (isometrically) $H^\infty(\mathbb{D}^d)$ and then one shows that the unconditional basis constant (see [22, chapitre 2, définition II.4], or [1, definition 3.1.4] for the definition) of the sequence $(K_{s_j})_{1 \leq j \leq n}$ of reproducing kernels associated to a finite sequence $S = (s_j)_{1 \leq j \leq n}$ is less than M_S (see also [23]). Also note that the reproducing kernel of $H^2(\mathbb{D}^d)$ is now, for $a = (a_1, \dots, a_d) \in \mathbb{D}^d$:

$$K_a(z_1, \dots, z_d) = \prod_{k=1}^d \frac{1}{(1 - \bar{a}_k z_k)},$$

with $\|K_a\|^2 = \prod_{k=1}^d (1 - |a_k|^2)^{-1}$.

PROPOSITION 4.2. *Let $\varphi: \mathbb{D}^d \rightarrow \mathbb{D}^d$ be a symbol inducing a compact composition operator $C_\varphi: H^2(\mathbb{D}^d) \rightarrow H^2(\mathbb{D}^d)$. Let $u = (u_1, \dots, u_N)$ be a finite sequence of points of \mathbb{D}^d and let $v_j = \varphi(u_j)$, $1 \leq j \leq N$. Assume that the v_j are distinct and let M_u and M_v be the interpolation constant of u and $v = (v_1, \dots, v_N)$, respectively. Then, setting:*

$$\mu_N^2 = \inf_{1 \leq j \leq N} \frac{\|K_{v_j}\|^2}{\|K_{u_j}\|^2} = \inf_{1 \leq j \leq N} \prod_{k=1}^d \frac{1 - |u_{j,k}|^2}{1 - |v_{j,k}|^2},$$

with $u_j = (u_{j,1}, \dots, u_{j,d})$ and $v_j = (v_{j,1}, \dots, v_{j,d})$, one has:

$$a_N(C_\varphi) \geq c' \mu_N M_u^{-1} M_v^{-1} \geq c' \mu_N M_v^{-2}. \tag{4.1}$$

The last inequality $M_u \leq M_v$ is proved as follows: let $\sup |w_j| \leq 1$ and choose $f \in H^\infty$ such that $f(v_j) = w_j$ and $\|f\|_\infty \leq M_v$; then $g = f \circ \varphi \in H^\infty$ and satisfies $\|g\|_\infty \leq M_v$ and $g(u_j) = f(v_j) = w_j$.

Particular case. Assume that $\varphi(z) = (\varphi_1(z_1), \dots, \varphi_d(z_d))$, with analytic functions $\varphi_1, \dots, \varphi_d: \mathbb{D} \rightarrow \mathbb{D}$. Then $v_{j,k} = \varphi_k(u_{j,k})$ and, if we set

$$\mu_{k,N}^2 = \inf_{1 \leq j \leq N} \frac{1 - |u_{j,k}|^2}{1 - |v_{j,k}|^2}$$

and denote by M_{v_k} the interpolation constant of the sequence $(v_{1,k}, \dots, v_{N,k})$, one has:

$$a_N(C_\varphi) \geq c' \prod_{k=1}^d \mu_{k,N} M_{v_k}^{-2}.$$

New proof of Theorem 3.1. It remains to choose u and v and to estimate the parameters of Proposition 4.2. As in the first proof, we may assume that $\varphi(0) = 0$ and that the differential $\varphi'(0)$ is invertible.

Since $\varphi'(0)$ is invertible, there exists $r \in (0, 1)$ and $c > 0$ such that, for any $v \in r\overline{\mathbb{D}}^d$, one may find $u \in \mathbb{D}^d$ with $\varphi(u) = v$ and $\|u\|_\infty \leq c \|v\|_\infty$. For $\mathbf{j} = (j_1, \dots, j_d) \in \{1, \dots, n\}^d$, we set $v_{\mathbf{j}} = (r\omega^{j_1}, \dots, r\omega^{j_d})$ where ω is a primitive n th-root of unity, e.g. $\omega = e^{2i\pi/n}$. The sequence $v = (v_{\mathbf{j}})_{\mathbf{j}}$ has length $N = n^d$. If we set $S = (s_k)_{k=1, \dots, n}$ with $s_k = r\omega^k$, we know [14, p. 284] that $M_S = r^{1-n}$, so that Theorem 4.1 gives us $M_v \leq r^{d(1-n)}$. We now write $v_{\mathbf{j}} = \varphi(u_{\mathbf{j}})$ with $\|u_{\mathbf{j}}\| \leq cr$. Finally,

$$\frac{\|K_{v_{\mathbf{j}}}\|^2}{\|K_{u_{\mathbf{j}}}\|^2} \geq \prod_{k=1}^d (1 - \|u_{\mathbf{j}_k}\|^2) \geq (1 - c^2 r^2)^d.$$

Collecting all those estimates and using (4.1), we obtain:

$$a_{N^d}(C_\varphi) \geq (1 - c^2 r^2)^d r^{2dn-2d} \gtrsim r^{2dn}.$$

Interpolating an arbitrary integer m between two consecutive d -powers, we clearly obtain Theorem 3.1.

5. Upper bounds

We will assume in this section that:

$$\Omega = B_{l_1} \times \dots \times B_{l_N}, \quad \text{with } l_1 + \dots + l_N = d \tag{5.1}$$

is the product of N unit balls. That covers the case of the unit ball of \mathbb{C}^d ($N = 1$) and the case of the polydisk of \mathbb{C}^d ($N = d$ and $l_1 = \dots = l_N = 1$). To save notation, we will assume in the sequel that $N = 2$.

A point $z = (z_j)_{1 \leq j \leq d} \in \Omega$ is of the form $z = (u, v)$ with $u = (u_j)_{1 \leq j \leq l_1}$, $v = (v_j)_{l_1 < j \leq d}$ and $\sum_{j=1}^{l_1} |u_j|^2 < 1$, $\sum_{j=l_1+1}^d |v_j|^2 < 1$. We see that Ω is the unit ball of \mathbb{C}^d equipped with the following norm:

$$\|z\| = \max \left[\left(\sum_{j=1}^{l_1} |u_j|^2 \right)^{1/2}, \left(\sum_{j=l_1+1}^d |v_j|^2 \right)^{1/2} \right], \tag{5.2}$$

where $z = (u, v)$ with $u \in \mathbb{C}^{l_1}$ and $v \in \mathbb{C}^{l_2}$.

The Shilov boundary of Ω is $S_\Omega = S_{l_1} \times S_{l_2}$ and the normalised invariant measure on S_Ω is $\sigma = \sigma_{l_1} \otimes \sigma_{l_2}$ where σ_{l_1} and σ_{l_2} denote respectively the area measure on the hermitian spheres S_{l_1} and S_{l_2} .

The following lemma can be found in [30, p. 16].

LEMMA 5.1. *The monomials e_α , with $e_\alpha(z) = z^\alpha$, form an orthogonal basis of $H^2(\Omega)$. Moreover if $\alpha = (\beta, \gamma)$ with $\beta = (\alpha_1, \dots, \alpha_{l_1})$ and $\gamma = (\alpha_{l_1+1}, \dots, \alpha_d)$, then writing $z = (u, v)$ we have:*

$$\|e_\alpha\|^2 = \int_{S_{l_1} \times S_{l_2}} |u^\beta|^2 |v^\gamma|^2 d\sigma_{l_1}(u) d\sigma_{l_2}(v) = \frac{(l_1 - 1)! \beta!}{(l_1 - 1 + |\beta|)!} \frac{(l_2 - 1)! \gamma!}{(l_2 - 1 + |\gamma|)!}.$$

Therefore, if $f = \sum_\alpha c_\alpha e_\alpha \in H^2(\Omega)$, one has:

$$\|f\|^2 = \sum_\alpha |c_\alpha|^2 \frac{(l_1 - 1)! \beta!}{(l_1 - 1 + |\beta|)!} \frac{(l_2 - 1)! \gamma!}{(l_2 - 1 + |\gamma|)!}.$$

We can now state the main result of that section, in which we set :

$$\|\varphi\|_\infty := \sup_{z \in \Omega} \|\varphi(z)\|.$$

THEOREM 5.2. *Let $\Omega = B_{l_1} \times B_{l_2}$, $d = l_1 + l_2$, and $\varphi: \Omega \rightarrow \Omega$ be a truly d -dimensional Schur map such that $\|\varphi\|_\infty < 1$. Then $\gamma_d^-(C_\varphi) > 0$, that is there exist some constants $0 < c \leq C < \infty$, independent of n , such that:*

$$a_n(C_\varphi) \leq C e^{-cn^{1/d}}, \quad n = 1, 2, \dots \tag{5.3}$$

Equivalently, one has $\beta_d^+(C_\varphi) \leq e^{-c} < 1$.

Proof. Let us set $r = \|\varphi\|_\infty < 1$. Let $f = \sum c_\alpha e_\alpha \in H^2(\Omega)$ with

$$c_\alpha = \widehat{f}(\alpha) \quad \text{and} \quad \|f\|^2 = \sum_\alpha |c_\alpha|^2 \|e_\alpha\|^2 \leq 1. \tag{5.4}$$

Then $C_\varphi f = \sum c_\alpha \varphi^\alpha$.

We approximate C_φ by the N_n -rank operator R defined by

$$Rf = \sum_{|\alpha| \leq n} c_\alpha \varphi^\alpha$$

and we set $g = C_\varphi(f) - R(f)$ as well as $\alpha = (\beta, \gamma)$. We begin with observing that $(l_1 - 1 + p)! / (l_1 - 1)! p! \leq (p + 1)^{l_1 - 1}$ and $(l_2 - 1 + q)! / (l_2 - 1)! q! \leq (q + 1)^{l_2 - 1}$. Since $|c_\alpha| \leq \|e_\alpha\|^{-1}$, we get by Lemma 5.1 and the multinomial formulae

$$\begin{cases} \sum_{|\beta|=p} \frac{p!}{\beta!} |\varphi^\beta(z)|^2 = \left(\sum_{j=1}^{l_1} |\varphi_j(z)|^2 \right)^p \\ \sum_{|\gamma|=q} \frac{q!}{\gamma!} |\varphi^\gamma(z)|^2 = \left(\sum_{j=l_1+1}^d |\varphi_j(z)|^2 \right)^q \end{cases} \tag{5.5}$$

that, setting $p + q = N$:

$$\begin{aligned} \sum_{\substack{|\beta|=p \\ |\gamma|=q}} \|e_\alpha\|^{-2} |\varphi^\alpha(z)|^2 &= \sum_{\substack{|\beta|=p \\ |\gamma|=q}} \frac{(l_1 - 1 + p)!}{\beta! (l_1 - 1)!} \frac{(l_2 - 1 + q)!}{\gamma! (l_2 - 1)!} |\varphi^\beta(z)|^2 |\varphi^\gamma(z)|^2 \\ &\leq (p + 1)^{l_1 - 1} (q + 1)^{l_2 - 1} \left(\sum_{j=1}^{l_1} |\varphi_j(z)|^2 \right)^p \left(\sum_{j=l_1+1}^d |\varphi_j(z)|^2 \right)^q \\ &\leq (p + 1)^{l_1 - 1} (q + 1)^{l_2 - 1} r^{2p} r^{2q} \leq (N + 1)^{l_1 + l_2 - 2} r^{2N}. \end{aligned}$$

We thus have for $z \in \Omega$ the pointwise estimate, by (5.4):

$$|g(z)|^2 \leq \sum_{|\alpha|>n} \|e_\alpha\|^{-2} |\varphi^\alpha(z)|^2 \leq \sum_{N>n} \sum_{p+q=N} (N+1)^{d-2} r^{2N} \leq C_d n^d r^{2n}$$

for all $z \in \Omega$. This now implies $\|(C_\varphi - R)f\|_{H^2} = \|g\|_{H^2} \leq C'_d n^{d/2} r^n$. Hence:

$$\|C_\varphi - R\| \leq C'_d n^{d/2} r^n.$$

Therefore:

$$a_{N_n+1}(C_\varphi) \leq C'_d n^{d/2} r^n.$$

Since $N_n \sim n^d$, we get, with $r < \rho < 1$:

$$a_{n^d}(C_\varphi) \lesssim \rho^n.$$

By interpolating an arbitrary integer N between two integers of the form n^d , we get $a_N(C_\varphi) \lesssim \rho^N$ for arbitrary indices, which ends the proof.

Remark 5.3. It follows that such a Schur map φ with $\|\varphi\|_\infty < 1$ induces a compact (hence bounded) composition operator $C_\varphi : H^2(\Omega) \rightarrow H^2(\Omega)$.

Remark 5.4. We proved in [23, theorem 3.4] (see also [25, theorem 3.14], for another proof) that, in the one-dimensional case ($d = 1$), one has the converse in Theorem 5.2: if $a_n(C_\varphi) \leq C e^{-cn}$ for some constants $0 < c \leq C < \infty$ independent of n , then $\|\varphi\|_\infty < 1$.

Our next result gives an upper bound for approximation numbers of composition operators on $H^2(\mathbb{D}^d)$ when the symbol φ appears as a tensor product of one-dimensional symbols.

THEOREM 5.5. *Let $\varphi_1, \dots, \varphi_d : \mathbb{D} \rightarrow \mathbb{D}$ be d Schur maps inducing compact composition operators on $H^2(\mathbb{D})$, and let $\varphi : \mathbb{D}^d \rightarrow \mathbb{D}^d$ be defined by:*

$$\varphi(z) = (\varphi_1(z_1), \dots, \varphi_d(z_d)), \quad \text{where } z = (z_1, \dots, z_d).$$

Then, for $C_\varphi : H^2(\mathbb{D}^d) \rightarrow H^2(\mathbb{D}^d)$, one has, for every $n \geq 1$:

$$a_n(C_\varphi) \leq \left(2^{d-1} \prod_{j=1}^d \|C_{\varphi_j}\| \right) \inf_{n_1 \cdots n_d \leq n} [a_{n_1}(C_{\varphi_1}) + \dots + a_{n_d}(C_{\varphi_d})].$$

For example, we know [24, theorem 2.4] that if each φ_j has image contained in a polygon, then $a_{n_j}(C_{\varphi_j}) \lesssim e^{-\beta_j \sqrt{n}}$; we get then $a_{n^d}(C_\varphi) \lesssim e^{-\beta \sqrt{n}}$ and, by interpolation, $a_n(C_\varphi) \lesssim e^{-\beta n^{1/(2d)}}$.

Proof of Theorem 5.5. Let us fix n_1, \dots, n_d with $n_1 \cdots n_d \leq n$ and consider, for each $j = 1, \dots, d$, an operator R_j on $H^2(\mathbb{D})$ with $\text{rank} < n_j$ such that $\|C_{\varphi_j} - R_j\| = a_{n_j}(C_{\varphi_j})$.

We define an operator R on $H^2(\mathbb{D}^d)$ by:

$$R(z^\alpha) = R_1(z_1^{\alpha_1}) \cdots R_d(z_d^{\alpha_d}),$$

where $\alpha = (\alpha_1, \dots, \alpha_d)$. This map R has clearly $\text{rank} < n_1 \cdots n_d \leq n$.

Now, fix $f \in H^2(\mathbb{D}^d)$.

One has $C_\varphi(f) - R(f) = \sum_{j=1}^d f_j$, where, writing $f(z) = \sum_{\alpha \in \mathbb{N}^d} c_\alpha z^\alpha$:

$$\begin{aligned} f_j &= \sum_{\alpha^{(j)} \in \mathbb{N}^{d-1}} \sum_{\alpha_j \in \mathbb{N}} c_\alpha \varphi_1(z_1)^{\alpha_1} \cdots \varphi_{j-1}(z_{j-1})^{\alpha_{j-1}} \\ &\quad \times [\varphi_j(z_j)^{\alpha_j} - R_j(z_j^{\alpha_j})] R_{j+1}(z_{j+1}) \cdots R_d(z_d^{\alpha_d}). \end{aligned}$$

Here and below, $\alpha^{(j)}$ stands for $(\alpha_1, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots, \alpha_d)$.

The norm of f_j may be estimated in the following way:

$$\begin{aligned} \|f_j\|_2^2 &= \int_{\mathbb{T}^{d-1}} \int_{\mathbb{T}} \left| \sum_{\alpha_j \in \mathbb{N}} \left[\sum_{\alpha^{(j)} \in \mathbb{N}^{d-1}} c_\alpha \varphi_1(z_1)^{\alpha_1} \cdots \varphi_{j-1}(z_{j-1})^{\alpha_{j-1}} \right. \right. \\ &\quad \times \left. \left. R_{j+1}(z_{j+1}^{\alpha_{j+1}}) \cdots R_d(z_d^{\alpha_d}) \right] [\varphi_j(z_j)^{\alpha_j} - R_j(z_j^{\alpha_j})] \right|^2 dz_j dz^{(j)} \\ &\leq [a_{n_j}(C_{\varphi_j})]^2 \int_{\mathbb{T}^{d-1}} \left[\sum_{\alpha^{(j)} \in \mathbb{N}^{d-1}} c_\alpha \varphi_1(z_1)^{\alpha_1} \cdots \varphi_{j-1}(z_{j-1})^{\alpha_{j-1}} \right. \\ &\quad \times \left. R_{j+1}(z_{j+1}^{\alpha_{j+1}}) \cdots R_d(z_d^{\alpha_d}) \right]^2 dz^{(j)}, \end{aligned}$$

where $z^{(j)}$ stands for $(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_d)$.

Integrating separately with respect to each variable, and using the fact that $\|R_k\| \leq a_{n_k}(C_{\varphi_k}) + \|C_{\varphi_k}\| \leq 2\|C_{\varphi_k}\|$, gives the result.

As another application of the separation of variables, we also get nearly exponential decay for some maps whose image touches the boundary, as was shown in dimension one [25].

THEOREM 5.6. *For every sequence of positive numbers $\varepsilon_n \downarrow 0$, there exists a continuous map $\varphi: \mathbb{D}^d \rightarrow \mathbb{D}^d$, analytic in \mathbb{D}^d , with $\varphi(1, 1, \dots, 1) = (1, 1, \dots, 1)$, such that $C_\varphi: H^2(\mathbb{D}^d) \rightarrow H^2(\mathbb{D}^d)$ is bounded and*

$$a_n(C_\varphi) \lesssim e^{-\varepsilon_n n^{1/d}}.$$

In particular, we can have $C_\varphi \in \bigcap_{p>0} S_p$ with $\varphi(\mathbb{T}^d) \cap \mathbb{T}^d \neq \emptyset$.

Proof. By [25], we can find an analytic self-map $\psi: \mathbb{D} \rightarrow \mathbb{D}$, belonging to the disk algebra $A(\mathbb{D})$, with $\psi(1) = 1$, and such that $a_n(C_\psi) \leq a e^{-n\varepsilon_n}$. Define $\varphi = \psi \oplus \psi \cdots \oplus \psi$ (d times). Then $\varphi: \mathbb{D}^d \rightarrow \mathbb{D}^d$ is continuous on \mathbb{D}^d , analytic in \mathbb{D}^d , and $\varphi(1, 1, \dots, 1) = (1, 1, \dots, 1)$. Theorem 5.5 with $n_1 = \dots = n_d = n$ shows that, if $n^d \leq N < (n+1)^d$, then

$$a_N(C_\varphi) \leq a_{n^d}(C_\varphi) \lesssim a_n(C_\psi) \lesssim e^{-n\varepsilon_n} \lesssim e^{-N^{1/d}\varepsilon_N},$$

completing the proof of Theorem 5.6.

6. Examples

6.1. Multi-lens maps

For $0 < \theta < 1$, the lens map λ_θ of parameter θ is defined by:

$$\lambda_\theta(z) = \frac{(1+z)^\theta - (1-z)^\theta}{(1+z)^\theta + (1-z)^\theta} \tag{6.1}$$

(see [31] or [20]).

Let $\lambda_1 = \lambda_{\theta_1}, \dots, \lambda_d = \lambda_{\theta_d}$ be lens maps of parameters $0 < \theta_1, \dots, \theta_d < 1$. We define a multi-lens map φ on the polydisk \mathbb{D}^d as:

$$\varphi(z_1, \dots, z_d) = (\lambda_1(z_1), \dots, \lambda_d(z_d)), \tag{6.2}$$

for $(z_1, \dots, z_d) \in \mathbb{D}^d$. We write it $\varphi = \lambda_1 \otimes \dots \otimes \lambda_d$.

To save notation, we will assume in the sequel that $\theta_1 = \dots = \theta_d = \theta$, and we will say that the multi-lens map $\varphi = \varphi_\theta$ has parameter θ .

THEOREM 6.1. *Let φ be a multi-lens map with parameter θ . Then, for positive constants a, b, a', b' depending only on θ and d , one has:*

$$a' e^{-b'n^{1/(2d)}} \leq a_n(C_\varphi) \leq a e^{-bn^{1/(2d)}} \tag{6.3}$$

In particular, $\gamma_d^-(C_\varphi) = 0$ even though C_φ belongs to all Schatten classes.

Proof. (1) The upper estimate follows from Theorem 5.5 since the lenses are contained in a polygon.

(2) To prove the lower bound, we will use Theorem 4.1 and Proposition 4.2.

First, the upper estimate implies that C_φ is a compact operator on $H^2(\mathbb{D}^d)$.

Let $\sigma > 0$ and, for $1 \leq j_k \leq n, 1 \leq k \leq d$:

$$u_{j_1, \dots, j_d} = (1 - e^{-j_1\sigma}, \dots, 1 - e^{-j_d\sigma}).$$

Let:

$$v_{j_1, \dots, j_d} = \varphi(u_{j_1, \dots, j_d}) = (\lambda_1(1 - e^{-j_1\sigma}), \dots, \lambda_d(1 - e^{-j_d\sigma})).$$

By (4.1), one has, with $N = n^d$:

$$a_N(C_\varphi) \geq c' \mu_N M_v^{-2}. \tag{6.4}$$

Actually, if

$$\mu_{k,n} = \inf_{1 \leq j_k \leq n} \frac{1 - |1 - e^{-j_k\sigma}|^2}{1 - |\lambda_k(1 - e^{-j_k\sigma})|^2},$$

one has:

$$a_N(C_\varphi) \geq c' \prod_{1 \leq k \leq d} \mu_{k,n} M_v^{-2}.$$

On the other hand, if $M_{k,v}$ is the interpolation constant of the sequence

$$(\lambda_k(1 - e^{-\sigma}), \dots, \lambda_k(1 - e^{-n\sigma})),$$

of points of \mathbb{D} , one has $M_v \leq M_{1,v} \dots M_{d,v}$, by Theorem 4.1; hence:

$$a_N(C_\varphi) \geq c' \prod_{1 \leq k \leq d} \mu_{k,n} M_{k,v}^{-2}.$$

But we proved in [26, see the proof of Proposition 2] that, for a suitable choice of σ (namely $\sigma = \frac{1}{\sqrt{n}}$), we have:

$$\mu_{k,n} M_{k,v}^{-2} \gtrsim e^{-\beta\sqrt{n}}$$

for some constant $\beta > 0$ depending only on θ . We get hence:

$$a_N(C_\varphi) \gtrsim e^{-\beta d \sqrt{n}}.$$

Since $N = n^d$, we get, by interpolation, that, for every $N \geq 1$:

$$a_N(C_\varphi) \gtrsim e^{-\beta d N^{1/(2d)}},$$

and that ends the proof of Theorem 6.1.

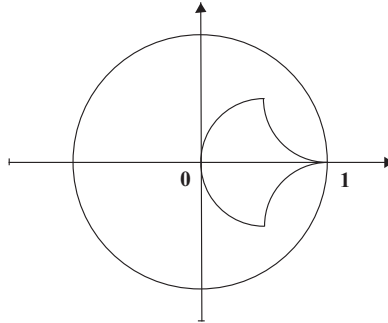


Fig. 1. Cusp map domain.

6.2. Multi-cusp map

We refer to [24, definition 4.1] for the definition of the cusp map. In short, it is the conformal map χ from the open unit disk onto the domain limited by three half-circles represented on Fig. 1, with $\chi(1) = 1$, $\chi(-1) = 0$ and $\chi(i) = (1 + i)/2$. We proved in [24, theorem 4.3], that $e^{-bn/\log n} \lesssim a_n(C_\chi) \lesssim e^{-an/\log n}$ for $n = 2, 3, \dots$, for some constants $b > a > 0$. It follows from Theorem 5.5 that if we define $\varphi: \mathbb{D}^d \rightarrow \mathbb{D}^d$ by $\varphi(z) = (\chi(z_1), \dots, \chi(z_d))$, then $a_n(C_\varphi) \leq C_d e^{-ad n^{1/d}/\log n}$.

THEOREM 6.2. *let $\chi: \mathbb{D} \rightarrow \mathbb{D}$ be the cusp map and $\varphi: \mathbb{D}^d \rightarrow \mathbb{D}^d$ defined by $\varphi(z) = (\chi(z_1), \dots, \chi(z_d))$. Then:*

$$c_d e^{-bn^{1/d}/\log n} \leq a_n(C_\varphi) \leq C_d e^{-an^{1/d}/\log n},$$

where $b > a > 0$ are absolute constants and $c_d < C_d$ depend only on d .

Proof. The upper bound is discussed above. Let us prove the lower bound.

As in [24, p. 556], we fix $0 < \sigma < 1$ and define inductively $u_j \in [0, 1)$ by $u_0 = 0$ and the relation

$$1 - \varphi(u_{j+1}) = \sigma[1 - \varphi(u_j)] \quad \text{with } 1 > u_{j+1} > u_j$$

(using the intermediate value theorem).

Setting $v_j = \varphi(u_j)$, we have $-1 < v_j < 1$, $(1 - v_{j+1})/(1 - v_j) = \sigma$, and $1 - v_n = \alpha \sigma^n$, with $\alpha = 1 - \varphi(0)$. We proved (see also [26, section 2.3, example 2]) that, if M_v is the interpolation constant of the sequence $v = (v_1, \dots, v_n)$, one has, for n large enough:

$$\frac{1 - |u_j|^2}{1 - |v_j|^2} M_v^{-1} \geq K \exp\left(-\frac{25n}{\log n}\right)$$

for all $j = 1, \dots, n$.

Using Proposition 4.2 for the sequence of $(u_{j_1}, \dots, u_{j_d})$, $1 \leq j_1, \dots, j_d \leq n$, we get that:

$$a_n^d(C_\varphi) \geq K^d \exp\left(-\frac{25dn}{\log n}\right).$$

By interpolation, we obtain:

$$a_n(C_\varphi) \geq c_d \exp\left(-\frac{25d^2 n^{1/d}}{\log n}\right)$$

for all $n \geq 1$.

6.3. A different example

Let $a, b > 0$ such that $a + b \leq 1$. Let us set $z = (z_1, z_2)$ and let $\varphi: \mathbb{D}^2 \rightarrow \mathbb{D}^2$ defined by:

$$\varphi(z) = (az_1 + bz_2, 0).$$

In this simple case, we are able to give a precise estimate of the approximation numbers.

THEOREM 6.3. *One has:*

$$c \frac{(a + b)^n}{n^{1/4}} \leq a_n(C_\varphi) \leq C \frac{(a + b)^n}{n^{1/4}},$$

where $0 < c < C < \infty$ are numerical constants.

In particular, if $a + b = 1$, the operator C_φ is compact, but is in the Schatten class S_p if and only if $p > 4$.

We will use two simple lemmas.

LEMMA 6.4. *The functions $f_n(z) = (az_1 + bz_2)^n$ are orthogonal with respect to the Haar measure $dm_2(z)$.*

Proof. Indeed, f_n is an homogeneous polynomial of degree n and homogeneous polynomials of different degrees are clearly orthogonal in $H^2(\mathbb{D}^d)$.

LEMMA 6.5. *One has:*

$$I_n := \int |az_1 + bz_2|^{2n} dm_2(z) \approx \frac{(a + b)^{2n}}{\sqrt{n}}.$$

Proof. We write:

$$I_n \approx \int_{-\pi}^{\pi} (a^2 + b^2 + 2ab - 2ab(1 - \cos t))^n dt$$

and, seeing $1 - \cos t$ as t^2 at the neighborhood of 0, we get:

$$\begin{aligned} I_n &\approx (a + b)^{2n} \int_0^{\pi} (1 - 2ab(a + b)^{-2}t^2)^n dt \\ &\approx (a + b)^{2n} \int_0^{\infty} e^{-\delta nt^2} dt \approx (a + b)^{2n} n^{-1/2}, \end{aligned}$$

with $\delta = 2ab(a + b)^{-2}$.

Proof of Theorem 6.3.

Lower bound. Recall that if $T: X \rightarrow Y$ is an operator between two Banach spaces X and Y , the n -th Bernstein number $b_n(T)$ of T , is defined as:

$$b_n(T) = \sup_{\substack{E \subset X \\ \dim E = n}} \inf_{x \in S_E} \|Tx\|, \tag{6.5}$$

where $S_E = \{x \in E; \|x\| = 1\}$ is the unit sphere of E , and that $a_n(T) = b_n(T)$ if X and Y are Hilbert spaces [29, theorem 2.1].

Let E be the $(n + 1)$ -dimensional subspace of $H^2(\mathbb{D}^2)$ generated by the functions $1, z_1, \dots, z_1^n$. Let $f \in E$ such that $\|f\| = 1$. If we show that $\|C_\varphi(f)\| \geq \delta(a + b)^n/n^{1/4}$, we get that the $(n + 1)$ -th Bernstein number $b_{n+1}(C_\varphi)$ is $\geq \delta(a + b)^n/n^{1/4}$. Since $H^2(\mathbb{D}^2)$ is a Hilbert space, one will get $a_n(C_\varphi) = b_n(C_\varphi) \geq \delta'(a + b)^n/n^{1/4}$.

Now writing $f(z) = \sum_{k=0}^n a_k z_1^k$, one has $1 = \|f\|^2 = \sum_{k=0}^n |a_k|^2$. On the other hand, $[C_\varphi(f)](z) = \sum_{k=0}^n a_k (az_1 + bz_2)^k$, so that, using Lemma 6.4:

$$\|C_\varphi(f)\|^2 = \sum_{k=0}^n |a_k|^2 \int_{\mathbb{T}^2} |az_1 + bz_2|^{2k} dz_1 dz_2 \gtrsim \sum_{k=0}^n |a_k|^2 \frac{(a+b)^{2n}}{n^{1/2}} = \frac{(a+b)^{2n}}{n^{1/2}},$$

by Lemma 6.5.

Upper bound. Let $f(z) = \sum_{k=0}^\infty a_{j,k} z_1^j z_2^k \in H^2(\mathbb{D}^2)$ such that $\|f\| = 1$; then:

$$[C_\varphi(f)](z) = \sum_{j=0}^\infty a_{j,0} (az_1 + bz_2)^j.$$

Let

$$(Rf)(z) = \sum_{j=0}^n a_{j,0} (az_1 + bz_2)^j.$$

This operator R has rank $\leq n + 1$ and, using Lemma 6.4 and Lemma 6.5:

$$\begin{aligned} \|C_\varphi(f) - R(f)\|^2 &= \sum_{j=n+1}^\infty |a_{j,0}|^2 \int_{\mathbb{T}^2} |az_1 + bz_2|^{2j} dz_1 dz_2 \\ &\lesssim \sum_{j=n+1}^\infty |a_{j,0}|^2 \frac{(a+b)^{2n}}{n^{1/2}} \leq \frac{(a+b)^{2n}}{n^{1/2}}; \end{aligned}$$

hence $a_n(C_\varphi) \lesssim (a+b)^n/n^{1/4}$.

Remark 6.6. The same proof works word for word in upper dimensions. Precisely: If $\varphi: z \in \mathbb{D}^d \mapsto (a_1 z_1 + \dots + a_d z_d, 0, \dots, 0) \in \mathbb{D}^d$ with $a_1, \dots, a_d > 0$ and $a_1 + \dots + a_d \leq 1$, then

$$a_n(C_\varphi) \approx \frac{(a_1 + \dots + a_d)^n}{n^{(d-1)/4}}.$$

Indeed, Lemma 6.5 then holds with the right-hand side equal to $\frac{(a_1 + \dots + a_d)^{2n}}{n^{(d-1)/2}}$ (see [13]).

7. An upper bound on the Bergman space

7.1. A general upper bound

In this section, we give an upper bound for “less trivial maps”, which is valid on the Bergman space $A^2(\mathbb{D}^d)$. Let us recall that $A^2(\mathbb{D}^d)$ is the space of all analytic functions $f: \mathbb{D}^d \rightarrow \mathbb{C}$ such that:

$$\int_{\mathbb{D}^d} |f(z)|^2 dm_d(z) < \infty,$$

where m_d is the normalised Lebesgue measure on \mathbb{D}^d . It is a Hilbert space with the norm defined by:

$$\|f\|_{A^2(\mathbb{D}^d)} = \left(\int_{\mathbb{D}^d} |f(z)|^2 dm_d(z) \right)^{1/2}.$$

Let us also recall some other definitions. If H is a Hilbert space of functions on some

measurable set Ω in \mathbb{C} , we say that a nonnegative Borel measure μ on Ω is a Carleson measure for H if there exists a positive constant C such that

$$\int_{\Omega} |f(z)|^2 d\mu(z) \leq C \|f\|_H^2$$

for every f in H . The smallest possible C in this inequality is called the Carleson norm of μ with respect to H . We denote it by $\|\mu\|_{C,H}$ or, when the context is clear, $\|\mu\|_C$.

Carleson measures on $A^2(\mathbb{D}^d)$ have been characterised by Hastings [18]. For any $\xi \in \mathbb{T}^d$ and any $\bar{\delta} = (\delta_1, \dots, \delta_d) \in (0, 2]^d$, we define

$$S(\xi, \bar{\delta}) = \{(z_1, \dots, z_d) \in \mathbb{D}^d; |z_k - \xi_k| < \delta_k, 1 \leq k \leq d\}.$$

The set $S(\xi, \bar{\delta})$ will be called a *Carleson box* at ξ . Then μ is a Carleson measure for $A^2(\mathbb{D}^d)$ if and only if, there is a constant $\kappa > 0$ such that, for any $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{T}^d$ and any $\bar{\delta} = (\delta_1, \dots, \delta_d) \in (0, 2]^d$, one has:

$$\mu(S(\xi, \bar{\delta})) \leq \kappa \delta_1^2 \cdots \delta_d^2. \tag{7.1}$$

Moreover, there exists some constant β_d depending only on d such that:

$$\|\mu\|_{C,A^2(\mathbb{D}^d)} \leq \beta_d \kappa.$$

Let now φ be a Schur map of \mathbb{D}^d and let us associate to it the following measure on \mathbb{D}^d :

$$\mu_{\varphi}(E) = m_d(\{z \in \mathbb{D}^d; \varphi(z) \in E\}).$$

Then φ induces a bounded composition operator C_{φ} on $A^2(\mathbb{D}^d)$ if and only if μ_{φ} is a Carleson measure for $A^2(\mathbb{D}^d)$, and then $\|C_{\varphi}\|^2 = \|\mu_{\varphi}\|_{C,A^2(\mathbb{D}^d)}$. Similarly, the compactness of C_{φ} may be characterised using the notion of vanishing Carleson measures, meaning that we can replace the big-oh condition in (7.1) by a small-oh condition.

Our aim in this section is to show that we can get upper bounds for the approximation numbers of C_{φ} (acting on $A^2(\mathbb{D}^d)$) if we can control the range of C_{φ} and “how much Carleson” μ_{φ} is. We need to introduce some terminology. Let $\rho \in (0, 1)$. We say that B is a Blaschke product on $\rho\mathbb{D}$ if it is a Blaschke product with zeros in $\rho\mathbb{D}$ and modulus 1 on $\rho\mathbb{T}$. In other words, if $B(\cdot\rho)$ is a usual Blaschke product. We define, for $j = 1, \dots, d$:

$$\mu_{\varphi,j,\rho}(E) = m_d(\{z \in \mathbb{D}^d; \varphi(z) \in E \text{ and } |\varphi_j(z)| > \rho\}). \tag{7.2}$$

Now, we have the following result.

THEOREM 7.1. *For $j = 1, \dots, d$, let $\eta_j \in (0, 1/2)$, let B_j be a Blaschke product on $(1 - \eta_j)\mathbb{D}$ with degree p_j and let $\Omega_j = \{\varphi_j(z); |\varphi_j(z)| \leq 1 - 2\eta_j\}$. Then:*

$$a_{p_1 \cdots p_{d+1}}(C_{\varphi}) \lesssim \sum_{j=1}^d \left(\frac{1}{\eta_j^2} \sup_{w \in \Omega_j} |B_j(w)|^2 \|C_{\varphi}\|^2 + \|\mu_{\varphi,j,1-2\eta_j}\|_{C,A^2(\mathbb{D}^d)} \right)^{1/2}.$$

Proof. In the course of the proof, we will just write dz or dw to designate the Lebesgue measure of the ambient hermitian space. Let K_j be the subspace of $A^2(\mathbb{D})$ of functions which are divisible by B_j and let P_j be the orthogonal projection from $A^2(\mathbb{D})$ onto $A^2(\mathbb{D}) \ominus K_j$. The map P_j has rank p_j . Let:

$$\begin{cases} P(z^{\alpha}) = P_1(z_1^{\alpha_1}) \cdots P_d(z_d^{\alpha_d}) \\ R = C_{\varphi} \circ P. \end{cases}$$

These maps have rank $p_1 \cdots p_d$. Write $f - P(f) = \sum_{j=1}^d f_j$ where:

$$f_j(z) = \sum_{\alpha^{(j)} \in \mathbb{N}^{d-1}} \sum_{\alpha_j \in \mathbb{N}} c_\alpha z_1^{\alpha_1} \cdots z_{j-1}^{\alpha_{j-1}} (z_j^{\alpha_j} - P_j(z_j^{\alpha_j})) P_{j+1}(z_{j+1}^{\alpha_{j+1}}) \cdots P_d(z_d^{\alpha_d}).$$

Observe that $\|f_j\|_{A^2(\mathbb{D}^d)} \leq \|f\|_{A^2(\mathbb{D}^d)}$.

For almost all $z^{(j)} \in \mathbb{D}^{d-1}$, the function $f_j(z_1, \dots, z_{j-1}, \cdot, z_{j+1}, \dots, z_d)$ is well-defined on \mathbb{D} and belongs to $A^2(\mathbb{D})$. Moreover, it may be factorised by B_j for $|z| \leq 1 - \eta_j$ (observe that B_j is in general not holomorphic in the whole of \mathbb{D}):

$$f_j(z_1, \dots, z_{j-1}, \cdot, z_{j+1}, \dots, z_d) = B_j(\cdot) g_j(z_1, \dots, z_{j-1}, \cdot, z_{j+1}, \dots, z_d).$$

For a fixed value of $z^{(j)}$, the functions

$$f_j(z_1, \dots, z_{j-1}, \cdot, z_{j+1}, \dots, z_d) \quad \text{and} \quad g_j(z_1, \dots, z_{j-1}, \cdot, z_{j+1}, \dots, z_d)$$

have the same modulus on $(1 - \eta_j)\mathbb{T}$. Hence, if $|z_j| = 1 - \eta_j$, using the norm of the pointwise evaluation in the Bergman space (or just subharmonicity), we get:

$$\begin{aligned} |g_j(z_1, \dots, z_{j-1}, z_j, z_{j+1}, \dots, z_d)|^2 &= |f_j(z_1, \dots, z_{j-1}, z_j, z_{j+1}, \dots, z_d)|^2 \\ &\lesssim \frac{1}{\eta_j^2} \int_{\mathbb{D}} |f_j(z_1, \dots, z_{j-1}, w, z_{j+1}, \dots, z_d)|^2 dw. \end{aligned} \tag{7.3}$$

By the maximum modulus principle with respect to $z_j, z^{(j)}$ remaining fixed, we get the same upper bound for $|z_j| \leq 1 - \eta_j$.

We now write $\|C_\varphi(f) - R(f)\| \leq \sum_{j=1}^d \|f_j \circ \varphi\|$. To control $\|f_j \circ \varphi\|$, we write:

$$\begin{aligned} \int_{\mathbb{D}^d} |f_j \circ \varphi(z)|^2 dz &= \int_{\{z \in \mathbb{D}^d; |\varphi_j(z)| \leq 1 - 2\eta_j\}} |f_j \circ \varphi(z)|^2 dz \\ &\quad + \int_{\{z \in \mathbb{D}^d; |\varphi_j(z)| > 1 - 2\eta_j\}} |f_j \circ \varphi(z)|^2 dz. \end{aligned}$$

The second integral is easily estimated using the Carleson norm of $\mu_{\varphi, j, 1 - 2\eta_j}$. For the first integral, (7.3) yields that, for any $z \in \mathbb{D}^d$ with $|\varphi_j(z)| \leq 1 - 2\eta_j$:

$$\begin{aligned} |f_j \circ \varphi(z)|^2 &\lesssim \sup\{|B_j(w)|^2; w \in \Omega_j\} \\ &\quad \times \frac{1}{\eta_j^2} \int_{\mathbb{D}} |f_j(\varphi_1(z), \dots, \varphi_{j-1}(z), w, \varphi_{j+1}(z), \dots, \varphi_d(z))|^2 dw. \end{aligned}$$

Fix for a while $w \in \mathbb{D}$ and define:

$$\begin{cases} h_{j,w}(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_d) = f_j(z_1, \dots, z_{j-1}, w, z_{j+1}, \dots, z_d) \\ \psi(z) = (\varphi_1(z), \dots, \varphi_{j-1}(z), \varphi_{j+1}(z), \dots, \varphi_d(z)). \end{cases}$$

We claim that $\mu_\psi = m_d(\{z \in \mathbb{D}^d; \psi(z) \in E\})$ is a Carleson measure for $A^2(\mathbb{D}^{d-1})$ and that:

$$\|\mu_\psi\|_{C, A^2(\mathbb{D}^{d-1})} \lesssim \|\mu_\varphi\|_{C, A^2(\mathbb{D}^d)}.$$

Indeed, let $\xi \in \mathbb{T}^{d-1}$ and $\bar{\delta} \in (0, 2]^{d-1}$ and define:

$$\xi' = (\xi_1, \dots, \xi_{j-1}, 1, \xi_{j+1}, \dots, \xi_d) \quad \text{and} \quad \bar{\delta}' = (\delta_1, \dots, \delta_{j-1}, 2, \delta_{j+1}, \dots, \delta_d).$$

Then (setting $\mu_\psi = m_d \circ \psi^{-1}$ as above)

$$\mu_\psi(S(\xi, \bar{\delta})) = \mu_\varphi(S(\xi', \bar{\delta}')) \leq 4\kappa_\varphi \delta_1^2 \cdots \delta_{j-1}^2 \delta_{j+1}^2 \cdots \delta_d^2.$$

This means that:

$$\begin{aligned} \int_{\mathbb{D}} \int_{\mathbb{D}^d} |f_j(\varphi_1(z), \dots, \varphi_{j-1}(z), w, \varphi_{j+1}(z), \dots, \varphi_d(z))|^2 dz dw \\ = \int_{\mathbb{D}} \int_{\mathbb{D}^d} |h_{j,w}(\psi(z))|^2 dz dw \\ \lesssim \|C_\varphi\|^2 \int_{\mathbb{D}} \int_{\mathbb{D}^{d-1}} |h_{j,w}(z)|^2 dz dw \\ \lesssim \|C_\varphi\|^2 \int_{\mathbb{D}} \int_{\mathbb{D}^{d-1}} |f_j(z_1, \dots, z_{j-1}, w, z_{j+1}, \dots, z_d)|^2 dz dw \\ = \|C_\varphi\|^2 \|f_j\|^2. \end{aligned}$$

We finally conclude by recalling that $\|f_j\|_{A^2(\mathbb{D}^d)} \leq \|f\|_{A^2(\mathbb{D})}$.

7.2. Applications

We shall now apply Theorem 7.1 to several examples. We shall need a couple of lemmas. The first one is [3, lemma 23]. If Γ is a curve contained in \mathbb{D} , we denote by $L_p(\Gamma)$ its hyperbolic length. Recall that, by definition, $L_p(\Gamma) = 2 \int_{\Gamma} \frac{|dz|}{1 - |z|^2}$.

LEMMA 7.2. *Let Ω be a bounded domain in \mathbb{D} whose boundary is a piecewise regular Jordan curve Γ , with $L_p(\Gamma) \geq 1$. Let s_1, \dots, s_n be points in Γ such that the hyperbolic length of the curve between any two points s_j and s_{j+1} is equal to $L_p(\Gamma)/n$, $1 \leq j \leq n$, where $s_{n+1} = s_1$. Let B be the Blaschke product of degree n whose zeros are precisely s_1, \dots, s_n . Then, for any $s \in \Omega$, one has:*

$$|B(s)| \leq \exp\left(-C \frac{n}{L_p(\Gamma)}\right),$$

where C is some absolute constant.

We also need to control the hyperbolic length of some curves.

LEMMA 7.3. *Let $\rho \in (0, 1)$ and let Γ_ρ be the circle $|z| = \rho$. Then $L_p(\Gamma_\rho) \lesssim 1/(1 - \rho)$.*

Proof. This follows from the definition of the hyperbolic length.

LEMMA 7.4. *For $\theta \in (0, \pi/2)$ and $\rho \in (0, 1)$, let $S(\theta)$ be a Stolz angle with angle θ and let $S_\rho(\theta) = \{z \in S(\theta) ; |z| < \rho\}$. Then $L_p(\partial S_\rho(\theta)) \lesssim \log[1/(1 - \rho)]$.*

Proof. Without loss of generality, we may assume that the vertex of $S(\theta)$ is 1. Recall that $S(\theta)$ is by definition the convex hull of 1 and the closed disk $\overline{D}(0, \sin \theta)$ and that

$$S(\theta) \subset \{1 - te^{i\varphi} : |\varphi| \leq \theta \text{ and } 0 \leq t < t_\theta\},$$

where $t_\theta < 2$ only depends on θ . The boundary of $S_\rho(\theta)$ can be divided into four parts: two line segments starting at $\rho e^{\pm i\alpha}$ and stopping at some fixed point, a circular arc of radius ρ joining $\rho e^{i\alpha}$ to $\rho e^{-i\alpha}$ and a part far from \mathbb{T} , independently of ρ . Each line segment can be parametrized as $1 - te^{\pm i\theta}$ with $t \gtrsim 1 - \rho$. Since $1 - |1 - te^{\pm i\theta}|^2 = 2t \cos \theta - t^2 \underset{0}{\sim} 2t \cos \theta$, it follows easily from the definition of the hyperbolic distance that the hyperbolic length of

each line segment is less than $-C \log(1 - \rho)$. Let us now turn to the circular arc $\rho e^{i\varphi}$ with $|\varphi| \leq \alpha$. It is easy to show that $\alpha \lesssim 1 - \rho$. Now,

$$\int_{-(1-\rho)}^{1-\rho} \frac{1}{1-\rho^2} d\varphi \lesssim 1.$$

This concludes the proof of this lemma.

As a first application of Theorem 7.1 and Lemma 7.3, we get the following corollary.

COROLLARY 7.5. *Let φ be a Schur map of \mathbb{D}^d inducing a continuous composition operator on $A^2(\mathbb{D}^d)$. Assume that there exist $\ell \in \{1, \dots, d\}$ and $k_1, \dots, k_\ell > 0$ such that:*

$$\|\mu_{\varphi, j, 1-\eta}\|_{C, A^2(\mathbb{D}^d)} \lesssim \eta^{k_j}, \quad j = 1, \dots, \ell; \eta \in (0, 1/2)$$

and $\|\varphi_j\|_\infty < 1$ for $j = \ell + 1, \dots, d$. Then:

$$a_q(C_\varphi) \lesssim \frac{(\log p)^{k/2}}{p},$$

where $q = p^{\frac{2}{k_1} + \dots + \frac{2}{k_\ell}} (\log p)^{d-1}$ and $k = \max_{j=1, \dots, \ell} k_j$

Proof. For $j \in \{1, \dots, \ell\}$, let $\eta_j \in (0, 1/2)$ to be defined later. Let Γ_j be the circle $|z| = (1 - 2\eta_j)/(1 - \eta_j)$ and let C_j be the Blaschke product defined in Lemma 7.2 with $\Gamma = \Gamma_j$ and degree $p_j \geq 1$. Let finally $B_j(\cdot) = C_j(\cdot(1 - \eta_j))$. Then, with the notations of Theorem 7.1 and the results of Lemmas 7.2 and 7.3, we get:

$$\begin{aligned} \frac{1}{\eta_j^2} \sup_{w \in \Omega_j} |B_j(w)|^2 \|C_\varphi\|^2 + \|\mu_{\varphi, j, 1-2\eta_j}\|_{C, A^2(\mathbb{D}^d)} \\ \lesssim \exp(-C p_j \eta_j - 2 \log \eta_j) + \exp(k_j \log \eta_j). \end{aligned}$$

We choose $\eta_j = \sigma_j \log(p_j)/p_j$ where σ_j is a numerical constant such that the last quantity is dominated by $(\log p_j/p_j)^{k_j}$.

For $j \geq \ell$, let $\eta_j = (1 - \|\varphi_j\|_\infty)/2$ so that $\|\mu_{\varphi, j, 1-2\eta_j}\|_{C, A^2(\mathbb{D}^d)} = 0$. With the same choice of B_j , we then have:

$$\frac{1}{\eta_j^2} \sup_{w \in \Omega_j} |B_j(w)|^2 \|C_\varphi\|^2 + \|\mu_{\varphi, j, 1-2\eta_j}\|_{C, A^2(\mathbb{D}^d)} \lesssim \exp(-C p_j).$$

Theorem 7.1 then yields:

$$a_{p_1 \dots p_{d+1}}(C_\varphi) \lesssim \sum_{j=1}^{\ell} \left(\frac{\log p_j}{p_j}\right)^{k_j/2} + \sum_{j=\ell+1}^d \exp(-C p_j).$$

Optimizing the choice of p_j by setting $p_j = p^{2/k_j}$ for $j = 1, \dots, \ell$ and $p_j = C \log p$ for $j = \ell + 1, \dots, d$ gives the result.

Remark 7.6. If φ_j is constant for $j > \ell$, we can dispense with the factor $(\log p)^{d-\ell}$ in the definition of q . Indeed, it is sufficient to take the Blaschke product of degree 1 which vanishes at $\varphi(0)$.

To be able to apply Theorem 7.1 or its Corollary 7.5, we need to control the quantity $\|\mu_{\varphi, j, 1-\eta}\|_{C, A^2(\mathbb{D}^d)}$ for some maps. We may appeal to the techniques of [2] when the coordinate functions are linear maps.

LEMMA 7.7. Let $a, b > 0$ with $a + b = 1$, $0 < \eta < \delta$ and $\xi \in \mathbb{T}$. Then:

$$m_2(\{z \in \mathbb{D}^2; |az_1 + bz_2| > 1 - \eta \text{ and } |az_1 + bz_2 - \xi| < \delta\}) \leq \delta^{3/2} \eta^2.$$

Proof. We may assume $\xi = 1$. The condition $|az_1 + bz_2 - 1| < \delta$ implies $\operatorname{Re}(z_i) \geq 1 - C\delta$ for some $C > 0$ independent of δ . The condition $|az_1 + bz_2| > 1 - \eta$ implies that $|z_i| > 1 - c\eta$ for some $c > 0$ independent of η . Indeed,

$$1 - \eta < |az_1 + bz_2| \leq b + a|z_1|$$

implies

$$|z_1| > \frac{1 - \eta - b}{a} = 1 - \frac{\eta}{a}.$$

Hence, z_2 belongs to the intersection of \mathbb{D} , the corona $1 - c\eta < |z| < 1$ and $\{z \in \mathbb{D}; \operatorname{Re}(z) > 1 - C\delta\}$. It is easy to see that the volume of this is $\lesssim \delta^{1/2}\eta$. A value of z_2 being fixed, z_1 belongs to the intersection of a ball of radius δ and a corona of size η ; this set has volume $\lesssim \delta\eta$. Fubini's theorem gives the conclusion.

COROLLARY 7.8. Let $\varphi(z) = (az_1 + bz_2, \varphi_2(z))$ with $a, b > 0$, $a + b = 1$ and $\|\varphi_2\|_\infty < 1$. Then:

$$\|\mu_{\varphi, 1, 1-\eta}\|_{C, A^2(\mathbb{D}^d)} \lesssim \eta^{3/2}.$$

Proof. We have to estimate $\sup_{\xi \in \mathbb{T}^2, \bar{\delta} \in (0, 2]^2} \mu_{\varphi, 1, 1-\eta}(S(\xi, \bar{\delta})) / \delta_1^2 \delta_2^2$.

Since $\mu_{\varphi, 1, 1-\eta}(S(\xi, \bar{\delta})) = 0$ when $\delta_2 < \|\varphi_2\|_\infty$, we have just to estimate

$$\sup_{\xi_1 \in \mathbb{T}, \delta_1 \in (0, 2]} \frac{m_2(\{z \in \mathbb{D}^2; |az_1 + bz_2 - 1| < \delta_1 \text{ and } |az_1 + bz_2| > 1 - \eta\})}{\delta_1^2}.$$

When $\delta_1 > \eta$, an appeal to Lemma 7.7 shows that the last quantity is dominated by $\delta_1^{-1/2} \eta^2$. On $(\eta, 2]$, its supremum is equal to $\eta^{3/2}$. When $\delta_1 < \eta$, the condition $|az_1 + bz_2| > 1 - \eta$ is automatically satisfied as soon as $|az_1 + bz_2 - 1| < \delta_1$, and the volume of the vectors $z \in \mathbb{D}^2$ satisfying this last inequality is $\delta_1^{3/2} \delta_1^2 = \delta_1^{7/2}$. Then,

$$\sup_{\xi_1 \in \mathbb{T}, \delta_1 \in (0, \eta]} \frac{m_2(\{z \in \mathbb{D}^2; |az_1 + bz_2 - 1| < \delta_1 \text{ and } |az_1 + bz_2| > 1 - \eta\})}{\delta_1^2} \lesssim \eta^{3/2}$$

and the corollary is proved.

We are ready to give a Bergman space analogue to Theorem 6.3.

COROLLARY 7.9. Let $\varphi(z) = (az_1 + bz_2, \varphi_2(z))$ with $a, b > 0$, $a + b = 1$ and $\|\varphi_2\|_\infty < 1$. Then:

$$a_n(C_\varphi) \lesssim \left(\frac{\log^2 n}{n}\right)^{3/4}.$$

Moreover, if φ_2 is constant, then:

$$a_n(C_\varphi) \lesssim \left(\frac{\log n}{n}\right)^{3/4}.$$

Proof. This follows from Corollary 7.5 and Corollary 7.8, in setting $p = (n / \log n)^{3/4}$ in the first case, and $p = n^{3/4}$ in the second one.

Remark 7.10. With this method, we lose a factor $\log n$ if we compare with the result obtained by adapting the proof of Theorem 6.3 to Bergman spaces (see the next section). However, it should be clear that our example is just a toy example, and that our method could be used for many maps, in particular for all examples given in [2]. Unfortunately, the estimates of volume could become quickly very complicated.

For more general maps, we have the following general estimates on $\|\mu_{\varphi, j, 1-\eta}\|_{C, A^2(\mathbb{D}^d)}$.

LEMMA 7.11. *Let φ be a Schur map of \mathbb{D}^d . Assume that there exist positive functions f_1, \dots, f_d such that $f_j(x)/x^2$ is non-decreasing and bounded on $(0, 2]$ and for which we have, for any $\xi \in \mathbb{T}^d$ and any $\bar{\delta} \in (0, 2]^d$:*

$$\mu_{\varphi}(S(\xi, \bar{\delta})) \lesssim f_1(\delta_1) \cdots f_d(\delta_d).$$

Then for any $\eta \in (0, 1)$ and any $j \in \{1, \dots, d\}$:

$$\|\mu_{\varphi, j, 1-\eta}\|_{C, A^2(\mathbb{D}^d)} \lesssim f_j(2\eta)/\eta^2.$$

Proof. For notational convenience, we assume $j = 1$. If $\delta_1 < \eta$, then

$$\mu_{\varphi, 1, 1-\eta}(S(\xi, \bar{\delta})) = \mu_{\varphi}(S(\xi, \bar{\delta})),$$

so that:

$$\sup \left\{ \frac{\mu(S(\xi, \bar{\delta}))}{\delta_1^2 \cdots \delta_d^2}; \xi \in \mathbb{T}^d, \bar{\delta} \in (0, 2]^d, \delta_1 \in (0, \eta) \right\} \lesssim f_1(\eta)/\eta^2 \leq f_1(2\eta)/\eta^2.$$

If $\delta_1 \geq \eta$, then there exists $N \lesssim \delta_1/\eta$ such that, for any $\xi_1 \in \mathbb{T}$, there exist $\xi_1^1, \dots, \xi_1^N \in \mathbb{T}$ satisfying

$$\{z \in \mathbb{D}^d; |z_1 - \xi_1| < \delta_1 \text{ and } |z_1| \geq 1 - \eta\} \subseteq \bigcup_{\ell=1}^N \{z \in \mathbb{D}^d; |z_1 - \xi_1^\ell| < 2\eta\}.$$

This means that, setting $\xi^\ell = (\xi_1^\ell, \xi_2, \dots, \xi_d)$ and $\bar{\delta}' = (2\eta, \delta_2, \dots, \delta_d)$,

$$\mu_{\varphi, 1, 1-\eta}(S(\xi, \bar{\delta})) \leq \sum_{\ell=1}^N \mu_{\varphi}(S(\xi^\ell, \bar{\delta}')) \lesssim (\delta_1/\eta) f_1(2\eta) f_2(\delta_2) \cdots f_d(\delta_d).$$

Dividing by $\delta_1^2 \cdots \delta_d^2$ and taking the supremum over $\bar{\delta} \in (0, 2]^d$ with $\delta_1 \in [\eta, 2]$ yield the result.

We now consider maps such that each coordinate function has its image contained in a polygon.

COROLLARY 7.12. *Let φ be a Schur map of \mathbb{D}^d inducing a continuous composition operator on $A^2(\mathbb{D}^d)$. Assume that each φ_j has its image contained in a polygon.*

(1) *If there exist $k_1, \dots, k_d > 2$ such that, for any $\xi \in \mathbb{T}^d$ and any $\bar{\delta} \in (0, 2]^d$:*

$$\mu_{\varphi}(S(\xi, \bar{\delta})) \lesssim \delta_1^{k_1} \cdots \delta_d^{k_d},$$

then:

$$a_n(C_{\varphi}) \lesssim \exp(-C n^{1/2d}).$$

(2) *If there exist $b > 0$ such that, for any $\xi \in \mathbb{T}^d$ and any $\bar{\delta} \in (0, 2]^d$:*

$$\mu_{\varphi}(S(\xi, \bar{\delta})) \lesssim \exp(-b[(1/\delta_1) + \cdots + (1/\delta_d)]),$$

then:

$$a_n(C_\varphi) \lesssim \exp(-C n^{1/d} / \log n).$$

Proof. (1) We follow the method of Corollary 7.5 but now Ω_j is the intersection of the Stolz angle with the disk $|z| < (1 - 2\eta_j)/(1 - \eta_j)$, so that the hyperbolic length of $\Gamma_j = \partial\Omega_j$ is dominated by $-\log(\eta_j)$. We now have:

$$\begin{aligned} \frac{1}{\eta_j^2} \sup_{w \in \Omega_j} |B_j(w)|^2 \|C_\varphi\|^2 + \|\mu_{\varphi, j, 1-2\eta_j}\|_{C, A^2(\mathbb{D}^d)} & \\ \lesssim \exp\left(C \frac{p_j}{\log \eta_j} - 2 \log \eta_j\right) + \exp(k_j \log \eta_j). & \end{aligned}$$

We choose $\eta_j = \exp(-\sigma_j p_j^{1/2})$ so that the right hand side is dominated by $\exp(-C' p_j^{1/2})$ if σ_j is small enough. We conclude by choosing $p_1 = \dots = p_d = p$ and by interpolating between p^d and $(p + 1)^d$.

(2) The proof is analogous: we have now $\|\mu_{\varphi, j, 1-\eta_j}\|_{C, A^2(\mathbb{D}^d)} \lesssim \exp(-b'/\eta_j)$ for $1 \leq j \leq d$ and $0 < \eta_j < 1$ and we choose $\eta_j = 1/p_j$.

Examples

(1) Let $\varphi = \lambda_1 \otimes \dots \otimes \lambda_d$, where λ_j is the lens map of parameter θ_j , $1 \leq j \leq d$. We know [20, lemma 3.3], that $\mu_{\lambda_j}(S(\xi, \delta)) \lesssim \delta^{1/\theta_j}$. We get hence that $a_n(C_\varphi) \lesssim \exp(-C n^{1/2d})$ for the operator $C_\varphi : A^2(\mathbb{D}^d) \rightarrow A^2(\mathbb{D}^d)$.

(2) Let $\chi_d = \chi \otimes \dots \otimes \chi$ be the multi-cusp map, as defined in Theorem 6.2. We know that assumption on the Carleson function is satisfied for the cusp map χ (see [21], bottom of the page 3967). We get hence that $a_n(C_{\chi_d}) \lesssim \exp(-C n^{1/d} / \log n)$, for the operator $C_{\chi_d} : A^2(\mathbb{D}^d) \rightarrow A^2(\mathbb{D}^d)$.

8. Concluding remarks

8.1. Toy examples

We can say a little more on our ‘‘toy examples’’, as evidenced by the following synthetic result: if the second component is a constant, we have the same estimates; and if the second component has a sup-norm < 1 , we lose at most a factor $\log n$, not a factor $\log^2 n$.

PROPOSITION 8.1. *Let $\varphi = (az_1 + bz_2, c)$ where $a, b > 0, a + b = 1$ and $|c| < 1, c$ a constant. Then, we have the two-sided estimate $a_n(C_\varphi) \approx n^{-3/4}$ without logarithmic factors.*

Similarly on the Hardy space: $a_n(C_\varphi) \approx n^{-1/4}$.

If now $\varphi = (az_1 + bz_2, \varphi_2)$ where $a, b > 0, a + b = 1$ and $\|\varphi_2\|_\infty < 1$, we have for $C_\varphi : H^2(\mathbb{D}^2) \rightarrow H^2(\mathbb{D}^2)$ the upper estimate: $a_n(C_\varphi) \lesssim (\log n/n)^{1/4}$.

Similarly, $a_n(C_\varphi) \lesssim (\log n/n)^{3/4}$ for the Bergman case.

Proof. We do not detail it. The key point is the following lemma.

LEMMA 8.2. *We have (dA denoting the area measure on \mathbb{D}):*

$$\int_{\mathbb{D}^2} |az_1 + bz_2|^{2n} dA(z_1)dA(z_2) \approx (n + 1)^{-5/2}.$$

The lemma follows from Lemma 6.5 (with a, b replaced by ar_1, br_2) and from an integration in polar coordinates (in two variables, $z_j = r_j e^{i\theta_j}, j = 1, 2$). We skip the details.

8.2. *Slow Blaschke products*

Some Blaschke products used before with help of the Poincaré metric can be replaced by the simpler, following version, *as far as Stolz angles are not concerned*.

We can proceed as follows (see also [20, lemma 3.3, p. 60]): let $0 < \rho < 1$, $r = \rho^2$, $\omega = e^{2i\pi/p}$ and B the Blaschke product of degree p on $\rho\mathbb{D}$ with zeros $r\omega^l$, $1 \leq l \leq p$. Namely:

$$B(z) = \prod_{1 \leq l \leq p} \frac{\rho(z - r\omega^l)}{\rho^2 - r\overline{\omega^l}z} =: C(z/\rho).$$

We observe that the usual Blaschke product C is

$$C(z) = \prod_{1 \leq l \leq p} \frac{z - \rho\omega^l}{1 - \rho\overline{\omega^l}z} = \frac{z^p - \rho^p}{1 - \rho^p z^p}.$$

For $|z| < 1$, the properties of the pseudo-hyperbolic metric give

$$|C(z)| \leq \frac{|z|^p + \rho^p}{1 + \rho^p|z|^p} \leq |z|^p + \rho^p. \quad (8.1)$$

This implies the following result.

PROPOSITION 8.3. *Let B be the previously defined Blaschke product. Then*

$$|z| = \rho^2 \Rightarrow |B(z)| \leq 2\rho^p. \quad (8.2)$$

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