



**HAL**  
open science

## Addendum and erratum to "Localization for a matrix-valued Anderson model"

Hakim Boumaza, Sylvain Zalczer

► **To cite this version:**

Hakim Boumaza, Sylvain Zalczer. Addendum and erratum to "Localization for a matrix-valued Anderson model". 2024. hal-04604162

**HAL Id: hal-04604162**

**<https://hal.science/hal-04604162v1>**

Submitted on 6 Jun 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

# ADDENDUM AND ERRATUM TO "LOCALIZATION FOR A MATRIX-VALUED ANDERSON MODEL"

HAKIM BOUMAZA AND SYLVAIN ZALCZER

ABSTRACT. In this short note we we give the proper rate of exponential decay for the Initial Length-Scale Estimate in the case of quasi-one-dimensional random operators of Schrödinger type. This corrects the statement and the demonstration of Proposition 5 in : H. Boumaza, Localization for a matrix-valued Anderson model, *Math. Phys. Anal. Geom.* 12(3), 255-286, 2009..

## 1. THE MODEL AND FIRST DEFINITIONS

In this note we give the proper rate of exponential decay for the ILSE in the case of quasi-one-dimensional operators of Schrödinger type. This corrects the statement and the demonstration of [2, Proposition 5], changing the  $\gamma_1(E)$  into a  $\gamma_N(E)$  in the exponential rate of decay. Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a complete probability space. Recall that in [2] one considers, for every  $\omega \in \Omega$ ,

$$(1.1) \quad H_{\omega, \ell} = -\frac{d^2}{dx^2} \otimes I_N + \sum_{n \in \mathbb{Z}} V_{\omega}^{(n)}(x - \ell n),$$

acting on  $L^2(\mathbb{R}) \otimes \mathbb{R}^N$ , where  $N \geq 1$  is an integer and  $\ell > 0$  is a real number. For every  $n \in \mathbb{Z}$ , the functions  $x \mapsto V_{\omega}^{(n)}(x)$  are symmetric matrix-valued functions, supported on  $[0, \ell]$  and bounded uniformly on  $x$ ,  $n$  and  $\omega$ . The sequence  $(V_{\omega}^{(n)})_{n \in \mathbb{Z}}$  is a sequence of *i.i.d* random variables on  $\Omega$ . We also assume that the potential  $x \mapsto \sum_{n \in \mathbb{Z}} V_{\omega}^{(n)}(x - \ell n)$  is such that  $\{H_{\omega, \ell}\}_{\omega \in \Omega}$  is  $\ell\mathbb{Z}$ -ergodic.

One considers the equation for the generalized eigenvalues, for every  $\omega \in \Omega$ ,

$$(1.2) \quad H_{\omega, \ell} u = E u, \quad \text{where } E \in \mathbb{R} \text{ and } u = \begin{pmatrix} u' \\ u \end{pmatrix} : \mathbb{R} \rightarrow \mathbb{R}^{2N}.$$

We introduce, for  $E \in \mathbb{R}$  and every  $x, y \in \mathbb{R}$ , the transfer matrix  $T_x^y(E)$  of  $H_{\omega, \ell}$  from  $x$  to  $y$  which maps a solution  $(u', u)$  at position  $x$  to the same solution at position  $y$ . It is defined by the relation

$$(1.3) \quad \begin{pmatrix} u'(y) \\ u(y) \end{pmatrix} = T_x^y(E) \begin{pmatrix} u'(x) \\ u(x) \end{pmatrix}$$

and in particular,  $T_x^x(E) = I_{2N}$  for every  $x \in \mathbb{R}$ . The transfer matrices are elements of the real symplectic group

$$(1.4) \quad \text{Sp}_N(\mathbb{R}) = \{M \in \mathcal{M}_{2N}(\mathbb{R}) \mid {}^t M J M = J\}$$

with  $J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$ .

---

The research of H.B was partially funded by the ANR Project RAW, ANR-20-CE40-0012.

The research of S.Z was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 258734477 – SFB 1173.

For  $E \in \mathbb{R}$ , we denote by  $\gamma_1(E), \dots, \gamma_{2N}(E)$  the *Lyapunov exponents* associated with the sequence  $(T_{\ell n}^{\ell(n+1)}(E))_{n \in \mathbb{Z}}$  (see [2]). Let us also define the *Furstenberg group*  $G(E)$  of  $\{H_{\omega, \ell}\}_{\omega \in \Omega}$  at  $E$  as the closed group generated by the support of  $\mu_E$ , where  $\mu_E$  is the common distribution of the random matrices  $T_{\ell n}^{\ell(n+1)}(E)$  and the closure is taken for the usual topology in  $\mathcal{M}_{2N}(\mathbb{R})$ :

$$G(E) = \overline{\langle \text{supp } \mu_E \rangle}.$$

## 2. A LARGE DEVIATION RESULT FOR BLOCKS OF PRODUCTS OF TRANSFER MATRICES

In order to estimate blocks of the products of transfer matrices as in [3], we introduce, given a vector subspace  $F$  of  $\mathbb{R}^{2N}$ , the orthogonal projection onto  $F$ ,  $\pi_F : \mathbb{R}^{2N} \rightarrow F$ , and we set

$$\pi_F^* : \begin{array}{l} F \rightarrow \mathbb{R}^{2N} \\ x \mapsto x \end{array}.$$

Recall that a subspace  $F \subset \mathbb{R}^{2N}$  is said *Lagrangian* if it is orthogonal to itself for  $J$  and of dimension  $N$ .

Let  $s_p(\cdot)$  denotes the  $p$ -th singular value of the considered matrix. We prove a Large Deviation Property for the singular values of the products of transfer matrices. For the definition of  $p$ -contractivity and  $L_p$ -strong irreducibility we refer to [1].

**Proposition 2.1.** *We fix a compact interval  $I \subset \mathbb{R}$ . We assume that for every  $E \in I$ :*

- (1) *the Furstenberg group  $G(E)$  is included in  $\text{Sp}_N(\mathbb{R})$  ;*
- (2) *for every  $p \in \{1, \dots, N\}$ ,  $G(E)$  is  $L_p$ -strongly irreducible.*

*Then for all  $\epsilon > 0$  and all  $E \in I$ , there exist  $C(\epsilon, E) > 0$  and  $c(\epsilon, E) > 0$  such that, for all  $p \in \{1, \dots, N\}$ , any Lagrangian subspace  $F$  and all integers  $m, n$ ,*

$$(2.1) \quad \mathbb{P} \left( \left\{ \left| \frac{1}{\ell(n-m)} \log s_p (T_{\ell m}^{\ell n}(E)) - \gamma_p(E) \right| \geq \epsilon \right\} \right) \leq C(\epsilon, E) e^{-c(\epsilon, E)\ell|n-m|}$$

and

$$(2.2) \quad \mathbb{P} \left( \left\{ \left| \frac{1}{\ell(n-m)} \log s_p (T_{\ell m}^{\ell n}(E)\pi_F^*) - \gamma_p(E) \right| \geq \epsilon \right\} \right) \leq C(\epsilon, E) e^{-c(\epsilon, E)\ell|n-m|}.$$

*Remark 2.2.* The constants  $C(\epsilon, E)$  and  $c(\epsilon, E)$  depend *a priori* on  $E$  and  $\epsilon$  but they can be taken uniform in  $\epsilon$  as it tends to 0 and uniform in  $E$  on the compact interval  $I$ . This is one of the reasons why we need to take the interval  $I$  compact.

*Proof.* Recall that for a symplectic matrix  $M \in \text{Sp}_N(\mathbb{R})$ ,  $s_p(M^{-1}) = s_p(M)$  for every  $p \in \{1, \dots, 2N\}$ . Hence one can assume that  $m \leq n$  without loss of generality since  $(T_{\ell m}^{\ell n}(E))^{-1} = T_{\ell n}^{\ell m}(E)$ .

Let  $p \in \{1, \dots, N\}$ . For each  $M \in \text{Sp}_N(\mathbb{R})$ , we denote by  $\widehat{M}$  the matrix of  $\text{GL}_k(\mathbb{R})$  such that

$$(2.3) \quad \widehat{M}_{ij} = \langle f_i, \Lambda^p M f_j \rangle.$$

where  $k$  is the dimension of  $L_p$  and  $(f_1, \dots, f_k)$  is an orthonormal basis of  $L_p$ , with  $f_1 = e_1 \wedge \dots \wedge e_p$ . Let us now denote by  $\widehat{G}(E)$  the subgroup of  $\text{GL}_k(\mathbb{R})$  which is generated by the matrices  $\widehat{M}$  for  $M \in G(E)$ . Then, since  $G(E)$  is  $L_p$ -strongly irreducible,  $\widehat{G}(E)$  is strongly

irreducible. Hence, applying [1, Theorem A.V.6.2], one gets the existence of  $\alpha > 0$  such that for any  $\epsilon > 0$  and any  $\bar{x} \in \mathbf{P}(L_p)$ ,

$$(2.4) \quad \limsup_{|n-m| \rightarrow +\infty} \frac{1}{\ell|n-m|} \log \mathbb{P} \left( \left| \frac{1}{\ell|n-m|} \log(\|\Lambda^p T_{\ell m}^{\ell n}(E)\bar{x}\|) - (\gamma_1 + \dots + \gamma_p)(E) \right| > \epsilon \right) \leq -\alpha$$

and

$$(2.5) \quad \limsup_{|n-m| \rightarrow +\infty} \frac{1}{\ell|n-m|} \log \mathbb{P} \left( \left| \frac{1}{\ell|n-m|} \log(\|\Lambda^p T_{\ell m}^{\ell n}(E)\|) - (\gamma_1 + \dots + \gamma_p)(E) \right| > \epsilon \right) \leq -\alpha.$$

Indeed, since the function  $x \mapsto V_\omega^{(n)}(x)$  is uniformly bounded in  $x$ ,  $n$  and  $\omega$ , and since the support of the common law of the transfer matrices is bounded, the assumption of finiteness of the integral in [1, Theorem A.V.6.2] is satisfied.

Now, let us take  $F$  a Lagrangian subspace of  $\mathbb{R}^{2N}$ . Then,

$$\begin{aligned} \|\Lambda^p(T_{\ell m}^{\ell n}(E)\pi_F^*)\| &= \sup_{\substack{u_1 \wedge \dots \wedge u_p \in \mathbf{P}(L_p) \\ u_i \in F}} \|(\Lambda^p T_{\ell m}^{\ell n}(E))(u_1 \wedge \dots \wedge u_p)\| \\ &= \|\Lambda^p T_{\ell m}^{\ell n}(E) \bar{u}\|, \quad \text{for some } \bar{u} \in \mathbf{P}(L_p) \end{aligned}$$

since the supremum is attained by compactness of  $\mathbf{P}(L_p)$ . Hence, (2.4) rewrites,

$$(2.6) \quad \limsup_{|n-m| \rightarrow +\infty} \frac{1}{\ell|n-m|} \log \mathbb{P} \left( \left| \frac{1}{\ell|n-m|} \log(\|\Lambda^p(T_{\ell m}^{\ell n}(E)\pi_F^*)\|) - (\gamma_1 + \dots + \gamma_p)(E) \right| > \epsilon \right) \leq -\alpha$$

for any Lagrangian  $F$ . Let, for  $n, m \in \mathbb{Z}$ ,  $p \in \{1, \dots, N\}$ ,  $\epsilon > 0$  and  $F$  a Lagrangian,

$$A_{n,m,p}(\epsilon, F) = \left\{ \left| \frac{1}{\ell|n-m|} \log(\|\Lambda^p(T_{\ell m}^{\ell n}(E)\pi_F^*)\|) - (\gamma_1 + \dots + \gamma_p)(E) \right| > \epsilon \right\},$$

$$A_{n,m,p}(\epsilon) = \left\{ \left| \frac{1}{\ell|n-m|} \log(\|\Lambda^p T_{\ell m}^{\ell n}(E)\|) - (\gamma_1 + \dots + \gamma_p)(E) \right| > \epsilon \right\},$$

$$B_{n,m,p}(\epsilon, F) = \left\{ \left| \frac{1}{\ell|n-m|} (\log(\|\Lambda^p(T_{\ell m}^{\ell n}(E)\pi_F^*)\|) - \log(\|\Lambda^{p-1}(T_{\ell m}^{\ell n}(E)\pi_F^*)\|)) - \gamma_p(E) \right| > \epsilon \right\}$$

and

$$B_{n,m,p}(\epsilon) = \left\{ \left| \frac{1}{\ell|n-m|} (\log(\|\Lambda^p T_{\ell m}^{\ell n}(E)\|) - \log(\|\Lambda^{p-1} T_{\ell m}^{\ell n}(E)\|)) - \gamma_p(E) \right| > \epsilon \right\}.$$

Then, one has

$$(2.7) \quad B_{n,m,p}(2\epsilon) \subset A_{n,m,p}(\epsilon) \cap A_{n,m,p-1}(\epsilon) \quad \text{and} \quad B_{n,m,p}(2\epsilon, F) \subset A_{n,m,p}(\epsilon, F) \cap A_{n,m,p-1}(\epsilon, F).$$

Since for any  $p \in \{1, \dots, N\}$ ,  $\|\Lambda^p T_{\ell m}^{\ell n}(E)\| = s_1(T_{\ell m}^{\ell n}(E)) \cdots s_p(T_{\ell m}^{\ell n}(E))$ , combining (2.7) and (2.5) one gets (2.1).

One also has, for any Lagrangian  $F$ ,  $\|\Lambda^p(T_{\ell m}^{\ell n}(E)\pi_F^*)\| = s_1(T_{\ell m}^{\ell n}(E)\pi_F^*) \cdots s_p(T_{\ell m}^{\ell n}(E)\pi_F^*)$ . Hence, combining (2.7) and (2.6) one gets (2.2). This achieves the proof.  $\square$

For any integer  $n \in [-L/3, L/3]$ , any  $T \in \mathcal{M}_{2N}(\mathbb{R})$  and any vector subspace  $F \subset \mathbb{R}^{2N}$ , we define

$$(2.8) \quad \Omega_\epsilon^F[T] := \left\{ \max_{1 \leq p \leq N} \left( \left| \frac{1}{\ell|n-L|} \log s_p(T) - \gamma_p(E) \right| + \left| \frac{1}{\ell|n-L|} \log s_p(T\pi_F^*) - \gamma_p(E) \right| \right) \leq \frac{\epsilon}{100N} \right\}.$$

Let

$$(2.9) \quad F_+ := \left\{ \begin{pmatrix} u \\ 0 \end{pmatrix} \mid u \in \mathbb{R}^N \right\} \subset \mathbb{R}^{2N} \quad \text{and} \quad F_- := \left\{ \begin{pmatrix} 0 \\ v \end{pmatrix} \mid v \in \mathbb{R}^N \right\} \subset \mathbb{R}^{2N}$$

and for any  $n \in \mathbb{Z}$ ,

$$(2.10) \quad F_n := \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{C}^{2N} \mid u = -\Phi_+^\downarrow(\ell n)(\Phi_+^\uparrow(\ell n))^{-1}v \right\}.$$

Note that since the transfer matrices are in  $\text{Sp}_N(\mathbb{R})$ , the vector subspace  $F_n$  is Lagrangian.

Finally, set

$$(2.11) \quad \Omega_\epsilon(n) := \Omega_\epsilon^{F_n}[{}^t T_{\ell L}^{\ell n}(E)] \cap \Omega_\epsilon^{F_+}[{}^t T_{\ell L}^{\ell n}(E)] \cap \Omega_\epsilon^{F_+}[T_{\ell L}^{\ell n}(E)] \cap \Omega_\epsilon^{F_-}[{}^t T_{\ell L}^{\ell n}(E)] \cap \Omega_\epsilon^{F_-}[T_{\ell L}^{\ell n}(E)].$$

### 3. INITIAL LENGTH-SCALE ESTIMATE

For  $x \in \mathbb{R}$  and  $L \geq 1$  an integer, we introduce the finite volume operators  $H_{\omega,x,L}$  which are the restrictions of  $H_\omega$  to intervals of the form  $\Lambda_L(x) := [x - \ell L, x + \ell L]$  with Dirichlet boundary conditions. We can then define  $R_{\omega,x,L}(z) := (H_{\omega,x,L} - z)^{-1}$  as the resolvent of  $H_{\omega,x,L}$ .

We will denote by  $\chi_{x,L}$  the characteristic function of  $\Lambda_L(x)$ . We also denote by  $\Gamma_{x,L}$  the characteristic function of the union of two regions near the boundary of  $\Lambda_L(x)$ :  $[x - \ell(L-1), x - \ell(L-3)] \cup [x + \ell(L-3), x + \ell(L-1)]$ .

**Definition 3.1.** Given  $E \in \mathbb{R}$ ,  $x \in \mathbb{Z}$  and  $L \in 6\mathbb{N}$  with  $E \notin \sigma(H_{\omega,x,L})$ , we say that the box  $\Lambda_L(x)$  is  $(\omega, m, E)$ -regular for a given  $m > 0$  if

$$(3.1) \quad \|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x,L/3}\| \leq e^{-mL}.$$

Denote  $\Lambda_L := \Lambda_L(0)$ .

**Proposition 3.2** (ILSE for Schrödinger). *Let  $I \subset \mathbb{R}$  be an open interval such that, for every  $E \in I$ , the Furstenberg group associated with  $\{H_{\omega,\ell}\}_{\omega \in \Omega}$  is  $p$ -contracting and  $L_p$ -strongly irreducible, for every  $p \in \{1, \dots, N\}$ . Let  $E \in I$ . For every  $\varepsilon > 0$ , there exist  $C, c > 0$  and  $L_0 \in \mathbb{N}$  such that, for every  $L \geq L_0$ ,*

$$(3.2) \quad \mathbb{P} \left( \left\{ \Lambda_L \text{ is } (\omega, \gamma_N(E) - \varepsilon, E) \text{-regular} \right\} \right) \geq 1 - Ce^{-c\ell L}.$$

To prove the correct ILSE for  $\{H_{\omega,\ell}\}_{\omega \in \Omega}$ , the first step is to give an explicit formula for the Green kernel of  $H_{\omega,0,\ell}$  in terms of the solutions  $\Phi_\pm$  of  $H_{\omega,\ell}\Phi_\pm = E\Phi_\pm$  satisfying

$$(3.3) \quad \Phi_-(-\ell L) = \begin{pmatrix} 0 \\ I_N \end{pmatrix} \quad \text{and} \quad \Phi_+(\ell L) = \begin{pmatrix} 0 \\ I_N \end{pmatrix},$$

which differs from the formula given in [2, Eq. (67)].

**Lemma 3.3.** *Let  $\omega \in \Omega$  and let  $x, y \in \Lambda_L$ . Assume that  $\Phi_+(x)$  and  $\Phi_-(x)$  are invertible as well as  $\Phi'_+(x)\Phi_+(x)^{-1} - \Phi'_-(x)\Phi_-(x)^{-1}$ . The Green kernel of  $H_{\omega,0,\ell}$  is given by*

$$(3.4) \quad G_{\Lambda_L}^\omega(E, x, y) = \begin{cases} \Phi_+(y)(\Phi_+(x))^{-1} (\Phi'_+(x)\Phi_+(x)^{-1} - \Phi'_-(x)\Phi_-(x)^{-1})^{-1} & \text{for } x \leq y \\ \Phi_-(y)(\Phi_-(x))^{-1} (\Phi'_+(x)\Phi_+(x)^{-1} - \Phi'_-(x)\Phi_-(x)^{-1})^{-1} & \text{for } x \geq y \end{cases}$$

*Proof.* We would like to have

$$(3.5) \quad G_{\Lambda_L}^\omega(E, x, y) = \begin{cases} \Phi_+(y)\alpha_+(x) & \text{for } x \leq y \\ \Phi_-(y)\alpha_-(x) & \text{for } x \geq y \end{cases}$$

In addition, we want  $G_{\Lambda_L}^\omega$  to be continuous, *i.e.* that for all  $x \in \Lambda_L$

$$(3.6) \quad \Phi_+(x)\alpha_+(x) = \Phi_-(x)\alpha_-(x).$$

By definition, we must have, for all  $\psi \in L^2(\Lambda_L)$  and almost all  $x \in \Lambda_L$ ,

$$(H_{\omega,0,\ell} - E) \int_{\Lambda_L} G_{\Lambda_L}^\omega(E, x, y)\psi(x)dx = \psi(y).$$

Computing this expression explicitly and using the fact that  $(H_{\omega,\ell} - E)\Phi_\pm = 0$ , we find that for all  $x \in \Lambda_L$

$$(3.7) \quad -\Phi_+(x)\alpha'_+(x) + \Phi_-(x)\alpha'_-(x) = I_N$$

The derivative of (3.6) combined with (3.7) implies that

$$(3.8) \quad \Phi'_+(x)\alpha_+(x) - \Phi'_-(x)\alpha_-(x) = I_N.$$

We can solve the system made of (3.6) and (3.8) to find that

$$(3.9) \quad \begin{cases} \alpha_+(x) = (\Phi_+(x))^{-1} (\Phi'_+(x)\Phi_+(x)^{-1} - \Phi'_-(x)\Phi_-(x)^{-1})^{-1} \\ \alpha_-(x) = (\Phi_-(x))^{-1} (\Phi'_+(x)\Phi_+(x)^{-1} - \Phi'_-(x)\Phi_-(x)^{-1})^{-1} \end{cases}$$

which concludes the proof.  $\square$

Our goal is now to bound  $\sup_{x,y \in \Lambda_{L_0}} |\Gamma_{0,L_0}(x)G_{\Lambda_{L_0}}^\omega(E, x, y)\chi_{0,L_0/3}(y)|$  with good probability. To this purpose, for any integer  $n \in [-L/3, L/3]$ , we consider the event  $\Omega_\epsilon(n)$  as defined in (2.11).

**Proposition 3.4.** *On  $\Omega_\epsilon := \cap_{n \in [-L/3, L/3]} \Omega_\epsilon(n)$ , we have that, for all  $x \in [-\ell L/3, \ell L/3]$  and all  $y \in [\ell L - \ell, \ell L]$ ,*

$$(3.10) \quad \|G_{\Lambda_L}^\omega(E, x, y)\| \leq C e^{-2(\gamma_N(E) - \epsilon)\ell L}.$$

*Proof.* We begin with proving that on  $\Omega_\epsilon$ , for all  $x \in [-\ell L/3, \ell L/3]$ ,  $\Phi_+(x)$  is invertible and we estimate its inverse. It corresponds to bounding from below its  $N$ -th singular value.

Let  $n$  be the unique integer such that  $x \in [n\ell, (n+1)\ell]$ . We first remark that, for all  $p = 1, \dots, 2N$ ,

$$(3.11) \quad s_p(T_{\ell L}^x) \geq s_p(T_{\ell L}^{\ell n})s_{2N}(T_{\ell n}^x) = s_p(T_{\ell L}^{\ell n})\|T_{\ell n}^x\|^{-1},$$

where the last equality comes from the fact that  $T_{\ell n}^x$  is symplectic. But we know from [2, Lemma 6] that there exists a constant  $C > 0$ , independent of  $x$  and  $\omega$ , such that  $\|T_{\ell n}^x\| \leq C$ . As a consequence,  $s_p(T_{\ell L}^x) \geq C^{-1}s_p(T_{\ell L}^{\ell n})$ .

We have the following singular value decomposition:

$$(3.12) \quad T_{\ell L}^x(E) = U\Sigma V,$$

where  $U$  and  $V$  are unitary and, since we have a symplectic matrix, we can write  $\Sigma = \begin{pmatrix} \Sigma_+ & 0 \\ 0 & \Sigma_- \end{pmatrix}$  with  $\Sigma_+ = \text{diag}(s_1, \dots, s_N)$  and  $\Sigma_- = \text{diag}(1/s_1, \dots, 1/s_N)$  with  $s_i \geq 1$  for every  $i \in \{1, \dots, N\}$ . We can write a block decomposition for  $U$  and  $V$ :  $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$  and  $V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$ . We have then

$$\Phi_+(x) = (I_N \ 0) \begin{pmatrix} \Phi_+(x) \\ \Phi'_+(x) \end{pmatrix} = (I_N \ 0) T_{\ell L}^x(E) \begin{pmatrix} 0 \\ I_N \end{pmatrix} = U_{11}\Sigma_+V_{12} + U_{12}\Sigma_-V_{22}.$$

But since, on the one hand, the blocks  $U_{ij}$ ,  $V_{ij}$  have norm less than 1 and, on the other hand, on the event  $\Omega_\epsilon(n)$ ,  $\|\Sigma_-\| \leq Ce^{-(\gamma_N(E)-\epsilon)\ell|L-n|}$ , we can write that

$$\begin{aligned} s_N(\Phi_+(x)) &\geq s_N(U_{11}\Sigma_+V_{12}) - Ce^{-(\gamma_N(E)-\epsilon)\ell|L-n|} \\ &\geq s_N(U_{11})s_N(\Sigma_+)s_N(V_{12}) - Ce^{-(\gamma_N(E)-\epsilon)\ell|L-n|}. \end{aligned}$$

Since  $s_N(\Sigma_+) \geq C^{-1}e^{(\gamma_N(E)-\epsilon)\ell|L-n|}$  (on the event  $\Omega_\epsilon(n)$ ), we are left with controlling  $s_N(U_{11})$  and  $s_N(V_{12})$ . We can prove as in Claim 3.4 and Remark 3.5 of [3] that, on  $\Omega_\epsilon(n)$ ,

$$(3.13) \quad s_N(V_{12}) \geq e^{-\frac{\epsilon}{25}\ell|L-n|} \text{ and } s_N(U_{11}) \geq e^{-\frac{\epsilon}{25}\ell|L-n|}.$$

As a consequence, on  $\Omega_\epsilon(n)$ ,

$$(3.14) \quad |s_N(\Phi_+(x))| \geq C^{-1}e^{(\gamma_N(E)-\frac{27\epsilon}{25})\ell|L-n|} - Ce^{-(\gamma_N(E)-\epsilon)\ell|L-n|}$$

For  $L$  large enough, one gets on  $\Omega_\epsilon$ ,

$$(3.15) \quad |s_N(\Phi_+(x))| \geq e^{(\gamma_N(E)-2\epsilon)\frac{2\ell L}{3}} > 0.$$

In particular,  $\Phi_+(x)$  is invertible.

The next step to be able to apply Lemma 3.3 is to prove that  $\Phi'_+(x)\Phi_+(x)^{-1} - \Phi'_-(x)\Phi_-(x)^{-1}$  is invertible. We prove as in Equation (33) of [3] that

$$(3.16) \quad s_N(\Phi'_+(x)\Phi_+(x)^{-1} - \Phi'_-(x)\Phi_-(x)^{-1}) \geq e^{-\frac{\epsilon}{2}\ell L}$$

Then, we have by Lemma 3.3 that

$$(3.17) \quad G_{\Lambda_L}^\omega(E, x, y) = \Phi_+(y)(\Phi_+(x))^{-1}(\Phi'_+(x)\Phi_+(x)^{-1} - \Phi'_-(x)\Phi_-(x)^{-1})^{-1}.$$

Finally, we remark that, for such a  $y$ , we have by [2, Lemma 6],

$$(3.18) \quad |\Phi_+(y)|^2 \leq N \exp\left(2 \int_y^{\ell L} |V_\omega(t)| dt\right) \leq C,$$

with  $C$  independent of  $\omega$  and  $L$ .

Together with (3.18), (3.14) and (3.16), it gives that on  $\Omega_\epsilon$

$$(3.19) \quad \|G_{\Lambda_L}^\omega(E, x, y)\| \leq Ce^{-2(\gamma_N(E)-\frac{7}{4}\epsilon)\ell L}.$$

□

The last step is to estimate the probability of  $\Omega_\epsilon$ . Since the sequence of transfer matrices of  $\{H_{\omega,\ell}\}_{\omega \in \Omega}$  and its Furstenberg group satisfy the hypothesis of Proposition 2.1, there exist  $C, c > 0$  such that  $\mathbb{P}(\Omega_\epsilon(n)) \leq Ce^{-c\ell|n-L|}$ . As a consequence,

$$(3.20) \quad \mathbb{P}^c(\Omega_\epsilon) \geq 1 - \sum_{n \in [-L/3, L/3]} Ce^{-c\ell|n-L|} \geq 1 - C'e^{-c'\ell L}$$

which prove the ILSE for  $\{H_{\omega,\ell}\}_{\omega \in \Omega}$ .

#### REFERENCES

- [1] Ph. Bougerol and J. Lacroix, *Products of Random Matrices with Applications to Schrödinger Operators*, Progress in probability and statistics, Birkhäuser, Boston, 1985.
- [2] H. Boumaza, Localization for a matrix-valued Anderson model, *Math. Phys. Anal. Geom.* 12(3), 255-286, 2009.
- [3] D. Macera and S. Sodin. Anderson Localisation for Quasi-One-Dimensional Random Operators, *Ann. Henri Poincaré* 23, No. 12, 4227–4247, 2022.

LAGA, UNIVERSITÉ SORBONNE PARIS NORD, 99 AVENUE J.B. CLÉMENT, F-93430 VILLETANEUSE  
*Email address:* boumaza@math.univ-paris13.fr

KIT, ENGLERSTRASSE 2, 76131, KARLSRUHE, GERMANY  
*Email address:* sylvain.zalczer@kit.edu