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Stabilization of a Beam-Heat Coupled Transmission System with Memory under Various Thermal Laws.

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Abstract

This study delves into the dynamic behavior of a coupled system, which consists of an Euler-Bernoulli beam equation and a heat equation with memory. The system's dynamics are influenced by different heat conduction laws, characterized by a parameter m . Specifically, we examine the Coleman-Gurtin and Gurtin-Pipkin laws, each of which exerts distinct effects on the temperature evolution within the material. By formulating a set of partial differential equations along with appropriate boundary and initial conditions, we lay the groundwork for analyzing the coupled system's behavior. Our investigation primarily centers on understanding the asymptotic properties of the solution semigroup within the framework of Dafermos history space. Through this analysis, we gain deeper insights into stability properties and potential applications under various heat conduction laws. The paper concludes with a conjecture and discussion regarding possible future extensions of this research.

Keywords: Beam and heat interactions; Memory effect by Dafermos; Coleman-Gurtin law; Gurtin-Pipkin law; Polynomial and exponential stability.

2020 MSC: 93D20, 47D60, 74F05.

1 Introduction

The continuous heat exchange processes experienced by structural components such as bridges, roofs, and asphalt are influenced by various factors including solar radiation, convection, and re-radiation to or from the surrounding environment. However, these structures also face the persistent risk of damage from fires, often exacerbated by frequent vehicle incidents. Of particular concern is the susceptibility of steel bridges to collapse or undergo significant damage during fires, emphasizing the urgent need for enhanced fire resistance measures in structural steel members [32].

Single-layer beams are typically represented by the Euler-Bernoulli partial differential equation (PDE), unlike the Rayleigh beam, which incorporates rotational inertia,

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$$u_{tt} + u_{xxxx} = 0,$$

where $u(x, t)$ represents the shape of the centerline of the beam. In many cases, multi-layer models are variations or bounded/compact perturbations of the Euler-Bernoulli beam model. These components, including single-layered and multi-layered “sandwich” beams [29, 39], are widely used in industrial applications. The Euler-Bernoulli-based beam model serves as a foundational framework, with multi-layer models often introduced as bounded or compact perturbations of this model.

Temperature loading can trigger a distinct form of instability known as “thermal buckling” [21]. When a temperature variation occurs in a beam, thermal stresses develop within the structure, potentially leading to significant lateral displacements once the temperature surpasses a critical threshold known as the “thermal bifurcation point”. This phenomenon is observed in various scenarios, including railway tracks, where steel tracks expand on hot days, generating compressive axial forces that may induce buckling due to constrained expansion. Similar challenges are encountered in sub-sea pipelines transporting hot products, where thermal expansion-induced compression can lead to buckling on the seabed. Thermal buckling poses a significant risk of catastrophic failure, as the buckled structure may no longer support loads as intended.

Consequently, the coupling between beams and heat has emerged as a focal point in civil, aerospace, and mechanical engineering. Thermal loadings often represent the most critical loading condition, highlighting the need to analyze beam behavior under thermal loads to enhance stability and resilience under diverse circumstances [24].

The coupling between mechanical deformation and the thermal energy of an elastic body has been recognized since the nineteenth century, with Lord Kelvin providing the first theoretical treatment in 1853. Subsequent research by [5] established stabilizing criteria for solutions of the Cauchy problem for dynamical linear thermoelasticity, while Green and Naghdi [25] proposed a thermoelasticity model of type II. Extensive examination of these models has contributed to a comprehensive understanding of their asymptotic behavior [28]. Moreover, physical problems coupling a heat equation with an elastic structure are addressed in works such as [14] and [33].

In recent literature, [45] and [47] investigate the stability of transmission systems consisting of Euler-Bernoulli beams and heat equations with boundary coupling. Their studies establish asymptotic formulas for eigenvalues and eigenfunctions, demonstrating exponential stability and fulfillment of the Riesz basis property.

1.1 Including Material’s Thermal Memory

Temperature variations within a material are related to heat flow in solids. The fundamental relation between the heat flux vector \mathbf{q} and the temperature gradient ∇u is formulated as linear relationship given by the classical Fourier constitutive law [23] that provides a reliable description of heat transmission via conduction in various materials, this relation is given by

$$\mathbf{q} = -\kappa \nabla u, \tag{1.1}$$

where $\kappa > 0$ is a material constant called the thermal conductivity. Take note of how heat transfers from hot to cold areas against the gradient in temperature. Hence, from the energy balance equation

$$\partial_t u + \operatorname{div} \mathbf{q} = 0, \tag{1.2}$$

we deduce the well-known heat equation (see [18])

$$\partial_t u - \kappa \nabla^2 u = 0. \tag{1.3}$$

It is highlighted in [38] the limitations of this parabolic heat equation, and two major issues are pointed out with this idea. To begin with, memory effects - which could be common in some materials, especially at low temperatures - can not be taken into consideration in the parabolic heat equation. Furthermore, this equation predicts that a thermal disturbance at one location in the body will immediately affect all other locations, which means that finite thermal discontinuities in Fourier heat

conductors have to spread at an infinite speed. This phenomena is unexpected and unobserved in real conductors, and frequently called the “paradox of heat conduction”, see e.g., [10, 22].

These observations suggest that Fourier’s law may be insufficient for elastic materials with memory, prompting efforts to develop more comprehensive constitutive assumptions to better relate heat flux to thermal history, as proposed by Nunziato [38]. This has led to various attempts to introduce hyperbolicity into heat conduction models to address these limitations. For instance, in piezoelectric materials like PolyVinylidene Fluoride (PVDF), theoretical descriptions have revealed shortcomings in mechanical properties such as creep [41]. Creep manifests as the drift of piezoelectric actuators’ output displacement under voltage, becoming noticeable during slow-speed and open-loop operations [26]. Consequently, dielectric relaxation is considered for predicting polymer behavior over longer timescales [44]. Linear viscoelastic theory better describes PVDF’s time response under applied stress [13, 37]. This underscores the need for more comprehensive constitutive assumptions in material models, especially those with memory effects, reinforcing the motivation for introducing hyperbolicity into heat conduction models to accurately represent material behavior [34].

In this paper, we primarily extend the classical Fourier constitutive law to more complex heat laws. One primary technique is represented by the Gurtin-Pipkin [27] and Coleman-Gurtin [11] laws, which utilize the ”memory kernel” g to modify the term ∇u present in the Fourier law [18]. Here, the convolution kernel $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a convex, summable function, whose precise properties will be elucidated later. Specifically, for $m \in (0, 1]$, we have

$$\mathbf{q} = -(1 - m)\kappa\nabla u - m\kappa \int_0^\infty g(s)\nabla u(t - s)ds, \quad (1.4)$$

where the initial past history of the temperature $u(x, t)|_{t \leq 0}$ is presumed to be known and regarded as an initial datum of the problem. Depending on the value of m , the heat conduction laws are as follows:

m	Heat Conduction Law
0	Fourier
$(0, 1)$	Coleman-Gurtin
1	Gurtin-Pipkin

Table 1: The heat conduction laws depending on m .

Inserting (1.4) into (1.2), we derive the integro-differential equation that will be the subject of our paper:

$$\partial_t u - (1 - m)\kappa\Delta u - m\kappa \int_0^\infty g(s)\Delta u(t - s)ds = 0. \quad (1.5)$$

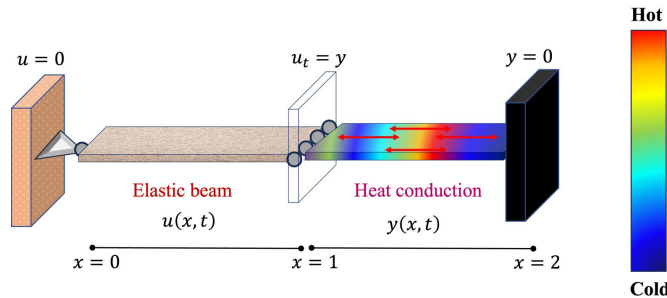


Figure 1: This illustration depicts an Euler-Bernoulli beam occupying the domain $(0,1)$, hinged at its left end, and free to slide at its right end. The heat equation governs the domain $(1,2)$. At the transmission point $x=1$, the velocity of the beam’s deflection is matched by that of the heat, showcasing the dynamic interplay between structural mechanics and thermal dynamics in the system.

To our knowledge, no previous work has explored the analysis of a system where a beam equation interacts with a heat equation involving a memory term. To address this gap, we investigate the system under various types of heat conduction. Although heat flow analysis in structures is typically a three-dimensional problem, simplifying assumptions can suffice for certain scenarios, such as concrete beams, frames, or bridge cross-sections, where one- or two-dimensional models can provide meaningful insights [20, 19, 24].

In this work, we investigate a one-dimensional system in which the Euler-Bernoulli beam equation is coupled with a heat equation incorporating memory effects. This coupling is established through transmission conditions at the interface in the transmission line configuration as below

$$\left\{ \begin{array}{ll} u_{tt} + u_{xxxx} = 0, & (x, t) \in Q_1, \\ y_t - c(1 - m)y_{xx} - cm \int_0^\infty g(s) y_{xx}(x, t - s) ds = 0, & (x, t) \in Q_2, \\ u_{xxx}(1, t) + c(1 - m)y_x(1, t) + cm \int_0^\infty g(s) y_x(1, t - s) ds = 0, & t \in \mathbb{R}_+^*, \\ u_t(1, t) = y(1, t), & t \in \mathbb{R}_+^*, \\ u(0, t) = y(2, t) = 0, & t \in \mathbb{R}_+^*, \\ u_{xx}(0, t) = 0, \quad u_x(1, t) = 0, & t \in \mathbb{R}_+^*, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = v_0(x), & x \in (0, 1), \\ y(x, 0) = y_0(x), \quad y(x, -s) = \varphi_0(x, s), & x \in (1, 2), \end{array} \right. \quad (\text{B-H}_m)$$

where $s > 0$, $Q_1 = (0, 1) \times (0, \infty)$, $Q_2 = (1, 2) \times (0, \infty)$, u_0, v_0, y_0 are assigned data, c is a strictly positive constant and φ_0 accounts for the so-called initial past history of y , and $m \in (0, 1]$ is a fixed parameter used to distinguish the two considered heat conduction laws: Coleman-Gurtin or Gurtin-Pipkin.

The convolution kernel $g : [0, +\infty[\mapsto [0, +\infty)$ is a convex integrable function (thus non-increasing and vanishing at infinity) of unit total mass (i.e. $\int_0^\infty g(s) ds = 1$), taking the explicit form $g(s) = \int_s^\infty \mu(r) dr$, $s \geq 0$,

where $\mu : (0, +\infty) \mapsto [0, +\infty)$ is the memory kernel satisfying the following conditions:

$$\left\{ \begin{array}{l} \mu \in L^1(0, \infty) \cap C^1(0, \infty), \quad \int_0^\infty \mu(r) dr = g(0) > 0, \\ \mu(0) = \lim_{s \rightarrow 0} \mu(s) < \infty, \text{ and } \mu \text{ satisfies the Dafermos} \\ \text{condition: } \mu'(s) \leq -K_\mu \mu(s) \text{ with } K_\mu > 0. \end{array} \right. \quad (\text{H})$$

The system shares similarities with fluid-structure interactions in linearized one-dimensional Navier-Stokes equations (heat equation-parabolic) and solid structures (elastic/piezoelectric beams-hyperbolic), as explored in [40, 46, 47, 48]. Such interactions are critical in scenarios, including aerodynamics in aircraft airflow [9] and piezoelectric systems in underwater environments [31].

Prior to detailing our primary findings, we offer a brief overview of recent advancements in different beam models employing various thermal laws.

In [16], the authors investigated the stability properties of a linear thermoelastic Timoshenko-Gurtin-Pipkin system wherein thermal coupling affects both bending moment and shear force. They demonstrate that the associated solution semigroup in the history space framework of Dafermos [12] is exponentially stable under either full Dirichlet or mixed Dirichlet-Neumann boundary conditions, irrespective of the model's structural parameter values.

In [15], three thermoelastic beam systems with hyperbolic heat conduction were analyzed. Beginning with the Bresse-Gurtin-Pipkin system, the author provides a necessary and sufficient condition for exponential stability and the optimal polynomial decay rate when the condition is violated. Subsequently, they continue an inquiry recently initiated in literature by drawing similar conclusions for the Bresse-Maxwell-Cattaneo system. Lastly, considering the Timoshenko-Gurtin-Pipkin system, they achieve the optimal polynomial decay rate when the known exponential stability condition is not satisfied.

In [1], the author investigated a piezoelectric beam under (Coleman or Pipkin)-Gurtin thermal law with magnetic effects. They demonstrate that under Coleman-Gurtin heat conduction, the system

exhibits exponential stability. Furthermore, by analyzing the system under Gurtin-Pipkin thermal law, they establish a polynomial energy decay rate of type t^{-1} . In [40], the consideration of boundary feedback controllers yielded an exponential decay rate. Moreover, a robust Finite Difference algorithm was proposed, maintaining the same decay rate.

The structure of this paper is as follows: In Section 2, we initiate our discussion by framing our problem in a semigroup setting to establish its well-posedness. Section 3 delves into the analysis of strong stability, demonstrating that the system's energy diminishes towards zero due to a diagonalization method. Section 4 establishes that under Coleman-Gurtin heat conduction, the system is polynomially stable with a decay rate of t^{-4} , and under Gurtin-Pipkin heat conduction, it achieves exponential stability. Finally, in Section 5, we conclude with a conjecture and discuss potential future extensions.

2 Functional Setting and Well-posedness

To rigorously define our problem, we introduce the functional space

$$V := H_*^2(0, 1) = \{u \in H^2(0, 1); u(0) = u_x(1) = 0\}$$

equipped with the inner product $\langle u, v \rangle_{2,*} = \langle u_{xx}, v_{xx} \rangle_{L^2(0,1)}$ for all $u, v \in H_*^2(0, 1)$. We also define

$$H_R^1(1, 2) := \{y \in H^1(1, 2); y(2) = 0\},$$

endowed with the norm $\|y\|_{H_R^1(1,2)}^2 = \int_1^2 |y_x|^2 dx$. Additionally, we introduce the memory space W defined as

$$W = L^2(\mathbb{R}^+, H_R^1(1, 2))$$

with the inner product

$$\langle \gamma_1, \gamma_2 \rangle_W = cm \int_1^2 \int_0^\infty \mu(s) \gamma_{1x} \overline{\gamma_{2x}} ds dx$$

for all $\gamma_1, \gamma_2 \in W$. For $s > 0$, we consider the auxiliary function

$$\gamma(x, s) = \int_0^s y(x, t - r) dr, \quad x \in (1, 2) \text{ and } s > 0.$$

Let

$$\zeta(x, t) := c(1 - m)y + cm \int_0^\infty \mu(s) \gamma(s) ds.$$

Now, we can rewrite the system as follows

$$\begin{cases} u_{tt} + u_{xxxx} = 0, & (x, t) \in Q_1, \\ y_t - \zeta_{xx} = 0, & (x, t) \in Q_2, \\ \gamma_t + \gamma_s - y = 0, & (x, t) \in Q_2, \\ u_{xxx}(1, t) = -\zeta_x(1), & t \in \mathbb{R}_+^*, \\ u_t(1, t) = y(1, t), & t \in \mathbb{R}_+^*, \\ u(0, t) = y(2, t) = 0, & t \in \mathbb{R}_+^*, \\ u_{xx}(0, t) = 0, u_x(1, t) = 0, & t \in \mathbb{R}_+^*, \\ u(x, 0) = u_0(x), u_t(x, 0) = v_0(x), & x \in (0, 1), \\ y(x, 0) = y_0(x), y(x, -s) = \varphi_0(x, s), & x \in (1, 2), \end{cases} \quad (2.1)$$

and $s > 0$. To compute the energy of this system, we begin by multiplying (2.1)₁ by $\overline{u_t}$, taking the real part, and integrating by parts over $(0, 1)$

$$\frac{1}{2} \frac{d}{dt} \int_0^1 (|u_t|^2 + |u_{xx}|^2) dx + \Re(u_{xxx}(1) \overline{u_t}(1)) = 0. \quad (2.2)$$

Next, we multiply (2.1)₂ by \bar{y} , integrate by parts over (1, 2), then using the fact that $y(2) = 0$, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_1^2 |y|^2 dx + \Re \left(cm \int_1^2 \int_0^\infty \mu(s) \gamma_x(s) \bar{y}_x dx \right) + \Re(c(1-m)y_x(1)\bar{y}(1)) \\ + c(1-m) \int_1^2 |y_x|^2 dx + \Re \left(cm \int_0^\infty \mu(s) \gamma_x(1, s) \bar{y}(1) ds \right) = 0. \end{aligned} \quad (2.3)$$

Let B denote the third term on the left-hand side of (2.3). To substitute B , we differentiate (2.1)₃ with respect to x , multiply by $cm\mu(s)\bar{\gamma}_x$, integrate over (1, 2) \times (0, ∞), we get

$$B = \frac{1}{2} \frac{d}{dt} cm \int_1^2 \int_0^\infty \mu(s) |\gamma_x|^2 ds dx - \frac{cm}{2} \int_1^2 \int_0^\infty \mu'(s) |\gamma_x|^2 ds dx + \frac{cm}{2} \left[\int_1^2 \mu(s) |\gamma_x|^2 dx \right]_0^\infty.$$

Substituting B into (2.3), we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_1^2 [|y|^2 + cm\mu(s) |\gamma_x|^2] dx + c(1-m) \int_1^2 |y_x|^2 dx + \Re(c(1-m)y_x(1)\bar{y}(1)) \\ - \frac{cm}{2} \int_1^2 \int_0^\infty \mu'(s) |\gamma_x|^2 ds dx + \Re \left(cm \int_0^\infty \mu(s) \gamma_x(1, s) \bar{y}(1) ds \right) = 0. \end{aligned}$$

Adding this equation to (2.2) and using the transmission conditions in (2.1) and the conditions (H), we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left\{ \int_1^2 |y|^2 dx + cm \int_1^2 \int_0^\infty \mu(s) |\gamma_x|^2 ds dx + \left[\int_0^1 (|u_t|^2 + |u_{xx}|^2) dx \right] \right\} \\ = -c(1-m) \int_1^2 |y_x|^2 dx + \frac{cm}{2} \int_1^2 \int_0^\infty \mu'(s) |\gamma_x|^2 ds dx. \end{aligned}$$

Hence, the energy of the system (2.1) can be written as

$$E(t) = \frac{1}{2} \int_0^1 (|u_t|^2 + |u_{xx}|^2) dx + \frac{1}{2} \int_1^2 |y|^2 dx + \frac{1}{2} cm \int_1^2 \int_0^\infty \mu(s) |\gamma_x|^2 ds dx,$$

and its time derivative as

$$\frac{dE(t)}{dt} = \frac{cm}{2} \int_1^2 \int_0^\infty \mu'(s) |\gamma_x|^2 ds dx - c(1-m) \int_1^2 |y_x|^2 dx.$$

Now, we define the following energy space

$$\mathcal{H} = V \times L^2(0, 1) \times L^2(1, 2) \times W,$$

equipped with the inner product

$$\langle U_1, U_2 \rangle_{\mathcal{H}} = \int_0^1 u_{1xx} \bar{u}_{2xx} dx + \int_0^1 v_1 \bar{v}_2 dx + \int_1^2 y_1 \bar{y}_2 dx + \langle \gamma_1, \gamma_2 \rangle_W,$$

where $U_i = (u_i, v_i, y_i, \gamma_i) \in \mathcal{H}$ for $i = 1, 2$. Let $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \mapsto \mathcal{H}$ be an unbounded linear operator defined by $\mathcal{A}(u, v, y, \gamma)^\top = (v, -u_{xxxx}, \zeta_{xx}, -\gamma_s + y)^\top$ for all $(u, v, y, \gamma)^\top \in D(\mathcal{A})$. The domain $D(\mathcal{A})$ is given by

$$D(\mathcal{A}) = \left\{ (u, v, y, \gamma) \in \mathcal{H}; v \in V, u \in H^4(0, 1) \cap V, y \in H_R^1(1, 2), \zeta \in H^2(1, 2), \right. \\ \left. \gamma_s \in W, u_{xx}(0) = 0, \gamma(\cdot, 0) = 0, u_{xxx}(1) = -\zeta_x(1), v(1) = y(1) \right\}.$$

Thus, we can rewrite the system (2.1) as an evolution equation on the Hilbert space \mathcal{H}

$$U_t = \mathcal{A}U \quad \text{and} \quad U(0) = U_0, \quad (2.4)$$

where $U_0 = (u_0, v_0, y_0, \gamma_0)^\top$ with $\gamma_0 = \int_0^s \varphi_0(x, r) dr$.

For all $U = (u, v, y, \gamma)^\top \in D(\mathcal{A})$, using Hypothesis (H) and the fact that $m \in [0, 1]$, we have

$$\Re(\langle \mathcal{A}U, U \rangle_{\mathcal{H}}) = \frac{cm}{2} \int_1^2 \int_0^\infty \mu'(s) |\gamma_x|^2 ds dx - c(1-m) \int_1^2 |y_x|^2 dx \leq 0,$$

which implies that \mathcal{A} is dissipative. By the Lax-Milgram Theorem, one can prove the existence of $U \in D(\mathcal{A})$ solving the equation $-\mathcal{A}U = F$ for all $F = (f_1, f_2, f_3, f_4(\cdot, s))^\top \in \mathcal{H}$. Therefore, the unbounded linear operator \mathcal{A} is m-dissipative, consequently, $0 \in \rho(\mathcal{A})$. Moreover, The Lumer-Phillips Theorem guarantees that \mathcal{A} generates a C_0 -semigroup of contractions $(e^{t\mathcal{A}})_{t \geq 0}$. Thus, the Cauchy problem (2.4) admits a unique solution having the following representation

$$U(t) = e^{t\mathcal{A}}U_0, \quad t \geq 0,$$

which leads to the well-posedness of (2.4). Hence, semigroup theory allows to show the following existence and uniqueness results.

Theorem 2.1. *For any $U_0 \in \mathcal{H}$, the problem (2.4) admits a unique weak solution $U \in C^0(\mathbb{R}^+, \mathcal{H})$. Moreover, if $U_0 \in D(\mathcal{A})$, the problem (2.4) admits a unique strong solution $U \in C^1(\mathbb{R}^+, \mathcal{H}) \cap C^0(\mathbb{R}^+, D(\mathcal{A}))$.*

3 Strong Stability

Theorem 3.1. *Let $m \in [0, 1]$ and assume that Hypothesis (H) holds. Then, the C_0 -semigroup of contractions $(e^{t\mathcal{A}})_{t \geq 0}$ is strongly stable in \mathcal{H} , i.e., for all $U_0 \in \mathcal{H}$, the solution of (2.4) satisfies*

$$\lim_{t \rightarrow +\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0.$$

According to Arendt-Batty [6], to prove Theorem 3.1, we need to show that \mathcal{A} has no pure imaginary eigenvalues and $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable. It will be sufficient to prove the following proposition.

Proposition 3.2. *Let $m \in [0, 1]$ and assume that Hypothesis (H) holds. We have*

$$i\mathbb{R} \subset \rho(\mathcal{A}). \quad (3.1)$$

Proof. We will proceed by contradiction. Suppose that (3.1) is false. Since we proved in Section 2 that $0 \in \rho(\mathcal{A})$, it remains to prove that there exists $\lambda_\infty \in \mathbb{R}^*$ such that $i\lambda_\infty \notin \rho(\mathcal{A})$. According to Remark A.1 in [2], [36, p.25], and Remark A.3 in [1], there exists $\{\lambda_n, U^n = (u^n, v^n, y^n, \gamma^n)^\top\}_{n \geq 1} \subset \mathbb{R}^* \times D(\mathcal{A})$ such that $\lambda_n \rightarrow \lambda_\infty$ as $n \rightarrow \infty$, $|\lambda_n| < |\lambda_\infty|$, and

$$\|U^n\|_{\mathcal{H}} = \|(u^n, v^n, y^n, \gamma^n)^\top\|_{\mathcal{H}} = 1, \quad (3.2)$$

such that

$$(i\lambda_n I - \mathcal{A})U^n = F^n := (f_n^1, f_n^2, f_n^3, f_n^4(\cdot, s)) \rightarrow 0 \quad (3.3)$$

in \mathcal{H} as $n \rightarrow \infty$. Detailing the equations in (3.3), we get the following system

$$\begin{cases} i\lambda_n u^n - v^n = f_n^1 & \text{in } V, \\ i\lambda_n v^n + u_{xxxx}^n = f_n^2 & \text{in } L^2(0, 1), \\ i\lambda_n y^n - \zeta_{xx}^n = f_n^3 & \text{in } L^2(1, 2), \\ i\lambda_n \gamma^n + \gamma_s^n - y^n = f_n^4(\cdot, s) & \text{in } W. \end{cases} \quad (3.4)$$

We will divide this proof into four steps. To prove (3.1), we look for a contradiction with (3.2) such as $\|U^n\|_{\mathcal{H}} \rightarrow 0$.

Step 1: By Lemma 3.5 in [3], for $m \in [0, 1]$ and when Hypothesis (H) holds, we get the two limits as $n \rightarrow \infty$

$$\int_1^2 |y^n|^2 dx \rightarrow 0, \quad \int_1^2 \int_0^\infty \mu(s) |\gamma_x^n|^2 ds dx \rightarrow 0. \quad (3.5)$$

Step 2: Inserting (3.4)₁ into (3.4)₂, we get

$$\lambda_n^2 u^n - u_{xxxx}^n = -[i\lambda_n f_n^1 + f_n^2]. \quad (3.6)$$

By the same computation as in Step 1 in the proof of [3, Lemma 3.6], we obtain

$$|\lambda_n u^n(1)| \rightarrow 0, \quad |u_{xxx}^n(1)| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.7)$$

Step 3: Let's define $w_n := \sqrt{|\lambda_n|}$ and $F_1^n := i\lambda_n f_n^1 + f_n^2$. Our goal is to establish that the solution of (3.6) satisfies the equation

$$u_{xx}^n(1) = -w_n^2 u^n(1) + \frac{\tanh(w_n)}{w_n} u_{xxx}^n(1) - \frac{1}{w_n \cosh(w_n)} \int_0^1 \sinh(w_n s) F_1^n(s) ds. \quad (3.8)$$

We define $U^n := (u^n, u_x^n, u_{xx}^n, u_{xxx}^n)^\top$ to reformulate (3.6) as a first order ODE

$$U_x^n = A_n U^n + F_2^n \quad (3.9)$$

where

$$A_n = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ w_n^4 & 0 & 0 & 0 \end{pmatrix}, \quad F_2^n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ F_1^n \end{pmatrix},$$

$U^n(1) = [u^n(1), 0, u_{xx}^n(1), u_{xxx}^n(1)]^\top$, and

$U^n(0) = [0, u_x^n(0), 0, u_{xxx}^n(0)]^\top$. The eigenvalues of A_n are $w_n, -w_n, iw_n, -iw_n$, all of which are simple. Therefore, the matrix A_n is diagonalizable and can be expressed as $A_n = P_n A_{1,n} P_n^{-1}$, where $A_{1,n} = \text{diag}[w_n, -w_n, iw_n, -iw_n]$ and P_n is the corresponding eigenvector matrix defined by

$$P_n := \begin{pmatrix} \frac{1}{w_n^3} & -\frac{1}{w_n^3} & \frac{i}{w_n^3} & -\frac{i}{w_n^3} \\ \frac{1}{w_n^2} & \frac{1}{w_n^2} & -\frac{1}{w_n^2} & -\frac{1}{w_n^2} \\ \frac{1}{w_n} & -\frac{1}{w_n} & \frac{i}{w_n} & \frac{i}{w_n} \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

For any $r \in \mathbb{R}$, let

$$E^n(r) = (E_{i,j}^n(r))_{1 \leq i,j \leq 4} := e^{A_n r} = P_n e^{A_{1,n} r} P_n^{-1}.$$

Here, the elements of $E^n(r)$ are given by

$$\begin{aligned} E_{ii}^n(r) &= \frac{1}{2} [\cosh(rw_n) + \cos(rw_n)], \quad \text{for } i = 1, 2, 3, 4, \\ E_{12}^n(r), E_{23}^n(r), E_{34}^n(r) &= \frac{1}{2w_n} [\sin(rw_n) + \sinh(rw_n)], \\ E_{13}^n(r) = E_{24}^n(r) &= \frac{1}{2w_n^2} [\cosh(rw_n) - \cos(rw_n)], \\ E_{14}^n(r) &= \frac{1}{2w_n^3} [\sinh(rw_n) - \sin(rw_n)], \\ E_{21}^n(r), E_{32}^n(r), E_{43}^n(r) &= \frac{w_n}{2} [\sinh(rw_n) - \sin(rw_n)], \\ E_{31}^n(r), E_{42}^n(r) &= w_n^4 E_{13}^n(r), \\ E_{41}^n(r) &= w_n^4 E_{12}^n(r). \end{aligned} \quad (3.10)$$

We employ classical ODE arguments to express the solution of (3.9) as

$$U^n(x) = e^{A_n(x-1)}U^n(1) - \int_x^1 e^{A_n(x-r)}F_1^n(r)dr.$$

Thus, at $x = 0$, we have

$$U^n(0) = e^{-A_n}U^n(1) - \int_0^1 e^{-A_nr}F_1^n(r)dr,$$

or equivalently,

$$\begin{pmatrix} 0 \\ u_x^n(0) \\ 0 \\ u_{xxx}^n(0) \end{pmatrix} = E^n(-1) \begin{pmatrix} u^n(1) \\ 0 \\ u_{xx}^n(1) \\ u_{xxx}^n(1) \end{pmatrix} - \int_0^1 E^n(-r) \begin{pmatrix} 0 \\ 0 \\ 0 \\ F_1^n(r) \end{pmatrix} dr.$$

Using (3.10), we derive the following equations

$$0 = E_{11}^n(-1)u^n(1) + E_{13}^n(-1)u_{xx}^n(1) + E_{14}^n(-1)u_{xxx}^n(1) - \int_0^1 E_{14}^n(-r)F_1^n(r)dr, \quad (3.11)$$

$$0 = w_n^4 E_{13}^n(-1)u^n(1) + E_{11}^n(-1)u_{xx}^n(1) + E_{12}^n(-1)u_{xxx}^n(1) - \int_0^1 E_{12}^n(-r)F_1^n(r)dr. \quad (3.12)$$

Multiplying (3.11) by w_n^2 and adding it to (3.12), we obtain

$$\begin{aligned} [w_n^2 E_{13}^n(-1) + E_{11}^n(-1)]u_{xx}^n(1) &= - [E_{11}^n(-1)w_n^2 + w_n^2 E_{13}^n(-1)w_n^2] u^n(1) \\ &\quad - [w_n^2 E_{14}^n(-1) + E_{12}^n(-1)] u_{xxx}^n(1) + \int_0^1 [w_n^2 E_{14}^n(-r) + E_{12}^n(-r)] F_1^n(r)dr. \end{aligned} \quad (3.13)$$

Here, we use the relations

$$\begin{aligned} w_n^2 E_{13}^n(-1) + E_{11}^n(-1) &= \cosh(w_n), \text{ and} \\ w_n^2 E_{14}^n(-1) + E_{12}^n(-1) &= \frac{-1}{w_n} \sinh(w_n). \end{aligned}$$

Substituting the above equations into (3.13), we arrive at

$$\cosh(w_n)u_{xx}^n(1) = -\cosh(w_n)w_n^2 u^n(1) + \frac{\sinh(w_n)}{w_n} u_{xxx}^n(1) - \frac{1}{w_n} \int_0^1 \sinh(w_nr) [i\lambda_n f_n^1(r) + f_n^2(r)] dr.$$

Since $\cosh(w_n) \neq 0$, we obtain the desired equality (3.8).

Step 4: The final step involves proving the estimation

$$\|\lambda_n u^n\|^2 + \|u_{xx}^n\|^2 \longrightarrow 0 \text{ as } n \rightarrow +\infty. \quad (3.14)$$

By passing to the limit in (3.8) and using (3.7), the fact that $w_\infty = \sqrt{|\lambda_\infty|} \neq 0$, and $\|F^n\|_{\mathcal{H}} \rightarrow 0$, we obtain

$$|u_{xx}^n(1)| \xrightarrow[n \rightarrow \infty]{} 0. \quad (3.15)$$

Next, multiplying (3.6) by $2x\overline{u_x^n}$, integrating over $(0, 1)$ and taking the real part, we get

$$-\int_0^1 |\lambda_n u^n|^2 dx + |\lambda_n u^n(1)|^2 - 3 \int_0^1 |u_{xx}^n|^2 dx + |u_{xx}^n(1)|^2 = 2 \Re \left(\int_0^1 x [i\lambda_n f_n^1 + f_n^2] \overline{u_x^n} dx \right).$$

Passing to the limit in the above equation, and using (3.7), (3.15), and the fact that u_x^n is uniformly bounded in $L^2(0, 1)$ and $\|F^n\|_{\mathcal{H}} \rightarrow 0$, we obtain the desired estimation (3.14).

Thus, we have shown that $\|U^n\|_{\mathcal{H}}^2 \xrightarrow[n \rightarrow \infty]{} 0$, which contradicts the assumption that $\|U^n\|_{\mathcal{H}} = 1$. This completes the proof. \square

4 Stability Results under Different Heat Conduction Models

In this section, we establish the polynomial stability of our system with decay rate of t^{-4} under the Coleman-Gurtin heat conduction (i.e., for $m \in (0, 1)$), and we achieve the exponential stability of the system under the Gurtin-Pipkin heat conduction (i.e., for $m = 1$). The main results are summarized in the following theorems

Theorem 4.1 (Polynomial Stability). *Assume that Hypothesis (H) holds and $m \in (0, 1)$. The C_0 -semigroup $(e^{t\mathcal{A}})_{t \geq 1}$ is polynomially stable; that is, there exists a constant $C > 0$ such that for every $U_0 \in D(\mathcal{A})$, we have*

$$E(t) \leq \frac{C}{t^4} \|U_0\|_{D(\mathcal{A})}, \quad t > 0. \quad (4.1)$$

Theorem 4.2 (Exponential Stability). *Assume that Hypothesis (H) holds and $m = 1$. Then, the C_0 -semigroup of contractions $(e^{t\mathcal{A}})_{t \geq 0}$ is exponentially stable, that is, there exist constants $C_1 \geq 1$ and $\tau_1 > 0$ independent of U_0 such that*

$$\|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} \leq C_1 e^{-\tau_1 t} \|U_0\|_{\mathcal{H}}, \quad t \geq 0.$$

To prove the polynomial stability, we will use a frequency domain approach given by Borichev and Tomilov [8] (see also [7] and [35]), and for the exponential stability, we will use a result of Huang [30] and Prüss [42]. Both approaches require verifying that $i\mathbb{R} \subset \rho(\mathcal{A})$ and

$$\limsup_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} \frac{1}{|\lambda|^\ell} \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(H)} < \infty. \quad (\text{H}_2)$$

The results are summarized in the following table:

m	ℓ	Stability Result
$(0, 1)$	$\frac{1}{2}$	polynomial energy decay rate of type t^{-4}
1	0	exponential stability

Table 2: The stability results obtained for $m \in (0, 1]$.

Since the condition $i\mathbb{R} \subset \rho(\mathcal{A})$ is verified in Proposition 3.2, it remains to prove that the condition (H_2) holds. By contradiction, suppose there exists $\{\lambda_n, U^n := (u^n, v^n, y^n, \gamma^n)^\top\} \subset \mathbb{R}_+^* \times D(\mathcal{A})$ with

$$\lambda_n \rightarrow +\infty \quad \text{and} \quad \|U^n\|_{\mathcal{H}} = 1, \quad (4.2)$$

such that

$$\lambda_n^\ell (i\lambda_n I - \mathcal{A})U^n = \mathcal{F}^n := (f_n^1, f_n^2, f_n^3, f_n^4(\cdot, s))^\top, \quad (4.3)$$

and $\|\mathcal{F}^n\| \rightarrow 0$ in \mathcal{H} . Let $F^n := \lambda^{-\ell} \mathcal{F}^n$ and $T^n = \|U^n\| \|F^n\| + \|F^n\|^2$. We shall find a contradiction with (4.2) such that $\|U^n\|_{\mathcal{H}} = o(1)$. For simplicity, we drop the index n for the remainder of this proof. Detailing (4.3), we obtain the following system

$$\begin{cases} i\lambda u - v = \lambda^{-\ell} f^1 & \text{in } V, \\ i\lambda v + u_{xxxx} = \lambda^{-\ell} f^2 & \text{in } L^2(0, 1), \\ i\lambda y - \zeta_{xx} = \lambda^{-\ell} f^3 & \text{in } L^2(1, 2), \\ i\lambda \gamma + \gamma_s - y = \lambda^{-\ell} f^4(\cdot, s) & \text{in } W. \end{cases} \quad (4.4)$$

For clarity, we will divide the proof into several Lemmas.

Lemma 4.3. *Assume that Hypotheses (H) holds and $m \in (0, 1]$. Then, the solution $(u, v, y, \gamma) \in D(\mathcal{A})$ of (4.3) satisfies the following estimates*

$$\int_1^2 \int_0^\infty \mu(s) |\gamma_x|^2 ds dx = o(\lambda^{-\ell}), \quad (4.5)$$

$$\|y\|_{L^2(1,2)}^2 = \begin{cases} o(\lambda^{-\ell-1}), & \text{if } m \in (0, 1), \\ o(\lambda^{-\ell}), & \text{if } m = 1. \end{cases} \quad (4.6)$$

$$|\xi_x(1)|^2, |y(1)|^2 = \begin{cases} o(\lambda^{-\ell+\frac{1}{2}}), & \text{if } m \in (0, 1), \\ o(\lambda^{-\ell}), & \text{if } m = 1. \end{cases} \quad (4.7)$$

Proof. By the results obtained in Lemma 4.3 , 4.4, 4.5, 5.5 and 5.6 in [3], and using (4.2), along with the facts that $\|U\| = 1$ and $\|F\| = o(1)$, we get the following results

$$\int_1^2 \int_0^\infty \mu(s) |\gamma_x|^2 ds dx \lesssim \lambda^{-\ell} \|F\|^2 = o(\lambda^{-\ell}),$$

$$\|y\|_{L^2(0,1)}^2 \lesssim \begin{cases} \lambda^{-\ell-1} T = o(\lambda^{-\ell-1}), & \text{if } m \in (0, 1), \\ \lambda^{-\ell} T = o(\lambda^{-\ell}), & \text{if } m = 1, \end{cases}$$

and

$$|\zeta_x(1)|^2, |y(1)|^2 \lesssim \begin{cases} \lambda^{-\ell+\frac{1}{2}} T = o(\lambda^{-\ell+\frac{1}{2}}), & m \in (0, 1), \\ \lambda^{-\ell} T = o(\lambda^{-\ell}), & m = 1, \end{cases}$$

□

Lemma 4.4. For $m \in (0, 1]$ and under Hypothesis (H), we have the following estimation

$$\int_0^1 |\lambda u|^2 dx + 3 \int_0^1 |u_{xx}|^2 dx = \begin{cases} o(\lambda^{-\ell+\frac{1}{2}}) + o(1), & \text{if } m \in (0, 1), \\ o(\lambda^{-\ell}), & \text{if } m = 1. \end{cases} \quad (4.8)$$

Proof. To find the above estimation, we will follow the coming steps.

Step 1: The continuity condition $v(1) = y(1)$, together with (4.7) gives

$$|v(1)|^2 = \begin{cases} o(\lambda^{-\ell+\frac{1}{2}}) & \text{if } m \in (0, 1) \\ o(\lambda^{-\ell}), & \text{if } m = 1. \end{cases}$$

From (4.4)₁, the above estimation, along with Young's and Poincaré's inequalities, and using the fact that $\|F\|_{\mathcal{H}} = o(1)$, we get

$$\begin{aligned} |\lambda u(1)|^2 &\leq 2 \left[|v(1)|^2 + \lambda^{-2\ell} |f^1(1)|^2 \right] \lesssim \left[|v(1)|^2 + \lambda^{-2\ell} \|f_{xx}^1\|^2 \right] \\ &= \begin{cases} o(\lambda^{-\ell+\frac{1}{2}}) & \text{if } m \in (0, 1), \\ o(\lambda^{-\ell}), & \text{if } m = 1. \end{cases} \end{aligned} \quad (4.9)$$

Additionally, using the transmission condition $u_{xxx}(1) = -\xi_x(1)$ with (4.7), we obtain the estimation

$$|u_{xxx}(1)|^2 = \begin{cases} o(\lambda^{-\ell+\frac{1}{2}}) & \text{if } m \in (0, 1) \\ o(\lambda^{-\ell}), & \text{if } m = 1. \end{cases} \quad (4.10)$$

Step 2: We aim to prove the following estimation

$$\int_0^1 |\lambda u|^2 dx + 3 \int_0^1 |u_{xx}|^2 dx - |u_{xx}(1)|^2 = \begin{cases} o(\lambda^{-\ell+\frac{1}{2}}) & \text{if } m \in (0, 1) \\ o(\lambda^{-\ell}), & \text{if } m = 1. \end{cases} \quad (4.11)$$

Inserting (4.4)₁ into (4.4)₂ we get

$$\lambda^2 u - u_{xxxx} = -\lambda^{-\ell} [i\lambda f^1 + f^2]. \quad (4.12)$$

Multiplying the above equation by $2x\bar{u}_x$, taking the real part, and then integrating by parts over $(0, 1)$, we obtain

$$\begin{aligned} \int_0^1 |\lambda u|^2 dx - |\lambda u(1)|^2 + 3 \int_0^1 |u_{xx}|^2 dx - |u_{xx}(1)|^2 &= \lambda^{-\ell} \Re \left(\int_0^1 x f^2 \bar{u}_x dx \right) \\ &+ 2\lambda^{-\ell} \Re [i\lambda f^1(1)\bar{u}(1)] - 2\lambda^{-\ell} \Re \left(i\lambda \int_0^1 (x f^1)_x \bar{u} dx \right). \end{aligned} \quad (4.13)$$

Given that λu and u_{xx} are uniformly bounded in $L^2(0, 1)$ and that $\|F\|_{\mathcal{H}} = o(1)$, we obtain

$$\left| \lambda^{-\ell} \Re \left(\int_0^1 x f^2 \bar{u}_x dx \right) \right| = o(\lambda^{-\ell}) \quad \text{and} \quad \left| \lambda^{-\ell} \Re \left(i\lambda \int_0^1 (x f^1)_x \bar{u} dx \right) \right| = o(\lambda^{-\ell}).$$

Now, using Young inequality and (4.9), we obtain

$$2 \left| \lambda^{-\ell} \Re (i\lambda f^1(1)\bar{u}(1)) \right| \leq 2\lambda^{-\ell} |f^1(1)| |\lambda u(1)| = \begin{cases} o(\lambda^{-\ell+\frac{1}{2}}) & \text{if } m \in (0, 1) \\ o(\lambda^{-\ell}), & \text{if } m = 1. \end{cases}$$

Finally, inserting the above three estimates in (4.13), we find the desired result (4.11).

Step 3: To finalize the proof of Lemma 4.4, it remains to estimate the term $|u_{xx}(1)|$ in (4.11). Precisely, we will prove that, for $m \in (0, 1]$, we have

$$|u_{xx}(1)| = \begin{cases} o(\lambda^{-\ell+\frac{1}{2}}) & \text{if } m \in (0, 1) \\ o(1) & \text{if } m = 1. \end{cases} \quad (4.14)$$

To achieve this, we need to solve the fourth-order PDE (4.12), which can be written as

$$-\lambda^2 u + u_{xxxx} = \lambda^{-\ell} [i\lambda f^1 + f^2]. \quad (4.15)$$

Let $w := \sqrt{\lambda}$, and denote $D := \partial_x$ and $F_1 := i\lambda f^1 + f^2$. Then, the above equation can be reformulated as $(D - iw)(D + iw)(D + w)(D - w)u = w^{-2\ell} F_1$. First, we set $U_1 := (D + iw)(D + w)(D - w)u$. The previous differential equation then becomes

$$(D - iw)U_1 = w^{-2\ell} F_1, \quad (4.16)$$

with a solution given by

$$U_1 = K_1 e^{iw(x-1)} + w^{-2\ell} \int_1^x e^{iw(x-z)} F_1(z) dz, \quad (4.17)$$

where $K_1 = u_{xxx}(1) + iw u_{xx}(1) - iw^3 u(1)$. Next, we set $U_2 := (D + w)(D - w)u$. Replacing U_2 in the definition of U_1 , we get $(D + iw)U_2 = U_1$, with a solution

$$U_2 = K_2 e^{-iw(x-1)} + \frac{K_1}{w} \sin((x-1)w) + w^{-2\ell-1} \int_1^x \sin(w(x-z)) F_1(z) dz,$$

where $K_2 = u_{xx}(1) - w^2 u(1)$. Finally, we set $U_3 := (D - w)u$. Inserting U_3 in the definition of U_2 , we obtain the following equation $(D + w)U_3 = U_2$. The solution is given by

$$\begin{aligned} U_3 &= K_3 e^{-w(x-1)} + \frac{1+i}{2w} K_2 (e^{-iw(x-1)} - e^{-w(x-1)}) + w^{-2\ell-1} \int_1^x \int_1^s e^{w(s-x)} \sin(w(s-z)) F_1(z) dz ds \\ &\quad - \frac{K_1}{2w^2} \left[\cos(w(x-1)) - \sin((x-1)w) - e^{-w(x-1)} \right], \end{aligned}$$

where $K_3 = -w u(1)$. Taking $x = 0$ and using K_1, K_2 , and the boundary conditions $u(0) = u_{xx}(0) = 0$, then multiplying the above equation by $w e^{-w}$, we obtain

$$\begin{aligned} -\frac{1}{2}u_{xx}(1) &= w e^{-w} u_x(0) - \frac{1+i}{2} e^{-w} e^{iw} u_{xx}(1) + w^2 u(1) + \frac{1+i}{2} w^2 (e^{iw} - e^w) e^{-w} u(1) \\ &\quad + e^{-w} w^{-2\ell} \int_0^1 \int_1^s e^{sw} \sin(w(s-z)) F_1(z) dz ds + \frac{i}{2} e^{-w} [\cos(w) + \sin(w)] u_{xx}(1) \\ &\quad + \frac{e^{-w}}{2w} (u_{xxx}(1) - iw^3 u(1)) [\cos(w) + \sin(w) - e^w]. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{1}{2}|u_{xx}(1)| &\leq \frac{w}{e^w} |u_x(0)| + \frac{1}{2}(3 + \sqrt{2})(1 + e^{-w}) |w^2 u(1)| \\ &\quad + \frac{\sqrt{2} + 2}{2e^w} |u_{xx}(1)| + \frac{e^{-w}}{2w} (2 + e^w) |u_{xxx}(1)| + |J|, \end{aligned} \quad (4.18)$$

where $J = e^{-w} w^{-2\ell} \int_0^1 \int_1^s e^{sw} \sin(w(s-z)) F_1(z) dz ds$.

To find the estimation (4.14), we need to give information on each term on the right-hand side of the inequality (4.18). Recall that $w = \sqrt{|\lambda|}$. Using the fact that $e^w \geq w^2$, and knowing that u_{xx} is uniformly bounded in $L^2(0, 1)$, we get

$$\frac{w}{e^w} |u_x(0)| \leq \frac{w}{e^w} \left| - \int_0^1 u_{xx} dx \right| \leq \frac{1}{w} \|u_{xx}\| \leq o(1). \quad (4.19)$$

From (4.11), we have $|u_{xx}(1)| = O(1)$, and using the fact that $e^w \geq w^2$, we get

$$\frac{1}{e^w} |u_{xx}(1)| \leq \frac{O(1)}{w^2} = \frac{O(1)}{\lambda} = o(1). \quad (4.20)$$

Using (4.9) and the fact that $e^w \geq w$, we get

$$w^2 [1 + e^{-w}] |u(1)| \leq \begin{cases} o(\lambda^{-\frac{\ell}{2} + \frac{1}{4}}) & \text{if } m \in (0, 1), \\ o(\lambda^{-\frac{\ell}{2}}), & \text{if } m = 1. \end{cases} \quad (4.21)$$

Using the fact that $e^w \geq w$, and using (4.9) and (4.10), we get

$$\begin{aligned} \frac{e^{-w}}{2w} [2 + e^w] [|u_{xxx}(1)| + w^3 |u(1)|] &\leq \left[\frac{1}{w} + \frac{1}{2} \right] \left[\frac{1}{w} |u_{xxx}(1)| + |\lambda u(1)| \right] \\ &= \begin{cases} o(\lambda^{-\frac{\ell}{2} + \frac{1}{4}}) & \text{if } m \in (0, 1), \\ o(\lambda^{-\frac{\ell}{2}}), & \text{if } m = 1. \end{cases} \end{aligned} \quad (4.22)$$

Finally, it remains to estimate the last term on the right hand side in (4.18). We observe that

$$J = e^{-w} w^{-2\ell} \int_0^1 \left[\int_0^z e^{ws} \sin(w(s-z)) ds \right] F_1(z) dz. \quad (4.23)$$

Let $I = \int_0^z e^{ws} \sin(w(s-z)) ds$. Integrating I by parts twice, we get

$$I = \frac{1}{2w} [\cos(wz) + \sin(wz) - e^{wz}]. \quad (4.24)$$

Inserting this into J , we get

$$J = \frac{1}{e^w w^{2\ell}} \int_0^1 \left[I f^2 + \frac{1}{2} [\cos(wz) + \sin(wz)] i w f^1 \right] dz - \frac{1}{2e^w w^{2\ell}} \int_0^1 e^{wz} i w f^1 dz. \quad (4.25)$$

Now, we will estimate each term on the right-hand side of the above equation. Using the fact that $e^w \geq w$ and that $\|F\|_{\mathcal{H}} = o(1)$, we get the following estimates

$$e^{-w} w^{-2\ell} \left| \int_0^1 I f^2 dx \right| = o(\lambda^{-\ell-\frac{1}{2}}), \quad \text{and} \quad e^{-w} w^{-2\ell} \left| \int_0^1 i w f^1 dz \right| = o(\lambda^{-\ell}) \quad (4.26)$$

For the remaining term in (4.25), integrating by parts over $(0, 1)$, then using (4.4)₁ and the fact $\|F\|_{\mathcal{H}} = o(1)$, we obtain

$$\left| \frac{1}{e^w w^{2\ell}} \int_0^1 e^{wz} f^1 dz \right| \leq \frac{1}{e^w w^{2\ell}} \left[\frac{e^w}{w} |f^1(1)| + \frac{1}{w} \int_0^1 e^{wz} |f_z^1| dz \right] \leq o(w^{-2\ell-1}) + o(1) = o(1).$$

Inserting the above estimation and (4.26) into (4.25), and then inserting (4.19), (4.20), (4.21), and (4.22) into (4.18), we find the desired estimation (4.14) for $m \in (0, 1]$. Thus, inserting (4.14) into (4.11), we complete the proof of this Lemma. \square

To conclude the proof of condition (H₂), we take $\ell = \frac{1}{2}$ if $m \in (0, 1)$ and $\ell = 0$ if $m = 1$ respectively in the estimates (4.6), (4.5) and (4.8). Then, adding the obtained estimates together, we obtain that $\|U\|_{\mathcal{H}} = o(1)$ for $m \in (0, 1]$, which contradicts the assumption (4.2). Thus, the condition (H₂) holds and the proofs of Theorem 4.1 and Theorem 4.2 are now completed.

Remark 4.5. *Considering a cantilevered beam, as illustrated in Fig. 2, instead of a hinged-sliding beam, i.e., if we replace the boundary condition $u_{xx}(0, t) = 0$ and $u_x(1, t) = 0$ in (B-H_m) with $u_x(0, t) = 0$ and $u_{xx}(1, t) = 0$, we still obtain the same stability results.*

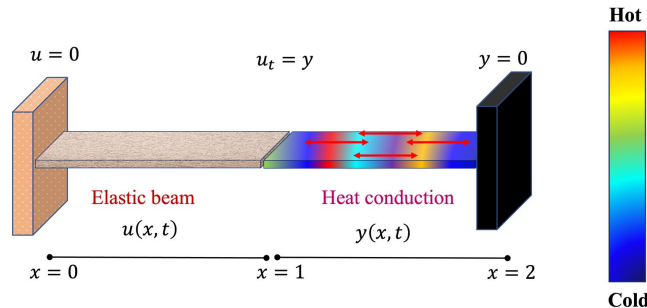


Figure 2: Schematic representation of a cantilevered Euler-Bernoulli beam occupying the domain $(0,1)$, clamped at its left end, and free at its right end. The heat equation governs the domain $(1,2)$.

5 Future Directions and Open Problems

The system outlined by (B-H_m) models fluid-structure interactions, encompassing the linearized one-dimensional Navier-Stokes equations (heat equation) and solid structures like elastic or piezoelectric beams. Consequently, the proposed methodology provides valuable insights into stabilization issues [40, 48]. Additionally, open problems arise in higher dimensions, especially regarding the geometric conditions on the domain and interface needed to achieve various types of stability.

Future research may explore transmission line configurations that integrate elastic and magnetizable piezoelectric beams, as discussed in [4]. Another intriguing direction for future research involves the Rayleigh beam equation, achieved by adding the rotational inertia term u_{ttxx} to the beam equation. This type of coupling will be the subject of an upcoming study.

Ongoing efforts are also directed towards demonstrating that the associated semigroup is polynomially stable with a decay rate similar to that established in [17]. It is anticipated that the decay rate

can be proven to be optimal for the classical solutions of the system, although this aspect lies beyond the scope of the current paper.

Looking ahead, immediate future work involves the analysis of model reduction techniques for the system described by (B-H_m). This includes the utilization of standard Finite Differences and Finite Elements approaches while incorporating numerical filtering, either directly or indirectly [43].

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