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ADDENDUM AND ERRATUM TO "LOCALIZATION FOR A MATRIX-VALUED ANDERSON MODEL"

HAKIM BOUMAZA AND SYLVAIN ZALCZER

ABSTRACT. In this short note we we give the proper rate of exponential decay for the Initial Length-Scale Estimate in the case of quasi-one-dimensional random operators of Schrödinger type. This corrects the statement and the demonstration of Proposition 5 in : H. Boumaza, Localization for a matrix-valued Anderson model, *Math. Phys. Anal. Geom.* 12(3), 255-286, 2009..

1. THE MODEL AND FIRST DEFINITIONS

In this note we give the proper rate of exponential decay for the ILSE in the case of quasi-one-dimensional operators of Schrödinger type. This corrects the statement and the demonstration of [2, Proposition 5], changing the $\gamma_1(E)$ into a $\gamma_N(E)$ in the exponential rate of decay. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space. Recall that in [2] one considers, for every $\omega \in \Omega$,

$$(1.1) \quad H_{\omega, \ell} = -\frac{d^2}{dx^2} \otimes I_N + \sum_{n \in \mathbb{Z}} V_{\omega}^{(n)}(x - \ell n),$$

acting on $L^2(\mathbb{R}) \otimes \mathbb{R}^N$, where $N \geq 1$ is an integer and $\ell > 0$ is a real number. For every $n \in \mathbb{Z}$, the functions $x \mapsto V_{\omega}^{(n)}(x)$ are symmetric matrix-valued functions, supported on $[0, \ell]$ and bounded uniformly on x , n and ω . The sequence $(V_{\omega}^{(n)})_{n \in \mathbb{Z}}$ is a sequence of *i.i.d* random variables on Ω . We also assume that the potential $x \mapsto \sum_{n \in \mathbb{Z}} V_{\omega}^{(n)}(x - \ell n)$ is such that $\{H_{\omega, \ell}\}_{\omega \in \Omega}$ is $\ell\mathbb{Z}$ -ergodic.

One considers the equation for the generalized eigenvalues, for every $\omega \in \Omega$,

$$(1.2) \quad H_{\omega, \ell} u = E u, \quad \text{where } E \in \mathbb{R} \text{ and } u = \begin{pmatrix} u' \\ u \end{pmatrix} : \mathbb{R} \rightarrow \mathbb{R}^{2N}.$$

We introduce, for $E \in \mathbb{R}$ and every $x, y \in \mathbb{R}$, the transfer matrix $T_x^y(E)$ of $H_{\omega, \ell}$ from x to y which maps a solution (u', u) at position x to the same solution at position y . It is defined by the relation

$$(1.3) \quad \begin{pmatrix} u'(y) \\ u(y) \end{pmatrix} = T_x^y(E) \begin{pmatrix} u'(x) \\ u(x) \end{pmatrix}$$

and in particular, $T_x^x(E) = I_{2N}$ for every $x \in \mathbb{R}$. The transfer matrices are elements of the real symplectic group

$$(1.4) \quad \text{Sp}_N(\mathbb{R}) = \{M \in \mathcal{M}_{2N}(\mathbb{R}) \mid {}^t M J M = J\}$$

with $J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$.

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For $E \in \mathbb{R}$, we denote by $\gamma_1(E), \dots, \gamma_{2N}(E)$ the *Lyapunov exponents* associated with the sequence $(T_{\ell n}^{\ell(n+1)}(E))_{n \in \mathbb{Z}}$ (see [2]). Let us also define the *Furstenberg group* $G(E)$ of $\{H_{\omega, \ell}\}_{\omega \in \Omega}$ at E as the closed group generated by the support of μ_E , where μ_E is the common distribution of the random matrices $T_{\ell n}^{\ell(n+1)}(E)$ and the closure is taken for the usual topology in $\mathcal{M}_{2N}(\mathbb{R})$:

$$G(E) = \overline{\langle \text{supp } \mu_E \rangle}.$$

2. A LARGE DEVIATION RESULT FOR BLOCKS OF PRODUCTS OF TRANSFER MATRICES

In order to estimate blocks of the products of transfer matrices as in [3], we introduce, given a vector subspace F of \mathbb{R}^{2N} , the orthogonal projection onto F , $\pi_F : \mathbb{R}^{2N} \rightarrow F$, and we set

$$\pi_F^* : \begin{array}{ccc} F & \rightarrow & \mathbb{R}^{2N} \\ x & \mapsto & x \end{array}.$$

Recall that a subspace $F \subset \mathbb{R}^{2N}$ is said *Lagrangian* if it is orthogonal to itself for J and of dimension N .

Let $s_p(\cdot)$ denotes the p -th singular value of the considered matrix. We prove a Large Deviation Property for the singular values of the products of transfer matrices. For the definition of p -contractivity and L_p -strong irreducibility we refer to [1].

Proposition 2.1. *We fix a compact interval $I \subset \mathbb{R}$. We assume that for every $E \in I$:*

- (1) *the Furstenberg group $G(E)$ is included in $\text{Sp}_N(\mathbb{R})$;*
- (2) *for every $p \in \{1, \dots, N\}$, $G(E)$ is L_p -strongly irreducible.*

Then for all $\epsilon > 0$ and all $E \in I$, there exist $C(\epsilon, E) > 0$ and $c(\epsilon, E) > 0$ such that, for all $p \in \{1, \dots, N\}$, any Lagrangian subspace F and all integers m, n ,

$$(2.1) \quad \mathbb{P} \left(\left\{ \left| \frac{1}{\ell(n-m)} \log s_p (T_{\ell m}^{\ell n}(E)) - \gamma_p(E) \right| \geq \epsilon \right\} \right) \leq C(\epsilon, E) e^{-c(\epsilon, E)\ell|n-m|}$$

and

$$(2.2) \quad \mathbb{P} \left(\left\{ \left| \frac{1}{\ell(n-m)} \log s_p (T_{\ell m}^{\ell n}(E)\pi_F^*) - \gamma_p(E) \right| \geq \epsilon \right\} \right) \leq C(\epsilon, E) e^{-c(\epsilon, E)\ell|n-m|}.$$

Remark 2.2. The constants $C(\epsilon, E)$ and $c(\epsilon, E)$ depend *a priori* on E and ϵ but they can be taken uniform in ϵ as it tends to 0 and uniform in E on the compact interval I . This is one of the reasons why we need to take the interval I compact.

Proof. Recall that for a symplectic matrix $M \in \text{Sp}_N(\mathbb{R})$, $s_p(M^{-1}) = s_p(M)$ for every $p \in \{1, \dots, 2N\}$. Hence one can assume that $m \leq n$ without loss of generality since $(T_{\ell m}^{\ell n}(E))^{-1} = T_{\ell n}^{\ell m}(E)$.

Let $p \in \{1, \dots, N\}$. For each $M \in \text{Sp}_N(\mathbb{R})$, we denote by \widehat{M} the matrix of $\text{GL}_k(\mathbb{R})$ such that

$$(2.3) \quad \widehat{M}_{ij} = \langle f_i, \Lambda^p M f_j \rangle.$$

where k is the dimension of L_p and (f_1, \dots, f_k) is an orthonormal basis of L_p , with $f_1 = e_1 \wedge \dots \wedge e_p$. Let us now denote by $\widehat{G}(E)$ the subgroup of $\text{GL}_k(\mathbb{R})$ which is generated by the matrices \widehat{M} for $M \in G(E)$. Then, since $G(E)$ is L_p -strongly irreducible, $\widehat{G}(E)$ is strongly

irreducible. Hence, applying [1, Theorem A.V.6.2], one gets the existence of $\alpha > 0$ such that for any $\epsilon > 0$ and any $\bar{x} \in \mathbf{P}(L_p)$,

$$(2.4) \quad \limsup_{|n-m| \rightarrow +\infty} \frac{1}{\ell|n-m|} \log \mathbb{P} \left(\left| \frac{1}{\ell|n-m|} \log(\|\Lambda^p T_{\ell m}^{\ell n}(E)\bar{x}\|) - (\gamma_1 + \dots + \gamma_p)(E) \right| > \epsilon \right) \leq -\alpha$$

and

$$(2.5) \quad \limsup_{|n-m| \rightarrow +\infty} \frac{1}{\ell|n-m|} \log \mathbb{P} \left(\left| \frac{1}{\ell|n-m|} \log(\|\Lambda^p T_{\ell m}^{\ell n}(E)\|) - (\gamma_1 + \dots + \gamma_p)(E) \right| > \epsilon \right) \leq -\alpha.$$

Indeed, since the function $x \mapsto V_\omega^{(n)}(x)$ is uniformly bounded in x , n and ω , and since the support of the common law of the transfer matrices is bounded, the assumption of finiteness of the integral in [1, Theorem A.V.6.2] is satisfied.

Now, let us take F a Lagrangian subspace of \mathbb{R}^{2N} . Then,

$$\begin{aligned} \|\Lambda^p(T_{\ell m}^{\ell n}(E)\pi_F^*)\| &= \sup_{\substack{u_1 \wedge \dots \wedge u_p \in \mathbf{P}(L_p) \\ u_i \in F}} \|(\Lambda^p T_{\ell m}^{\ell n}(E))(u_1 \wedge \dots \wedge u_p)\| \\ &= \|\Lambda^p T_{\ell m}^{\ell n}(E) \bar{u}\|, \quad \text{for some } \bar{u} \in \mathbf{P}(L_p) \end{aligned}$$

since the supremum is attained by compactness of $\mathbf{P}(L_p)$. Hence, (2.4) rewrites,

$$(2.6) \quad \limsup_{|n-m| \rightarrow +\infty} \frac{1}{\ell|n-m|} \log \mathbb{P} \left(\left| \frac{1}{\ell|n-m|} \log(\|\Lambda^p(T_{\ell m}^{\ell n}(E)\pi_F^*)\|) - (\gamma_1 + \dots + \gamma_p)(E) \right| > \epsilon \right) \leq -\alpha$$

for any Lagrangian F . Let, for $n, m \in \mathbb{Z}$, $p \in \{1, \dots, N\}$, $\epsilon > 0$ and F a Lagrangian,

$$A_{n,m,p}(\epsilon, F) = \left\{ \left| \frac{1}{\ell|n-m|} \log(\|\Lambda^p(T_{\ell m}^{\ell n}(E)\pi_F^*)\|) - (\gamma_1 + \dots + \gamma_p)(E) \right| > \epsilon \right\},$$

$$A_{n,m,p}(\epsilon) = \left\{ \left| \frac{1}{\ell|n-m|} \log(\|\Lambda^p T_{\ell m}^{\ell n}(E)\|) - (\gamma_1 + \dots + \gamma_p)(E) \right| > \epsilon \right\},$$

$$B_{n,m,p}(\epsilon, F) = \left\{ \left| \frac{1}{\ell|n-m|} (\log(\|\Lambda^p(T_{\ell m}^{\ell n}(E)\pi_F^*)\|) - \log(\|\Lambda^{p-1}(T_{\ell m}^{\ell n}(E)\pi_F^*)\|)) - \gamma_p(E) \right| > \epsilon \right\}$$

and

$$B_{n,m,p}(\epsilon) = \left\{ \left| \frac{1}{\ell|n-m|} (\log(\|\Lambda^p T_{\ell m}^{\ell n}(E)\|) - \log(\|\Lambda^{p-1} T_{\ell m}^{\ell n}(E)\|)) - \gamma_p(E) \right| > \epsilon \right\}.$$

Then, one has

$$(2.7) \quad B_{n,m,p}(2\epsilon) \subset A_{n,m,p}(\epsilon) \cap A_{n,m,p-1}(\epsilon) \quad \text{and} \quad B_{n,m,p}(2\epsilon, F) \subset A_{n,m,p}(\epsilon, F) \cap A_{n,m,p-1}(\epsilon, F).$$

Since for any $p \in \{1, \dots, N\}$, $\|\Lambda^p T_{\ell m}^{\ell n}(E)\| = s_1(T_{\ell m}^{\ell n}(E)) \cdots s_p(T_{\ell m}^{\ell n}(E))$, combining (2.7) and (2.5) one gets (2.1).

One also has, for any Lagrangian F , $\|\Lambda^p(T_{\ell m}^{\ell n}(E)\pi_F^*)\| = s_1(T_{\ell m}^{\ell n}(E)\pi_F^*) \cdots s_p(T_{\ell m}^{\ell n}(E)\pi_F^*)$. Hence, combining (2.7) and (2.6) one gets (2.2). This achieves the proof. \square

For any integer $n \in [-L/3, L/3]$, any $T \in \mathcal{M}_{2N}(\mathbb{R})$ and any vector subspace $F \subset \mathbb{R}^{2N}$, we define

$$(2.8) \quad \Omega_\epsilon^F[T] := \left\{ \max_{1 \leq p \leq N} \left(\left| \frac{1}{\ell|n-L|} \log s_p(T) - \gamma_p(E) \right| + \left| \frac{1}{\ell|n-L|} \log s_p(T\pi_F^*) - \gamma_p(E) \right| \right) \leq \frac{\epsilon}{100N} \right\}.$$

Let

$$(2.9) \quad F_+ := \left\{ \begin{pmatrix} u \\ 0 \end{pmatrix} \mid u \in \mathbb{R}^N \right\} \subset \mathbb{R}^{2N} \quad \text{and} \quad F_- := \left\{ \begin{pmatrix} 0 \\ v \end{pmatrix} \mid v \in \mathbb{R}^N \right\} \subset \mathbb{R}^{2N}$$

and for any $n \in \mathbb{Z}$,

$$(2.10) \quad F_n := \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{C}^{2N} \mid u = -\Phi_+^\downarrow(\ell n)(\Phi_+^\uparrow(\ell n))^{-1}v \right\}.$$

Note that since the transfer matrices are in $\text{Sp}_N(\mathbb{R})$, the vector subspace F_n is Lagrangian.

Finally, set

$$(2.11) \quad \Omega_\epsilon(n) := \Omega_\epsilon^{F_n}[{}^t T_{\ell L}^{\ell n}(E)] \cap \Omega_\epsilon^{F_+}[{}^t T_{\ell L}^{\ell n}(E)] \cap \Omega_\epsilon^{F_+}[T_{\ell L}^{\ell n}(E)] \cap \Omega_\epsilon^{F_-}[{}^t T_{\ell L}^{\ell n}(E)] \cap \Omega_\epsilon^{F_-}[T_{\ell L}^{\ell n}(E)].$$

3. INITIAL LENGTH-SCALE ESTIMATE

For $x \in \mathbb{R}$ and $L \geq 1$ an integer, we introduce the finite volume operators $H_{\omega,x,L}$ which are the restrictions of H_ω to intervals of the form $\Lambda_L(x) := [x - \ell L, x + \ell L]$ with Dirichlet boundary conditions. We can then define $R_{\omega,x,L}(z) := (H_{\omega,x,L} - z)^{-1}$ as the resolvent of $H_{\omega,x,L}$.

We will denote by $\chi_{x,L}$ the characteristic function of $\Lambda_L(x)$. We also denote by $\Gamma_{x,L}$ the characteristic function of the union of two regions near the boundary of $\Lambda_L(x)$: $[x - \ell(L-1), x - \ell(L-3)] \cup [x + \ell(L-3), x + \ell(L-1)]$.

Definition 3.1. Given $E \in \mathbb{R}$, $x \in \mathbb{Z}$ and $L \in 6\mathbb{N}$ with $E \notin \sigma(H_{\omega,x,L})$, we say that the box $\Lambda_L(x)$ is (ω, m, E) -regular for a given $m > 0$ if

$$(3.1) \quad \|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x,L/3}\| \leq e^{-mL}.$$

Denote $\Lambda_L := \Lambda_L(0)$.

Proposition 3.2 (ILSE for Schrödinger). *Let $I \subset \mathbb{R}$ be an open interval such that, for every $E \in I$, the Furstenberg group associated with $\{H_{\omega,\ell}\}_{\omega \in \Omega}$ is p -contracting and L_p -strongly irreducible, for every $p \in \{1, \dots, N\}$. Let $E \in I$. For every $\varepsilon > 0$, there exist $C, c > 0$ and $L_0 \in \mathbb{N}$ such that, for every $L \geq L_0$,*

$$(3.2) \quad \mathbb{P} \left(\left\{ \Lambda_L \text{ is } (\omega, \gamma_N(E) - \varepsilon, E) \text{-regular} \right\} \right) \geq 1 - Ce^{-c\ell L}.$$

To prove the correct ILSE for $\{H_{\omega,\ell}\}_{\omega \in \Omega}$, the first step is to give an explicit formula for the Green kernel of $H_{\omega,0,\ell}$ in terms of the solutions Φ_\pm of $H_{\omega,\ell}\Phi_\pm = E\Phi_\pm$ satisfying

$$(3.3) \quad \Phi_-(-\ell L) = \begin{pmatrix} 0 \\ I_N \end{pmatrix} \quad \text{and} \quad \Phi_+(\ell L) = \begin{pmatrix} 0 \\ I_N \end{pmatrix},$$

which differs from the formula given in [2, Eq. (67)].

Lemma 3.3. *Let $\omega \in \Omega$ and let $x, y \in \Lambda_L$. Assume that $\Phi_+(x)$ and $\Phi_-(x)$ are invertible as well as $\Phi'_+(x)\Phi_+(x)^{-1} - \Phi'_-(x)\Phi_-(x)^{-1}$. The Green kernel of $H_{\omega,0,\ell}$ is given by*

$$(3.4) \quad G_{\Lambda_L}^\omega(E, x, y) = \begin{cases} \Phi_+(y)(\Phi_+(x))^{-1} (\Phi'_+(x)\Phi_+(x)^{-1} - \Phi'_-(x)\Phi_-(x)^{-1})^{-1} & \text{for } x \leq y \\ \Phi_-(y)(\Phi_-(x))^{-1} (\Phi'_+(x)\Phi_+(x)^{-1} - \Phi'_-(x)\Phi_-(x)^{-1})^{-1} & \text{for } x \geq y \end{cases}$$

Proof. We would like to have

$$(3.5) \quad G_{\Lambda_L}^\omega(E, x, y) = \begin{cases} \Phi_+(y)\alpha_+(x) & \text{for } x \leq y \\ \Phi_-(y)\alpha_-(x) & \text{for } x \geq y \end{cases}$$

In addition, we want $G_{\Lambda_L}^\omega$ to be continuous, *i.e.* that for all $x \in \Lambda_L$

$$(3.6) \quad \Phi_+(x)\alpha_+(x) = \Phi_-(x)\alpha_-(x).$$

By definition, we must have, for all $\psi \in L^2(\Lambda_L)$ and almost all $x \in \Lambda_L$,

$$(H_{\omega,0,\ell} - E) \int_{\Lambda_L} G_{\Lambda_L}^\omega(E, x, y)\psi(x)dx = \psi(y).$$

Computing this expression explicitly and using the fact that $(H_{\omega,\ell} - E)\Phi_\pm = 0$, we find that for all $x \in \Lambda_L$

$$(3.7) \quad -\Phi_+(x)\alpha'_+(x) + \Phi_-(x)\alpha'_-(x) = I_N$$

The derivative of (3.6) combined with (3.7) implies that

$$(3.8) \quad \Phi'_+(x)\alpha_+(x) - \Phi'_-(x)\alpha_-(x) = I_N.$$

We can solve the system made of (3.6) and (3.8) to find that

$$(3.9) \quad \begin{cases} \alpha_+(x) = (\Phi_+(x))^{-1} (\Phi'_+(x)\Phi_+(x)^{-1} - \Phi'_-(x)\Phi_-(x)^{-1})^{-1} \\ \alpha_-(x) = (\Phi_-(x))^{-1} (\Phi'_+(x)\Phi_+(x)^{-1} - \Phi'_-(x)\Phi_-(x)^{-1})^{-1} \end{cases}$$

which concludes the proof. \square

Our goal is now to bound $\sup_{x,y \in \Lambda_{L_0}} |\Gamma_{0,L_0}(x)G_{\Lambda_{L_0}}^\omega(E, x, y)\chi_{0,L_0/3}(y)|$ with good probability. To this purpose, for any integer $n \in [-L/3, L/3]$, we consider the event $\Omega_\epsilon(n)$ as defined in (2.11).

Proposition 3.4. *On $\Omega_\epsilon := \cap_{n \in [-L/3, L/3]} \Omega_\epsilon(n)$, we have that, for all $x \in [-\ell L/3, \ell L/3]$ and all $y \in [\ell L - \ell, \ell L]$,*

$$(3.10) \quad \|G_{\Lambda_L}^\omega(E, x, y)\| \leq C e^{-2(\gamma_N(E) - \epsilon)\ell L}.$$

Proof. We begin with proving that on Ω_ϵ , for all $x \in [-\ell L/3, \ell L/3]$, $\Phi_+(x)$ is invertible and we estimate its inverse. It corresponds to bounding from below its N -th singular value.

Let n be the unique integer such that $x \in [n\ell, (n+1)\ell]$. We first remark that, for all $p = 1, \dots, 2N$,

$$(3.11) \quad s_p(T_{\ell L}^x) \geq s_p(T_{\ell L}^{\ell n})s_{2N}(T_{\ell n}^x) = s_p(T_{\ell L}^{\ell n})\|T_{\ell n}^x\|^{-1},$$

where the last equality comes from the fact that $T_{\ell n}^x$ is symplectic. But we know from [2, Lemma 6] that there exists a constant $C > 0$, independent of x and ω , such that $\|T_{\ell n}^x\| \leq C$. As a consequence, $s_p(T_{\ell L}^x) \geq C^{-1}s_p(T_{\ell L}^{\ell n})$.

We have the following singular value decomposition:

$$(3.12) \quad T_{\ell L}^x(E) = U\Sigma V,$$

where U and V are unitary and, since we have a symplectic matrix, we can write $\Sigma = \begin{pmatrix} \Sigma_+ & 0 \\ 0 & \Sigma_- \end{pmatrix}$ with $\Sigma_+ = \text{diag}(s_1, \dots, s_N)$ and $\Sigma_- = \text{diag}(1/s_1, \dots, 1/s_N)$ with $s_i \geq 1$ for every $i \in \{1, \dots, N\}$. We can write a block decomposition for U and V : $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$ and $V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$. We have then

$$\Phi_+(x) = (I_N \ 0) \begin{pmatrix} \Phi_+(x) \\ \Phi'_+(x) \end{pmatrix} = (I_N \ 0) T_{\ell L}^x(E) \begin{pmatrix} 0 \\ I_N \end{pmatrix} = U_{11}\Sigma_+V_{12} + U_{12}\Sigma_-V_{22}.$$

But since, on the one hand, the blocks U_{ij} , V_{ij} have norm less than 1 and, on the other hand, on the event $\Omega_\epsilon(n)$, $\|\Sigma_-\| \leq Ce^{-(\gamma_N(E)-\epsilon)\ell|L-n|}$, we can write that

$$\begin{aligned} s_N(\Phi_+(x)) &\geq s_N(U_{11}\Sigma_+V_{12}) - Ce^{-(\gamma_N(E)-\epsilon)\ell|L-n|} \\ &\geq s_N(U_{11})s_N(\Sigma_+)s_N(V_{12}) - Ce^{-(\gamma_N(E)-\epsilon)\ell|L-n|}. \end{aligned}$$

Since $s_N(\Sigma_+) \geq C^{-1}e^{(\gamma_N(E)-\epsilon)\ell|L-n|}$ (on the event $\Omega_\epsilon(n)$), we are left with controlling $s_N(U_{11})$ and $s_N(V_{12})$. We can prove as in Claim 3.4 and Remark 3.5 of [3] that, on $\Omega_\epsilon(n)$,

$$(3.13) \quad s_N(V_{12}) \geq e^{-\frac{\epsilon}{25}\ell|L-n|} \text{ and } s_N(U_{11}) \geq e^{-\frac{\epsilon}{25}\ell|L-n|}.$$

As a consequence, on $\Omega_\epsilon(n)$,

$$(3.14) \quad |s_N(\Phi_+(x))| \geq C^{-1}e^{(\gamma_N(E)-\frac{27\epsilon}{25})\ell|L-n|} - Ce^{-(\gamma_N(E)-\epsilon)\ell|L-n|}$$

For L large enough, one gets on Ω_ϵ ,

$$(3.15) \quad |s_N(\Phi_+(x))| \geq e^{(\gamma_N(E)-2\epsilon)\frac{2\ell L}{3}} > 0.$$

In particular, $\Phi_+(x)$ is invertible.

The next step to be able to apply Lemma 3.3 is to prove that $\Phi'_+(x)\Phi_+(x)^{-1} - \Phi'_-(x)\Phi_-(x)^{-1}$ is invertible. We prove as in Equation (33) of [3] that

$$(3.16) \quad s_N(\Phi'_+(x)\Phi_+(x)^{-1} - \Phi'_-(x)\Phi_-(x)^{-1}) \geq e^{-\frac{\epsilon}{2}\ell L}$$

Then, we have by Lemma 3.3 that

$$(3.17) \quad G_{\Lambda_L}^\omega(E, x, y) = \Phi_+(y)(\Phi_+(x))^{-1}(\Phi'_+(x)\Phi_+(x)^{-1} - \Phi'_-(x)\Phi_-(x)^{-1})^{-1}.$$

Finally, we remark that, for such a y , we have by [2, Lemma 6],

$$(3.18) \quad |\Phi_+(y)|^2 \leq N \exp\left(2 \int_y^{\ell L} |V_\omega(t)| dt\right) \leq C,$$

with C independent of ω and L .

Together with (3.18), (3.14) and (3.16), it gives that on Ω_ϵ

$$(3.19) \quad \|G_{\Lambda_L}^\omega(E, x, y)\| \leq Ce^{-2(\gamma_N(E)-\frac{7}{4}\epsilon)\ell L}.$$

□

The last step is to estimate the probability of Ω_ϵ . Since the sequence of transfer matrices of $\{H_{\omega,\ell}\}_{\omega \in \Omega}$ and its Furstenberg group satisfy the hypothesis of Proposition 2.1, there exist $C, c > 0$ such that $\mathbb{P}(\Omega_\epsilon(n)) \leq Ce^{-c\ell|n-L|}$. As a consequence,

$$(3.20) \quad \mathbb{P}^c(\Omega_\epsilon) \geq 1 - \sum_{n \in [-L/3, L/3]} Ce^{-c\ell|n-L|} \geq 1 - C'e^{-c'\ell L}$$

which prove the ILSE for $\{H_{\omega,\ell}\}_{\omega \in \Omega}$.

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