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Projective and Carrollian geometry at time/space-like infinity on projectively compact Ricci flat Einstein manifolds.

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Abstract

In this article we discuss how to construct canonical *strong* Carrollian geometries at time/space like infinity of projectively compact Ricci flat Einstein manifolds (M, g) and discuss the links between the underlying projective structure of the metric g . The obtained Carrollian geometries are determined by the data of the projective compactification. The key idea to achieve this is to consider a new type of Cartan geometry based on a non-effective homogeneous model for projective geometry. We prove that this structure is a general feature of projectively compact Ricci flat Einstein manifolds.

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1 Introduction

A geometric structure on a manifold with boundary M can induce distinct geometries on the interior $\overset{\circ}{M}$ and on the boundary ∂M . An important example of this phenomenon is when M is equipped with a Cartan geometry admitting a holonomy reduction. As proved by Čap–Gover–Hammerl in [ČGH14] this indeed leads to a curved orbit decomposition of M , that one can take to be compatible with the canonical decomposition $M = \overset{\circ}{M} \cup \partial M$. This provides a framework in which one can explore, from a geometric perspective, different compactification procedures which appear in modern day physics. These are of interest since such compactifications typically underlie holographic duality principles.

Holonomy reductions of projective structures [ČGM14; ČG14; ČG16; FG18] provide particularly rich orbit decompositions in which there is an interesting interplay between conformal, projective and pseudo-riemannian geometry. This is nicely illustrated by the diagram in Figure 1a which illustrates the *projective* compactification of Minkowski spacetime $(\mathbb{M}^{d+1}, \eta = -dt^2 + \text{Euc}_{\mathbb{R}^d})$, as obtained from a curved orbit decomposition of the projective sphere (which generalises to an appropriate curved setting [FG18]). Čap and Gover have shown (see [ČG14, Proposition 2.4]) that boundary points of this compactification are, in a precise sense, endpoints of the geodesics of \mathbb{M}^{d+1} . The decomposition of the boundary into three orbits then corresponds to the three possible types of geodesic curves reaching it, which are distinguished by the “norm” $\eta(u, u)$ of their tangent vector: timelike $\eta(u, u) < 0$, null $\eta(u, u) = 0$ and spacelike $\eta(u, u) > 0$ curves. Accordingly, the three boundary orbits correspond to “timelike”, “null” and “spacelike” infinity respectively.

The *projective* picture outlined above is markedly different from that of the widely known *conformal* compactification of Penrose [Pen63]. Conformal compactification has proven to be a valuable conceptual tool that formalised the notion of isolated system in general relativity [Ger77; Ash15] and in general helped to phrase and attack global problems in mathematical relativity [Val16; Fri18]. The conformal boundary of asymptotically flat spacetime is null, with topology $\mathbb{R} \times S^2$, and usually referred to as (conformal) null infinity or \mathcal{I} . This, however, differs from projective null infinity, that is, the part of the projective boundary composed of endpoints to null geodesics and which we denoted by ς^\pm in Figure 1a, which is topologically a sphere S^2 . One can roughly say

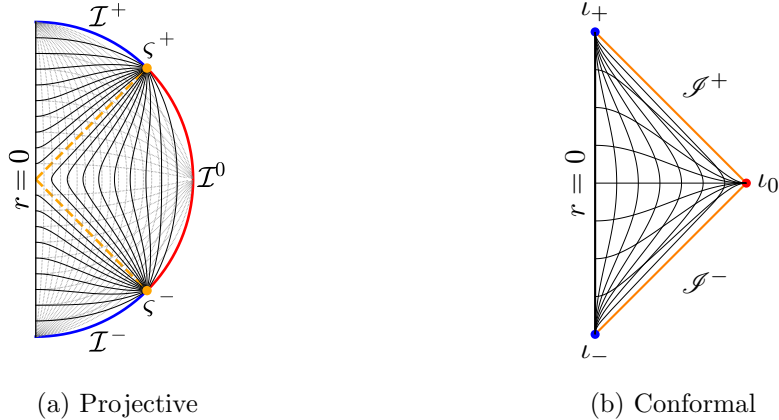


Figure 1: The projective and conformal compactifications of \mathbb{M}^{d+1} .

On the left (a), the projective compactification: $\mathcal{I}^\pm \simeq \mathbb{H}^d$ denotes future/past projective infinity, isometric to hyperbolic space. $\mathcal{I}^0 \simeq dS^d$ denotes projective spatial infinity, isometric to de Sitter space. Projective null infinities $\varsigma^\pm \simeq S^{d-1}$ are (Riemannian) conformal spheres. Altogether the boundary is topologically S^d . The lightcone of the origin is represented by the dotted orange line, it divides the compactified spacetime into different regions that are foliated with the level hypersurfaces of $\rho = \frac{1}{\sqrt{|-t^2+x^2|}}$. The faint dotted horizontal (resp. vertical) curves are the $t = \text{cst}$ (resp. $r = \text{cst}$) slices.

On the right (b), the conformal compactification: the inside is foliated horizontally (resp. vertically) by the $t = \text{cst}$ (resp. $r = \text{cst}$) hypersurfaces. $\mathcal{S}^\pm \simeq S^{d-1} \times \mathbb{R}$ denotes future/past conformal null infinity while conformal future/past ι^\pm and spatial ι^0 infinities are reduced to points. The total boundary is topologically $S^1 \times S^{d-1}$.

that the projective construction at null infinity is missing a 1-dimensional fibre.

On the other hand, in the timelike and spacelike regions (reduced to points in the conformal picture, see Figure 1b), the projective boundary is richer than that of the conformal compactification. This feature seems to suggest that one might hope to be able to capture information about their asymptotics and, indeed, there is a rich literature on asymptotic flatness at spatial infinity [AH78; BS82; Chr89; AR92; Fri98; FK00; Val04a; Val04b; Val05; CD11; Pra19; MV21; PS22; CNP23; CGW23; AK24a; AK24b; MPV24], which uses assumptions closely related to projective compactness [ČG14].

However, there is not enough room at $\mathcal{I}^\pm, \mathcal{I}^0$ to encode, for example, asymptotic symmetries in a way which is intrinsic to the boundary and, accordingly, any form of gravitational data in a geometrical manner. In fact, it was already suggested in the seminal work of Ashtekar–Hansen [AH78] to circumvent this problem by introducing a line bundle $\tilde{\mathcal{I}}^\epsilon \simeq \mathcal{I}^\epsilon \times \mathbb{R}$ over the projective boundaries $\mathcal{I}^0 \simeq dS^3$, $\mathcal{I}^\pm \simeq H^3$ - thus imitating the model of conformal null infinity $\mathcal{S}^\pm \simeq \varsigma^\pm \times \mathbb{R}$, thought of as a line bundle over projective null infinity $\varsigma^\pm \simeq S^2$. This idea resurfaced in the work of Figueroa-

O’Farril and collaborators [FHPS22] where such extensions were naturally constructed as homogeneous spaces.

The essential feature of the homogeneous model studied by Figueroa-O’Farril, Have, Prohazka and Salzer in [FHPS22] is the following: the action of the Poincaré group $ISO(n-1, 1)$ on \mathbb{RP}^{n+1} fixes a point. Once this point is removed, the decomposition of $\mathbb{RP}^{n+1} \setminus \{pt\}$ into orbits, given by the top line of (1.1), *fibres over the projective compactification*, given by the bottom line of (1.1), of n -dimensional Minkowski space \mathbb{M}^n .

$$\begin{array}{ccc} \mathbb{RP}^{n+1} \setminus \{pt\} & \simeq & (\mathbb{R} \times \mathbb{M}^n) \sqcup (\text{Ti}^n \sqcup \mathcal{S}^{n-1} \sqcup \text{Spi}^n) \\ \downarrow & & \downarrow \qquad \qquad \qquad \downarrow \\ \mathbb{RP}^n & \simeq & \mathbb{M}^n \sqcup (H^{n-1} \sqcup S^{n-2} \sqcup dS^{n-1}). \end{array} \quad (1.1)$$

Crucially, the action of $ISO(n-1, 1)$ on the homogeneous spaces Ti , Spi and \mathcal{S} is non-trivial along the 1-dimensional fibres and this suggests that Ti and Spi should be thought of as “extended” boundaries.

In previous work [BH24], we studied in detail the curved picture corresponding to this homogeneous model. More precisely, we investigated the curved orbit decomposition of compact $(n+1)$ -manifolds \widetilde{M} equipped with a projective structure $[\nabla]$ whose corresponding normal Cartan connection [Car24; Sha97; Kob95] admits a holonomy reduction [ČGH14] to a specific Poincaré group described by the existence of two parallel tractors I^A, H^{AB} . The resulting picture is that \widetilde{M} is naturally endowed with a distinguished 1-dimensional foliation given by “integral curves”¹ of a distinguished densified vector field n^a . Assuming, after excluding a discrete number of singular points $\mathcal{Z}(n^a)$, that the leaf space M of the foliation was a manifold, we found that it inherited a projective structure whose normal Cartan connection also admits a holonomy reduction such that the curved orbit decomposition of $\widetilde{M} \setminus \mathcal{Z}(n^a)$ fibers over that of M according to the following diagram:

$$\begin{array}{ccc} \widetilde{M} \setminus \mathcal{Z}(n^a) & = & \mathcal{O} \sqcup \overbrace{(\widetilde{\mathcal{H}}_- \sqcup \widetilde{\mathcal{H}}_0 \sqcup \widetilde{\mathcal{H}}_+)}^{\partial \widetilde{M}} \\ \downarrow & & \downarrow \\ M & \simeq & \mathring{M} \sqcup \underbrace{(\mathcal{H}_- \sqcup \mathcal{H}_0 \sqcup \mathcal{H}_+)}_{\partial M} \end{array} .$$

This directly generalises² the homogeneous model (1.1): \mathring{M} splits into two parts \mathring{M}_\pm and $\mathring{M}_\pm \cup \partial M$ is a projective compactification [ČG14] of order 1 of an Einstein metric inherited on \mathring{M}_\pm . This model therefore provides a geometric situation in which we

¹Since we have a densified vector field, we point out that the parametrisation of these curves is undetermined.

²Note that here \mathcal{H}_+ stands for spatial infinity and \mathcal{H}_- for timelike infinity. Thus both \mathcal{H}_- and \mathcal{H}_0 are possibly disconnected with past and future components.

naturally have a line bundle over the projective compactification, directly generalising [FHPS22].

Moreover, as we will recall in Definition 2.1, the line bundles $\tilde{\mathcal{H}}_{\pm} \rightarrow \mathcal{H}_{\pm}$ can in fact be defined in terms of intrinsic data on \mathcal{H}_{\pm} , i.e. only in terms of the data of the projective compactification and without referring to the higher dimensional manifold. What is more, asymptotic symmetries are then naturally identified with automorphisms of \mathcal{H}_{\pm} . Consequently, it seems justified to refer to the total space $\tilde{\mathcal{H}}_{\pm}$ of the line bundle as the *extended (projective) boundary* of M .

Nevertheless, at this stage, it was unclear if there was any other geometric structure induced by the higher dimensional manifold \tilde{M} that could in fact be intrinsic to the projectively compact Ricci flat manifolds M . The main goal of this article is to provide a *positive* answer to this question³:

Theorem. *Given a projectively compact Ricci flat metric \hat{g} on a manifold with boundary $M = \mathring{M} \cup \partial M$, then there is a natural way to equip the extended boundary $\tilde{\mathcal{H}}^{\pm} \rightarrow \mathcal{H}^{\pm}$ with a projective structure that admits a holonomy reduction to strong Carrollian geometries.*

This is the content of Theorem 4.2 and Proposition 4.3. Here, the term *Carrollian geometries* refers to the degenerate geometries introduced in [DGHZ14], see [Her22] for details on some aspects of the related Cartan geometries.

The strategy towards obtaining this result is to again consider a curved orbit decomposition, closely related to the homogeneous model (1.1), but instead coming from a holonomy reduction of a *non-effective* Cartan geometry on a n -dimensional manifold M based on non-effective realisation of projective space⁴

$$\mathbb{R}P^n \simeq G/P \approx \frac{SL(n+1) \ltimes \mathbb{R}^{n+1}}{\text{Aff}(n) \ltimes \mathbb{R}^{n+1}}. \quad (1.2)$$

The details of this realisation, as well as the corresponding Cartan geometry and its holonomy reduction will be discussed in Sections 3.1 and 3.2.

One can always recover an effective Cartan geometry from a non-effective one by quotienting by the kernel K (in the case at stake $K \approx \mathbb{R}^{n+1}$). In general, trying to use a non-effective geometry, such as (1.2), to study the underlying effective geometry is nevertheless likely to be a futile task, as it is not clear that the extra degrees of freedom in the kernel will carry any meaningful intrinsic information. However, in the case at hand we will prove that the holonomy reduction will completely fix the non-effective geometry in terms of the effective one.

More precisely, we will prove (see Theorem 3.1) that, in the interior \mathring{M} , the non-effective Cartan geometry, modelled on (1.2), is completely determined by the projectively compact metric \hat{g} . We then prove that it extends to points of the usual projective boundary \mathcal{H}^{\pm} , see Theorem 3.3. Finally, we prove that this non-effective Cartan

³In this work, we are interested in the space/time like parts of the boundary and will work away from the null part, \mathcal{H}_0 , which will be the object of future work.

⁴Here $\text{Aff}(n) = SL(n) \ltimes \mathbb{R}^n$ is the affine group, so that $\mathbb{R}P^n \simeq \frac{(G/K)}{(P/K)} \approx \frac{SL(n+1)}{\text{Aff}(n)}$ is the usual, effective, realisation of projective space.

geometry, canonically induced on \mathcal{H}^\pm by the projective compactification, is equivalent to an effective Carrollian geometry on the extended boundary $\widetilde{\mathcal{H}}^\pm$.

The paper is organised as follows: in Section 2 we review elements of projective tractor calculus and related results for projectively compact metrics that we will need on the rest of this article. We then discuss in Section 3 the curved orbit decomposition induced by the holonomy reduction of a non-effective Cartan geometry modelled on (1.2). Finally, in Section 4, we derive the geometry induced on the extended boundary \widetilde{H}^\pm from this construction.

Notations

We will make extensive use of the abstract index notation. In this context, we will denote the sheaf of vector fields, \mathcal{E}^a , the sheaf of one forms, \mathcal{E}_a , etc. To simplify notations, when the context is clear, we will abuse notation and not specify over which open set we are working, or, only specify the manifold, e.g. \mathcal{E}_M^a .

The bundle of projective densities of weight w (defined below) will be written $\mathcal{E}(w)$. Once more, for readability, we will abuse notation and not distinguish between $\mathcal{E}(w)$ and the modules of local sections; which interpretation is appropriate should always be clear from the context. In cases where confusion could arise, will use the notation Γ to indicate sections.

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2 Brief review of the standard projective tractor calculus

For the convenience of the reader we review elements of the usual projective tractor calculus, developed in [BEG94]. We then recall from [ČGM14; ČG14; ČG16] Čap-Gover’s definition for projectively compact Ricci flat manifold and some associated results.

2.1 Basic definitions

Let M be a manifold⁵ of dimension n endowed with an equivalence class of projectively equivalent torsion-free connections $[\widetilde{\nabla}]$. It is a standard result⁶ that two torsion-free

⁵With or without boundary, we refer the reader, for instance, to [BK23] for a detailed discussion about defining projective structures on manifolds with boundary.

⁶See, for instance, [Kob95, Proposition 7.2]

affine connections $\tilde{\nabla}$ and ∇ are projectively equivalent if and only if there is a 1-form Υ_a such that for any vector field ξ^b :

$$\tilde{\nabla}_a \xi^b = \nabla_a \xi^b + \Upsilon_a \xi^b + \Upsilon_c \xi^c \delta_a^b. \quad (2.1)$$

We will use $\tilde{\nabla} = \nabla + \Upsilon$ as shorthand for this condition.

We denote by $\mathcal{E}(w)$ the associated vector bundle to the frame bundle $P^1(M)$ determined by the 1-dimensional representation of $GL_n(\mathbb{R})$: $A \mapsto |\det A|^{\frac{w}{n+1}}$. Sections of this bundle will be called *projective densities* of weight w . Their definition guarantees that under the projective change of connection $\nabla = \tilde{\nabla} + \Upsilon$ we have, for any section $\sigma \in \mathcal{E}(w)$,

$$\nabla \sigma = \tilde{\nabla} \sigma + w \Upsilon \sigma.$$

For any vector bundle \mathcal{B} with base M , we will write: $\mathcal{B}(w) = \mathcal{B} \otimes \mathcal{E}(w)$ and we will say that sections of $\mathcal{B}(w)$ are of weight w .

Let $G = \text{PGL}(n, \mathbb{R})$ and $H = (\mathbb{R}^n)^* \rtimes GL(n, \mathbb{R})$ the isotropy subgroup of a given line. The class $[\nabla]$ determines a reduction of the second-order frame bundle $P^2(M)$ to a H -principle bundle P , on which there is a unique *normal* Cartan connection ω [Car24; Kob95]; this is a normal Cartan geometry modelled on projective geometry as described in [Sha97].

The *standard dual (projective) tractor bundle*, is defined [BEG94] to be the 1-jet prolongation of $\mathcal{E}(1)$:

$$\mathcal{T}^* = J^1 \mathcal{E}(1).$$

In the abstract index notation sections of tractor bundles will be indicated by indices $A, B, C \dots$.

\mathcal{T}^* fits into a canonical short exact sequence of vector bundles:

$$0 \longrightarrow T^*M(1) \xrightarrow{Z} \mathcal{T}^* \xrightarrow{X} \mathcal{E}(1) \longrightarrow 0.$$

$\begin{array}{ccc} \text{---} \curvearrowright & & \text{---} \curvearrowright \\ & W & Y \\ \text{---} \curvearrowleft & & \text{---} \curvearrowleft \end{array}$

A choice of connection in the projective class $[\nabla]$ provides a non-canonical isomorphism $\mathcal{T}^* \simeq TM(1) \oplus \mathcal{E}(1)$, described above by the non-canonical maps Y and W . When thinking of these maps as sections of $\mathcal{T}^* \otimes \mathcal{E}(-1)$ and $T^*M(1) \otimes \mathcal{T}$ respectively, they transform under change of connection $\tilde{\nabla} = \nabla + \Upsilon$ according to:

$$\tilde{Y}_A = Y_A - \Upsilon_a Z_A^a, \quad \tilde{W}_a^A = W_a^A + \Upsilon_a X^A. \quad (2.2)$$

Fixing a choice of connection ∇ in the projective class, sections of \mathcal{T}^* will be written:

$$T_A = \sigma Y_A + \mu_a Z_A^a,$$

similar expressions will be used with sections of tensor powers of the tractor bundle.

We also recall the standard decomposition of the curvature tensor $R_{ab}{}^c{}_d$:

$$R_{ab}{}^c{}_d = W_{ab}{}^c{}_d + 2\delta_{[a}^c P_{b]d} + \beta_{ab} \delta_d^c,$$

where the Weyl tensor W is trace-free, P_{ab} is the projective Schouten tensor and β_{ab} can be thought of as the curvature tensor of the density bundle $\mathcal{E}(1)$. They are given by

$$(n-1)P_{ab} = R_{ab} + \beta_{ab}, \quad \beta_{ab} = -\frac{2}{n+1}R_{[ab]} = -P_{[ab]},$$

where $R_{ab} = R_{da}{}^d{}_b$ is the Ricci tensor. We note that the tensor quantities P and β are not projectively invariant. In particular, under a projective change of connection $\tilde{\nabla} = \nabla + \Upsilon$:

$$\tilde{P}_{ab} = P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b. \quad (2.3)$$

A connection ∇ is said to be *special* if it preserves a nowhere vanishing density σ . In this case $\beta_{ab} = 0$ (in particular all density bundles are flat) and we say that ∇ is the *scale* determined by σ . Even when it does not preserve a nowhere vanishing density, we will refer to a choice of connection in the class as a choice of scale.

The normal Cartan connection ω induces a linear connection on \mathcal{T}^* . Having fixed a projective connection ∇ in the class, and split the short-exact sequence, the action of this connection can be summarised conveniently by the relations:

$$\nabla_a X^A = W_a^A, \quad \nabla_a W_b^B = -P_{ab} X^B, \quad \nabla_a Y_A = P_{ab} Z_A^b, \quad \nabla_a Z_B^b = -\delta_a^b Y_B. \quad (2.4)$$

The tractor curvature, defined by the identity $\Omega_{ab}{}^C{}_D T^D = 2\nabla_{[a} \nabla_{b]} T^C$, is known to be given in an arbitrary scale by:

$$\Omega_{ab}{}^C{}_D = W_{ab}{}^c{}_d W_c^C Z_D^d - Y_{abd} X_C Z_D^d \quad (2.5)$$

where: $Y_{abd} = 2\nabla_{[a} P_{b]d}$ is the projective Cotton tensor.

For any weighted tractor bundle, $T^{\mathcal{A}}(w)$, where \mathcal{A} is an arbitrary set of abstract indices, the normal Cartan connection induces a natural projectively invariant differential operator:

$$D : T^{\mathcal{A}}(w) \rightarrow \mathcal{T}_A \otimes T^{\mathcal{A}}(w-1)$$

known as the Thomas D -operator and defined by:

$$D_A t^{\mathcal{A}} = w t^{\mathcal{A}} Y_A + \nabla_a t^{\mathcal{A}} Z_A^a. \quad (2.6)$$

D behaves much like a covariant derivative with a cotractor index, in particular, *it satisfies the Leibniz rule*; this can be useful in computations. Also observe that one has

$$D_A X^B = \delta_A^B. \quad (2.7)$$

2.2 Metric projective structures

A projective class $[\nabla]$ on an n -manifold M is said to be *metric* if it admits a solution $\zeta^{ab} \in \mathcal{E}^{(ab)}(-2)$ to the Mikeš-Sinjukov metrisability equation [Mik96; Sin79]:

$$\nabla_c \zeta^{ab} - \frac{2}{n+1} \delta_c^{(a} \nabla_d \zeta^{b)d} = 0. \quad (2.8)$$

If we introduce⁷

$$\mathbf{det} : \begin{array}{ll} \mathcal{E}^{ab}(w) & \longrightarrow \mathcal{E}[n(w+2)+2] \\ h^{ab} & \longmapsto \frac{1}{n!} \epsilon_{a_1 \dots a_n b_1 \dots b_n}^2 h^{a_1 b_1} \dots h^{a_n b_n} \end{array}$$

then solutions of Eq. (2.8) yield inverse metrics $g^{ab} = |\mathbf{det}(\zeta)|\zeta^{ab}$ on $\{\pm \mathbf{det}(\zeta) > 0\}$ whose Levi-Civita connection ∇^g belongs to the projective class $[\nabla]$. ∇^g is equivalently characterised as the special connection that preserves the density $\mathbf{det}(\zeta)$.

Solutions of the metrisability equation been shown [EM08; FG18] to be in one-to-one correspondence with tractors H^{AB} that satisfy:

$$\nabla_c H^{AB} + \frac{2}{n} X^{(A} W_{cE}{}^B) H^{EF} = 0, \quad (2.9)$$

where $W_{cE}{}^B{}_F = Z_c^E \Omega_{ce}{}^B{}_F$ and $\Omega_{ce}{}^B{}_F$ is the tractor curvature. In this case H^{AB} is of the form⁸

$$H^{AB} = \zeta^{ab} W_a^A W_b^B + \lambda^b X^{(A} W_b^{B)} + \nu X^A X^B, \quad (2.10)$$

with

$$\lambda^a = -\frac{2}{n+1} \nabla_c \zeta^{ca}, \quad \nu = \frac{\nabla_a \nabla_b \zeta^{ab}}{n(n+1)} + \frac{1}{n} P_{ab} \zeta^{ab}. \quad (2.11)$$

If $\nabla_c H^{AB} = 0$, then it can be shown that Eq. (2.9) is automatically satisfied; these solutions are said to be *normal*. This is equivalent to

$$\nabla_c \zeta^{ab} + 2\delta_c^{(a} \lambda^{b)} = 0, \quad \nabla_c \lambda^a + \delta_c^a \nu - P_{cb} \zeta^{ba} = 0. \quad (2.12)$$

(and implies (2.11)). Normal solutions are known to correspond to Einstein metrics [ČGM14]. Introducing⁹

$$\mathbf{det} : \begin{array}{ll} \mathcal{T}^{AB} & \longrightarrow C^\infty(M) \\ H^{AB} & \longmapsto \frac{(n+1)}{n!} \epsilon_{A_0 A_1 \dots A_n B_0 B_1 \dots B_n}^2 H^{A_0 B_0} \dots H^{A_n B_n} \end{array}$$

then $|\mathbf{det}(\zeta)|\zeta^{ab} \in \mathcal{E}^{ab}$ is Einstein of scalar curvature

$$R = n(n-1) \operatorname{sgn}(\mathbf{det} \zeta) \det H. \quad (2.13)$$

In the scale associated to $|\mathbf{det}(\zeta)|$ this can be rewritten as $R = n(n-1)|\mathbf{det} \zeta| \nu$.

⁷ ϵ^2 is the section that realises the identification $(\Lambda^n TM)^{\otimes 2} \simeq \mathcal{E}(2(n+1))$; if M is oriented, then it can be thought of as the square of the volume form.

⁸In what follows we will have $\lambda^a = -\hat{\sigma} N^a$.

⁹We recall from [FG18] that the projective tractor volume form is defined by

$$\epsilon_{A_0 A_1 \dots A_n B_1 B_1 \dots B_n}^2 := \epsilon_{a_1 \dots a_n b_1 \dots b_n}^2 Y_{[A_0} Z_{A_1}^{a_1} \dots Z_{A_n}^{a_n} Y_{B_0} Y_{B_1}^{b_1} \dots Z_{B_n}^{b_n}]$$

2.3 Projective compactification

The notion of projective compactification was first introduced in the work of Čap-Gover [ČGM14; ČG14; ČG16]. We recall that an affine connection $\hat{\nabla}$ on the interior \mathring{M} of a manifold with boundary M , is said to be *projectively compact of order α* if each boundary point $x \in \partial M$ admits a neighbourhood $x \in U \subset M$ and a boundary defining function $\rho : U \rightarrow \mathbb{R}_+$, such that the projectively equivalent connection $\nabla = \hat{\nabla} + \frac{d\rho}{\alpha\rho}$ on $U \cap \mathring{M}$ admits a smooth extension to U . By extension, a metric \hat{g} on \mathring{M} is said to be projectively compact of order α , if its Levi-Civita connection, $\hat{\nabla}$, is projectively compact in this sense.

In this article, our main focus is the case of a *Ricci flat* projectively compact *Lorentzian*¹⁰ metric \hat{g} , therefore $\alpha = 1$ [ČG14, Theorem 3.3]. As discussed in [ČGH14; ČG14; FG18] the boundary ∂M is then totally geodesic and inherits a projective structure which admits a holonomy reduction to the Poincaré group. This gives a curved orbit decomposition, generalising the flat model depicted in Figure 1a:

$$\partial M = \mathcal{H}_\pm \cup \mathcal{H}_0, \quad (2.14)$$

where \mathcal{H}_\pm are closed hypersurfaces that inherit Einstein metrics which are projectively compact of order 2: spatial infinity \mathcal{H}_+ is a Lorentzian manifold with positive scalar curvature while past/future time infinity \mathcal{H}_- is Riemannian with negative scalar curvature. The projective boundary for these metrics coincides with projective null infinity \mathcal{H}_0 ; it is a $(n - 2)$ -dimensional submanifold inheriting a conformal structure.

We will review here from these references some essential results. We will make extensive use of these in the rest of the articles and this will also be useful to fix notations.

Define the density and densified inverse metric¹¹:

$$\hat{\sigma} = |\text{Vol}_{\hat{g}}|^{\frac{1}{n+1}}, \quad \zeta^{ab} = \hat{\sigma}^{-2} \hat{g}^{ab},$$

where $\hat{\sigma}$ is extended by 0 to M and is a *canonical boundary defining density* in the sense of [ČG14, Proposition 2.3], in particular:

$$\hat{\sigma} = 0, \quad \nabla_a \hat{\sigma} \neq 0, \quad \text{along } \partial M.$$

On the other hand, ζ^{ab} is clearly a normal solution to the metrisability equation (2.8).

From the results in [ČG14], Ricci flatness implies that ζ^{ab} and $\hat{\sigma}$ define tractors H^{AB} and I_B such that:

$$\nabla_c H^{AB} = 0, \quad \nabla_c I_B = 0, \quad I_B = D_B \hat{\sigma}, \quad H^{AB} I_A = 0. \quad (2.15)$$

Since the projective structure extends to M , H^{AB} and I_B extend to parallel tractors on M .

¹⁰Signature $(n, 1)$, for definiteness, as our discussion is essentially independent of this choice.

¹¹It then follows from this definition that $|\det(\zeta^{ab})| = \hat{\sigma}^2$ and therefore the data of ζ^{ab} is strictly equivalent to that of \hat{g}^{ab} .

From (2.10), (2.6), the expression of H^{AB} and I_A in the Levi-Civita scale $\hat{\nabla}$ is readily seen to be simply:

$$H^{AB} = \zeta^{ab} \hat{W}_a^A \hat{W}_b^B, \quad I_A = \hat{\sigma} \hat{Y}_A. \quad (2.16)$$

It follows then from the transformation rules (2.2) that, since any other scale ∇ is related to $\hat{\nabla}$ by

$$\nabla = \hat{\nabla} + \hat{\sigma}^{-1} \nabla_a \hat{\sigma},$$

we have $I_B = \hat{\sigma} Y_A + \nabla_a \hat{\sigma} Z_A^a$ and

$$H^{AB} = \zeta^{ab} W_a^A W_b^B - 2\hat{\sigma}^{-1} \zeta^{ab} \nabla_b \hat{\sigma} W_a^{(A} X^{B)} + \hat{\sigma}^{-2} \zeta^{ab} \nabla_a \hat{\sigma} \nabla_b \hat{\sigma} X^A X^B. \quad (2.17)$$

Comparing with (2.10) gives expressions for λ^a and ν . If ∇ extends to the boundary then the components of H^{AB} must also have smooth extensions to ∂M and therefore ζ^{ab} , λ^a , ν must extend to boundary points. In particular

$$\zeta^{ab} \nabla_a \hat{\sigma} = 0, \quad \hat{\sigma}^{-1} \zeta^{ab} \nabla_a \hat{\sigma} \nabla_b \hat{\sigma} = 0, \quad \text{along } \partial M,$$

for any connection ∇ that extends to the boundary. It follows that the restriction of ζ^{ab} to ∂M defines a canonical tensor

$$\bar{h}^{ab} \in \mathcal{E}_{\partial M}^{(ab)}(-2) \quad (2.18)$$

which is itself a solution to the metrisability equation (2.8) on ∂M , with associated tractor \bar{H}^{AB} . A key point is that it gives a canonical section¹²

$$\lambda_0 := -\mathbf{det} \bar{h}^{ab} \in \mathcal{E}(2)_{\partial M}$$

of the intrinsic density bundle on ∂M that provides the orbit decomposition (2.14). In particular, it is a boundary defining density for \mathcal{H}_0 , $\mathcal{H}_0 = \mathcal{Z}(\lambda_0)$, and has a fixed sign on \mathcal{H}_{\pm} :

$$\lambda_0 = 0 \quad \text{on } \mathcal{H}_0, \quad \lambda_0 = \pm |\lambda_0| \quad \text{on } \mathcal{H}_{\pm}.$$

As was proved in [ČG14], near points of \mathcal{H}_{\pm} , one can in fact always choose a scale such that both

$$N^a := \hat{\sigma}^{-2} \zeta^{ab} \nabla_b \hat{\sigma}, \quad \nu := \hat{\sigma}^{-2} \zeta^{ab} \nabla_a \hat{\sigma} \nabla_b \hat{\sigma}, \quad (2.19)$$

are finite along the boundary. We reformulate this fact in:

Theorem 2.1. *A special scale ∇ , defined on a neighbourhood U of a point $x_0 \in \mathcal{H}_{\pm}$, is such that*

$$N^a := \hat{\sigma}^{-2} \zeta^{ab} \nabla_b \hat{\sigma}$$

¹²Due to different possible signatures, λ_0 is positive on \mathcal{H}_+ and negative on \mathcal{H}_- .

extends smoothly to $\partial M \cap U$ if and only if it preserves a density τ satisfying

$$i^* \tau = \sqrt{|\lambda_0|}.$$

In this case, the canonical density λ_0 is given indifferently on U by any of the following,

$$\lambda_0 = -\mathbf{det} \bar{h}^{ab} = i^* \nu^{-1} = \pm i^* \tau^2 \quad \in \mathcal{E}_{\partial M}(2). \quad (2.20)$$

When evaluated in such scales the metric can then be rewritten as

$$\zeta_{ab} = \frac{\nu^{-1}}{\hat{\sigma}^2} \nabla_a \hat{\sigma} \nabla_b \hat{\sigma} + q_{ab}, \quad (2.21)$$

for some finite tensorial density q_{ab} .

Proof. First, let ∇ preserve a density τ and satisfy the condition on N^a . Then it follows from results in [ČG14] (see in particular Eq. (2.22) below) that $\tau^2 \nu$ is constant along the boundary and (rescaling τ if necessary by a constant factor) we can achieve $i^* \tau^2 = \pm i^* \nu^{-1}$. By definition of N^a and ν , it also follows that in this scale

$$\zeta^{ab} = \nu^{-1} \hat{\sigma}^2 N^a N^b + q^{ab},$$

with q^{ab} restricting to \bar{h}^{ab} along the boundary and in particular $q^{ab} \nabla_b \hat{\sigma} = 0$. We deduce, since ζ^{ab} has Lorentzian signature,

$$\hat{\sigma}^2 := |\mathbf{det} \zeta| = -\mathbf{det} \zeta = -\nu^{-1} \hat{\sigma}^2 (N^a \nabla_a \hat{\sigma})^2 \mathbf{det} \bar{h} + O(\hat{\sigma})$$

and thus $i^* \nu^{-1} = -\mathbf{det} \bar{h}$.

Assume now that $i^* \tau = |\lambda_0|^{\frac{1}{2}}$. By [ČG14, Lemma 3.13, Proposition 3.14] near every point $x_0 \in \mathcal{H}_\pm$ there exists a density $\tilde{\tau}$ such that $\tilde{N}^a = \hat{\sigma}^{-2} \zeta^{ab} \tilde{\nabla}_b \hat{\sigma}$ extends smoothly to $U \cap \partial M$, where $\tilde{\nabla}$ is the projective scale determined by $\tilde{\tau}$. Near x_0 , one may write: $\tau = \tilde{\tau} + \hat{\sigma} \omega$ so that

$$\hat{\sigma}^{-2} \zeta^{ab} \nabla_b \hat{\sigma} = \tilde{N}^a - \tau^{-1} \zeta^{ab} \tilde{\nabla}_b \omega - \tau^{-1} \omega \hat{\sigma} \tilde{N}^a.$$

Since the right-hand side is finite by definition of $\tilde{\tau}$, it follows that the left-hand side is also, and consequently, N^a also extends smoothly to boundary points near x_0 . \square

When using these particularly important scales, the Einstein equations also imply the following useful properties on the projective Schouten tensor:

$$\begin{aligned} \nabla_a \nu &= -2\hat{\sigma} N^b P_{ba} = -\hat{\sigma} (2\nu^{-1} P_{bc} N^b N^c) \nabla_a \hat{\sigma} + O(\hat{\sigma}), \\ P_{ab} &= \nu \bar{h}_{ab} + (\nu^{-2} P_{cd} N^c N^d) \nabla_a \hat{\sigma} \nabla_b \hat{\sigma} + O(\hat{\sigma}) \end{aligned} \quad (2.22)$$

Proof.

We briefly justify these crucial statements. The first equation is immediate from $\nabla_c H^{AB} = 0$ and (2.4). The second follows from (2.12):

$$P_{cb}\zeta^{ba} = \delta_c^a \nu - \nabla_c \hat{\sigma} N^a + \hat{\sigma} \nabla_c N^a.$$

Indeed, decomposing P_{cb} :

$$P_{cb} = \nu^{-2} P_{cd} N^c N^d \nabla_a \hat{\sigma} \nabla_b \hat{\sigma} + 2\nu^{-1} v_{(a} \nabla_{b)} \hat{\sigma} + \tilde{P}_{ab}$$

where $N^a \tilde{P}_{ab} = 0$ and $N^a v_a = 0, v_a \in \mathcal{E}_a(-3)$. Since we have the block-diagonal decomposition of the inverse metric:

$$\zeta^{ab} = \hat{\sigma}^2 \nu^{-1} N^a N^b + q^{ab},$$

where $q^{ab} \nabla_b \hat{\sigma} = 0$, Eq. (2.12) shows that:

$$\nu \delta_c^a - \nabla_c \hat{\sigma} N^a = \nu^{-1} q^{ab} v_b \nabla_a \hat{\sigma} + q^{ab} \tilde{P}_{bc} + O(\hat{\sigma}).$$

The first term on the right-hand side is off-diagonal and is consequently vanishing, $q^{ab} v_b \rightarrow 0$, which implies that $v_b \rightarrow 0$ because q is invertible on the restriction to $T\partial M$. \square

We have already seen that the weight-1 density $|\lambda_0|^{\frac{1}{2}} \in \mathcal{E}_{\partial M}(1)$ is very useful in this context. However, and despite the fact that $\nabla_a \sigma$ provides a natural identification

$$\mathcal{E}_{\partial M}(1) \simeq \iota^* \mathcal{E}_M(1), \quad (2.23)$$

this boundary density $|\lambda_0|^{\frac{1}{2}}$ has no canonical extension to a section $|\lambda|^{\frac{1}{2}}$ of $\mathcal{E}_M(1)$. This remark was the basis for our definition of S_{pi} and T_{i} in [BH24]: these are the total space of line bundles,

$$S_{\text{pi}} \rightarrow \mathcal{H}_+, \quad T_{\text{i}} \rightarrow \mathcal{H}_-,$$

which are tailored to deal with this ambiguity.

Since the Thomas derivative of $|\lambda_0|^{\frac{1}{2}}$ provides a map

$$\mathcal{H}_{\pm} \longrightarrow \mathcal{T}_{\partial M}^* \simeq i^* J^1 \mathcal{E}(1)_M / j^1 \hat{\sigma} = \mathcal{T}^* / I_A,$$

we can indeed define the following.

Definition 2.1. *The **extended boundaries** $S_{\text{pi}}/T_{\text{i}}$, that we will note uniformly $\tilde{\mathcal{H}}_+/\tilde{\mathcal{H}}_-$, are defined to be the pullback to \mathcal{H}_{\pm} of $i^* J^1 \mathcal{E}(1)_M = i^* \mathcal{T}_M^*$ (viewed as a bundle with base $i^* J^1 \mathcal{E}(1)_M / J^1 \hat{\sigma} \simeq \mathcal{T}_{\partial M}^*$) along the Thomas derivative $D_A |\lambda_0|^{\frac{1}{2}}$:*

$$\begin{array}{ccc} \tilde{\mathcal{H}}_{\pm} & \dashrightarrow & i^* \mathcal{T}_M^* \\ \downarrow \pi & & \downarrow \text{Can. Proj.} \\ \mathcal{H}_{\pm} & \xrightarrow{D_A |\lambda_0|^{\frac{1}{2}}} & \mathcal{T}_{\partial M}^* \simeq \mathcal{T}_M^* / I \end{array}$$

These are principal \mathbb{R} -bundles over \mathcal{H}_\pm , with \mathbb{R} -action defined by

$$(D_A|\lambda|^{\frac{1}{2}}, k) \mapsto D_A|\lambda|^{\frac{1}{2}} \mp kI_A.$$

In practice, it can be enlightening to think of sections of this bundle as formal asymptotic expansions to order 1 in $\hat{\sigma}$ for an extension of $|\lambda_0|^{\frac{1}{2}}$ to a 1-density on M . In other words a formal expression of the type:

$$|\lambda|^{\frac{1}{2}} = |\lambda_0|^{\frac{1}{2}} \mp u\hat{\sigma} + O(\hat{\sigma}^2),$$

where u is a function on ∂M .

3 Non-effective Cartan projective geometry and the extended tractor calculus

3.1 Non-effective homogeneous model

The starting point for our discussion is the Cartan geometry modelled on the *non-effective* description of projective space in n dimensions $\mathbb{R}P^n$ as the set of all lines passing through a fixed point $[I] \in \mathbb{R}P^{n+1}$ in projective space in $n+1$ dimensions. Here $I \in \mathbb{R}^{n+2}$ is a fixed vector and this leads to the description of projective space as a homogeneous space

$$\mathbb{R}P^n = G/P \tag{3.1}$$

where G and P are the following Lie subgroups of $PSL(n+2)$:

$$G = \left\{ \begin{pmatrix} 1 & \chi_\mu & \chi_0 \\ 0 & A_\nu^\mu & \omega^\mu \\ 0 & \Upsilon_\nu & a \end{pmatrix} \text{ mod } Z(SL(n+2)) \right\}, \quad P = \{g \in G, \omega^\mu = 0\}. \tag{3.2}$$

Contrary to the usual description of projective space, the action of G on $\mathbb{R}P^n$ is not faithful and its kernel (the largest normal subgroup of G in P) is given by the normal subgroup:

$$K = \left\{ \begin{pmatrix} 1 & \chi_\mu & \chi_0 \\ 0 & \varepsilon\delta_\nu^\mu & 0 \\ 0 & 0 & \varepsilon \end{pmatrix} \text{ mod } Z(SL(n+2)), \quad \varepsilon I_{n+1} \in Z(SL(n+1)) \right\},$$

we denote $\mathfrak{g}, \mathfrak{p}, \mathfrak{k}$ respectively the Lie algebras of the groups G, P, K .

The non-effectivity we consider here should be distinguished from that, for instance, present in spin geometry, as the kernel is large enough to be visible in the infinitesimal structure; K is not a discrete group.

Let M be a n -dimensional manifold equipped with a Cartan geometry $(\mathcal{C} \rightarrow M, \omega)$ modelled on G/P . A general \mathfrak{g} -valued one-form can be parametrised as follows:

$$\begin{pmatrix} 0 & v_\nu & v_0 \\ 0 & \varpi^\mu{}_\nu - \frac{1}{n+1}\varpi^\rho{}_\nu\delta^\mu{}_\rho & \theta^\mu \\ 0 & P_\nu & -\frac{1}{n+1}\varpi^\rho{}_\nu \end{pmatrix}. \tag{3.3}$$

We define $\mathcal{T} \rightarrow M$ the associated vector bundle associated to the restriction to P of the fundamental representation of $PSL(n+2, \mathbb{R})$. In the abstract index notation, we will use the notation $\mathcal{A}, \mathcal{B}, \dots$ to indicate sections of \mathcal{T} .

Quotienting out the kernel K , the Cartan geometry $(\mathcal{C} \rightarrow M, \omega)$ induces a projective connection on M . We shall make the assumption that the connection obtained from this procedure is *normal* and write $[\nabla]$ for the corresponding projective class; observe that this implies that the Cartan connection ω is also torsion-free. By extension, we shall also say that these connections are *normal*.

The bundle \mathcal{T} has the decomposition structure given in Figure 2, realising it as an extension of the usual projective tractor bundle \mathcal{T} with its normal Cartan connection.

$$0 \longrightarrow \mathbb{R} \xrightarrow{I} \mathcal{T} \xrightarrow{\Pi} \mathcal{T} \longrightarrow 0.$$

Figure 2: Decomposition sequence of \mathcal{T}

This filtration corresponds to the existence of distinguished parallel sections $I^A, \Pi^A_{\mathcal{A}}$,

$$\nabla_a I^A = 0, \quad \nabla_a \Pi^A_{\mathcal{A}} = 0 \quad (3.4)$$

respectively realising the first injection and the projection. The sequence can be split by any choice of section $L_{\mathcal{A}}$ such that $L_{\mathcal{C}} I^{\mathcal{C}} = 1$; two such sections differ by a projective cotractor $\chi_A = \chi_a Z^a_A + \chi_0 Y_A$ and each choice realises a non-canonical isomorphism,

$$\mathcal{T} \simeq \mathbb{R} \oplus \mathcal{T}. \quad (3.5)$$

The parameters χ_μ and χ_0 of the realisation (3.2) of the group G , correspond to cycling through different choices of splitting or, equivalently, to cycling through different realisations of the isomorphism (3.5). As this degree of freedom is at this point abstract, for the remainder of this section we shall simply assume the existence of such splittings.

Similarly to the usual tractor calculus, we define the non-canonical splitting operators $\Pi^A_{\mathcal{A}}$ and $L_{\mathcal{A}}$ so that sections of \mathcal{T} and \mathcal{T}^* can be written

$$T^A = f I^A + \Pi^A_{\mathcal{A}} t^A, \quad T_{\mathcal{A}} = g L_{\mathcal{A}} + \Pi^A_{\mathcal{A}} \mu_A. \quad (3.6)$$

We shall refer to such choice of splitting as “choosing a gauge”. This is to be distinguished from the projective freedom $\nabla \in [\nabla]$; that we refer to as a “choice of scale”. In the following we will repeatedly consider general changes of gauge $L_{\mathcal{B}} \mapsto \hat{L}_{\mathcal{B}}$ and parametrise these as

$$L_{\mathcal{B}} \quad \mapsto \quad \hat{L}_{\mathcal{B}} = L_{\mathcal{B}} + \chi_A \Pi^A_{\mathcal{A}}. \quad (3.7)$$

The linear connection induced by the Cartan connection on \mathcal{T} is summarised in the following Proposition.

Proposition 3.1. *The linear connection induced on the bundle \mathcal{T} by the Cartan connection is, given a choice of gauge $L_{\mathcal{A}}$, described by:*

$$\nabla_c L_{\mathcal{C}} = -v_{Bc} \Pi^B_{\mathcal{C}}, \quad \nabla_c I^{\mathcal{C}} = 0, \quad \nabla_c \Pi^B_{\mathcal{B}} = v_{Bc} I^B, \quad \nabla_c \Pi^B_{\mathcal{B}} = 0. \quad (3.8)$$

In the above, v_{Bc} is a cotractor valued 1-form parametrising the connection and depending on the choice of splitting. Under a change of gauge (3.7) it transforms according to:

$$\hat{v}_{Bc} = v_{Bc} - \nabla_c \chi_B. \quad (3.9)$$

The curvature tractor is readily computed in terms of the connection coefficient v_{cC} and the usual projective curvature tractor:

$$\Omega_{ab}{}^A{}_B = 2\nabla_{[a} v_{|A|b]} \Pi_B^A I^A + \Omega_{ab}{}^A{}_B \Pi_B^B \Pi_A^A. \quad (3.10)$$

To summarise this section, we have seen that the non-effective Cartan geometries that we consider, and in particular modelled on (3.1), are parametrised (see Eqs. (3.3), (3.8)) by a projective connection together with a cotractor valued 1-form v_{Bc} . The latter is related to the non-effectiveness of the model and unconstrained at this stage. In the coming sections we will see that this freedom can be fixed by requiring a holonomy reduction to the Poincaré group $ISO(1, n) \subset G$.

3.2 Metric holonomy reduction

We shall now additionally assume that there is a non-degenerate *parallel* section H^{AB} of signature $(n+1, 2)$,

$$\nabla_a H^{AB} = 0, \quad (3.11)$$

such that the canonical section I^A is null;

$$\Phi_{AB} I^A I^B = 0.$$

Here Φ_{AB} denotes the pointwise inverse of H^{AB} .

This amounts to having a holonomy reduction to the Poincaré group $ISO(1, n) \subset G$ and, in this section, we will draw some immediate consequences of these equations.

3.2.1 Decomposition in a gauge

Making use of the invariant projection $\Pi : \mathcal{T} \rightarrow \mathcal{T}$ and the canonical tractor I^A , one obtains a pair of projective tractors (H^{AB}, I_A) ,

$$H^{AB} = H^{AB} \Pi_A^A \Pi_B^B, \quad I_B = \Phi_{AB} I^A \Pi_B^B. \quad (3.12)$$

These are gauge invariant quantities; note that the invariance of I_B relies on the fact that I^A is null. This condition also implies that H^{AB} is degenerate of kernel I_A ,

$$H^{AB} I_B = 0. \quad (3.13)$$

It will be useful to introduce the following notations for *gauge-dependent* decompositions of H^{AB} . These notations will be used throughout.

Definition 3.1. Any gauge L_A uniquely defines a smooth function f , a projective tractor J^B , and a symmetric bilinear form φ_{BC} on \mathcal{T} such that

$$\begin{aligned} H^{AB} &= H^{AB} \Pi_A^A \Pi_B^B + 2J^B I^{(A} \Pi_{B)}^B + f I^A I^B, \\ \Phi_{BC} &= 2I_C L_{(B} \Pi_{C)}^C + \varphi_{BC} \Pi_B^B \Pi_C^C. \end{aligned} \quad (3.14)$$

Since $H^{AC} \Phi_{CB} = \delta_B^A$, they satisfy the relations

$$H^{AC} \varphi_{CB} + J^A I_B = \delta_B^A, \quad f I_A = -J^B \varphi_{AB}, \quad J^C I_C = 1. \quad (3.15)$$

We can now turn to the consequences of imposing that H^{AB} is parallel transported. First, it immediately follows from (3.4) and (3.11) that

$$\nabla_c H^{AB} = 0, \quad \nabla_c I_B = 0. \quad (3.16)$$

In general one has the proposition.

Proposition 3.2. In any gauge L_A , the equations $\nabla_c H^{AB} = 0$ are equivalent to (3.16) together with

$$\nabla_c J^A = -H^{AB} \nu_{Bc}, \quad \nabla_c f = -2J^B \nu_{Bc}, \quad \nabla_c \varphi_{AB} = 2\nu_{c(A} I_{B)}. \quad (3.17)$$

Proof. The desired relations are obtained directly using $\nabla_c H^{AB} = 0$ and Proposition 3.1. \square

By the results in [ČG14; FG18], equations (3.16) imply that the projective class $[\nabla]$ is the projective class of a projectively compact Ricci-flat Lorentzian metric. Recall, from the results reviewed in section 2.3, that (H^{AB}, I_A) will then be parametrised in terms of ζ^{ab} , $\hat{\sigma}$ as in (2.15) and (2.17). For the other components of H^{AB} we have the following proposition.

Proposition 3.3. Under our assumption that H^{AB} is parallel, any gauge L_A uniquely defines a projective density $\bar{\lambda} \in \mathcal{E}(2)$ and a projective tractor $\lambda_A \in \mathcal{T}_A(1)$ such that

$$\bar{\lambda} = \varphi_{AB} X^A X^B, \quad \lambda_A = \varphi_{AB} X^B. \quad (3.18)$$

Moreover,

$$\begin{aligned} \varphi_{AB} &= D_B \lambda_A - \hat{\sigma} \nu_{Ab} Z_B^b - \nu_{Cb} Z_B^b X^C I_A, \\ \hat{\sigma} J^A &= X^A - H^{AB} \lambda_B, \\ \hat{\sigma}^2 f &= -\bar{\lambda} + H^{AB} \lambda_A \lambda_B. \end{aligned} \quad (3.19)$$

In other words, in a gauge L_A , the triplet (∇, H^{AB}, I^A) is entirely parametrised by $(\zeta^{ab}, \hat{\sigma})$ and (ν_{Cb}, λ_A) . As we shall see, these last two quantities are essentially pure gauge and everything can in fact be determined by the metric data $(\zeta^{ab}, \hat{\sigma})$.

Proof.

By definition, $\lambda_A = \varphi_{AB}X^B$, hence taking the Thomas derivative:

$$D_C\lambda_A = (D_C\varphi_{AB})X^B + \varphi_{AC} = \nabla_c\varphi_{AB}Z_C^cX^B + \varphi_{AC}.$$

Using Proposition 3.2 this becomes

$$D_C\lambda_A = 2v_{c(A}I_{B)}Z_C^cX^B + \varphi_{AC} = \hat{\sigma}v_{cA}Z_C^c + v_{cB}X^B I_A Z_C^c + \varphi_{AC},$$

which gives the desired expression for φ_{AB} .

Second, contracting the equation on the left of (3.15) with X^B gives

$$H^{AC}\varphi_{CB}X^B + J^A\hat{\sigma} = X^A$$

and making use of the definition $\lambda_A = \varphi_{AB}X^B$ this yields the second relation in (3.19). In order to obtain the third, we contract the second with λ_A

$$\hat{\sigma}J^A\lambda_A = \bar{\lambda} - H^{AB}\lambda_A\lambda_B$$

and rewrite the left-hand side as

$$\hat{\sigma}J^A\lambda_A = \hat{\sigma}J^A\varphi_{AB}X^B = -\hat{\sigma}fI_AX^B = -\hat{\sigma}^2f,$$

where we made use of the second equation in (3.15). \square

Orbit decomposition

It follows from (3.16), and the results from [ČG14; FG18] that the projective class $[\nabla]$ is the projective class of a Ricci-flat projectively compact of order 1 Einstein metric:

$$\hat{g}_{ab} = \hat{\sigma}^{-2}\zeta_{ab} \quad (3.20)$$

with $\zeta^{ab} = H^{AB}Z_A^aZ_B^b$, $\hat{\sigma} = X^AI_A$. We have already summarised in Section 2.3 some consequences of this fact.

Note that whilst $\hat{\sigma} \in \mathcal{E}(1)$ is a gauge-independent quantity, and gives the curved orbit decomposition

$$M = \mathring{M} \cup \partial M,$$

$\bar{\lambda} \in \mathcal{E}(2)$, defined in Eq. (3.18), is a gauge-dependent quantity: under the change of gauge (3.7) one has

$$\bar{\lambda} \mapsto \bar{\lambda} - 2\hat{\sigma}\chi_AX^A. \quad (3.21)$$

However, its restriction to $\partial M = \mathcal{Z}(\hat{\sigma})$,

$$\lambda_0 := i^*\bar{\lambda} \in \mathcal{E}_{\partial M}(2), \quad (3.22)$$

is gauge independent. Consequently, the decomposition of ∂M according to the sign of λ_0

$$\partial M = \mathcal{H}^- \cup \mathcal{H}^0 \cup \mathcal{H}^+ \quad (3.23)$$

is geometrically relevant. We will see that λ_0 in fact coincides with the distinguished scale on ∂M discussed in Section 2.3 (see the discussion before Theorem 2.1).

3.2.2 The extended connection on the interior $\overset{\circ}{M}$

Our first result is to prove that, on $\overset{\circ}{M}$, the extended connection is in fact completely fixed by the holonomy reduction (3.11).

Theorem 3.1.

On $\overset{\circ}{M}$, both the connection ∇_c on \mathcal{T} and H^{AB} are completely determined by ζ_{ab} and $\hat{\sigma}$. More precisely:

1. On $\overset{\circ}{M}$ there is a unique gauge \hat{L}_A such that

$$\hat{\lambda}_A := \hat{\varphi}_{AB} X^B = 0. \quad (3.24)$$

2. In this preferred gauge, the connection components (3.8) are given by

$$\hat{v}_{Ac} = -\hat{\sigma}^{-1} \zeta_{cb} Z^b_A, \quad (3.25)$$

and the metric components (3.14) by

$$\hat{f} = 0, \quad \hat{J}^B = \hat{\sigma}^{-1} X^B, \quad \hat{\varphi}_{BC} = \zeta_{bc} Z^b_B Z^c_C. \quad (3.26)$$

We shall refer to this specific choice of splitting on $\overset{\circ}{M}$ as the *metric gauge*; the components of tractors in this gauge will be written with hatted letters. Note that, even though at this stage it is natural to also fix the scale ∇ to be the Levi-Civita connection of $\sigma^{-2}\zeta_{ab}$, one is not forced in anyway to do that: the above condition fixes the gauge L_A but leaves a complete freedom in the scale ∇ .

Proof of Theorem 3.1.

1. *Existence and uniqueness of metric gauge.*

Under a generic change of gauge (3.7), φ_{AB} transforms according to

$$\varphi_{AB} \mapsto \hat{\varphi}_{AB} = \varphi_{AB} - 2\chi_{(A} I_{B)},$$

contracting with X^B , one sees that we are trying to achieve

$$\varphi_{AB} \mapsto 0 = \varphi_{AB} X^B - \chi_A \hat{\sigma} - \chi_B X^B I_A.$$

On $\overset{\circ}{M}$ this is uniquely solved by

$$\chi_A = \varphi_{AB} X^B \hat{\sigma}^{-1} - \frac{1}{2} \varphi_{BC} X^B X^C \hat{\sigma}^{-2} I_A = \lambda_A \hat{\sigma}^{-1} - \frac{1}{2} \bar{\lambda} \hat{\sigma}^{-2} I_A.$$

2. *Computation of components of H^{AB} and the connection coefficient \hat{v}_{Ab}*

Taking $\hat{\lambda}_A = 0$ in (3.19) yields $\hat{f} = 0$, $\hat{J}^A = \hat{\sigma}^{-1} X^A$ and

$$\hat{\varphi}_{AB} = -\hat{\sigma} \hat{v}_{Ab} Z^b_B - \hat{v}_{Cb} Z^b_B X^C I_A. \quad (3.27)$$

Since $0 = \hat{\lambda}_A = \hat{\varphi}_{AB}X^B$, we must have $\hat{\varphi}_{BC} = \xi_{bc}Z^b_B Z^c_C$ for some ξ_{bc} and the above equation is uniquely solved as

$$v_{Ab} = -\hat{\sigma}^{-1}\xi_{bc}Z^c_A.$$

Finally making use of $\hat{\varphi}_{BC} = \xi_{bc}Z^b_B Z^c_C$, (2.17) and (3.15) one finds

$$\begin{aligned}\delta^A_B &= H^{AC}\xi_{bc}Z^b_B Z^c_C + \hat{\sigma}^{-1}X^A D_B \hat{\sigma} \\ &= \zeta^{ac}\xi_{bc}Z^b_B W_a^A - \hat{\sigma}^{-1}\zeta^{ac}\xi_{bc}\nabla_a \hat{\sigma} Z^b_B X^A + \hat{\sigma}^{-1}X^A D_B \hat{\sigma}\end{aligned}$$

which is equivalent to $\zeta^{ac}\xi_{bc} = \delta^a_b$.

3. The expressions that we derived for the connection form and the components of H^{AB} only involve $\hat{\sigma}$ and ζ_{ab} . Since H^{AB} is also completely determined by ζ , we conclude that the structure is completely determined by $\hat{\sigma}$ and ζ_{ab} on the \mathring{M} .

□

Clearly, the formulae in Theorem 3.1 do not extend to the boundary and hence do not teach us anything about how such a structure should look there. This will be important when studying the question of existence of these structures in Section 3.4. In the next section, we shall devise a natural way to fix the gauge at \mathcal{H}^\pm , however, we will observe that it implicitly requires an additional non-canonical choice, that we will later understand to be a section of the extended boundary $\tilde{\mathcal{H}}_\pm$.

3.3 Projective compactification

The aim of this section is to construct a distinguished gauge at boundary points; throughout this section we will only be interested in boundary points *away* from null infinity $\mathcal{H}^0 = \partial M \cap \mathcal{Z}(\bar{\lambda})$, where conformal geometry and conformal compactness provide the relevant tools, and concentrate on \mathcal{H}^\pm where $\bar{\lambda} \neq 0$.

In Theorem 3.1, it was established that there was a uniquely determined gauge on \mathring{M} such that the gauge dependent fields,

$$\lambda_A := \varphi_{AB}X^B, \quad \lambda := \varphi_{AB}X^B X^A,$$

vanish entirely. This gauge is however singular at points where $\hat{\sigma} = 0$: this is because λ becomes gauge invariant along the boundary, see Eq. (3.21), and by definition cannot vanish on \mathcal{H}^\pm . We are going to use λ_A and $\bar{\lambda}$ to define alternative gauges, better suited than the metric gauge near boundary points of \mathcal{H}^\pm .

Our first observation is:

Lemma 3.1. *For any everywhere regular gauge L_A there is, at points where $\lambda \neq 0$, a unique projective scale $\nabla_a \in [\nabla_a]$ such that*

$$\lambda_A = \bar{\lambda} Y_A. \quad (3.28)$$

In order to preserve this condition under a change of gauge $L_{\mathcal{A}} \mapsto L_{\mathcal{A}} + \chi_A \Pi_{\mathcal{A}}^A$, then one must perform simultaneously a change of projective scale $\nabla_a \mapsto \nabla_a + \Upsilon_a$ such that

$$\Upsilon_a = \frac{\chi_0 \nabla_a \hat{\sigma} + \hat{\sigma} \chi_a}{\bar{\lambda} - 2\chi_0 \hat{\sigma}} \quad (3.29)$$

where $\chi_A = \chi_0 Y_A + \chi_a Z_A^a$ in the original projective scale.

Proof. Let at first ∇_a be an arbitrary projective scale. We shall show that there exists a unique 1-form Υ_a such that $\nabla_a + \Upsilon_a$ satisfies the required condition. Using Equation (2.2), Υ_a must solve:

$$\lambda_A W_a^A \mapsto 0 = \lambda_A W_a^A + \Upsilon_a \varphi_{AB} X^B X^A$$

which has a unique solution

$$\Upsilon_a = -\bar{\lambda}^{-1} \lambda_A W_a^A. \quad (3.30)$$

In order to prove (3.29), we need to study how $\lambda_A W_a^A$ changes under a simultaneous change of gauge $L_{\mathcal{A}} \mapsto L_{\mathcal{A}} + \chi_A \Pi_{\mathcal{A}}^A$ and projective connection $\nabla \mapsto \nabla + \Upsilon$.

Under a change of gauge $\varphi_{AB} \mapsto \varphi_{AB} - 2\chi_{(A} I_{B)}$ and thus

$$\bar{\lambda} \mapsto \bar{\lambda} - 2\chi_0 \hat{\sigma}, \quad 0 = \lambda_A W_a^A \mapsto -\chi_0 \nabla_a \hat{\sigma} - \hat{\sigma} \chi_a.$$

It then follows from (3.30) that we now need to make the change of scale

$$\Upsilon_a = -(\bar{\lambda} - 2\chi_0 \hat{\sigma})^{-1} (-\chi_0 \nabla_a \hat{\sigma} - \hat{\sigma} \chi_a)$$

to preserve the condition $\lambda_A W_a^A = 0$. \square

Remark 3.1. Observe that if a gauge is regular at points of $\mathcal{H}^0 := \partial M \cap \mathcal{Z}(\bar{\lambda})$, then the scale ∇ given by the proposition cannot extend there. Indeed, if it could, then the condition $\lambda_A = \bar{\lambda} Y_A$ would also extend by continuity. Plugging this into the second equation of (3.19), we would have $\hat{\sigma} J^A \rightarrow X^A$ at these points, but this can only happen if $J^A \sim \hat{\sigma}^{-1} X^A$, contradicting the regularity of H^{AB} . More generally, if any gauge and scale $(L_{\mathcal{A}}, \nabla_a)$ are such that together $\lambda_A = \bar{\lambda} Y_A$ holds then at least one of them must be singular near points of \mathcal{H}^0 .

We recall that, in any scale ∇ , H^{AB} must be of the form (2.17). The choice of scale (3.28) will allow to have similar expressions for the extended metric.

Proposition 3.4. *In a choice of gauge and scale $(L_{\mathcal{A}}, \nabla)$ satisfying (3.28), and therefore excluding points of \mathcal{H}^0 , the components (3.14) of the extended metric H^{AB} read*

$$\begin{aligned} \varphi_{AC} &= \left(\bar{\lambda} P_{ca} - \frac{1}{2} \hat{\sigma}^{-1} \nabla_c \bar{\lambda} \nabla_a \hat{\sigma} - \hat{\sigma} v_{Bc} W_a^B \right) Z_A^a Z_C^c + \bar{\lambda} Y_A Y_B, \\ J^A &= \left(\frac{\bar{\lambda}}{\hat{\sigma}^2} \zeta^{ab} \nabla_b \hat{\sigma} \right) W_a^A + \hat{\sigma}^{-1} \left(1 - \frac{\bar{\lambda}}{\hat{\sigma}^2} \zeta^{ab} \nabla_a \hat{\sigma} \nabla_b \hat{\sigma} \right) X^A, \\ f &= -\frac{\bar{\lambda}}{\hat{\sigma}^2} \left(1 - \frac{\bar{\lambda}}{\hat{\sigma}^2} \zeta^{ab} \nabla_a \hat{\sigma} \nabla_b \hat{\sigma} \right). \end{aligned} \quad (3.31)$$

In particular, since the extended metric is finite along the boundary, we recover (see the discussion around (2.19)), that the vector field N^a and density ν

$$N^a = \hat{\sigma}^{-2} \zeta^{ab} \nabla_b \hat{\sigma}, \quad \nu = N^a \nabla_a \hat{\sigma},$$

extend smoothly to points of ∂M . On top of this, we obtain that the restriction of ν^{-1} and $\bar{\lambda}$ must coincide at the boundary to give the preferred boundary density

$$\lambda_0 = \iota^* \nu^{-1} = \iota^* \bar{\lambda}. \quad (3.32)$$

Finally, the component v_{Ab} of the connection must satisfy

$$v_{Cb} X^C = \hat{\sigma}^{-1} \frac{1}{2} \nabla_b \bar{\lambda}. \quad (3.33)$$

and since it must be finite this means that our scales ∇ satisfy $\iota^* \nabla_c \bar{\lambda} = 0$. In particular, by Theorem 2.1, along the boundary, it is a special scale defined by the density $\sqrt{|\lambda_0|} = |\det \bar{h}|^{\frac{1}{2}}$.

Proof. Taking (3.19) from Proposition 3.3 gives, making use of our choice of scales satisfying (3.28),

$$\begin{aligned} \varphi_{AB} &= D_B(Y_A \bar{\lambda}) - \hat{\sigma} v_{Ab} Z^b{}_B - v_{Cb} Z^b{}_B X^C I_A, \\ \hat{\sigma} J^A &= X^A - \bar{\lambda} H^{AB} Y_B, \\ \hat{\sigma}^2 f &= -\bar{\lambda} + \bar{\lambda}^2 H^{AB} Y_A Y_B. \end{aligned}$$

Making use of (2.17) one then obtains the desired expressions for J^A and f . Appealing to the definitions (2.6) and (2.4) for the Thomas D -operator and the tractor connection, the expression of φ_{AB} can be rewritten as

$$\varphi_{AB} = \bar{\lambda} Y_A Y_B + \nabla_b \bar{\lambda} Y_A Z^b{}_B + \bar{\lambda} P_{ab} Z^a{}_A Z^b{}_B - \hat{\sigma} v_{Ab} Z^b{}_B - v_{Cb} Z^b{}_B X^C I_A$$

since, in our scale, $\varphi_{AB} X^B = \bar{\lambda} Y_A$, this implies that

$$v_{Cb} X^C = \hat{\sigma}^{-1} \frac{1}{2} \nabla_b \bar{\lambda}$$

and therefore

$$\varphi_{AB} W_a^A W_b^B = \bar{\lambda} P_{ab} - \hat{\sigma} v_{Ab} W_a^A - \hat{\sigma}^{-1} \frac{1}{2} \nabla_b \bar{\lambda} \nabla_b \hat{\sigma}.$$

□

This Proposition means that the extended metric is entirely parametrised by ζ^{ab} , $\hat{\sigma}$, $\bar{\lambda}$ and the coefficient v_{Ab} of the connection. We will now impose a condition on the gauge, which will leave us with a very small amount of freedom:

Proposition 3.5. *Let x_0 be a point which does not belong to \mathcal{H}^0 . Then there always exists a neighbourhood of x_0 and local choice of gauge L_A such that*

$$f := H^{AB}L_AL_B = 0. \quad (3.34)$$

Proof.

If x_0 is in the interior $\overset{\circ}{M}$, one can always pick the metric gauge (see Theorem 3.1).

If x_0 is in \mathcal{H}^\pm then, since $\bar{\lambda} \neq 0$ at x_0 , we may assume, restricting if necessary to a smaller neighbourhood, that $\bar{\lambda} \neq 0$. The expression for the gauge shows that what we are trying to achieve only depends on a choice of gauge and does not imply anything on the choice of scale: it is therefore consistent to require (3.28). We now consider the change of gauge

$$\begin{aligned} H^{AB}L_AL_B &\mapsto 0 = H^{AB}\tilde{L}_A\tilde{L}_B \\ &= H^{AB}(L_A + \chi_A\Pi_A^A)(L_B + \chi_B\Pi_B^B) \\ &= f + H^{AB}\chi_A\chi_B + 2J^B\chi_B. \end{aligned}$$

Making use of (3.31) and the expression (2.19) for N and ν , this can be rewritten as

$$0 = f + \chi_0^2\nu + \zeta^{ab}\chi_a\chi_b - 2\hat{\sigma}\chi_0N^a\chi_a + 2\bar{\lambda}N^a\chi_a + 2\hat{\sigma}^{-1}\chi_0(1 - \bar{\lambda}\nu). \quad (3.35)$$

Since $\bar{\lambda} \neq 0$ note that $\hat{\sigma}^{-1}(1 - \bar{\lambda}\nu) = -\hat{\sigma}f\bar{\lambda}^{-1}$, so:

$$0 = f + \chi_0^2\nu + \zeta^{ab}\chi_a\chi_b - 2\hat{\sigma}\chi_0N^a\chi_a + 2\bar{\lambda}N^a\chi_a - 2\bar{\lambda}^{-1}\hat{\sigma}\chi_0f.$$

We only need to prove that a solution to this equation exists; let us take as an ansatz $\chi_0 = 0$ and $\chi_a = \chi\nabla_a\hat{\sigma}$ we find:

$$\begin{aligned} 0 &= f + \chi^2\zeta^{ab}\nabla_a\hat{\sigma}\nabla_b\hat{\sigma} + 2\chi N^a\nabla_a\hat{\sigma}\bar{\lambda} \\ &= \chi^2\hat{\sigma}^2\nu + \chi 2\bar{\lambda}\nu + f. \end{aligned}$$

Now the (reduced) discriminant of this quadratic equation is $\bar{\lambda}^2\nu^2 - \hat{\sigma}^2f\nu$: it must be positive in a neighbourhood of the boundary $\hat{\sigma} = 0$ and therefore one can always achieve the gauge with a term of the form

$$\chi = -\frac{\bar{\lambda}}{\hat{\sigma}^2} \left(1 - \sqrt{1 - f\bar{\lambda}^{-2}\hat{\sigma}^2\nu^{-1}} \right) = -\frac{1}{2}f\bar{\lambda}^{-1}\nu^{-1} + O(\hat{\sigma}). \quad (3.36)$$

□

Remark 3.2. We point out that there is a slight subtlety at this point, it is not clear from the above that there is a gauge and *special* projective scale in which we have these properties. In other words, there may be no density τ for which $\nabla_a\tau = 0$.

We shall now require both (3.28) and (3.34) and draw the consequences of these choices. We recall that N^a , ν are given by the expressions (2.19).

Proposition 3.6. *Let x_0 be a point away from \mathcal{H}^0 . Let $(L_{\mathcal{A}}, \nabla)$ be a choice of gauge and scale in a neighbourhood of x_0 satisfying both (3.28) and (3.34). Then, the components (3.14) of the extended metric H^{AB} satisfy*

$$f = 0, \quad J^A = \bar{\lambda} N^a W_a^A + \hat{\sigma}^{-1}(1 - \bar{\lambda}\nu)X^A, \quad \bar{\lambda}(1 - \bar{\lambda}\nu) = 0, \quad (3.37)$$

$$\varphi_{AB} = (\zeta_{ab} - \bar{\lambda}\hat{\sigma}^{-2}\nabla_a\hat{\sigma}\nabla_b\hat{\sigma}) Z_A^a Z_B^b + \bar{\lambda} Y_A Y_B. \quad (3.38)$$

In particular, $q_{ab} := \zeta_{ab} - \bar{\lambda}\hat{\sigma}^{-2}\nabla_a\hat{\sigma}\nabla_b\hat{\sigma}$ extends smoothly to points of ∂M in this neighbourhood. What is more, it also follows from (3.28) and (3.34) that, when $\bar{\lambda} \neq 0$, $N^a q_{ab} = 0$. Finally the components v_{Ab} of the connection are given by

$$v_{Ab}X^A = \hat{\sigma}^{-1}\frac{1}{2}\nabla_b\bar{\lambda}, \quad (3.39)$$

$$v_{Ab}W^A{}_a = -\hat{\sigma}^{-1}\left(q_{ab} - \bar{\lambda}P_{ab} + \frac{1}{2}\hat{\sigma}^{-1}\nabla_a\bar{\lambda}\nabla_b\hat{\sigma}\right).$$

In particular, since the extended connection is finite along the boundary, we recover, at points where $\bar{\lambda} \neq 0$, the identities (2.22) on the projective Schouten tensor.

Proof.

Equations (3.37) are directly obtained from (3.31) under the assumption that $f = 0$. It also implies

$$\varphi_{AB} = q_{ab} Z_A^a Z_B^b + \bar{\lambda} Y_A Y_B,$$

where $q_{ab} = (\bar{\lambda}P_{ba} - \frac{1}{2}\hat{\sigma}^{-1}\nabla_b\bar{\lambda}\nabla_a\hat{\sigma} - \hat{\sigma}v_{Cb}W_a^C)$. Then the first equation in (3.15) gives, when contracted with $W^B{}_b Z_A^a$,

$$\zeta^{ac}q_{bc} + \bar{\lambda}N^a\nabla_b\hat{\sigma} = \delta_b^a,$$

or equivalently at points where $\hat{\sigma} \neq 0$

$$q_{ab} + \bar{\lambda}\hat{\sigma}^{-2}\nabla_a\hat{\sigma}\nabla_b\hat{\sigma} = \zeta_{ab}.$$

Therefore one has

$$q_{ab} = \zeta_{ab} - \bar{\lambda}\hat{\sigma}^{-2}\nabla_a\hat{\sigma}\nabla_b\hat{\sigma} = \left(\bar{\lambda}P_{ba} - \frac{1}{2}\hat{\sigma}^{-1}\nabla_b\bar{\lambda}\nabla_a\hat{\sigma} - \hat{\sigma}v_{Cb}W_a^C\right)$$

which gives the required expression for $v_{Cb}W_a^C$.

Finally, it follows immediately from $f = 0$ and the second equation in (3.15) that

$$0 = J^A \varphi_{AB} W^B{}_b = \bar{\lambda} N^a q_{ab}.$$

□

Remark 3.3. If the gauge $L_{\mathcal{A}}$ is such that $\bar{\lambda} = 0$, which is however impossible on \mathcal{H}^{\pm} , then the gauge is in fact the metric gauge and the expression in Proposition 3.6 coincide with those of Theorem 3.1.

The message from Proposition 3.6 is that under the joint conditions (3.28), (3.34) for $(L_{\mathcal{A}}, \nabla)$, which we reproduce here for convenience,

$$f = 0, \quad \lambda_A = \bar{\lambda} Y_A, \quad (3.40)$$

both the extended metric and the extended connection are parametrised by ζ^{ab} and $\hat{\sigma}$. The combined conditions (3.40) on the gauge and scale can always be achieved locally, in a neighbourhood away from \mathcal{H}^0 , and it follows from Lemma 3.1 and the details of the proof of Proposition 3.5 (see eq. (3.35)) that, if $L_{\mathcal{A}} \mapsto L_{\mathcal{A}} + \chi_A \Pi_{\mathcal{A}}^A$, $\nabla_a \mapsto \nabla_a + \Upsilon_a$ describes a transformation from $(L_{\mathcal{A}}, \nabla_a)$ to another couple satisfying (3.40) then the parameters (χ_A, Υ_a) must satisfy

$$\Upsilon_a = \frac{\chi_0 \nabla_a \hat{\sigma} + \hat{\sigma} \chi_a}{\bar{\lambda} - 2\chi_0 \hat{\sigma}}, \quad (3.41)$$

$$0 = \chi_0^2 \nu - 2\chi_0 (\hat{\sigma} N^a \chi_a - \hat{\sigma}^{-1} (1 - \bar{\lambda} \nu)) + \zeta^{ab} \chi_a \chi_b + 2\bar{\lambda} N^a \chi_a. \quad (3.42)$$

The second equation can be solved for χ_0 and substituted into the first to give a meaningful value for Υ . In this sense, one should think of χ_a as being a free parameter determining the others.

It follows from Lemma 3.1 that a gauge $L_{\mathcal{A}}$ such that $f = 0$, existence being guaranteed by Proposition 3.5, uniquely defines, at point where $\lambda \neq 0$, a pair $(L_{\mathcal{A}}, \nabla)$ satisfying our gauge fixing condition (3.40) (if $\lambda = 0$, then the gauge is the metric gauge and any scale ∇ will then satisfy (3.40)). We can now investigate the converse.

Lemma 3.2. *Let ∇ be some scale and suppose that both $(L_{\mathcal{A}}, \nabla_a)$ and $(L_{\mathcal{A}} + \chi_A \Pi_{\mathcal{A}}^A, \nabla_a)$ satisfy (3.40). Then we must have $\chi_a = -\chi_0 \hat{\sigma}^{-1} \nabla_a \hat{\sigma}$ and there are only two possibilities for χ_0 , depending on whether $\bar{\lambda}$ was zero for $L_{\mathcal{A}}$; i.e. depending on whether $L_{\mathcal{A}}$ was in fact the metric gauge:*

$$\text{if } \bar{\lambda} \neq 0 \quad \text{then} \quad \chi_0 = \frac{\nu^{-1}}{2\hat{\sigma}} \quad \text{and} \quad \bar{\lambda} = \nu^{-1} \mapsto 0, \quad (3.43)$$

$$\text{if } \bar{\lambda} = 0 \quad \text{then} \quad \chi_0 = -\frac{\nu^{-1}}{2\hat{\sigma}} \quad \text{and} \quad \bar{\lambda} = 0 \mapsto \nu^{-1}. \quad (3.44)$$

In other terms, for a given scale ∇ , and away from \mathcal{H}^0 , there is \mathbb{Z}_2 ambiguity on the space of gauges satisfying (3.40): This \mathbb{Z}_2 action takes us back and forth between the metric gauge, $\lambda_A = 0$, and another gauge uniquely associated to ∇ . (We recall that the metric gauge does not fix any scale at all and thus there is no contradiction).

Proof. Let ∇ be some scale and suppose that $(L_{\mathcal{A}}, \nabla)$ satisfies (3.40). If $(L_{\mathcal{A}} + \chi_A \Pi_{\mathcal{A}}^A, \nabla_a)$ also satisfies (3.40) then, from (3.41), one should have:

$$0 = \chi_0 \nabla_a \hat{\sigma} + \hat{\sigma} \chi_a, \quad 0 = \chi_0^2 \nu - 2\chi_0 (\hat{\sigma} N^a \chi_a - \hat{\sigma}^{-1} (1 - \bar{\lambda} \nu)) + \zeta^{ab} \chi_a \chi_b + 2\bar{\lambda} N^a \chi_a.$$

Solving the first equation gives $\chi_a = -\chi_0 \hat{\sigma}^{-1} \nabla_a \hat{\sigma}$ on \mathring{M} and injecting this in the second yields

$$0 = 2\chi_0 (2\chi_0 \nu + \hat{\sigma}^{-1} (1 - 2\bar{\lambda} \nu)) \quad (3.45)$$

Therefore, there is only one non-trivial admissible choice which is $\chi_0 = \hat{\sigma}^{-1}\nu^{-1}(\bar{\lambda}\nu - \frac{1}{2})$. Now since, under a general change of gauge,

$$\lambda_A = \bar{\lambda}Y_A \quad \mapsto \quad (\bar{\lambda} - 2\hat{\sigma}\chi_0)Y_A - (\chi_0\nabla_a\hat{\sigma} + \hat{\sigma}\chi_a)Z_A^a, \quad (3.46)$$

and given the constraint $0 = \bar{\lambda}(1 - \bar{\lambda}\nu)$ (see (3.37)), there is a unique (non-trivial) possibility, which however depends on whether or not $\bar{\lambda}$ was zero in the first place :

$$\text{if } \bar{\lambda} \neq 0 \quad \text{then} \quad \chi_0 = \frac{\bar{\lambda}}{2\hat{\sigma}} = \frac{\nu^{-1}}{2\hat{\sigma}} \quad \text{and} \quad \bar{\lambda} = \nu^{-1} \mapsto 0, \quad (3.47)$$

$$\text{if } \bar{\lambda} = 0 \quad \text{then} \quad \chi_0 = -\frac{\nu^{-1}}{2\hat{\sigma}} \quad \text{and} \quad \bar{\lambda} = 0 \mapsto \nu^{-1}. \quad (3.48)$$

□

Now we remind the reader that the metric gauge $\bar{\lambda} = 0$ does not extend smoothly to the boundary ∂M and therefore requiring a smooth extension will in fact uniquely fix the gauge.

Proposition 3.7. *Let x_0 be a point in \mathcal{H}^\pm . Let ∇ be any scale which extends smoothly near x_0 such that N^a , as defined by (2.19), also has a smooth extension with non-vanishing boundary value.*

Then there exists a unique gauge L_A satisfying (3.40) and extending smoothly to a neighbourhood of x_0 . It is obtained from the metric gauge $\hat{L}_A \mapsto L_A = \hat{L}_A + \chi_A\Pi^A_{\mathcal{A}}$ by taking

$$\chi_A = -\frac{1}{2}\nu^{-1}\hat{\sigma}^{-1}\left(Y_A - \hat{\sigma}^{-1}\nabla_a\hat{\sigma}Z_A^a\right). \quad (3.49)$$

Proof.

Let us start by proving unicity. First consider the metric gauge \hat{L}_A . The pair (∇, \hat{L}_A) satisfies (3.40); however, since we identically have $\bar{\lambda} = 0$ in this gauge, this is not an admissible gauge along the boundary. We therefore look for another pair (∇, L_A) of gauge and scale, with $L_A = \hat{L}_A + \chi_A\Pi^A_{\mathcal{A}}$, satisfying the gauge fixing conditions (3.40). In other terms, we are looking for a solution to (3.41) where $\Upsilon_a = 0$. As we already saw, under the assumption that $\bar{\lambda} = 0$, this has a unique solution given by

$$\chi_a = -\chi_0\hat{\sigma}^{-1}\nabla_a\hat{\sigma}, \quad \chi_0 = -\frac{1}{2}\hat{\sigma}^{-1}\nu^{-1}.$$

We shall now prove that given ∇ such a gauge always exists in a neighbourhood of x_0 , we emphasise that it should be understood that we work locally. Let \tilde{L}_A be some regular gauge in a neighbourhood of x_0 . Since it is regular at x_0 , we can assume, restricting the neighbourhood if necessary, $\bar{\lambda} \neq 0$. Now, by assumption, $\nu^{-1} = (\hat{\sigma}^{-2}\zeta^{ab}\nabla_a\hat{\sigma}\nabla_b\hat{\sigma})^{-1}$ is finite and we recall that its boundary value $\lambda_0 = \iota^*\bar{\lambda} = \iota^*\nu^{-1}$ does not depend on the gauge. Making use of (3.21), one may always achieve $\bar{\lambda} = \nu^{-1}$. From the proof of Proposition 3.5 one then sees that it is possible to modify this gauge in order to achieve

$f = 0$ and that, in addition, one can do so while preserving the condition $\bar{\lambda} = \nu^{-1}$. We call L'_A the gauge achieved in this way and ∇' the unique connection which, by Lemma 3.1 and our assumption, now gives $\lambda_A = \nu^{-1}Y_A$. This gives a pair (∇', L'_A) satisfying (3.40) in a neighbourhood of x_0 ; we also have that, by construction and from (3.37), $\nu = \nu'$.

From this pair, we would now like to construct L_A such that the pair (∇, L_A) also satisfies (3.40). Introduce the gauge/scale transformation

$$L'_A \mapsto L'_A + \chi_A \Pi^A_A, \quad \nabla'_a \mapsto \nabla'_a - \Upsilon_a = \nabla_a,$$

where χ_A is the only unknown.

Since $\bar{\lambda} = \bar{\lambda}' = \bar{\lambda} - 2\hat{\sigma}\chi_0$ we must have $\chi_0 = 0$ and we find, from (3.41), that $\chi_a = \chi_A W'^A_a$:

$$-\Upsilon_a = \nu \hat{\sigma} \chi_a, \quad 0 = \zeta^{ab} \chi_a \chi_b + 2\nu^{-1} N^a \chi_a. \quad (3.50)$$

We will be able to solve the first equation if and only if we can show that Υ_a vanishes along ∂M . Furthermore, since $N'^a = N^a + \hat{\sigma}^{-1} \zeta^{ab} \Upsilon_a$, and both ∇' and ∇ are regular along ∂M (the first by Proposition 3.4, the second by hypothesis) we may write:

$$\Upsilon_a = f_0 \nabla_a \hat{\sigma} + f_1 \hat{\sigma} \Upsilon'_a$$

with $\Upsilon'_a N^a = 0$. However, by assumption,

$$\nu' = \nu + 2\hat{\sigma} N^a \Upsilon_a + \zeta^{ab} \Upsilon_a \Upsilon_b = \nu \Rightarrow 2\hat{\sigma} N^a \Upsilon_a + \zeta^{ab} \Upsilon_a \Upsilon_b = 0,$$

and thus if we can prove the first condition of (3.50) the second will follow immediately. Now, using this equation and the decomposition of Υ we see that:

$$2\hat{\sigma} \nu f_0 = -\Upsilon_a \Upsilon_b \zeta^{ab} = -f_0^2 \hat{\sigma}^2 \nu - f_1^2 \hat{\sigma}^2 \zeta^{ab} \Upsilon'_a \Upsilon'_b,$$

which shows that f_0 must be of the form $\hat{\sigma} f'_0$ and hence we can solve for χ_a in the first equation (3.50). \square

Proposition (3.7) is the main result of this section and closes it. Let us however record the following.

Proposition 3.8. *If ∇ is a scale such that N^a is finite and non-vanishing then $|\nu|^{-\frac{1}{2}} N^a \in \mathcal{E}^a(-2)$ is a solution of the projectively invariant geodesic equation, i.e.*

$$|\nu|^{-\frac{1}{2}} N^c \nabla_c (|\nu|^{-\frac{1}{2}} N^a) = 0,$$

in particular the integral lines of N^a are projective geodesics for ∇ .

Proof.

By the previous proposition, for any scale ∇ such that N^a is finite, we can always find a unique compatible gauge L_A . Recall that, in such choices of gauge, $\bar{\lambda} N^a = J^A Z^a_A$, hence:

$$\nabla_c (\bar{\lambda} N^a) = (\nabla_c J^A) Z^a_A - J^A Y_A \delta^a_c = (\nabla_c J^A) Z^a_A.$$

Lemma 3.2 then shows that:

$$\nabla_c(\bar{\lambda}N^a) = -Z_A^a H^{AB} \nu_{Bc} = -(\zeta^{ab} \nu_{bc} - \hat{\sigma} N^a X^B \nu_{Bc}).$$

Observe that since $f = 0$ in our gauge, we also have:

$$J^B \nu_{Bc} = \bar{\lambda} N^b \nu_{bc} = 0,$$

which shows that

$$\bar{\lambda} N^c \nabla_c(\bar{\lambda} N^a) = \hat{\sigma} \bar{\lambda} X^B \nu_{Bc} N^c N^a.$$

Now, using equation (3.39), we deduce that

$$\bar{\lambda} N^c \nabla_c(\bar{\lambda} N^a) = N^a \frac{1}{2} \bar{\lambda} N^c \nabla_c \bar{\lambda} = N^a |\bar{\lambda}|^{\frac{3}{2}} N^c \nabla_c |\bar{\lambda}|^{\frac{1}{2}},$$

and thus

$$|\bar{\lambda}|^{\frac{1}{2}} N^c \nabla_c (|\bar{\lambda}|^{\frac{1}{2}} N^a) = -|\bar{\lambda}|^{\frac{1}{2}} N^c (\nabla_c |\bar{\lambda}|^{\frac{1}{2}}) N^a + N^c \nabla_c (|\bar{\lambda}| N^a) = 0.$$

□

3.4 Existence for Ricci-flat projectively compact Einstein metrics

In the previous discussion, we explored the structures induced by a holonomy reduction of a non-effective projective geometry on a manifold M . In particular, we studied a gauge fixing condition (3.40), that, for every projective scale ∇ satisfying the condition that N^a is regular at the boundary, singled out two possible distinguished choices of gauge. At interior points, both solutions were possible and corresponded either to the metric gauge (see Theorem 3.1) or another gauge $L_{\mathcal{A}}$ that is uniquely associated with the scale (see Proposition 3.6 and Proposition 3.7). Near points of the boundary ∂M where $\bar{\lambda} \neq 0$ (which we recall is a gauge invariant condition at boundary points contrary to other points) the metric gauge is always singular and the second solution is the only regular gauge satisfying our conditions. Note that these results have been obtained under the assumptions that the initial non-effective Cartan geometry was well-defined and smooth everywhere and one might be worried that this could constrain the underlying projectively compact Einstein manifold. As we shall now see, this is not the case.

The work of the previous section indeed puts us in a position to study the question of the existence, on $M \setminus \mathcal{H}_0$ and for any projectively compact metrics as reviewed in Section 2.3, of such non-effective Cartan geometries: we only need to check that the expressions given by Theorem 3.1 and Proposition 3.6 lead, when applied to projectively compact metrics, to a well-defined structure.

On the interior $\overset{\circ}{M}$, the metric gauge makes things particularly easy because the components of H^{AB} and the connection, as given by Theorem 3.1, are clearly well-defined on $\overset{\circ}{M}$. We can see these objects as living on a trivial bundle $\mathcal{T}|_{\overset{\circ}{M}} \oplus \mathbb{R}$ over $\overset{\circ}{M}$.

Near any boundary point $x_0 \in \mathcal{H}_\pm$, the key observation is that any scale associated to a density τ given by Theorem 2.1 satisfies the hypothesis of Proposition 3.7 in a neighbourhood U . Hence, we can locally define the components using the expressions of Proposition 3.6 (or, equivalently, by starting from the metric gauge and applying the change of gauge (3.49)). It only remains to check that the corresponding expressions extend smoothly to boundary points:

Theorem 3.2.

For any scale τ as given in Theorem 2.1, we define the extended connection modelled on (3.1) by (3.8) and

$$v_{Ab} = \nu^{-2} N^c P_{cb} Y_A + \hat{\sigma}^{-1} \left(\nu^{-1} P_{ab} - \nu^{-2} \nabla_a \hat{\sigma} N^d P_{db} - q_{ab} \right) Z_A^a. \quad (3.51)$$

Because of the identities (2.22), which are valid for any projectively compact Einstein metric, this expression is finite along the boundary and admits the following asymptotic behaviour:

$$\begin{aligned} v_{Ab} &= (2\nu^{-3} N^c N^d P_{cd}) \nabla_b \hat{\sigma} Y_A \\ &\quad + \hat{\sigma}^{-1} \left(\nu^{-1} P_{ab} - \nu^{-2} \nabla_a \hat{\sigma} N^d P_{db} - q_{ab} \right) Z_A^a \\ &\quad + O(\hat{\sigma}). \end{aligned}$$

This connection admits a holonomy reduction given by a parallel transported non-degenerate tractor pairing H^{AB} :

$$H^{AB} = H^{AB} \Pi_A^A \Pi_B^B + 2\nu^{-1} N^a W_a^A I^{(B} \Pi^A)_{A}. \quad (3.52)$$

It extends smoothly to the boundary and is invertible.

Proof. From Equation (3.9), equation (3.49) and Proposition 3.1 we have:

$$\hat{v}_{Ab} = -\frac{1}{2} \nu^{-2} \hat{\sigma}^{-1} \nabla_b \nu Y_A + \left(-\hat{\sigma}^{-1} \zeta_{ba} + \hat{\sigma}^{-2} \nabla_b \hat{\sigma} \nabla_a \hat{\sigma} + \nu^{-1} \hat{\sigma}^{-1} P_{ba} + \frac{1}{2} \nu^{-2} \hat{\sigma}^{-2} \nabla_b \nu \nabla_a \hat{\sigma} \right) Z_A^a.$$

This also follows from Proposition 3.6. The desired expression in the statement is then obtained from Equation (2.21) and the identities (2.22). The identities (2.22) also justify that it extends to boundary points.

The expression for H^{AB} follows from Proposition 3.6. Since $N^a \nu^{-1}$ is non-vanishing, H^{AB} is non-degenerate. \square

At this point there are still two potential difficulties that we will elucidate in one go. The first is that Theorem 3.2 makes use of a scale τ and since we could have chosen instead any other scale $\tilde{\tau}$, we need to check that all such connections only differ by a (finite) gauge transformation. This will ensure that the resulting connection is a well defined (i.e. invariant) object. Second, we have only shown that the structure exists locally, and need to justify that the local constructions can be glued together in a coherent fashion.

Since there is a canonical way to move between any of these gauges $L_{\mathcal{A}}$, $\tilde{L}_{\mathcal{A}}$ and the metric gauge $\hat{L}_{\mathcal{A}}$ (which is well-defined in \mathring{M}), we can define a canonical way to move between two local gauges determined by two densities τ and $\tilde{\tau}$ on the intersection of two local neighbourhoods $U \cap V$ by the successive gauge/scale transformations:

$$(L_{\mathcal{A}}, \nabla_a) \rightarrow (\hat{L}_{\mathcal{A}}, \nabla_a) \rightarrow (\hat{L}_{\mathcal{A}}, \tilde{\nabla}_a) \rightarrow (\tilde{L}_{\mathcal{A}}, \tilde{\nabla}).$$

If the overall gauge transformation cotractor $\chi_{\mathcal{A}}$, i.e. defined such that $\tilde{L}_{\mathcal{B}} = L_{\mathcal{B}} + \chi_{\mathcal{A}} \Pi_{\mathcal{A}}^A$, extends smoothly to boundary points, then this gives a valid gauge transformation between the local structures that can be used to extend the structure to $U \cup V$. Along the way it will also prove that the extended connection as defined in Theorem 3.2 is an invariant object.

Lemma 3.3. *Let $\tau, \tilde{\tau}$ be two scales as given by Theorem 2.1, and defined in a neighbourhood of \mathcal{H}_{\pm} . Denote by $L_{\mathcal{A}}, \tilde{L}_{\mathcal{A}}$ the corresponding gauges determined by Proposition 3.7. Let us write $\tilde{\tau} = \tau \mp \omega \hat{\sigma}$, then in the splitting determined by τ :*

$$\tilde{L}_{\mathcal{A}} = L_{\mathcal{A}} + D_{\mathcal{A}}(\omega \tau) \Pi_{\mathcal{A}}^A + O(\hat{\sigma}, \nabla \hat{\sigma})$$

From which we deduce:

$$\tilde{v}_{ab} = v_{ab} + (\tau \nabla_a \nabla_b \omega \pm \tau^{-1} \bar{h}_{ab} \omega) + O(\hat{\sigma}, \nabla \hat{\sigma}). \quad (3.53)$$

Proof. The proof of the first equation can be found in Appendix A. It is essentially a computation that provides no interesting insight at this point. One can also check the result by solving the gauge fixing conditions (3.41) for $(\chi_{\mathcal{A}}, \Upsilon_a = -\tilde{\tau}^{-1} \nabla_a \tilde{\tau})$ at leading order.

To obtain the second equation it is sufficient to take the derivative of the previous equation. Since $\nabla_c I_B = 0$, it follows that $\nabla_a \nabla_b \hat{\sigma} = -P_{ab} \hat{\sigma}$, hence:

$$\nabla O(\hat{\sigma}, \nabla \hat{\sigma}) = O(\hat{\sigma}, \nabla \hat{\sigma}).$$

Now:

$$\nabla_a D_B(\omega \nu^{-1} \tau^{-1}) = \nu^{-1} \tau^{-1} (P_{ab} \omega + \nabla_a \nabla_b \omega) Z_B^b + O(\hat{\sigma}),$$

and Equation (3.53) then follows directly from using (2.22) and (2.20). \square

The general spirit of this construction is simply that we can deduce from this a Cartan atlas $\{U^i, \omega^i\}$ and transition functions $\{f_{ij}\}$ that can be used to reconstruct the P -principal bundle equipped with its global Cartan connection. We then recover the extended tractor bundle by the standard associated bundle construction. This construction is at this point ad-hoc and a more satisfying geometric description of the extended tractor bundle \mathcal{S} is the object of work in progress. This proves the following theorem.

Theorem 3.3.

Given a projectively compact Ricci flat Einstein metric \hat{g}_{ab} of signature (p, q) on the interior \mathring{M} of a manifold with boundary M , there is a uniquely determined non-effective Cartan connection modelled on (3.1) and extending to $M \setminus \mathcal{H}_0$.

This Cartan geometry is such that, by construction, each choice of scale $\tau \in \mathcal{E}(1)$ as given by Theorem 2.1, uniquely determines a gauge close to any point $x_0 \in \partial M \setminus \mathcal{H}_0$.

What is more, it admits a holonomy reduction to the subgroup $ISO(p, q)$ of $PSL(n+2)$:

$$\begin{pmatrix} 1 & -\omega^\rho \eta_{\rho\sigma} A^\sigma_\nu & -\frac{1}{2} \eta_{\mu\nu} \omega^\mu \omega^\nu \\ 0 & A^\mu_\nu & \omega^\mu \\ 0 & 0 & 1 \end{pmatrix} \text{ mod } Z(SL(n+2)), \quad A \in SO(p, q).$$

4 Projective structure on the extended boundary

4.1 Induced non effective Cartan geometry on the boundary

Let us recall, from [ČGM14; ČG14], that the projective boundary \mathcal{H}_\pm inherits a projective Cartan geometry together with a holonomy reduction to $SO(1, n-1)$. This induces on \mathcal{H}_\pm a metric of signature¹³ (r, s) , and a Cartan geometry modelled on

$$SO(1, n-1) / SO(r, s). \quad (4.1)$$

Now, in the previous sections, we found that every choice of scale τ given by Theorem 2.1, i.e. such that¹⁴ $\iota^* \tau = |\lambda_0|^{\frac{1}{2}}$ gives rise to a distinguished boundary gauge $L_{\mathcal{A}}$ for the non-effective projective geometry. We obtain in this way, along the boundary, and after making use of the holonomy reduction to $ISO(1, n-1)$, a non-effective Cartan geometry modelled on

$$SO(1, n-1) \times \mathbb{R}^n / SO(r, s) \times \mathbb{R}^n, \quad (4.2)$$

and related to the effective one by “forgetting” about the non-effective part \mathbb{R}^n ; in practice this will mean quotienting by $I^{\mathcal{A}}$ in the expressions below.

The induced connection can be read off from Eqs. (3.8), (2.4), (2.22) and Theorem 3.2: fixing a section τ , and denoting by $\bar{W}_a^{\mathcal{A}}$ the splitting of the projective tractor bundle on \mathcal{H}_\pm determined by $\tau_0 = |\lambda_0|^{\frac{1}{2}}$, it is given by

$$\begin{aligned} \nabla_a (\tau_0^{-1} X^A \Pi_A^{\mathcal{A}}) &= \tau_0^{-1} \bar{W}_a^{\mathcal{A}} \Pi_A^{\mathcal{A}}, \\ \nabla_a (\tau_0^{-1} \bar{W}_b^{\mathcal{A}} \Pi_A^{\mathcal{A}}) &= \mp \tau_0^{-2} \bar{h}_{ab} (\tau_0^{-1} X^A \Pi_A^{\mathcal{A}}) + \tau_0^{-1} \iota^* v_{ab} I^{\mathcal{A}}, \\ \nabla_a I^{\mathcal{A}} &= 0. \end{aligned} \quad (4.3)$$

with $\iota^* v_{ab} = \iota^* (\hat{\sigma}^{-1} (\nu^{-1} P_{ab} - q_{ab}))$.

There is some interplay between Lemma 3.3 and this induced Cartan connection on which we will now elaborate.

Indeed, this lemma shows that, although all the projective scales τ coincide in restriction to the boundary, the restriction of the corresponding gauges $L_{\mathcal{A}}$ actually depends on the details of their extensions in the interior. Nevertheless, the dependence is relatively tame: it only depends on the section of $\tilde{\mathcal{H}}_\pm$ determined by τ .

¹³ $(r, s) = (1, n-2)$ on \mathcal{H}_+ and $(r, s) = (0, n-1)$ on \mathcal{H}_- .

¹⁴In this section we will also frequently use the notation $\tau_0 := \iota^* \tau = |\lambda_0|^{\frac{1}{2}}$ for this density.

Theorem 4.1.

Let $\tau, \tilde{\tau}$ be two distinguished projective scales as given by Theorem 2.1 and let us write

$$\tilde{\tau} = \tau \mp \omega \hat{\sigma}.$$

for some function ω in a neighbourhood of \mathcal{H}_\pm . Then

- The connection component v_{ab} of the extended tractor connection has, when **pulled back to the boundary** along the canonical injection $\iota : \partial M \hookrightarrow M$, the following transformation rule:

$$\iota^* \tilde{v}_{ab} = \iota^* v_{ab} + \iota^* \left(\tau_0 \nabla_a \nabla_b \omega \pm \tau_0^{-1} \bar{h}_{ab} \omega \right). \quad (4.4)$$

- In particular, if τ and $\tilde{\tau}$ define the same section

$$\hat{\tau} : \mathcal{H}_\pm \rightarrow \tilde{\mathcal{H}}_\pm$$

of $\tilde{\mathcal{H}}_\pm \rightarrow \mathcal{H}_\pm$, i.e. $\omega = \sigma \omega_1$, $\omega_1 \in \mathcal{E}(-1)$, then the $\iota^* \tilde{v}_{ab} = \iota^* v_{ab}$.

Hence, each section of $\tilde{\mathcal{H}}_\pm$ globally defines $v_{ab} \in \mathcal{E}_{(ab)}^{\partial M}(1)$ and the boundary value of the extended tractor connection only depends on this section.

Proof. The first point is just obtained by pulling back the result of Lemma 3.3 along the canonical injection. To prove the second point, observe that if $\omega = \hat{\sigma} \omega_1$ then:

$$\nabla_a \nabla_b \omega = \hat{\sigma} \nabla_a \nabla_b \omega_1 + 2 \nabla_{(a} \hat{\sigma} \nabla_{b)} \omega_1 + \underbrace{(\nabla_a \nabla_b \hat{\sigma})}_{-P_{ab} \hat{\sigma}} \omega_1.$$

□

4.2 Carrollian structure on the extended boundary

The simple transformation rule (4.4) for the component v_{ab} of the induced tractor connection (4.3) suggests that it would perhaps more naturally live as a structure on the extended boundary $\tilde{\mathcal{H}}_\pm$. In this section, we shall show that this intuition is correct and use the properties of the non-effective connection on \mathcal{H}_\pm to construct an *effective* Cartan geometry on $\tilde{\mathcal{H}}_\pm$.

First, let us make some general remarks related to the fact that $\tilde{\mathcal{H}}_\pm \rightarrow \mathcal{H}_\pm$ is a principal \mathbb{R} -bundle and that it is equipped with a preferred degenerate metric $h_{ab} := \pi^*(\tau_0^{-2} \bar{h}_{ab})$ of signature $(1, r, s)$, where $(r, s) = (1, n - 2)$ on \mathcal{H}_+ and $(r, s) = (0, n - 1)$ on \mathcal{H}_- . In a local trivialisation we shall write points of $\tilde{\mathcal{H}}_\pm$ as couples $p = (x, u)$ with $n^a = \partial_u$ the generator of the \mathbb{R} -action. The pair (h_{ab}, n^a) satisfies

$$n^b h_{ab} = 0, \quad \mathcal{L}_n h_{ab} = 0,$$

and is a (weak) Carrollian geometry in the sense of [DGHZ14]. It naturally defines a reduction of the frame bundle:

Definition 4.1. The pair (h_{ab}, n^a) on $\tilde{\mathcal{H}}_{\pm}$ defines a reduction of the structure group of the frame bundle from $GL(n)$ to $SO(r, s) \times \mathbb{R}^{n-1}$:

Let \mathcal{F} denote the bundle of 1-jets at $(0, 0)$ of local diffeomorphisms $\Phi : \mathbb{R}^{n-1} \times \mathbb{R} \rightarrow \tilde{\mathcal{H}}_{\pm}$ such that there is a local diffeomorphism $\phi : \mathbb{R}^{n-1} \rightarrow \partial M$ making the following diagram commute:

$$\begin{array}{ccc} \mathbb{R}^{n-1} \times \mathbb{R} & \xrightarrow{\Phi} & \tilde{\mathcal{H}}_{\pm} \\ \downarrow \text{Proj}_1 & & \downarrow \pi \\ \mathbb{R}^{n-1} & \xrightarrow{\phi} & \partial M \end{array}$$

and:

1. $\Phi_* \frac{\partial}{\partial x^n} = \frac{\partial}{\partial u}$,
2. $\Phi^* h = \eta_{ij} dx^i dx^j$, $\eta = \text{diag}(\mp, 1, \dots, 1)$

This is a principal bundle with structure group $SO(r, s) \times \mathbb{R}^{n-1}$ formed by matrices of the form:

$$\begin{pmatrix} 1 & \chi \\ 0 & A \end{pmatrix}, \quad A \in SO(r, s).$$

The right action of $(\mathbb{R}, +)$ on $\tilde{\mathcal{H}}_{\pm}$ lifts to an action on \mathcal{F} which commutes with the action of $SO(r, s) \times \mathbb{R}^{n-1}$.

It is important to observe at this point that this bundle admits specific types of (local) sections t , coming from sections $\hat{\tau}$ of $\tilde{\mathcal{H}}_{\pm}$, that have the equivariance property:

$$t \circ R_{\lambda} = R_{\lambda} \circ t, \quad \lambda \in \mathbb{R}.$$

Throughout, we shall only work with sections of this type; that we will refer to as *admissible frames*. Any two such admissible frames are related by a map $\mathcal{H}_{\pm} \rightarrow SO(r, s) \times \mathbb{R}^{n-1}$ (so they are transported by the right action along the fibre). By construction these frames must be of the form (n^a, m_i^a) and the corresponding dual co-frames $(\nabla_a u, m^i_a)$.

Since, under our assumption, $\nabla_n u = 1$, the zero set of u always define a section $\hat{\tau} : \mathcal{H}_{\pm} \rightarrow \tilde{\mathcal{H}}_{\pm}$. The other way round a section $\hat{\tau}$ of $\tilde{\mathcal{H}}_{\pm} \rightarrow \mathcal{H}_{\pm}$ uniquely defines a function u as the unique function such that $\nabla_n u = 1$ and $u \circ \hat{\tau} = 0$. Therefore, one sees that a section $\hat{\tau}$ defines, up to $SO(r, s)$, an admissible frame $(\nabla_a u, m^i_a)$.

This remark will serve to state the following.

Theorem 4.2.

The non-effective Cartan geometry (4.3) on \mathcal{H}_{\pm} induces on the extended boundary $\tilde{\mathcal{H}}_{\pm}$ a Cartan geometry modelled on

$$SO(1, n-1) \times \mathbb{R}^n / SO(r, s) \times \mathbb{R}^{n-1}. \quad (4.5)$$

These homogenous spaces are the (pseudo)-Carrollian spaces T_i and S_{pi} from [FHPS22] and in particular yield effective Cartan geometries.

In an admissible frame, given by a section $\hat{\tau}$ of $\tilde{\mathcal{H}}_{\pm} \rightarrow \mathcal{H}_{\pm}$, the connection is given by

$$\begin{aligned}\nabla_a \left(\tau_0^{-1} X^A \Pi_A^A \right) &= \tau_0^{-1} \pi^* \bar{W}_a^A \Pi_A^A + \nabla_a u I^A, \\ \nabla_a \left(\tau_0^{-1} \pi^* \bar{W}_b^A \Pi_A^A \right) &= \mp h_{ab} \left(\tau_0^{-1} X^A \Pi_A^A \right) + \left(\tau_0^{-1} v_{ab} \pm u h_{ab} \right) I^A, \\ \nabla_a I^A &= 0.\end{aligned}\tag{4.6}$$

By results from [Her22, Section 2] these Cartan geometries are equivalent to strong Carrollian geometries in the sense of [DGHZ14] (see also appendix of [BM18] for more details on these geometry).

Proof. Let τ and $\tilde{\tau} = \tau \pm u \hat{\sigma}$, we only need to check that the transformation rules of the above objects are consistent with those of a Cartan connection. In fact, only the term containing v might pose a problem.

Since by definition $u \circ \tau = 0$ and $u \circ \tilde{\tau} = \omega$, one has $u \mapsto u - \omega$ and, from (4.4),

$$\begin{aligned}\tau_0^{-1} v_{ab} \pm u h_{ab} &\mapsto \tau_0^{-1} v_{ab} + \nabla_a \nabla_b \omega \pm h_{ab} \omega \pm (u - \omega) h_{ab} \\ &= \tau_0^{-1} v_{ab} \pm u h_{ab} + \nabla_a \nabla_b \omega\end{aligned}$$

which is the correct transformation rule, see e.g. [Her22, Section 2]. We only need to check that this geometry is torsion free, this can be done by directly evaluating the curvature and making use of $v_{[ab]} = 0$ (see the explicit expression below (4.3)). \square

The precise relation between this effective Cartan geometry on $\tilde{\mathcal{H}}_{\pm}$ and the non-effective Cartan geometry realised on \mathcal{H}_{\pm} is given by the following.

Proposition 4.1. *The curvature $F^A_{\mathcal{B}cd}$ of the Cartan geometry given by Theorem 4.2 satisfies $n^c F^A_{\mathcal{B}cd} = 0$. In particular, one can quotient by the action of $(\mathbb{R}, +)$ and obtain in this way a (non-effective) Cartan geometry on \mathcal{H}_{\pm} . The resulting connection then coincides with (4.3).*

Proof. The constraint on the curvature can be checked by a direct computation. Then a tractor field

$$\tilde{T}^{\mathcal{A}} = \tilde{T}^0 I^{\mathcal{A}} + \tilde{T}^b \left(\tau_0^{-1} \pi^* \bar{W}_b^A \Pi_A^A \right) + \tilde{T}^- \left(\tau_0^{-1} X^A \Pi_A^A \right)$$

is parallel transported along the fibres of $\tilde{\mathcal{H}}_{\pm} \rightarrow \mathcal{H}_{\pm}$ if $n^c \nabla_c T^{\mathcal{A}} = 0$ i.e. iff

$$\partial_u \tilde{T}^0 + \tilde{T}^- = 0, \quad \partial_u \tilde{T}^a = 0, \quad \partial_u \tilde{T}^- = 0 \quad .$$

In other words, vertically constant tractors are related to tractors on \mathcal{H}_{\pm}

$$T^{\mathcal{A}} = T^0 I^{\mathcal{A}} + T^b \left(\tau_0^{-1} \bar{W}_b^A \Pi_A^A \right) + T^- \left(\tau_0^{-1} X^A \Pi_A^A \right)$$

via $(\tilde{T}^0, \tilde{T}^a, \tilde{T}^-) = (T^0 - u T^-, T^a, T^-)$. One can now check that differentiating $\tilde{T}^{\mathcal{A}}$ in the horizontal direction defines on \mathcal{H}_{\pm} a connection identical with (4.3). \square

4.3 Projective structure on the extended boundary

From Theorem 4.2, and results of [Her22, Section 2], we obtained that $\tilde{\mathcal{H}}_{\pm}$ is equipped with a torsion free linear connection ∇ satisfying

$$\nabla_c h_{ab} = 0, \quad \nabla_c n^a = 0. \quad (4.7)$$

(This is the strong Carrollian from [DGHZ14]). In particular $\tilde{\mathcal{H}}_{\pm}$ is equipped with a projective connection $[\nabla]$.

We now want to investigate aspects of this projective geometry. It will prove useful to introduce the following densified objects¹⁵,

$$\bar{n}^a := \tau_0^{-1} n^a, \quad \bar{h}_{ab} = \tau_0^2 h_{ab}.$$

Recall that then $|\det(\bar{h}^{ab})| = \tau_0^2$ and therefore the pair $(\bar{n}^a, \bar{h}_{ab})$ really is equivalent to (n^a, h_{ab}) .

Using \bar{n}^a one may reproduce the steps of the proof of [BH24, Proposition 5.1] and obtain the isomorphism:

Lemma 4.1.

$$\mathcal{E}(1)_{\tilde{\mathcal{H}}_{\pm}} \simeq \pi^* \mathcal{E}_{\partial M}(1).$$

One may also show that:

Lemma 4.2. *There is a well-defined Lie derivative operator $\mathcal{L}_n : \mathcal{E}_{\tilde{\mathcal{H}}_{\pm}}(1) \rightarrow C^\infty(\mathbb{R})$.*

1. *Projective densities that are pullbacks of densities on ∂M will be called **adapted scales**, they can be characterised by:*

$$\tau = \pi^* \bar{\tau} \Leftrightarrow \mathcal{L}_{\bar{n}} \tau = 0.$$

2. *Since dual tractors $\mathcal{T}_{\tilde{\mathcal{H}}_{\pm}}^*$ are identified with $J^1 \mathcal{E}(1)_{\tilde{\mathcal{H}}_{\pm}}$, the Lie derivative $\mathcal{L}_{\bar{n}}$ defines, via the bundle map $j_p^1 \sigma \mapsto (\mathcal{L}_{\bar{n}} \sigma)_p$, a canonical tractor field $I^{\mathbf{A}}$. Furthermore, we have the short exact sequence of vector bundles:*

$$0 \longrightarrow \pi^* \mathcal{T}_{\partial M}^* \xrightarrow{\Pi} \mathcal{T}_{\tilde{\mathcal{H}}_{\pm}}^* \xrightarrow{I} \mathbb{R} \longrightarrow 0.$$

Proposition 4.2. *The pair $(\bar{n}^a, \bar{h}_{ab})$ defines on $\tilde{\mathcal{H}}_{\pm}$ two projective tractor field $I^{\mathbf{A}}$ and $H_{\mathbf{AB}}$ given, in an adapted scale, by*

$$I^{\mathbf{A}} = \bar{n}^a W_a^{\mathbf{A}}, \quad H_{\mathbf{AB}} = (\bar{h}_{ab} - \nabla_a \tau_0 \nabla_b \tau_0) Z_{\mathbf{A}}^a Z_{\mathbf{B}}^b - \tau_0 \nabla_a \tau_0 2Z^a_{(\mathbf{A}} Y_{\mathbf{B})} + \tau_0^2 Y_{\mathbf{A}} Y_{\mathbf{B}}. \quad (4.8)$$

The inner product $H_{\mathbf{AB}}$ is degenerate and $I^{\mathbf{A}} H_{\mathbf{AB}} = 0$.

¹⁵In order not to proliferate notation, we abusively identify the densified objects on \mathcal{H}_{\pm} and their pullback on $\tilde{\mathcal{H}}_{\pm}$.

Proof. One only needs to check that, upon changing adapted scale, these tractors are invariant. However this is the case since then

$$\begin{aligned}\nabla_a \tau_0 &\mapsto \nabla_a \tau_0 + \Upsilon_a \tau_0, & W_a^{\mathbf{A}} &\mapsto W_a^{\mathbf{A}} + \Upsilon_a X^{\mathbf{A}}, \\ Z_{\mathbf{A}}^a &\mapsto Z_{\mathbf{A}}^a, & Y_{\mathbf{A}} &\mapsto Y_{\mathbf{A}} - \Upsilon_a Z_{\mathbf{A}}^a,\end{aligned}$$

and $n^a \Upsilon_a = 0$ due to the fact that we are restricting to adapted scales. \square

As was already suggested by Proposition 4.1, the Cartan geometry from Theorem 4.2 is not generic, rather one has.

Proposition 4.3.

The Cartan geometry given by Theorem 4.2 is obtained from the projective Cartan geometry on $\tilde{\mathcal{H}}_{\pm}$ by a holonomy reduction to $ISO(1, n-1)$.

This follows from the fact that $I^{\mathbf{A}}$ and $H_{\mathbf{AB}}$ are parallel transported

$$\nabla_c I^{\mathbf{A}} = 0, \qquad \nabla_c H_{\mathbf{AB}} = 0.$$

Proof. Since the content of the above proposition is written in a projectively invariant form, we can prove it in any scale that we like. In the adapted scale τ_0 , the projective Cartan connection is simply given by (4.6) and one can check that it satisfies $\nabla_c I^{\mathbf{A}} = 0$, $\nabla_c H_{\mathbf{AB}} = 0$. \square

To conclude this section, we observe that it would have also been possible to first construct the projective structure and deduce the Carrollian structures from the holonomy reduction. This point of view would emphasise the strong inspiration of the present work from the picture developed in [BH24]. The projective structure obtained here on the extended boundary, and the related question of its projective compactness, would be a natural starting point for recovering conformal null infinity \mathcal{I} from S_{pi}/T_1 as the projective boundary; this will be the object of future investigations.

A Proof of Lemma 3.3

In this section we resume the notations introduced in Section 3. We recall that, in the statement of Lemma 3.3, we set $\tilde{\tau} = \tau \mp \omega \hat{\sigma}$ for some arbitrary function ω , to simplify the presentation we will only treat the case with “+”. Let ∇_a (resp. $\tilde{\nabla}_a$) be the projective scale determined by τ (resp. $\tilde{\tau}$), for convenience, introduce:

$$\Upsilon_a = \tilde{\tau}^{-1} \nabla_a \tilde{\tau} \Rightarrow \tilde{\nabla}_a = \nabla_a - \Upsilon_a,$$

and:

$$\eta^a = \bar{\lambda} N^a = \nu^{-1} \hat{\sigma}^{-2} \zeta^{ab} \nabla_b \hat{\sigma} \Rightarrow \eta^a \nabla_a \hat{\sigma} = 1.$$

Our goal is to determine the relationship between the distinguished gauges $L_{\mathcal{A}}$ and $\tilde{L}_{\mathcal{A}}$ these scales determine by Proposition 3.7. To this end it is sufficient to compute the difference

$$\bar{\chi}_A = \tilde{\chi}_A - \chi_A$$

where χ_A (resp. $\tilde{\chi}_A$) are defined by Eq. (3.49).

Now:

$$\nabla_a \tilde{\tau} = \omega \nabla_a \hat{\sigma} + \nabla_a \omega \hat{\sigma}.$$

Hence:

$$\Upsilon_a = \tau^{-1} (\omega \nabla_a \hat{\sigma} + (\nabla_a \omega - \omega^2 \tau^{-1} \nabla_a \hat{\sigma}) \hat{\sigma}) + (\omega^3 \nabla_a \hat{\sigma} \tau^{-2} - \omega \nabla_a \omega \tau^{-1}) \hat{\sigma}^2 + O(\hat{\sigma}^3).$$

We will also need to compute, $\zeta^{ab} \Upsilon_a$. Using that $\zeta^{ab} \nabla_a \hat{\sigma} = \nu \hat{\sigma}^2 \eta^a$:

$$\begin{aligned} \zeta^{ab} \Upsilon_a &= \tau^{-1} \left(\hat{\sigma}^2 \nu \omega \eta^b + \zeta^{ab} \nabla_a \omega \hat{\sigma} - \omega \zeta^{ab} \nabla_a \omega \tau^{-1} \hat{\sigma}^2 + O(\hat{\sigma}^3) \right) \\ &= \tau^{-1} \hat{\sigma} \left(\zeta^{ab} \nabla_a \omega + (\nu \omega \eta^b - \omega \zeta^{ab} \nabla_a \omega \tau^{-1}) \hat{\sigma} + O(\hat{\sigma}^2) \right) \end{aligned}$$

By definition, we have:

$$\tilde{\nu} = \nu - 2\hat{\sigma} \nu \eta^b \Upsilon_b + \zeta^{ab} \Upsilon_a \Upsilon_b.$$

We evaluate the two terms separately, introducing the *weighted* quantities:

$$\nabla_\eta = \eta^a \nabla_a \quad \text{and} \quad |\nabla \omega|^2 = \zeta^{ab} \nabla_a \omega \nabla_b \omega,$$

one may write:

$$\eta^a \Upsilon_a = \tau^{-1} (\omega + (\nabla_\eta \omega - \omega^2 \tau^{-1}) \hat{\sigma} + (\omega^3 \tau^{-2} - \omega \nabla_\eta \tau^{-1}) \hat{\sigma}^2 + O(\hat{\sigma}^3))$$

and

$$\zeta^{ab} \Upsilon_a \Upsilon_b = \tau^{-2} \hat{\sigma}^2 (\omega^2 \nu + |\nabla \omega|^2) + O(\hat{\sigma}^3)$$

Hence:

$$\tilde{\nu} = \nu - 2\nu \hat{\sigma} \tau^{-1} \omega + (3\nu \tau^{-2} \omega^2 + \tau^{-2} |\nabla \omega|^2 - 2\nu \nabla_\eta \tau^{-1}) \hat{\sigma}^2 + O(\hat{\sigma}^3)$$

from which it follows that

$$\tilde{\nu}^{-1} = \nu^{-1}(1 + 2\hat{\sigma}\tau^{-1}\omega + (\omega^2\tau^{-2} - \nu^{-1}\tau^{-2}|\nabla\omega|^2 + 2\nabla_\eta\omega\tau^{-1})\hat{\sigma}^2 + O(\hat{\sigma}^3)).$$

It is now straightforward to compute $\tilde{\chi}_A$ in terms of χ_A . Working in the splitting determined by ∇ , we have:

$$\tilde{\chi}_A = -\frac{1}{2}\hat{\sigma}^{-1}\tilde{\nu}^{-1}Y_A + \frac{1}{2}\hat{\sigma}^{-2}\tilde{\nu}^{-1}(\nabla_a\hat{\sigma} - 2\Upsilon_a\hat{\sigma})Z_A^a.$$

The first component is easy seen to be:

$$-\frac{1}{2}\hat{\sigma}^{-1}\nu^{-1} - \tau^{-1}\nu^{-1}\omega + O(\hat{\sigma}).$$

For the second, we work order by order, ignoring the factor of $\frac{1}{2}$ and writing:

$$\nabla_a\hat{\sigma} - 2\Upsilon_a\hat{\sigma} = \nabla_a\hat{\sigma} - 2\hat{\sigma}\omega\nabla_a\hat{\sigma}\tau^{-1} - 2\hat{\sigma}^2\tau^{-1}(\nabla_a\omega - \omega^2\tau^{-1}\nabla_a\hat{\sigma}) + O(\hat{\sigma}^3)$$

we obtain the following expressions at each order:

Order	
$\hat{\sigma}^{-2}$	$\nu^{-1}\hat{\sigma}^{-2}\nabla_a\hat{\sigma}$
$\hat{\sigma}^{-1}$	$2\tau^{-1}\omega\nu^{-1}\nabla_a\hat{\sigma} - 2\nu^{-1}\omega\nabla_a\hat{\sigma}\tau^{-1} = 0$
$\hat{\sigma}^0$	$\nu^{-1}(\omega^2\tau^{-2} - \nu^{-1}\tau^{-2} \nabla\omega ^2 + 2\nabla_\eta\omega\tau^{-1})\nabla_a\hat{\sigma}$ $-4\tau^{-1}\omega^2\nabla_a\hat{\sigma}\nu^{-1}$ $-2\nu^{-1}\tau^{-1}(\nabla_a\omega - \omega^2\tau^{-1}\nabla_a\hat{\sigma})$

which yields the desired result, after observing that $\nabla\nu = O(\hat{\sigma})$ and $\nabla\tau = O(\hat{\sigma})$.

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