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Abstract

Acknowledging the economic role of knowledge, education, training and health, can preserve and make them thrive without assuming that preferences depend on any of them. Here, living capital is defined as their aggregate.

In a planned economy, we find that the optimal sequence of total capital is always monotonic. Depending on the productivity of total capital (defined as the sum of physical and living capital), three different regimes hold: bounded growth; asymptotically balanced unbounded growth or unbalanced unbounded growth.

Only at the very early stages of development the economy devotes all its investment effort to increase the stock of physical capital and only physical capital. As the economy develops, it will start using living capital in production. In the first regime, total capital converges to a steady state with a positive stock of total capital, which is larger than the one without living capital. In the second regime, growth becomes unbounded, and consumption grows at a constant rate. Total capital grows at the same rate but only asymptotically. In the third case, living capital is used increasingly at the beginning. Once the economy is sufficiently rich, physical capital starts growing faster than living capital.

To close, we consider a market economy with externalities from the living. In this case, if the government levies taxes to finance the accumulation of living capital and implements exactly the optimal sequence of living capital as in the planner's program, then the equilibrium market prices decentralize exactly the planner's solution.

Keywords: Ramsey model, physical capital, living capital, bounded and unbounded growth, global analysis.

JEL codes: C61, C62, O44.

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1 Introduction

Mankind is currently facing tremendous challenges directly or indirectly caused by economic activities. Among these challenges, we find climate change, the exhaustion of natural resources, the rapid loss in biodiversity, and the diffusion of new lethal diseases. Obviously, one feels compelled to immediately reduce production and consumption, hoping that the virtues of degrowth would re-establish a healthier (sustainable) equilibrium as praised by Georgescu-Roegen (1979). Tradition would suggest to consider human, social and natural capital as productive inputs and as utils, and then to model how production damages them. While this approach is extremely important and informative, we propose a more comprehensive measure to contain the living: living capital. Living capital is defined as a comprehensive measure of all economic inputs that are alive, and which are not raw labor, to acknowledge that all living things form a unity. The fact of acknowledging the economic role of all living things can change investment behaviors, privilege their preservation, and bring degrowth even without introducing any negative feedback from production to living capital.

Living capital includes education, knowledge, social interactions, and a broad definition of health, which recognizes that environmental and human health are one and the same thing.[1] Besides, living capital includes all bonds and feedbacks among all its components, and this is exactly what distinguishes living capital from natural and social capital, and even their sum. Obviously, there are positive and negative feedbacks, but we shall later assume that the net feedback effect is positive making living capital resilient. The three essential characteristics of living capital are that, first, it is not essential; second, it may be irreversible; and third, it depreciates at a slower pace than physical capital. Living capital is not essential because economies can produce without any living capital, using physical capital, labor and land. And still, we prove that, once the role of living capital is accounted for, agents, policy makers and the market will invest and secure its growth. Living capital is irreversible when it is, for instance, knowledge or technology embedded in the physical. Finally, regarding the third assumption: Why does living capital depreciate slower than physical capital? It cannot be otherwise since living capital is resilient because of the connections among all its components. Hence it is as if we were assuming that the positive feedbacks outperform the negative.

From a historical modelling perspective, we have observed the sequential introduction of various types of productive capitals in economic models. The first growth models of Ramsey (1928) and Solow (1956) focused on the role of physical capital in human development. In the Sixties, the new growth the-

[1] Here we are in line with the World Health Organization, which provides with a comprehensive measure of health, defined by One Health as follows: "One Health is an integrated, unifying approach that aims to sustainably balance and optimize the health of people, animals and ecosystems. It recognizes that the health of humans, domestic and wild animals, plants, and the wider environment (including ecosystems) are closely linked and interdependent" (see https://www.who.int/health-topics/one-health#tab=tab_1).

ories pioneered by Arrow (1962) and Uzawa (1965) stress instead the roles of knowledge and human capital. The notion of natural capital was sequentially introduced in different disciplines in the early Seventies.[2] According to Holdren and Ehrlich (1974) natural capital is the sum of nature's services which range from food production, to biodiversity warrant, conversion of waste and disease control. For Arrow et al. (2004) natural capital includes all contributions of nature to present and future utility. Contributions can be direct and indirect as preserving biodiversity, flood control, water purification, etc. Only since 2012, international organizations like the United Nations and the European Union started collecting data to account for natural resources and nature services. Although evidence is still scarce, there seems to be a strong will to render nature, namely in terms of renewable resources and ecosystemic services, its complete importance, including its economic role. The most recent productive capital to enter the economic modelling scene is social capital. Coleman (1990) defines social capital as the institutional relations between people understood widely to include both as formal and informal relations. Soon after, Putnam (1993) tested the concept using evidence on the Seventies Italian reform of the regional administration, to underline in particular the connection between social capital and the performance of institutions.

In an otherwise standard Ramsey setup in discrete time, we obtain the optimal trajectories for both physical and living capital assuming that the relative "price" between the two capitals is constant. Following Bruno et al. (2009), Le Van et al. (2010) and Le Van et al. (2018), we characterize the dynamics of total capital defined as the sum of the value of physical and living capital. Depending on the productivity of total capital, three different regimes can arise: (1) bounded growth; (2) asymptotically balanced unbounded growth; and (3) unbalanced unbounded growth. Let us briefly describe each of them:

(1) If the sum of the productivity parameters of physical and living capital is lower than one, then total capital converges to a steady state with a positive stock of living capital.

(2) When this sum is equal to one, growth becomes unbounded and consumption grows at a constant rate. Interestingly, total capital grows at the same rate as consumption but only asymptotically. In fact, the economy will optimally follow an Asymptotically Balanced Growth Path (ABGP).

(3) Finally, when the sum is greater than one, growth is unbounded for both physical and living capital. When total capital trespasses a given threshold, the economy starts accumulating living capital. The ratio of physical to living capital first decreases and then it grows to infinity, after total capital grows beyond a critical threshold.

We can read these results as both an alternative to degrowth theories and a challenge to the literature supporting the Environmental Kuznets Curve (EKC hereafter). Indeed, we show in cases (2) and (3) that an economy which recognizes the productive role of living capital will preserve and promote it along the

[2] See for instance Holdren and Ehrlich (1974), Solow (1974), Hartwick (1977) and the numerous references there on.

optimal path. Only at the very early stages of development, the economy will devote all its investment effort to increase the stock of physical capital and only physical capital. As the economy develops, it will start using living capital in production in an increasing manner, relatively reducing investment in physical capital. Once the economy is sufficiently rich and the stock of living capital sufficiently large, but only then, physical capital will start growing faster than living capital. Hence, our model describes an endogenous production shift that preserves and makes thrive all living capital without assuming that preferences depend on the living nor that production degrades the living.

We also mentioned that our model can challenge the EKC because, in case (3), it actually describes a U-shaped relationship between economic development and physical capital. Here, physical capital does not decrease in the long term to privilege living capital. On the contrary, both capitals will increase together. The premise being that the economy reached a critical development stage in which living capital was sufficiently abundant. Among the early works that unearthed the possibility of an inverted U-shaped relationship between production and environmental quality when environmental damage affected negatively welfare, let us mention John and Pecchenino (1994), Selden and Song (1995), Stokey (1998), Jaeger (1998), Andreoni and Levinson (2001) and Bosi and Desmarchelier (2018), among others. Regarding empirical support, see Grossman and Krueger (1995), Harbaoug et al. (2002) or Haberl et al. (2020). Our results could even be related to the imbalance effect revealed in Ouattara et al. (2016) in the context of tourism-based economies. Like us, they find that economies should first build a sufficiently large stock of capital and infrastructure that will ensure the optimal economic trajectory, before devoting resources to nature preservation.

Let us close our brief review of the related literature by mentioning some complementary works, which have adopted different perspectives. As mentioned, ours is a standard Ramsey framework and it does not deal with the problem of sustainability. We could have claimed to do so by trivially summarizing some of the findings in the literature in social welfare considering a sufficiently small discount factor, or a scrap function in the objective function to signal that the policy maker cared about the final state of living capital.[3] However, sustainability deserves a more careful treatment, possible only once living capital is fully understood as a productive factor. Regarding agents' predator behavior, Van der Ploeg (2010) shows that cooperation among agents is crucial for environmental preservation when natural capital can be converted into physical capital. Institutions are also pointed out as key warrants of natural preservation in Veeman and Politylo (2003) and Arrow et al. (2004) (among others). We also understand from the later that we should rest cautious about technological progress since it can increase production without bounds and without accounting for all its consequences. As Arrow et al. (2004), Ehrlich et al. (2012) also highlight the effects that uncontrolled population growth could have

[3] See for an introduction to the subject Chichilnisky (1977, 1982, 1996, 1997), Fleurbaey (2009), Fleurbaey and Zuber (2013) or Cairns et al. (2019), and all the references therein and thereafter. Obviously, our reference list is not exhaustive.

on the environment. While all these channels remain key, we have intentionally excluded them from our analysis to study how economic recognition alone can secure living capital.

Finally, let us mention that from a strict technical point of view, and regarding the dynamic behavior of the optimal trajectories, our results are in line with Le Van et al. (2010). It is shown that if human capital and new technologies are sufficiently efficient, then investment in human capital and new technologies turn out to be larger than investment in physical capital. Conversely, in Le Van et al. (2018), investment in physical capital remains larger than investment in social capital in the long run.

After studying the social planner problem, we consider the market economy with living externalities. In this case, we show that if the government levies taxes to finance the accumulation of living capital and implements exactly the optimal sequence of living capital obtained in the planner's program, then the equilibrium market prices exactly decentralize the planner's solution. Note that we are able to provide the explicit sequences of these market prices.

The rest of the paper is organized as follows. In section 2, we solve the social planner's problem by proving the existence of an optimal path and providing a global analysis of economic trajectories. These results are complemented by numerical simulations. In section 3, we consider a market economy with living externalities and we compute the competitive equilibrium. Section 4 concludes. All proofs are gathered in the Appendix.

2 Social planner

Let us begin with the policy maker's problem. The policy maker needs to decide on both the sequences of physical and living capital that will maximize overall social welfare.

2.1 Fundamentals

Our economy is endowed with three factors to produce a unique final good, which is used for all purposes. These three productive factors are labor, physical capital and living capital. Denoting by K_t and L_t physical and living capital, let us assume that their evolution in time is described as follows:

$$K_{t+1} = I_t^K + (1 - \delta_K) K_t \tag{1}$$

$$L_{t+1} = I_t^L + (1 - \delta_L) L_t \tag{2}$$

where I_t^K and I_t^L stand for investment in each of the capitals at time t, and δ_K and δ_L are their depreciation rates. Plausibly, we can suppose that living capital depreciates less than physical capital. Let us also assume that output and physical capital are the same good.

Let q be the constant "price" of living capital in terms of output. Then, total capital at time t is given by:

$$X_t \equiv K_t + qL_t$$

Note that there is not an *a priori* reason why the growth processes of physical and living capital could not follow different trajectories and trends. Furthermore, physical capital could experience degrowth while total capital grows. This is exactly the situation we analyze and characterize here, and we will delve with it in detail once the model is fully presented and completely solved analytically.

The final good sector utilizes both capitals together with labor, N_t, blending their specific services. However, as we will see next, each input has a very different role in production. Let us denote by Y_t total output at time t. Then,

$$Y_t = A\left(1+\lambda L_t\right)^{\theta} K_t^{\alpha} N_t^{1-\alpha} \tag{3}$$

Our approach departs from the literature on climate change and, instead of modelling environmental damages from production, the production function in (3) reveals how living capital enhances productivity.[4]

Let us introduce some plausible parameter restrictions on technology and capital depreciation, which are assumed to hold throughout the paper, even if not mentioned explicitly. Our first assumption ensures that positive production is possible without any living capital, while production is zero without physical capital.

Assumption 1 $0<\theta<1$, $0<\alpha<1$, $\lambda>0$, $A>0$ and $0<\delta_L\leq\delta_K\leq 1$.

Regarding labor, let us assume an inelastic labor supply: $N_t=\bar{N}=1$ for any t. Then we can write Y_t as $Y_t = A\left(1+\lambda L_t\right)^{\theta} K_t^{\alpha}$.

There exists a planner who maximizes a classic time discounted intertemporal utility function $\sum_{t=0}^{\infty}\beta^t \ln c_t$, where β is the time discount parameter with $0<\beta<1$. The policy maker needs to take into account a sequence of constraints:

$$c_t + I_t^K + qI_t^L \leq A\left(1+\lambda L_t\right)^{\theta} K_t^{\alpha}$$

for all $t\geq 0$, given $K_0, L_0 > 0$. Note that investment in natural capital needs to be multiplied by q, the transformation price. Using (1) and (2), we can write the overall resource constraint of the economy at time t as

$$c_t + K_{t+1} - \left(1-\delta_K\right)K_t + q\left[L_{t+1} - \left(1-\delta_L\right)L_t\right] \leq A\left(1+\lambda L_t\right)^{\theta} K_t^{\alpha}$$

Remark 1 *Consumption good and physical capital are the same good, while the investment in living capital may be irreversible. As a result, physical capital can actually be consumed, and in that case* $I_t^K \equiv K_{t+1} - \left(1-\delta_K\right)K_t < 0$ *at some* t*. We have also assumed that living capital cannot be transformed into the consumption good, that is* $I_t^L \equiv L_{t+1} - \left(1-\delta_L\right)L_t \geq 0$ *for all* t*. Note that this*

[4] Alternatively, living capital could also be interpreted as a part of the Total Factor Productivity (TFP) of physical capital and labor. A higher stock of living capital would make both physical capital and labor more productive. Our results do not depend on this specific interpretation.

assumption is compatible with degrowth in living capital at some period t, that is $L_{t+1} < L_t$, provided that $I_t^L < \delta_L L_t$. However, the irreversibility assumption becomes superfluous, because as we will see, the economy always experiences growth in living capital along the optimal trajectory.

We can define gross production as

$$F(K_t, L_t) \equiv A(1+\lambda L_t)^{\theta} K_t^{\alpha} + (1-\delta_K) K_t + q(1-\delta_L) L_t \tag{4}$$

so that gross savings at time t coincides with X_{t+1}, that is

$$X_{t+1} \equiv K_{t+1} + qL_{t+1}$$

Accordingly, the resource constraint at t can be rewritten as:

$$c_t + X_{t+1} \leq F(K_t, L_t)$$

Hence, the social planner's problem becomes:

$$\max \sum_{t=0}^{\infty} \beta^t \ln c_t$$

subject to

$$c_t + X_{t+1} \leq \mathcal{F}(X_t) \tag{5}$$

given the initial condition $X_0 > 0$, where

$$\mathcal{F}(X_t) \equiv \max\{F(K_t, L_t) : K_t + qL_t \leq X_t\} \tag{6}$$

is the maximal gross output given the aggregate capital X_t.

Our analysis begins by proving the existence of a (unique) critical level of total capital, $\bar{X}$, with the following characteristic: if the economy's stock of total capital lies below $\bar{X}$, then production does not use any living capital, that is $X_t = K_t$. However, beyond $\bar{X}$, production will always include some living capital. We will come back to this result and its implications after we actually prove it. For now, let us start by proving that this critical value $\bar{X}$ is the unique solution to equation

$$f(X) = q(\delta_K - \delta_L) \geq 0 \tag{7}$$

where $f(X) \equiv (\alpha q - \lambda\theta X) A X^{\alpha-1}$. Note that function f is continuous for any $X > 0$, strictly decreasing from ∞ to 0 in $(0, \alpha q/(\lambda\theta)]$, and negative for any $X > \alpha q/(\lambda\theta)$. Therefore, equation (7) has a unique solution $\bar{X}$ and, clearly, this solution satisfies

$$0 < \bar{X} \leq \frac{q}{\lambda}\frac{\alpha}{\theta} \tag{8}$$

Before presenting our first results, we would like to underline that the model's dynamics are driven by α and θ, the productivity parameters for physical and living capital. Furthermore, their sum,

$$\gamma \equiv \alpha + \theta > 0$$

captures the productivity of total capital, and the ratio α/θ, the relative productivity of physical to living capital. As we will see, the magnitude of γ, less or greater than one, does determine whether the economy will experience bounded or unbounded growth.

Lemma 2 (maximal gross output) *Function $\mathcal{F}$ has the following properties:*
(1) If $X_t \leq \bar{X}$, then $L^*(X_t) = 0$ and $K^*(X_t) = X_t$, and $\mathcal{F}$ is given by

$$\mathcal{F}(X_t) = AX_t^{\alpha} + (1 - \delta_K) X_t \tag{9}$$

(2) If $X_t > \bar{X}$, then $L_t = L^*(X_t) \in (0, X_t/q)$ is the solution to $\mathcal{L}(L_t) = \mathcal{R}(L_t)$ with

$$\mathcal{L}(L_t) \equiv \lambda\theta A (1 + \lambda L_t)^{\theta-1} (X_t - qL_t)^{\alpha} + q\delta_K \tag{10}$$

$$\mathcal{R}(L_t) \equiv \alpha q A (1 + \lambda L_t)^{\theta} (X_t - qL_t)^{\alpha-1} + q\delta_L \tag{11}$$

and $K_t = K^*(X_t) = X_t - qL^*(X_t) \in (0, X_t)$. $\mathcal{F}$ is given by

$$\begin{aligned} \mathcal{F}(X_t) &= F(K^*(X_t), L^*(X_t)) \\ &= A[1 + \lambda L^*(X_t)]^{\theta} [X_t - qL^*(X_t)]^{\alpha} \\ &\quad + (1 - \delta_K)[X_t - qL^*(X_t)] + (1 - \delta_L) qL^*(X_t) \end{aligned} \tag{12}$$

(3) Function $\mathcal{F}$ is continuous, strictly increasing and differentiable for any $X_t \geq 0$, and so it is at $X_t = \bar{X}$.
(4) Functions $K^*(X_t)$ and $L^*(X_t)$ are strictly increasing for $X_t > \bar{X}$.
(5) $\mathcal{F}$ is strictly concave if $X_t < \bar{X}$. If $X_t > \bar{X}$, $\mathcal{F}$ is strictly concave if and only if $\gamma < 1$.

Lemma 2 shows that poor economies, those whose total capital lies below the threshold, do not consider living capital as a production factor. In particular, Lemma 2 proves that economies below the threshold will specialize in "heavy" industries, relying uniquely on physical capital. This result could be even more consequential if industrial pollution and its damage was introduced in the model. Indeed, most probably we would see a "living poverty trap" emerge in which poor countries would not contemplate living capital as linked to production, and could never do so since they would induce irreversible damages to living and physical capital that would retain them below the threshold, in the trap.

In order to gain further understanding on the structure of the economy, we define the ratio of physical to living capital as

$$\zeta(X_t) \equiv \frac{K^*(X_t)}{qL^*(X_t)}$$

As proven in Lemma 2, this ratio is only defined on $\left(\bar{X}, \infty\right)$ since $L^*(X_t) = 0$ for values of X_t below $\bar{X}$.

Lemma 3 (asymptotic properties of capitals) *Physical and living capital have the following asymptotic properties:*

(1) When aggregate capital is unbounded, its components are unbounded as well:

$$\begin{aligned} \lim_{X_t \to \infty} L^* (X_t) &= \infty \\ \lim_{X_t \to \infty} K^* (X_t) &= \infty \end{aligned}$$

(2) If $\gamma < 1$, then

$$\lim_{X_t \to \infty} \frac{\mathcal{F}(X_t)}{X_t} \leq 1 - \delta_L < 1$$

(3) If $\gamma = 1$, there exists $\mu \in (0, \alpha/\theta]$ such that, for any sequence (X_n) with $\lim_{n\to\infty} X_n = \infty$, we have

$$\mu \leq \lim \inf_{n \to \infty} \zeta (X_n) \leq \lim \sup_{n \to \infty} \zeta (X_n) \leq \frac{\alpha}{\theta}$$

(4) If $\gamma > 1$, function ζ has the following properties:

$$\begin{aligned} \lim_{X_t \to \bar{X}^-} \zeta (X_t) &= \infty \\ \lim_{X_t \to \infty} \zeta (X_t) &= \frac{\alpha}{\theta} \end{aligned}$$

Moreover, if $\delta_K > \delta_L$, $\zeta (X_t) < \alpha/\theta$ for any X_t large enough.

Some comments are in order. Although point (1) could seem *a priori* a trivial result, it is absolutely not. Point (1) proves that if total capital increases to infinite, then both physical and living capital necessarily grow boundless. That is to say, if the optimal solution implies that total capital will grow forever, it will be impossible to observe a trajectory along which, for instance, physical capital will grow while living capital will decrease to reach zero. Both capitals need to grow in order to sustain the infinite accumulation of total capital.

Points (2), (3) and (4) study the limit behavior of some ratios depending on the productivity of total capital. Note that these limits are taken with respect to total capital or time. If total capital productivity is smaller than 1, then case (2) shows that production per unit of total capital will be bounded by $1 - \delta_L$, that is by the persistence of living capital.

According to (3), if $\gamma = 1$, that is, if $\theta = 1 - \alpha$ and production can be understood as a Cobb-Douglas production function in physical and living capital, then the ratio of physical capital to living capital will remain with time between μ and $\alpha/\theta = \alpha/(1-\alpha)$. The upper limit of the ratio of physical to living capital is the ratio between their productivity parameters. That is to say, the ratio of the capitals does not necessarily converge towards the ratio of their productivities, it could remain "stuck" below. If on the contrary the production function exhibits increasing returns to scale, i.e. $\gamma > 1$ then we have that first, the ratio of capitals does converge to α/θ when total capital tends to infinite.

Second, if total capital converges (from above) towards $\bar{X}$, then the ratio of physical to living capital will tend to infinite (as living capital will tend to zero as shown in Lemma 2).

On the technical side, we observe that the maximization in (6) takes place at any period, included $t = 0$. Thus, given (K_0, L_0) and, then, $X_0 = K_0 + qL_0$, the initial physical and natural capitals available for production are possibly different and equal to $(K^*(X_0), L^*(X_0)) = (X_0, 0)$ if $X_0 \leq \bar{X}$, and to $(K^*(X_0), L^*(X_0)) = (X_0 - qL^*(X_0), L^*(X_0))$ with $L^*(X_0) \in (0, X_0/q)$ if $X_0 > \bar{X}$.

2.2 Existence of an optimal path

The following two lemmas prove that, first, the utility function either converges to a finite real number or diverges to $-\infty$, and second and most importantly, they also show that the utility function has a maximum.

As a preliminary step, we need to characterize the steady state of an economy where living capital does not play any role. Then we tackle the more general case.

When living capital does not have any role in production, that is, when $\lambda = 0$, then

$$\bar{X} = \left(\frac{\alpha A}{\delta_K - \delta_L}\right)^{\frac{1}{1-\alpha}} \tag{13}$$

If we further assume that both capitals depreciate at the same rate, that is, if $\delta_K = \delta_L$, then $X_t \leq \bar{X} = \infty$, $L^*(X_t) = 0$ and $K^*(X_t) = X_t$. The maximum of production is $\mathcal{F}(X_t) = AX_t^\alpha + (1 - \delta_K) X_t$. Since there is only one capital in the model now, we naturally recover the traditional Ramsey model. Then, recall that the Modified Golden Rule is the Euler equation at the steady state, that is: $\alpha A\tilde{X}^{\alpha-1} + 1 - \delta_K = 1/\beta$. As a result, the steady state without living capital is given by

$$\hat{X} \equiv \left[\frac{\alpha\beta A}{1 - \beta(1 - \delta_K)}\right]^{\frac{1}{1-\alpha}} \tag{14}$$

Let us deal next with the general case in which $\lambda \neq 0$. We shall assume in the following that the threshold of total capital is below the Modified Golden Rule:

Assumption 2 $\bar{X} < \hat{X}$.

Assumption 2 is satisfied when living capital is very efficient, that is, when both of L_t's productivity parameters λ and θ are high, or when the "relative price" of living capital q is low. Indeed, when λ and θ are large enough or q is sufficiently small, we have that

$$\bar{X} \leq \frac{\alpha q}{\lambda\theta} < \hat{X}$$

In order to prove the following lemma, we introduce a third critical value for

total capital $\hat{X}$ defined as:

$$\hat{X} \equiv \left(\frac{A}{\delta_K}\right)^{\frac{1}{1-\alpha}} \tag{15}$$

and compare the three critical values for total capital defined in (13), (14) and (15). Since $\alpha, \beta \in (0,1)$, we have $\tilde{X} < \hat{X}$ and, because of Assumption 2,

$$\bar{X} < \tilde{X} < \hat{X} \tag{16}$$

Lemma 4 (convergence of intertemporal utility) *Let $0 < \gamma < 1/\beta$. Under Assumption 2, the objective function of the policy maker is finite, that is $\sum_{t=0}^{\infty} \beta^t \ln c_t < \infty$ for any $X_0 \geq 0$. Moreover, the limit of the utility function along any feasible path exists. It converges to a finite real number or to $-\infty$: $\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \ln\left[\mathcal{F}\left(X_t\right) - X_{t+1}\right] \in \mathbb{R} \cup \{-\infty\}$.*

We can re-write the policy maker's objective as $\sum_{t=0}^{\infty} \beta^t \ln\left[\mathcal{F}\left(X_t\right) - X_{t+1}\right]$ using (5). Note that the objective function is now an infinite sum which depends exclusively on the sequence of total capital. Let $\Pi\left(X_0\right)$ denote the set of sequences of composite capital stocks $\mathbf{X} \equiv \left(X_t\right)_{t=0}^{\infty}$, feasible from X_0, that is the set of $\mathbf{X} \equiv \left(X_t\right)_{t=0}^{\infty}$ such that $0 \leq X_{t+1} \leq \mathcal{F}\left(X_t\right)$ for any $t \geq 0$. We can prove that there exists an optimal solution to the planner's problem.

Proposition 5 (existence of an optimal path) *Let $0 < \gamma < 1/\beta$. Under Assumption 2, the function $U\left(\mathbf{X}\right) \equiv \sum_{t=0}^{\infty} \beta^t \ln\left[\mathcal{F}\left(X_t\right) - X_{t+1}\right]$ is upper semi-continuous in $\Pi\left(X_0\right)$ with respect to the product topology and it attains a maximum in $\Pi\left(X_0\right)$.*

2.3 Dynamics

Having proved the existence of a solution to the social planner problem, and having characterized some of the asymptotic properties of feasible trajectories for X_t, let us study next the problem of the convergence of the optimal solution. Although we have made an effort to keep our setting as simple as possible on the technical side, production involves two different types of capital which differ not only in their definition but also in their role in production. Hence, our problem is essentially complex.

The following lemma provides us with interesting and extremely useful results: the optimal path of total capital is monotonic. Furthermore, it also shows that the stock of living capital is positive from a critical date on.

Lemma 6 (monotonicity and threshold) *Let $X_0 > 0$. The optimal sequence $\left(X_t\right)_{t=0}^{\infty}$ is monotonic. Furthermore, the optimal solution does not converge to zero when $t \to \infty$. Moreover, there exists T such that, for any $t \geq T$, the sequence remains above the threshold, that is, total capital is always larger than $\bar{X}$: $X_t > \bar{X}$, and the optimal natural capital is positive, $L_t > 0$, for any $t \geq T$.*

In particular, the previous lemma implies that if the economy is sufficiently well endowed at $t = 0$, that is, if $X_0 > \bar{X}$, then $T = 0$, implying that living capital is always strictly positive along the optimal solution.

Remark 7 *Our lemma can also be read as follows. If $X_0 < \bar{X}$, then the optimal sequence is monotonically increasing and there will be a time T at which total capital will be equal to $\bar{X}$, when the economy will start employing living capital in production. After T, total capital will continue growing so that living capital will always be positive after T. Otherwise, if $X_0 > \bar{X}$, then the optimal sequence of capital could be either monotonic increasing or decreasing. What we know for sure is that the sequence remains above the threshold for all t, and that the economy always involves living capital in production.*

Whether the optimal sequence reaches a steady state or whether it increases forever is the subject of the next series of three propositions. These propositions distinguish three cases depending on whether the productivity parameter of total capital is smaller, equal or larger than one.

Proposition 8 (bounded growth) *If $\gamma < 1$, the optimal sequence $(X_t)_{t=0}^{\infty}$ converges to a steady state X^*. This steady state is larger than the modified golden rule steady state value for X_t without living capital: $X^* > \tilde{X}$.*

We know from Lemma 6 that any optimal sequence includes some living capital from a time T onwards, so that the limit value of total capital does include a positive amount of living capital. What Proposition 8 further offers is the proof that taking into account the services of nature and human capital as part of the production function leads the economy towards a higher level of total capital. However, since the maintenance of living capital requires some of the final good, the increase in living capital could come in detriment of physical capital. Although this increase could also come from the extra production generated, or from an extraordinary effort in savings, truth is we do not have the answer this far and the comparison established with the standard Ramsey model could just stop at total capital. The following lemma sheds light on the asymptotic composition of total capital.

Lemma 9 *Let $\gamma < 1$. If $\delta_K = \delta_L$, then*

$$\lim_{t\to\infty} \zeta_t = \frac{\alpha}{\theta}\left(1 + \frac{1}{\lambda L^*}\right) > \frac{\alpha}{\theta}$$

where L^* is the stationary stock of living capital.

Then, when $\delta_K = \delta_L$, living capital does enhance the accumulation of physical capital in the long run since the ratio physical to living capital is strictly larger than the ratio of their productivities. In other words, physical capital is accumulated beyond its "relative productivity".

When the parameter of total productivity of both capitals is equal to one, we obtain a balanced growth path as one would have expected in light of similar

results in the standard Ramsey model. The following proposition characterizes the optimal trajectories and growth rates for consumption and each of the capitals.

Proposition 10 (asymptotically balanced unbounded growth) *If $\gamma = 1$, then the optimal paths are given by*

$$\begin{aligned} c_t &= \bar{c}_0 \left[\beta \mathcal{F}'\left(\bar{X}\right)\right]^t \\ X_{t+1} &= \bar{c}_0 \left[\beta \mathcal{F}'\left(\bar{X}\right)\right]^t \frac{\beta}{1-\beta} - z \\ L_{t+1} &= \left(\bar{c}_0 \left[\beta \mathcal{F}'\left(\bar{X}\right)\right]^t \frac{\beta}{1-\beta} - z - \bar{X}\right) \frac{1}{q+\lambda\bar{X}} \\ K_{t+1} &= \left(\bar{c}_0 \left[\beta \mathcal{F}'\left(\bar{X}\right)\right]^t \frac{\beta}{1-\beta} - z + \frac{q}{\lambda}\right) \frac{\lambda\bar{X}}{q+\lambda\bar{X}} \end{aligned}$$

where

$$z \equiv \frac{q}{\lambda}\left[1 + \frac{\delta_L}{\mathcal{F}'\left(\bar{X}\right) - 1}\right] \tag{17}$$

and $\bar{c}_0$ is the value of initial consumption which maximizes

$$V\left(c_0\right) = \sum_{t=0}^{\infty} \beta^t \ln\left(c_0 \left[\beta \mathcal{F}'\left(\bar{X}\right)\right]^t\right)$$

The maximum is attained at

$$\bar{c}_0 \equiv \left(1-\beta\right) \mathcal{F}'\left(\bar{X}\right)\left(X_0 + \frac{q}{\lambda}\left[1 + \frac{\delta_L}{\mathcal{F}'\left(\bar{X}\right) - 1}\right]\right)$$

Therefore, growth is unbounded

$$\lim_{t\to\infty} K_t = \lim_{t\to\infty} L_t = \infty$$

with

$$\lim_{t\to\infty} \frac{K_t}{qL_t} = \bar{X}\frac{\lambda}{q} \leq \frac{\alpha}{\theta} \tag{18}$$

but asymptotically balanced

$$\frac{L_{t+1}}{L_t} \sim \frac{K_{t+1}}{K_t} \sim \frac{X_{t+1}}{X_t} \sim \frac{c_{t+1}}{c_t} = \beta \mathcal{F}'\left(\bar{X}\right) > 1 \tag{19}$$

along the ABGP (Asymptotically Balanced Growth Path).

Let us close this section by characterizing the optimal solution when $1 < \gamma < 1/\beta$. In this case, the optimal solution exhibits unbalanced unbounded growth:

Proposition 11 (unbalanced unbounded growth) *Let $1 < \gamma < 1/\beta$. The optimal sequence $(X_t)_{t=0}^{\infty}$ converges to infinity with $\lim_{t\to\infty} L_t = \lim_{t\to\infty} K_t = \infty$ and*

$$\lim_{t\to\infty} \zeta_t = \lim_{t\to\infty} \frac{K_t}{qL_t} = \frac{\alpha}{\theta} \tag{20}$$

(1) If $\delta_K = \delta_L$, the capital ratio always stays above its limit, that is, the ratio physical to living capital converges towards its limit value from above. Then, we have that for any t:

$$\zeta_t \equiv \frac{K_t}{qL_t} = \frac{\alpha}{\lambda\theta}\frac{1}{L_t} + \frac{\alpha}{\theta} > \frac{\alpha}{\theta}$$

(2) If $\delta_K > \delta_L$, then the sequence of capital ratios (ζ_t) is uniformly bounded away from zero and from ∞. Moreover, when X_t is sufficiently close to $\bar{X}$ from the right, $\zeta'(X_t) < 0$, and when X_t is sufficiently large, $\zeta'(X_t) > 0$. If $\delta_K > \delta_L$ and X_t is large enough, then $\zeta(X_t) < \alpha/\theta$.

Proposition 11 shows that when total capital is very efficient, $1 < \gamma < 1/\beta$, both physical and living capital grow without bound ($\lim_{t\to\infty} K_t^* = \lim_{t\to\infty} L_t^* = \infty$), and so does total capital ($\lim_{X\to\infty} X_t^* = \infty$). When $1 < \gamma < 1/\beta$ and living capital is more efficient than physical capital, i.e. $\theta > \alpha$, then the economy ends up accumulating relatively more living capital in the long run:

$$\lim_{t\to\infty} \frac{K_t}{qL_t} < 1$$

Now, let us come back to Proposition 11. The proposition clearly shows that when physical capital and living capital wear down at the same rate, so that capitals are indistinguishable from this point of view, then $\zeta_t > \alpha/\theta$ for all t. As mentioned, the economy will optimally accumulate more physical capital in relative terms than its relative productivity. The more productive physical capital, the more the economy will favor it. However, the most interesting result is presented in point (2). Since $1 < \gamma < 1/\beta$, we know that total capital will be monotonically increasing. What point (2) brings anew is that, when physical capital deteriorates faster than living capital, the composition of total capital will vary with time. When total capital is low, close to $\bar{X}$ and that the economy has recently started incorporating nature and human capital services into account for production purposes, the economy privileges investment in living capital so that the ratio ζ_t decreases. Then, when total capital becomes large enough, then its relative composition changes letting physical capital dominate the mix from that moment onwards. In this respect, stressing the role of the natural component in the living capital, we can say that there our model generates a reverse environmental Kuznets curve.

Before closing this section, let us study the role of the relative price q on the growth rate of both capitals when $\gamma = 1$, and the ratio between physical and living capital when $1 < \gamma < 1/\beta$:

Corollary 12 (impact of the relative price on growth) *Let us distinguish two cases:*

(1) When $\gamma = 1$ the growth factor $K_{t+1}/K_t \sim L_{t+1}/L_t$ asymptotically decreases with q. Moreover, in the limit, the capital ratio K_t/L_t increases in q, that is, the economy uses more physical than living capital in the long run.

(2) When $1 < \gamma < 1/\beta$, the ratio K_t/L_t asymptotically increases with q.

These results deserve some conclusive comments:

(1) According to Proposition 8, when $\gamma < 1$, degrowth is possible when the initial value of total capital is larger then the steady state, $X_0 > X^*$, as in the standard Ramsey model.

(2) Corollary 12 shows that an increase in the relative price of living capital will slow down the economy when $\gamma = 1$. In this case, q has a negative impact on growth because, as shown in the proof, in the long run, the critical value $\bar{X}$ of total capital increases, lowering in turn total productivity, $\mathcal{F}'\left(\bar{X}\right)$.

(3) When γ is larger than one, Corollary 12 proves that an increase in the relative price of living capital q increases the relative amount of physical capital with respect to living capital in the long run. Indeed, the limit ratio $q\alpha/\theta$ depends both on this price and α/θ, the comparative advantage of physical capital in terms of productivity (see also the proof).

2.4 Simulations

We close this section providing some numerical illustrations to offer some answers to the remaining open questions.

Note that while illustrating the cases when $\gamma \leq 1$ has little interest since all trajectories are explicit, it is extremely informative to obtain more information on the transition dynamics in the more challenging case of unbalanced and unbounded growth when $1 < \gamma < 1/\beta$. Here, we define adequate ratios for each case that converge in the long-run. Then, from that converging sequence we will be able to recover (numerically this time) some further information on the original capitals or ratios. Let us consider the following subcases: (1) $\delta_K = \delta_L$ and (2) $\delta_K > \delta_L$.

When the depreciation rates are equal, it is possible to write the dynamics in variable ζ_t, which, we know, converges to a limit α/θ according to Proposition 11.

Proposition 13 (simulations with equal depreciation rates) *Let $1 < \gamma < 1/\beta$ and $\delta_K = \delta_L$. The optimal dynamics are driven by the Euler equation*

$$\frac{\psi\left(\zeta_{t+1}, \zeta_{t+2}\right)}{\psi\left(\zeta_t, \zeta_{t+1}\right)} = \beta \mathcal{F}'\left(X\left(\zeta_{t+1}\right)\right) \tag{21}$$

where

$$\psi\left(\zeta_t, \zeta_{t+1}\right) \equiv \mathcal{F}\left(X\left(\zeta_t\right)\right) - X\left(\zeta_{t+1}\right) = c_t \tag{22}$$

represents the resource constraint and

$$X_t = X\left(\zeta_t\right) \equiv \frac{q}{\lambda} \frac{\alpha\zeta_t + \alpha}{\theta\zeta_t - \alpha}$$

total capital.

The optimal trajectory for ζ_t is determined by (21) together with the transversality condition:

$$\lim_{t\to\infty} \zeta_t = \frac{\alpha}{\theta}$$

Using (69) and (22), we will obtain the sequences of total capital $(X_t)_{t=0}^{\infty}$ and consumption $(c_t)_{t=0}^{\infty}$.

When the depreciation rates are different, it is possible to write the optimal dynamics in terms of a new variable φ_t, which also converges to a limit $\lambda\theta/(\alpha q)$.

Proposition 14 (simulations with different depreciation rates) *Let $1 < \gamma < 1/\beta$ and $\delta_K > \delta_L$. Economic dynamics are driven by a recursive transition function*

$$\varphi_{t+2} = \mathcal{T}\left(\varphi_t, \varphi_{t+1}\right) \tag{23}$$

where

$$\varphi_t = \varphi\left(X_t\right) \equiv \frac{1 + \lambda L^*\left(X_t\right)}{X_t - qL^*\left(X_t\right)} \tag{24}$$

and

$$\mathcal{T}\left(\varphi_t, \varphi_{t+1}\right) \equiv \varphi\left(\mathcal{F}\left(\varphi^{-1}\left(\varphi_{t+1}\right)\right) - \beta\mathcal{F}'\left(\varphi^{-1}\left(\varphi_{t+1}\right)\right)\left[\mathcal{F}\left(\varphi^{-1}\left(\varphi_t\right)\right) - \varphi^{-1}\left(\varphi_{t+1}\right)\right]\right)$$

The trajectory is determined by (23) jointly with the transversality condition:

$$\lim_{t\to\infty} \varphi_t = \frac{\lambda\theta}{\alpha q}$$

We close this paper by using the theoretical results presented in the previous subsection in some numerical illustrations that will complete the long-term description of the dynamics of our model.

In the first simulation, we assume that both physical and living capital depreciate at the same rate: $\delta_K = \delta_L = 0.04$. The calibration is summarized in the following table.

A	TFP	$1/2$
α	physical capital productivity parameter	$1/3$
θ	living capital productivity parameter	$2/3 + 0.01$
β	time discount	0.99
δ_K	physical capital depreciation rate	0.04
δ_L	living capital depreciation rate	0.04
q	relative price of living capital	$1/2$
λ	living capital parameter	$1/2$

Figures 1 and 2 present the dynamics corresponding to point (1) of Propo-

sition 11, that is to Proposition 13.

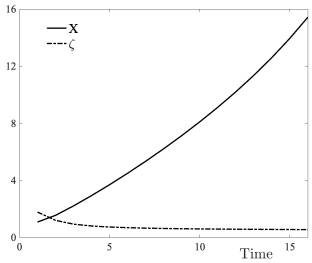

Fig. 1

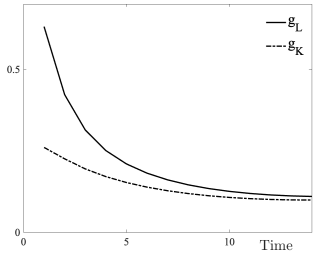

Fig. 2

Our program uses the fact that the ratio ζ converges towards α/θ and recovers backwards the trajectory for ζ (Figure 1). Using this trajectory, we are able to compute the optimal trajectories for X, K and L. Our results show that under our calibration, when $1 < \gamma < 1/\beta$, all original variables grow with time. Besides, natural capital grows faster than physical capital until ζ approaches the steady state. Once ζ reaches its steady state, both capitals grow at the same rate forever (Figure 2).

Now, let us assume different depreciation rates for physical and living capital: $\delta_K = 0.04 > \delta_L = 0.039$. We have also modified the calibration to better show the dynamics according to the following table.

A	TFP	1
α	physical capital productivity parameter	$2/3$
θ	living capital productivity parameter	$1/3 + 0.0001$
β	time discount	0.99
δ_K	physical capital depreciation rate	0.04
δ_L	living capital depreciation rate	0.039
q	relative price of living capital	$1/4$
λ	living capital parameter	$1/2$

Using equation (7), we compute $\bar{X} = 0.9982$ and we fix $X_0 = 0.9983$, just slightly larger than $\bar{X}$. Moreover, we choose $K_0 = (3/4)\, X_0$ so that $L_0 = (X_0 - K_0)/q = 0.9983$. In this set-up, Figures 3 and 4 present the physical and living capital dynamics corresponding to point (2) of Proposition 11, that is to

Proposition 14.

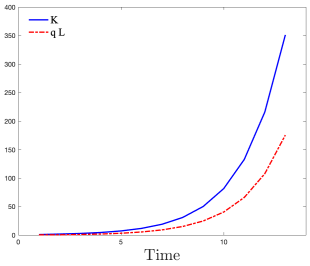

Fig. 3

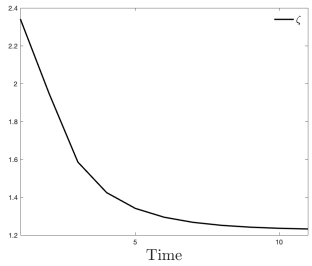

Fig. 4

As predicted by Proposition 11, when X_0 is close to $\bar{X}$, ζ decreases with time. Both physical and living capital grow forever.

3 Optimal growth and competitive equilibrium with living externalities

To complete our analysis, we consider in this section alternative problems where the decision maker takes as given the trajectory for living capital. Moreover, living capital coincides with the optimal trajectory from the planner's program (P0). We will say that there are externalities from the living capital or "living externalities".

In the first subsection, we compare (P0) to the problem of a policy maker identical to that of the previous section except that, as mentioned, she takes living capital to be equal to its optimal trajectory in (P0). We prove that both problems have the same optimal solution for physical capital, and hence consumption. In the final subsection, we consider instead a market economy. There, the policy maker will also take living capital as given and will levy taxes to finance investment in living capital. We prove in our last theorem that there exists a sequence for prices which decentralizes the policy maker's problem and ensures that the economy will follow the optimal solution for (P0).

3.1 Optimal growth with living externalities

Reconsider gross output (4) and maximal output (6) with total capital $X_t \equiv K_t + qL_t$ given. A solution $\left(K^*\left(X_t\right), L^*\left(X_t\right)\right)_{t=0}^{\infty}$ satisfies

$$\mathcal{F}\left(X_t\right) = A\left[1+\lambda L^*\left(X_t\right)\right]^{\theta} K^*\left(X_t\right)^{\alpha} + \left(1-\delta_K\right) K^*\left(X_t\right) + q\left(1-\delta_L\right) L^*\left(X_t\right)$$

Let us redefine gross output as a function of total capital and living capital:

$$G\left(X_t, L_t\right) \equiv A\left(1+\lambda L_t\right)^{\theta}\left(X_t - qL_t\right)^{\alpha} + \left(1-\delta_K\right)\left(X_t - qL_t\right) + q\left(1-\delta_L\right) L_t$$

and denote the first derivatives of G by G_X and G_L. We know that $L^*(X_t)$ is solution to $G_L(X_t, L_t) = 0$. Invoking the Envelope Theorem, we have

$$\mathcal{F}'(X_t) = G_X(X_t, L^*(X_t)) + G_L(X_t, L^*(X_t)) L^{*\prime}(X_t) = G_X(X_t, L^*(X_t))$$

when $X_t > \bar{X}$ (that is $L^* > 0$). Hence, if $X_t > \bar{X}$,

$$\mathcal{F}'(X_t) = G_X(X_t, L^*(X_t)) = \alpha A [1 + \lambda L^*(X_t)]^{\theta} [X_t - qL^*(X_t)]^{\alpha-1} + 1 - \delta_K$$

We observe that $\mathcal{F}'(X_t) = F_K(K_t, L^*(X_t)) > 0$.

In the following subsections, we compare the initial planner's program (P0) to the new program (P1), where the planner chooses the optimal sequence $(K_t^*)_{t=0}^{\infty}$ taking as given the sequence $(L_t^*)_{t=0}^{\infty}$ solution to (P0). In program (P1), $(L_t^*)_{t=0}^{\infty}$ is just a particular sequence of externalities from the living. From now on, for simplicity, we denote a sequence $(z_t)_{t=0}^{\infty}$ by (z_t).

3.1.1 Optimal growth

Let us begin by recalling the initial program (P0) and the most salient asymptotic properties of its optimal solution.

(P0) Optimal growth.

$$\max_{(K_t, L_t)} \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$c_t + K_{t+1} + qL_{t+1} \leq F(K_t, L_t)$$

for any $t \geq 0$.

Let $(K_t^*, L_t^*)_{t=0}^{\infty}$ be the optimal path of program (P0) with $(K_0^*, L_0^*) = (K_0, L_0)$. Assuming $u(c) \equiv \ln c$, we have obtained the following properties of the optimal sequence (K_t^*, L_t^*) depending on the "total productivity" parameter $\gamma \equiv \alpha + \theta$.

(1) If $\gamma < 1$, then (K_t^*, L_t^*) converges to the steady state (K^*, L^*) with

$$\lim_{t \to \infty} \frac{K_t^*}{qL_t^*} = \frac{K^*}{qL^*} > 0 \tag{25}$$

(2) If $\gamma = 1$, growth is unbounded and asymptotically balanced with

$$\lim_{t \to \infty} \frac{K_t^*}{qL_t^*} = \bar{X}\frac{\lambda}{q} \leq \frac{\alpha}{\theta} \tag{26}$$

(3) If $1 < \gamma < 1/\beta$, growth is unbounded and unbalanced with

$$\lim_{t \to \infty} \frac{K_t^*}{qL_t^*} = \frac{\alpha}{\theta} \tag{27}$$

3.1.2 Optimal growth with living externalities

Let us assume that the planner takes as given the optimal sequence of living externalities (L_t^*), which is the optimal solution to (P0), and that she maximizes her objective function with respect to the sequence of physical capital levels (K_t). This problem is denoted by (P1), and it is defined and solved below.

(P1) Optimal growth with living externalities.

$$\max_{(K_t)} \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$c_t + K_{t+1} \leq M\left(K_t, L_t^*, L_{t+1}^*\right)$$

for any $t \geq 0$, where $M\left(K_t, L_t^*, L_{t+1}^*\right) \equiv F(K_t, L_t^*) - qL_{t+1}^*$.

There exists an optimal solution $(\tilde{K}_t)$. Furthermore, this solution is unique because of the strict concavity of $F(K, L)$ in K and the strict concavity of $\ln c$ in c.

One can prove that the optimal solutions to (P1) and (P0) coincide, that is $(\tilde{K}_t) = (K_t^*)$, where (K_t^*) is the optimal solution to (P0). Indeed, maximizing the Lagrangian function

$$\sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t \left[F(K_t, L_t^*) - c_t - K_{t+1} - qL_{t+1}^*\right]$$

we obtain the first-order conditions $\lambda_t = \beta^t u'(c_t)$ and $\lambda_t = \lambda_{t+1} F_K\left(K_{t+1}, L_{t+1}^*\right)$ for any t. As usual, the first order conditions can be combined to get the Euler equation

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta F_K\left(K_{t+1}, L_{t+1}^*\right) = \beta \mathcal{F}'(X_{t+1})$$

which coincides with the Euler equation of program (P0).

3.2 Competitive equilibrium with living externalities

Next, we consider a market economy where the sequence of living capital (L_t) is given. In particular, let us assume that the market economy takes the optimal sequence (L_t^*) as given. In other terms, it is as if the government chooses $(L_t) = (L_t^*)$, which is attainable by levying an opportune sequence of real taxes (τ_t^*) to finance the first-best accumulation of living capital, that is $\tau_t^* = q\left[L_{t+1}^* - (1 - \delta_L) L_t^*\right]$.

The firm chooses the sequence of physical capital demand (K_t) that maximizes its intertemporal profit:

$$\pi^* = \max_{(K_t)} \left(\sum_{t=0}^{\infty} p_t^* \left[A\left(1 + \lambda L_t^*\right)^{\theta} K_t^{\alpha} + (1 - \delta_K) K_t - K_{t+1} - \tau_t^*\right] - r^* K_0 \right)$$

At the same time, the representative agent maximizes her intertemporal utility function under the intertemporal budget constraint: $\sum_{t=0}^{\infty} p_t^* c_t = \pi^* + r^* K_0$. The associated competitive equilibrium (P2) is defined as follows:

(P2) Competitive equilibrium (p_t^*, q^*, K_t^*) with (L_t^*) as externality.

(P2) comprehends the firm's and the representative consumer's problems and the clearing of all markets:

(1) The firm chooses the sequence of physical capital demand (K_t) to maximize the profit:

$$\pi^* = \max_{(K_t)} \left(\sum_{t=0}^{\infty} p_t^* \left[M\left(K_t, L_t^*, L_{t+1}^*\right) - K_{t+1} \right] - r^* K_0 \right)$$

Profit maximization gives

$$p_t^* = p_{t+1}^* M_K\left(K_{t+1}, L_{t+1}^*, L_{t+2}^*\right) = p_{t+1}^* F_K\left(K_{t+1}, L_{t+1}^*\right)$$

for any $t \geq 0$.

Remark 15 *We observe that there is only one firm acting as a competitive firm without market power. In other terms, the firm takes prices as given and does not play strategically by exploiting its market power. In the case of constant returns to scale, a zero profit condition holds and maximizing the profits of a large number of firms yields the same solution in terms of optimal aggregate demand for capital than maximizing the profit of an aggregate firm. However, in our case, returns to scale are not constant and we assume that the unique existing firm is a price taker. Even if this is a strong assumption, we can partially justify it by noticing that the capital good and the consumption good are the same and the firms shares the market power with many consumers.*

(2) The representative consumer maximizes her intertemporal utility function under an intertemporal budget constraint:

$$\max_{(c_t)} \sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$
$$\sum_{t=0}^{\infty} p_t^* c_t \leq \pi^* + r^* K_0$$

where π^* is the profit she earns as (unique) shareholder. r^* is the price of K_0 at period -1. Indeed, note that physical capital is bought one period before being used.

(3) All markets clear:

$$c_t + K_{t+1}^* = M\left(K_t^*, L_t^*, L_{t+1}^*\right)$$

with $K_0^* = K_0$.

Our final theorem provides with the prices that allow to decentralize the policy maker's solution.

Theorem 16 *Let $0 < \gamma < 1/\beta$ and let the sequence (c_t^*, K_t^*) be solution to (P1). Define*

$$\begin{aligned} p_t^* &\equiv \beta^t u'(c_t^*) = \beta^t / c_t^* \\ r^* &\equiv p_0^* M_K(K_0, L_0^*, L_1^*) = \alpha A (1 + \lambda L_0)^{\theta} K_0^{\alpha - 1} + 1 - \delta_K \end{aligned}$$

Then $(r^*, (p_t^*, c_t^*, K_t^*)_{t=0}^{\infty})$ is a competitive equilibrium where (L_t^*), solution to (P0), is a sequence of externalities.

Note that the price the firm has to pay for K_0 increases with the relative productivity of living capital, λ, and with θ, its share in production. Additionally, r^* also increases with the initial level of living capital.

4 Conclusion

We have defined living capital as a comprehensive measure of all economic inputs that are alive. The three characteristics of living capital are that: first, it is not essential; second, it is irreversible; third, it depreciates at a slower pace than physical capital. Among these, the one that makes living capital truly different from any other productive capital is its non-essentiality, which means that the economy could very well produce without living capital.

Then, we defined total capital as a sum of physical and living capital in terms of value and we have considered, first, the planner's solution and, then, the market economy.

In the case of the policy maker's problem, we have shown that the optimal sequence of capital is always monotonic. Depending on the productivity of total capital, three different regimes hold: (1) bounded growth; (2) asymptotically balanced unbounded growth; (3) unbalanced unbounded growth. In case (1), when the sum of the productivity parameters of physical and living capital is lower than one, total capital converges to a steady state with a positive stock of living capital, which is larger than the one without living capital. In the second case, when this sum is equal to one, growth becomes unbounded and the economy experiences an asymptotically balanced growth path. Finally, in case (3), when the sum of the productivity parameters of physical and living capital is greater than one, growth is unbounded for both physical and living capital. Clearly, this implies that total capital grows forever. Beyond a critical point of total capital, living capital starts to grow but the ratio physical to living capital first decreases and then, after a critical date, it grows to infinity.

In the case of a market economy, we show that if the government levies taxes to finance the accumulation of living capital and implements exactly the optimal sequence of living capital as in the planner's program, then the equilibrium market prices decentralize exactly the planner's solution. We provide the explicit sequence of market prices.

5 Appendix

Proof of Lemma 2

We maximize $F(K_t, L_t) \equiv A(1+\lambda L_t)^{\theta} K_t^{\alpha} + (1-\delta_K) K_t + (1-\delta_L) qL_t$ under the constraint $K_t + qL_t \leq X_t$ with $L_t, K_t \geq 0$.

We observe that $F(X_t - qL_t, L_t)$ is strictly concave in L_t:

$$\frac{[F(X_t - qL_t, L_t)]''}{A(1+\lambda L_t)^{\theta} K_t^{\alpha}} = -\alpha(1-\alpha)\frac{q^2}{K_t^2} - \theta(1-\theta)\frac{\lambda^2}{(1+\lambda L_t)^2} - 2\alpha\theta\frac{q\lambda}{K_t(1+\lambda L_t)} < 0$$

that is $[F(X_t - qL_t, L_t)]'' < 0$.

Equivalently, we can maximize $A(1+\lambda L_t)^{\theta}(X_t - qL_t)^{\alpha} + (\delta_K - \delta_L) qL_t$ with respect to $L_t \geq 0$.

If $L_t > 0$, we obtain the first-order condition:

$$\lambda\theta A(1+\lambda L_t)^{\theta-1}(X_t - qL_t)^{\alpha} - \alpha q A(1+\lambda L_t)^{\theta}(X_t - qL_t)^{\alpha-1} + q(\delta_K - \delta_L) = 0 \tag{28}$$

or, equivalently, $\mathcal{L}(L_t) = \mathcal{R}(L_t)$, where $\mathcal{L}(L_t)$ and $\mathcal{R}(L_t)$ are given by (10) and (11).

We observe that, for $L \in [0, X/q)$, $\mathcal{L}'(L) < 0$ and $\mathcal{R}'(L) > 0$.

Moreover, $\lim_{L_t \to 0} \mathcal{L}(L_t) = \lambda\theta A X_t^{\alpha} + q\delta_K$, $\lim_{L_t \to 0} \mathcal{R}(L_t) = \alpha q A X_t^{\alpha-1} + q\delta_L$, $\lim_{L_t \to X_t/q} \mathcal{L}(L_t) = q\delta_K$ and $\lim_{L_t \to (X_t/q)^-} \mathcal{R}(L_t) = \infty$.

Thus, there exists a unique value $L^* \in [0, X_t/q)$ such that $\mathcal{L}(L^*) = \mathcal{R}(L^*)$ if and only if $\lambda\theta A X_t^{\alpha} + q\delta_K \geq \alpha q A X_t^{\alpha-1} + q\delta_L$.

The critical value $\bar{X}$ is defined by $\lambda\theta A X^{\alpha} + q\delta_K = \alpha q A X^{\alpha-1} + q\delta_L$, that is by (7).

When $X_t = \bar{X}$, then $L^* = 0$; when $X_t > \bar{X}$, then $L^* > 0$ because $\lambda\theta A X_t^{\alpha} + q\delta_K - \alpha q A X_t^{\alpha-1} - q\delta_L > 0$ (indeed, the LHS of this inequality is strictly increasing in X_t).

Therefore, we obtain the following.

(1) If $X_t \leq \bar{X}$, then $L^*(X_t) = 0$ and $K^*(X_t) = X_t$, and the maximum is given by (9).

(2) If $X_t > \bar{X}$, then $L^*(X_t) \in (0, X_t/q)$ and the maximum is given by (12).

(3) Functions (9) and (12) are continuous and differentiable in $\left[0, \bar{X}\right)$ and $\left(\bar{X}, \infty\right)$ respectively. Moreover, they are also continuous at $X = \bar{X}$:

$$\begin{aligned} & \lim_{X_t \to \bar{X}^+} \mathcal{F}(X_t) \\ = & \lim_{L^* \to 0^+} \left[A(1+\lambda L^*)^{\theta}(X_t - qL^*)^{\alpha} + (1-\delta_K)(X_t - qL^*) + (1-\delta_L) qL^*\right] \\ = & AX_t^{\alpha} + (1-\delta_K) X_t = \lim_{X_t \to \bar{X}^-} \mathcal{F}(X_t) \end{aligned}$$

We apply the Envelope Theorem to prove the differentiability of $\mathcal{F}$ at $X_t = \bar{X}$. Let us define

$$G(X_t, L_t) \equiv A(1+\lambda L_t)^{\theta}(X_t - qL_t)^{\alpha} + (1-\delta_K)(X_t - qL_t) + (1-\delta_L) qL_t$$

and denote the first derivatives by G_X and G_L.

We know that $L^*(X_t)$ is solution to $G_L(X_t, L_t) = 0$. We observe that, if $X_t > \bar{X}$ (that is $L^* > 0$), then by the Envelope Theorem,

$$\mathcal{F}'(X_t) = G_X(X_t, L^*(X_t)) + G_L(X_t, L^*(X_t)) L^{*\prime}(X_t) = G_X(X_t, L^*(X_t))$$

Hence, if $X_t > \bar{X}$,

$$\mathcal{F}'(X_t) = G_X(X_t, L^*(X_t)) = \alpha A [1 + \lambda L^*(X_t)]^{\theta} [X_t - qL^*(X_t)]^{\alpha-1} + 1 - \delta_K > 0 \tag{29}$$

which proves that, $\mathcal{F}$ is strictly increasing.

Additionally, we have that

$$\begin{aligned} \lim_{X_t \to \bar{X}^-} \mathcal{F}'(X_t) &= \alpha A X_t^{\alpha-1} + 1 - \delta_K \\ \lim_{X_t \to \bar{X}^+} \mathcal{F}'(X_t) &= \lim_{X_t \to \bar{X}^+} G_X(X_t, L^*(X_t)) = G_X(X_t, 0) = \alpha A X_t^{\alpha-1} + 1 - \delta_K \end{aligned}$$

which proves that $\mathcal{F}$ is differentiable at $X_t = \bar{X}$.

(4) Taking the derivative of (28) with $L_t = L^*(X_t)$ with respect to X_t, we obtain

$$qL^{*\prime}(X_t) = \frac{\alpha q \lambda \theta + \alpha(1-\alpha) q^2 \frac{P(X_t)}{K^*(X_t)}}{\theta(1-\theta)\lambda^2 \frac{K^*(X_t)}{P(X_t)} + 2\alpha q \lambda \theta + \alpha(1-\alpha) q^2 \frac{P(X_t)}{K^*(X_t)}} > 0 \tag{30}$$

$$K^{*\prime}(X) = 1 - qL^{*\prime}(X) = \frac{\theta(1-\theta)\lambda^2 \frac{K^*(X_t)}{P(X_t)} + \alpha q \lambda \theta}{\theta(1-\theta)\lambda^2 \frac{K^*(X_t)}{P(X_t)} + 2\alpha q \lambda \theta + \alpha(1-\alpha) q^2 \frac{P(X_t)}{K^*(X_t)}} > 0 \tag{31}$$

where $P(X_t) \equiv 1 + \lambda L^*(X_t)$.

(5) Focus on the derivative for $X_t > \bar{X}$:

$$\mathcal{F}'(X_t) = \alpha A [1 + \lambda L^*(X_t)]^{\theta} [X_t - qL^*(X_t)]^{\alpha-1} + 1 - \delta_K > 0 \tag{32}$$

The second derivative when $X_t > \bar{X}$ is given by

$$\mathcal{F}''(X_t) = \alpha A \frac{\lambda \theta K^*(X_t) L^{*\prime}(X) - (1-\alpha) P(X_t) [1 - qL^{*\prime}(X_t)]}{P(X_t)^{1-\theta} K^*(X_t)^{2-\alpha}}$$

Then, $\mathcal{F}''(X_t) < 0$ if and only if

$$\lambda \theta K^*(X_t) qL^{*\prime}(X_t) - (1-\alpha) qP(X_t) [1 - qL^{*\prime}(X_t)] < 0 \tag{33}$$

Replacing expressions (30) and (31) for $qL^{*\prime}(X_t)$ and $1 - qL^{*\prime}(X_t) = K^{*\prime}(X_t)$ in (33), we find that $\mathcal{F}''(X_t) < 0$ if and only if $\gamma < 1$. ■

Proof of Lemma 3

(1) First, we show that $\lim_{X_t \to \infty} L^*(X_t) = \infty$.

If it was not the case, there should exist ν and a sequence (X_n) with $\lim_{n\to\infty} X_n = \infty$ such that $L^*(X_n) \leq \nu$ for any n.

Let $L_n^* \equiv L^*(X_n)$. According to (28), for any n, we have

$$\begin{aligned} 0 &= \lambda\theta A\left(1+\lambda L_n^*\right)^{\theta-1}\left(X_n - qL_n^*\right)^{\alpha} \\ &\quad -\alpha qA\left(1+\lambda L_n^*\right)^{\theta}\left(X_n - qL_n^*\right)^{\alpha-1} + q\left(\delta_K - \delta_L\right) \end{aligned} \tag{34}$$

and

$$\begin{aligned} \lambda\theta A\left(1+\lambda L_n^*\right)^{\theta-1}\left(X_n - qL_n^*\right)^{\alpha} &\geq \lambda\theta A\left(1+\lambda\nu\right)^{\theta-1}\left(X_n - q\nu\right)^{\alpha} \\ \alpha qA\left(1+\lambda L_n^*\right)^{\theta}\left(X_n - qL_n^*\right)^{\alpha-1} &\leq \alpha qA\left(1+\lambda\nu\right)^{\theta}\left(X_n - q\nu\right)^{\alpha-1} \end{aligned}$$

We observe that

$$\begin{aligned} \lim_{n\to\infty}\left[\lambda\theta A\left(1+\lambda L_n^*\right)^{\theta-1}\left(X_n - qL_n^*\right)^{\alpha}\right] &\geq \lim_{n\to\infty}\left[\lambda\theta A\left(1+\lambda\nu\right)^{\theta-1}\left(X_n - q\nu\right)^{\alpha}\right] = \infty \\ \lim_{n\to\infty}\left[\alpha qA\left(1+\lambda L_n^*\right)^{\theta}\left(X_n - qL_n^*\right)^{\alpha-1}\right] &\leq \lim_{n\to\infty}\left[\alpha qA\left(1+\lambda\nu\right)^{\theta}\left(X_n - q\nu\right)^{\alpha-1}\right] = 0 \end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\left[\lambda\theta A\left(1+\lambda L_n^*\right)^{\theta-1}\left(X_n - qL_n^*\right)^{\alpha} - \alpha qA\left(1+\lambda L_n^*\right)^{\theta}\left(X_n - qL_n^*\right)^{\alpha-1} + q\left(\delta_K - \delta_L\right)\right] = \infty$$

which is in contradiction with (34), and which proves that $\lim_{X_t\to\infty} L^*(X_t) = \infty$.

Next, let us show in a similar manner that $\lim_{X_t\to\infty} K^*(X_t) = \infty$.

If it was not the case, then there should exist $\kappa > 0$ and a sequence (X_n) with $\lim_{n\to\infty} X_n = \infty$ such that $K^*(X_n) \leq \kappa$ for any n.

Let $K_n^* \equiv K^*(X_n)$. According to (28), for any n, we have

$$\lambda\theta A\left(1+\lambda L_n^*\right)^{\theta-1} K_n^{*\alpha} - \alpha qA\left(1+\lambda L_n^*\right)^{\theta} K_n^{*\alpha-1} + q\left(\delta_K - \delta_L\right) = 0 \tag{35}$$

Since $\lim_{n\to\infty} L_n^* = \infty$ and $K_n^* \in [0,\kappa]$, a bounded set, we have

$$\lim_{n\to\infty}\left[\lambda\theta A\left(1+\lambda L_n^*\right)^{\theta-1} K_n^{*\alpha}\right] = 0$$

Moreover, $\alpha qA\left(1+\lambda L_n^*\right)^{\theta} K_n^{*\alpha-1} \geq \alpha qA\left(1+\lambda L_n^*\right)^{\theta}\kappa^{\alpha-1}$ for any n and

$$\lim_{n\to\infty}\left[\alpha qA\left(1+\lambda L_n^*\right)^{\theta} K_n^{*\alpha-1}\right] \geq \lim_{n\to\infty}\left[\alpha qA\left(1+\lambda L_n^*\right)^{\theta}\kappa^{\alpha-1}\right] = \infty$$

Thus,

$$\lim_{n\to\infty}\left[\lambda\theta A\left(1+\lambda L_n^*\right)^{\theta-1} K_n^{*\alpha} - \alpha qA\left(1+\lambda L_n^*\right)^{\theta} K_n^{*\alpha-1} + q\left(\delta_K - \delta_L\right)\right] = -\infty$$

which is in contradiction with (35).

Observe that these results hold for any $\gamma > 0$.

(2) We have

$$\mathcal{F}(X_t) = A\left[1+\lambda L^*(X_t)\right]^{\theta}\left[X_t - qL^*(X_t)\right]^{\alpha} + (\delta_K - \delta_L)\, qL^*(X_t) + (1-\delta_K)\, X_t$$

Hence,

$$\begin{aligned}
\frac{\mathcal{F}(X_t)}{X_t} &= A\left[1+\lambda L^*(X_t)\right]^{\theta}\frac{\left[X_t - qL^*(X_t)\right]^{\alpha}}{X_t} + 1 - \delta_L \\
&\quad + (\delta_K - \delta_L)\left[\frac{qL^*(X_t)}{X_t} - 1\right] \\
&\leq A\left[1+\frac{\lambda}{q}qL^*(X_t)\right]^{\theta}\frac{\left[X_t - qL^*(X_t)\right]^{\alpha}}{X_t} + 1 - \delta_L \\
&< A\left(1+\frac{\lambda}{q}X_t\right)^{\theta} X_t^{\alpha-1} + 1 - \delta_L
\end{aligned}$$

since $X_t > \bar{X}$ entails $L^*(X_t) > 0$.

$\gamma < 1$ implies

$$\lim_{X_t\to\infty}\left(1+\frac{\lambda}{q}X_t\right)^{\theta} X_t^{\alpha-1} = 0$$

Therefore,

$$\lim_{X_t\to\infty}\frac{\mathcal{F}(X_t)}{X_t} \leq 1 - \delta_L < 1$$

(3) Let $\gamma = 1$.

First we prove that there exists $\mu \in (0, \alpha/\theta]$ such that, for any sequence (X_n) with $\lim_{n\to\infty} X_n = \infty$, we have

$$\mu \leq \lim\inf_{n\to\infty}\zeta(X_n)$$

From equation (28), we have

$$\begin{aligned}
&\lambda\theta A\left[1+\lambda L^*(X_t)\right]^{\theta-1}\left[X_t - qL^*(X_t)\right]^{\alpha} - \alpha qA\left[1+\lambda L^*(X_t)\right]^{\theta}\left[X_t - qL^*(X_t)\right]^{\alpha-1} \\
= \;& q(\delta_L - \delta_K) \qquad (36)
\end{aligned}$$

Since, by definition,

$$\zeta(X_t) = \frac{X_t - qL^*(X_t)}{qL^*(X_t)}$$

then, we have

$$L^*(X_t) = \frac{1}{q}\frac{X_t}{1+\zeta(X_t)}$$

and (36) becomes $\mathcal{L}_1(X_t) + \mathcal{L}_2(X_t) = \mathcal{R}(X_t)$, where

$$\begin{aligned}
\mathcal{L}_1(X_t) &\equiv \lambda A X_t^{\alpha}\left[1+\frac{\lambda}{q}\frac{X_t}{1+\zeta(X_t)}\right]^{\theta-1}\left[\theta-\frac{\alpha+\theta}{1+\zeta(X_t)}\right] \\
\mathcal{L}_2(X_t) &\equiv -\alpha q A X_t^{\alpha-1}\left[1+\frac{\lambda}{q}\frac{X_t}{1+\zeta(X_t)}\right]^{\theta-1} \\
\mathcal{R}(X_t) &\equiv q(\delta_L-\delta_K)\left[\frac{\zeta(X_t)}{1+\zeta(X_t)}\right]^{1-\alpha}
\end{aligned}$$

We want to prove that there exist $\mu > 0$ and $\nu > 0$ such that, for any $X_t \geq \nu$,

$$\zeta(X_t) \geq \mu$$

Suppose the contrary. In this case, there would exist a sequence (X_n) with $\lim_{n\to\infty} X_n = \infty$ such that $\lim_{n\to\infty} \zeta(X_n) = 0$.

We observe that $\lim_{n\to\infty} \mathcal{L}_1(X_n) = -\infty$ because $\gamma \geq 1$,

$$\lim_{n\to\infty}\left(\lambda A X_n^{\alpha}\left[1+\frac{\lambda}{q}\frac{X_n}{1+\zeta(X_n)}\right]^{\theta-1}\right) = \infty$$

and

$$\lim_{n\to\infty}\left[\theta-\frac{\gamma}{1+\zeta(X_n)}\right] = -\alpha$$

Moreover, since $\alpha < 1$ and $\theta < 1$,

$$\lim_{n\to\infty}\mathcal{L}_2(X_n) = -\alpha q A \lim_{n\to\infty}\left(X_n^{\alpha-1}\left[1+\frac{\lambda}{q}\frac{X_n}{1+\zeta(X_n)}\right]^{\theta-1}\right) = 0$$

Finally, since $\alpha < 1$,

$$\lim_{n\to\infty}\mathcal{R}(X_n) = q(\delta_L-\delta_K)\lim_{n\to\infty}\left[\frac{\zeta(X_n)}{1+\zeta(X_n)}\right]^{1-\alpha} = 0$$

Then, $0 = \lim_{n\to\infty}\mathcal{R}(X_n) = \lim_{n\to\infty}\mathcal{L}_1(X_n) + \lim_{n\to\infty}\mathcal{L}_2(X_n) = -\infty$, a contradiction.

Now, let us prove that, for any sequence (X_n) with $\lim_{n\to\infty} X_n = \infty$, we have

$$\limsup_{n\to\infty}\zeta(X_n) \leq \frac{\alpha}{\theta}$$

Equation (28) writes:

$$\lambda\theta A\left[1+\lambda L^*(X_t)\right]^{\theta-1}K^*(X_t)^{\alpha} + q(\delta_K-\delta_L) = \alpha q A\left[1+\lambda L^*(X_t)\right]^{\theta}K^*(X_t)^{\alpha-1}$$

Since $\delta_K \geq \delta_L$, we have

$$\lambda\theta A\left[1+\lambda L^*(X_t)\right]^{\theta-1}K^*(X_t)^{\alpha} \leq \alpha q A\left[1+\lambda L^*(X_t)\right]^{\theta}K^*(X_t)^{\alpha-1}$$

that is

$$\frac{K^*(X_t)}{qL^*(X_t)} \leq \frac{\alpha}{\lambda\theta}\frac{1+\lambda L^*(X_t)}{L^*(X_t)} \tag{37}$$

Since $\lim_{X_t\to\infty} L^*(X_t) = \infty$, for any sequence (X_n) such that $\lim_{n\to\infty} X_n = \infty$, we find

$$\limsup_{n\to\infty} \frac{K^*(X_n)}{qL^*(X_n)} \leq \frac{\alpha}{\theta}$$

(4) Finally, let $\gamma > 1$.

It is easy to show that $\lim_{X_t\to\bar{X}^-} \zeta(X_t) = \infty$. Indeed, $L^*(X_t)$ converges to 0 continuously and $K^*(X_t)$ to $\bar{X}$ also continuously ($L^*(X_t)$ and $K^*(X_t)$ are differentiable).

Focus now on the asymptotic behavior of $\zeta(X_t)$ when $X_t \to \infty$. We want to prove that $\lim_{X_t\to\infty} \zeta(X_t) = \alpha/\theta$.

Let us to show first that, if $\varepsilon > 0$, then there exists $\mu > 0$ such that $\zeta(X_t) \leq \mu$ for any $X_t \geq \bar{X} + \varepsilon$.

According to (37), we have

$$\zeta(X_t) \leq \frac{\alpha}{\lambda\theta}\frac{1+\lambda L^*(X_t)}{L^*(X_t)}$$

for any $X_t \geq \bar{X} + \varepsilon$. Since $\lim_{X_t\to\infty} L^*(X_t) = \infty$,

$$\lim_{X_t\to\infty}\left[\frac{\alpha}{\lambda\theta}\frac{1+\lambda L^*(X_t)}{L^*(X_t)}\right] = \frac{\alpha}{\theta}$$

Let $\eta > 0$. Then, there exists $Y > \bar{X} + \varepsilon$ such that

$$\frac{\alpha}{\lambda\theta}\frac{1+\lambda L^*(X_t)}{L^*(X_t)} \leq \frac{\alpha}{\theta} + \eta$$

for any $X_t > Y$. Let

$$\mu \equiv \max\left\{\max_{X_t\in[\bar{X}+\varepsilon,Y]}\left[\frac{\alpha}{\lambda\theta}\frac{1+\lambda L^*(X_t)}{L^*(X_t)}\right], \frac{\alpha}{\theta} + \eta\right\}$$

Therefore, we obtain

$$\zeta(X_t) \leq \mu \tag{38}$$

for any $X_t \geq \bar{X} + \varepsilon$.

(28) is equivalent to

$$\lambda\theta A K^*(X_t) - \alpha q A\left[1+\lambda L^*(X_t)\right] = q\left(\delta_L - \delta_K\right)\left[1+\lambda L^*(X_t)\right]^{1-\theta} K^*(X_t)^{1-\alpha}$$

$$\frac{K^*(X_t)}{qL^*(X_t)} = \frac{\alpha}{\lambda\theta}\frac{1+\lambda L^*(X_t)}{L^*(X_t)} + \frac{\delta_L - \delta_K}{\lambda\theta A}\frac{\left[1+\lambda L^*(X_t)\right]^{1-\theta} K^*(X_t)^{1-\alpha}}{L^*(X_t)} \tag{39}$$

Note that

$$\frac{\left[1+\lambda L^{*}\left(X_{t}\right)\right]^{1-\theta} K^{*}\left(X_{t}\right)^{1-\alpha}}{L^{*}\left(X_{t}\right)}=\left[\frac{1+\lambda L^{*}\left(X_{t}\right)}{L^{*}\left(X_{t}\right)}\right]^{1-\theta}\left[\frac{L^{*}\left(X_{t}\right)}{K^{*}\left(X_{t}\right)}\right]^{-\theta} K^{*}\left(X_{t}\right)^{1-\gamma} \tag{40}$$

According to (38), we have that, for any $X_t \geq \bar{X}+\varepsilon$,

$$\begin{aligned}\frac{K^{*}\left(X_{t}\right)}{L^{*}\left(X_{t}\right)} &\leq \mu q \\ \left[\frac{L^{*}\left(X_{t}\right)}{K^{*}\left(X_{t}\right)}\right]^{-\theta} &\leq \left(\frac{1}{\mu q}\right)^{-\theta}\end{aligned}$$

Then, for any $X_t \geq \bar{X}+\varepsilon$,

$$\begin{aligned}0 &\leq \left[\frac{1+\lambda L^{*}\left(X_{t}\right)}{L^{*}\left(X_{t}\right)}\right]^{1-\theta}\left[\frac{L^{*}\left(X_{t}\right)}{K^{*}\left(X_{t}\right)}\right]^{-\theta} K^{*}\left(X_{t}\right)^{1-\alpha-\theta} \\ &\leq \left[\frac{1+\lambda L^{*}\left(X_{t}\right)}{L^{*}\left(X_{t}\right)}\right]^{1-\theta}\left(\frac{1}{\mu q}\right)^{-\theta} K^{*}\left(X_{t}\right)^{1-\alpha-\theta}\end{aligned}$$

If $\gamma > 1$, we have

$$\lim_{X_{t}\to\infty}\left(\left[\frac{1+\lambda L^{*}\left(X_{t}\right)}{L^{*}\left(X_{t}\right)}\right]^{1-\theta}\left(\frac{1}{\mu q}\right)^{-\theta} K^{*}\left(X_{t}\right)^{1-\gamma}\right)=0$$

and, thus, the limit of (40) also exists with

$$\lim_{X_{t}\to\infty}\frac{\left[1+\lambda L^{*}\left(X_{t}\right)\right]^{1-\theta} K^{*}\left(X_{t}\right)^{1-\alpha}}{L^{*}\left(X_{t}\right)}=0$$

In addition, according to (39), we obtain

$$\lim_{X_{t}\to\infty}\frac{K^{*}\left(X_{t}\right)}{qL^{*}\left(X_{t}\right)}=\frac{\alpha}{\lambda\theta}\lim_{X_{t}\to\infty}\frac{1+\lambda L^{*}\left(X_{t}\right)}{L^{*}\left(X_{t}\right)}=\frac{\alpha}{\theta} \tag{41}$$

Since

$$\frac{qL^{*}\left(X_{t}\right)}{X_{t}}=\frac{1}{1+\zeta\left(X_{t}\right)}$$

then

$$\frac{qL^{*}\left(X_{t}\right)}{X_{t}}<\frac{\theta}{\alpha+\theta}$$

for any X and

$$\lim_{X_{t}\to\infty}\frac{qL^{*}\left(X_{t}\right)}{X_{t}}=\frac{\theta}{\alpha+\theta}$$

Finally, consider $\delta_K > \delta_L$. According to (28), we have

$$\zeta\left(X_{t}\right)\equiv\frac{K^{*}\left(X_{t}\right)}{qL^{*}\left(X_{t}\right)}=\frac{\alpha}{\theta}+\frac{\alpha A+\left(\delta_{L}-\delta_{K}\right)\left[1+\lambda L^{*}\left(X_{t}\right)\right]^{1-\theta} K^{*}\left(X_{t}\right)^{1-\alpha}}{\lambda\theta A L^{*}\left(X_{t}\right)}$$

Notice that

$$\lim_{X\to\infty}\left(\left(\delta_L-\delta_K\right)\left[1+\lambda L^*\left(X_t\right)\right]^{1-\theta}K^*\left(X_t\right)^{1-\alpha}\right)=-\infty$$

For X_t large enough, $\zeta\left(X_t\right)<\alpha/\theta$ and, according to (41), $\lim_{X\to\infty}\zeta\left(X_t\right)=\alpha/\theta$. ■

Proof of Lemma 4

First, we want to prove that, under Assumption 2, $\sum_{t=0}^{\infty}\beta^t\ln c_t<\infty$ for any $X_0\geq 0$.

We observe that $c_t\leq\mathcal{F}\left(X_t\right)$ for any t. If $X_0=0$, then $c_t=0$ for any $t\geq 0$ and $\sum_{t=0}^{\infty}\beta^t\ln c_t<\infty$.

Let $X_0>0$. If the feasible sequence $\left(X_t\right)_{t=0}^{\infty}$ is bounded, the sequences $\left(\mathcal{F}\left(X_t\right)\right)_{t=0}^{\infty}$ and $\left(c_t\right)_{t=0}^{\infty}$ are bounded as well. Thus, $\sum_{t=0}^{\infty}\beta^t\ln c_t<\infty$.

Now, let the sequence $\left(X_t\right)_{t=0}^{\infty}$ be unbounded.

A sequence $\left(X_t\right)_{t=0}^{\infty}$ is feasible if $X_{t+1}\leq\mathcal{F}\left(X_t\right)$ for any t. Define the maximal sequence $\left(\tilde{X}_t\right)_{t=0}^{\infty}$ by $\tilde{X}_{t+1}=\mathcal{F}\left(\tilde{X}_t\right)$ for any t with $\tilde{X}_0=X_0$. According to point (3) in Lemma 2, $\mathcal{F}$ is strictly increasing and $X_t\leq\tilde{X}_t$ for any t. Then, if $\left(c_t\right)_{t=0}^{\infty}$ is a sequence of feasible consumptions, $c_t\leq\mathcal{F}\left(X_t\right)\leq\mathcal{F}\left(\tilde{X}_t\right)=\tilde{X}_{t+1}$ for any t, and $\sum_{t=0}^{\infty}\beta^t\ln c_t\leq\sum_{t=0}^{\infty}\beta^t\ln\mathcal{F}\left(\tilde{X}_t\right)$.

Let us show that there exists T such that $\tilde{X}_t>\bar{X}$ for any $t\geq T$.

The case $\tilde{X}_t\leq\bar{X}$ for any t is impossible. If $X_0=\tilde{X}_0\leq\bar{X}$, there exists T such that $\tilde{X}_T>\bar{X}$. Indeed, if not, $\tilde{X}_{t+1}=A\tilde{X}_t^{\alpha}+\left(1-\delta_K\right)\tilde{X}_t\equiv\xi\left(\tilde{X}_t\right)$ for any $t\geq 0$. Since ξ is continuous and strictly concave with $\xi'\left(0\right)=\infty$, $\xi'\left(\infty\right)=1-\delta_K<1$, it crosses the line $\tilde{X}_{t+1}=\tilde{X}_t$ at $\tilde{X}_t=0$ and $\tilde{X}_t=\hat{X}$. Thus, according to (16), $\lim_{t\to\infty}\tilde{X}_t=\hat{X}>\bar{X}$, a contradiction.

Let $\tilde{X}_T>\bar{X}$ for some T. Since $\hat{X}=A\hat{X}^{\alpha}+\left(1-\delta_K\right)\hat{X}$ and, according to (16), $\bar{X}<\hat{X}$, we have $\bar{X}<A\bar{X}^{\alpha}+\left(1-\delta_K\right)\bar{X}$. Moreover, by definition of $\bar{X}$, $\mathcal{F}\left(\bar{X}\right)=A\bar{X}^{\alpha}+\left(1-\delta_K\right)\bar{X}$. Since $\mathcal{F}$ is strictly increasing, if $\tilde{X}_T>\bar{X}$, then $\tilde{X}_{T+1}=\mathcal{F}\left(\tilde{X}_T\right)>\mathcal{F}\left(\bar{X}\right)=A\bar{X}^{\alpha}+\left(1-\delta_K\right)\bar{X}>\bar{X}$. By induction, we obtain $\tilde{X}_t>\bar{X}$ for any $t\geq T$. Clearly, if $T=0$, we have $\tilde{X}_t>\bar{X}$ for any $t\geq 0$.

In the following, without loss of generality, we focus on the case $\tilde{X}_t>\bar{X}$ for any $t\geq 0$.

We consider three cases: (1) $\gamma<1$, (2) $\gamma=1$, (3) $1<\gamma<1/\beta$.

(1) $\gamma<1$.

According to point (3) of Lemma 2, $\mathcal{F}$ is C^1, and, since $\gamma<1$, according to point (5), $\mathcal{F}$ is strictly concave. Then, $\mathcal{F}\left(X_t\right)/X_t$ is a continuous and strictly decreasing function. Moreover, according to point (2) of Lemma 3,

$$\lim_{X_t\to 0^+}\frac{\mathcal{F}\left(X_t\right)}{X_t}=\infty\text{ and }\lim_{X_t\to\infty}\frac{\mathcal{F}\left(X_t\right)}{X_t}\leq 1-\delta_L<1$$

Thus, $\mathcal{F}\left(X_t\right)/X_t$ crosses 1 once and, therefore, there exists a unique fixed point $Z>\bar{X}$ such that $Z=\mathcal{F}\left(Z\right)$.

If $\tilde{X}_T \leq Z$ for some T, then $\tilde{X}_{T+1} = \mathcal{F}\left(\tilde{X}_T\right) < \mathcal{F}(Z) = Z$. By induction, we can conclude that $\tilde{X}_t \leq Z$ for any $t \geq T$.

If $\tilde{X}_T > Z$ for some T, then $\tilde{X}_{T+1} = \mathcal{F}\left(\tilde{X}_T\right) < \tilde{X}_T$ and, again by induction, we conclude that $\tilde{X}_t \leq \tilde{X}_T$ for any $t \geq T$.

Therefore, $\tilde{X}_t \leq \max\left\{Z, \tilde{X}_T\right\}$ for any t. This implies $\sum_{t=0}^{\infty} \beta^t \ln c_t \leq \sum_{t=0}^{\infty} \beta^t \ln \tilde{X}_{t+1} < \infty$.

(2) $\gamma = 1$.

As above, $\tilde{X}_t > \bar{X}$ for any $t \geq 0$. Then, according to point (5) of Lemma 2, $\mathcal{F}''(X_t) = 0$ for any $X_t > \bar{X}$ and, thus, $\mathcal{F}(X_t) = aX_t + b$ for any $X_t > \bar{X}$. Therefore, $\tilde{X}_{t+1} = \mathcal{F}\left(\tilde{X}_t\right) = a\tilde{X}_t + b$ for any $t \geq 0$.

We can distinguish two subcases depending on a: (2.1) $0 < a < 1$, (2.2) $a \geq 1$.

(2.1) Let $0 < a < 1$ and Z satisfy $Z = aZ + b$.

If $\tilde{X}_t \leq Z$ then $\tilde{X}_{t+1} \leq Z$ and, by induction, $\tilde{X}_t \leq Z$ for all t. The sequence $\left(\tilde{X}_t\right)_{t=0}^{\infty}$ is bounded and we are fine:

$$\begin{aligned}\sum_{t=T}^{\infty} \beta^t \ln c_t &= \sum_{t=T}^{\infty} \beta^t \ln\left[\mathcal{F}(X_t) - X_{t+1}\right] \leq \sum_{t=T}^{\infty} \beta^t \ln \mathcal{F}\left(\tilde{X}_t\right) \\ &\leq \sum_{t=T}^{\infty} \beta^t \ln \mathcal{F}(Z) = \frac{\beta^T}{1-\beta} \ln \mathcal{F}(Z) < \infty\end{aligned}$$

If $\tilde{X}_T > Z$ for some T, then $\tilde{X}_{T+1} = a\tilde{X}_T + b < \tilde{X}_T$ and, by induction, $\tilde{X}_t \leq \tilde{X}_T$ for any $t \geq T$. Therefore, $\tilde{X}_t \leq \max\left\{Z, \tilde{X}_T\right\}$ for any t. This implies $\sum_{t=0}^{\infty} \beta^t \ln c_t \leq \sum_{t=0}^{\infty} \beta^t \ln \tilde{X}_{t+1} < \infty$.

(2.2) Let $a \geq 1$. Then, $\tilde{X}_{t+1} = a\tilde{X}_t + b > \tilde{X}_t$ and the sequence $\left(\tilde{X}_t\right)_{t=0}^{\infty}$ converges to infinity when t goes to infinity.

Let $\varepsilon > 0$. There exists T such that

$$\tilde{X}_t > \frac{1}{\varepsilon}\frac{b}{a}$$

for any $t \geq T$. Clearly,

$$\tilde{X}_{t+1} = \left(1 + \frac{1}{\tilde{X}_t}\frac{b}{a}\right) a\tilde{X}_t < (1+\varepsilon)\, a\tilde{X}_t$$

for any $t \geq T$.

Since

$$\tilde{X}_{T+1} > \tilde{X}_T > \frac{1}{\varepsilon}\frac{b}{a}$$

we get also $\tilde{X}_{T+2} < (1+\varepsilon)\, a\tilde{X}_{T+1}$. By induction, $\tilde{X}_{T+t} < [(1+\varepsilon)\, a]^t \tilde{X}_T$ and, thus,

$$\begin{aligned}
\sum_{t=T+1}^{\infty} \beta^t \ln c_t &\leq \sum_{t=T+1}^{\infty} \beta^t \ln X_{t+1} = \beta^{T+1} \sum_{t=0}^{\infty} \beta^t \ln X_{T+t+2} \\
&\leq \beta^{T+1} \sum_{t=0}^{\infty} \beta^t \ln \tilde{X}_{T+t+2} < \beta^{T+1} \sum_{t=0}^{\infty} \beta^t \ln \left([(1+\varepsilon)\, a]^{t+2} \tilde{X}_T \right) \\
&= \beta^{T+1} \sum_{t=0}^{\infty} \beta^t \left((t+2) \ln [(1+\varepsilon)\, a] + \ln \tilde{X}_T \right) < \infty
\end{aligned}$$

because $\sum_{t=0}^{\infty} t\beta^t = \beta/(1-\beta)^2 < \infty$. Then, $\sum_{t=0}^{\infty} \beta^t \ln c_t = \sum_{t=0}^{T} \beta^t \ln c_t + \sum_{t=T+1}^{\infty} \beta^t \ln c_t < \infty$.

(3) $1 < \gamma < 1/\beta$.

We know that $\tilde{X}_t > \bar{X}$ for any $t \geq 0$ and that

$$\begin{aligned}
\tilde{X}_{t+1} &\equiv \mathcal{F}\left(\tilde{X}_t\right) \\
&= A\left[1 + \lambda L^*\left(\tilde{X}_t\right)\right]^{\theta} \left[\tilde{X}_t - qL^*\left(\tilde{X}_t\right)\right]^{\alpha} \\
&\quad + (1-\delta_K)\left[\tilde{X}_t - qL^*\left(\tilde{X}_t\right)\right] + (1-\delta_L)\, qL^*\left(\tilde{X}_t\right)
\end{aligned}$$

We observe that $\tilde{X}_t \equiv \tilde{K}^*\left(\tilde{X}_t\right) + qL^*\left(\tilde{X}_t\right)$ and

$$\begin{aligned}
\tilde{X}_{t+1} &= \mathcal{F}\left(\tilde{X}_t\right) \\
&= A\left[1 + \frac{\lambda}{q} qL^*\left(\tilde{X}_t\right)\right]^{\theta} \left[\tilde{X}_t - qL^*\left(\tilde{X}_t\right)\right]^{\alpha} \\
&\quad + (1-\delta_K)\left[\tilde{X}_t - qL^*\left(\tilde{X}_t\right)\right] + (1-\delta_L)\, qL^*\left(\tilde{X}_t\right) \\
&\leq A\left(1 + \frac{\lambda}{q}\tilde{X}_t\right)^{\theta} \tilde{X}_t^{\alpha} + (1-\delta_K)\,\tilde{X}_t + (1-\delta_L)\,\tilde{X}_t \\
&= A\left(1 + \frac{\lambda}{q}\tilde{X}_t\right)^{\theta} \tilde{X}_t^{\alpha} + (2-\delta_K-\delta_L)\,\tilde{X}_t \\
&= A\left(\frac{\lambda}{q}\right)^{\theta} \left(\tilde{X}_t + \frac{q}{\lambda}\right)^{\theta} \tilde{X}_t^{\alpha} + (2-\delta_K-\delta_L)\,\tilde{X}_t \\
&= a\left(\tilde{X}_t + b\right)^{\theta} \tilde{X}_t^{\alpha} + c\tilde{X}_t
\end{aligned}$$

where $a \equiv A\,(\lambda/q)^{\theta} > 0$, $b \equiv q/\lambda > 0$ and $c \equiv 2 - \delta_K - \delta_L$.

Define

$$\tilde{Y}_{t+1} \equiv a\left(\tilde{Y}_t + b\right)^{\gamma} + c\tilde{Y}_t$$

with $\tilde{Y}_0 = \tilde{X}_0$. Clearly,

$$\tilde{Y}_{t+1} = a\left(\tilde{Y}_t + b\right)^{\alpha}\left(\tilde{Y}_t + b\right)^{\theta} + c\tilde{Y}_t \geq a\left(\tilde{Y}_t + b\right)^{\theta}\tilde{Y}_t^{\alpha} + c\tilde{Y}_t$$

and

$$\begin{aligned}
\tilde{Y}_1 &\geq a\left(\tilde{Y}_0 + b\right)^{\theta}\tilde{Y}_0^{\alpha} + c\tilde{Y}_0 = a\left(\tilde{X}_0 + b\right)^{\theta}\tilde{X}_0^{\alpha} + c\tilde{X}_0 \geq \mathcal{F}\left(\tilde{X}_0\right) = \tilde{X}_1 \\
\tilde{Y}_2 &\geq a\left(\tilde{Y}_1 + b\right)^{\theta}\tilde{Y}_1^{\alpha} + c\tilde{Y}_1 \geq a\left(\tilde{X}_1 + b\right)^{\theta}\tilde{X}_1^{\alpha} + c\tilde{X}_1 \geq \mathcal{F}\left(\tilde{X}_1\right) = \tilde{X}_2
\end{aligned}$$

Thus, by induction, $\tilde{Y}_t \geq \tilde{X}_t$ for any $t \geq 0$.

Since $\gamma > 1$ we can find $\varepsilon > 0$ such that $a\left(b/\varepsilon\right)^{\gamma} > b/\varepsilon$. Clearly, $\tilde{Y}_t > b/\varepsilon$ implies

$$\tilde{Y}_{t+1} \equiv a\left(\tilde{Y}_t + b\right)^{\gamma} + c\tilde{Y}_t > a\tilde{Y}_t^{\gamma} > a\left(\frac{b}{\varepsilon}\right)^{\gamma} > \frac{b}{\varepsilon}$$

and

$$\begin{aligned}
\tilde{Y}_{t+1} &\equiv a\left(\tilde{Y}_t + b\right)^{\gamma} + c\tilde{Y}_t \\
&= \left[a\left(1 + \frac{b}{\tilde{Y}_t}\right)^{\gamma} + \frac{c}{\tilde{Y}_t^{\gamma-1}}\right]\tilde{Y}_t^{\gamma} < a\left(1+\varepsilon\right)^{\gamma}\tilde{Y}_t^{\gamma}
\end{aligned}$$

If $\tilde{Y}_t \leq b/\varepsilon$ for any t, since $c_t \leq F\left(\tilde{X}_t\right) \leq F\left(\tilde{Y}_t\right) \leq F\left(b/\varepsilon\right)$, the sum $\sum_{t=0}^{\infty}\beta^t \ln c_t$ is finite-valued.

Suppose instead that $\tilde{Y}_T > b/\varepsilon$ for some T. Then, $\tilde{Y}_t > b/\varepsilon$ for any $t \geq T$ and $\tilde{Y}_{t+1} < a\left(1+\varepsilon\right)^{\gamma}\tilde{Y}_t^{\gamma}$ for any $t \geq T$. Hence, $\tilde{X}_{T+1} \leq \tilde{Y}_{T+1} < a\left(1+\varepsilon\right)^{\gamma}\tilde{Y}_T^{\gamma}$ and

$$\tilde{X}_{T+t} \leq \tilde{Y}_{T+t} < \left[a\left(1+\varepsilon\right)^{\gamma}\right]^{\frac{\gamma^t-1}{\gamma-1}}\tilde{Y}_T^{\gamma^t} \tag{42}$$

for any $t \geq 1$. In particular, we get

$$\tilde{X}_{T+t+2} \leq \tilde{Y}_{T+t+2} < \left[a\left(1+\varepsilon\right)^{\gamma}\right]^{\frac{\gamma^{t+2}-1}{\gamma-1}}\tilde{Y}_T^{\gamma^{t+2}}$$

and, thus,

$$\begin{aligned}
\sum_{t=T+1}^{\infty}\beta^t \ln c_t &\leq \sum_{t=T+1}^{\infty}\beta^t \ln X_{t+1} = \beta^{T+1}\sum_{t=0}^{\infty}\beta^t \ln X_{T+t+2} \\
&\leq \beta^{T+1}\sum_{t=0}^{\infty}\beta^s \ln \tilde{X}_{T+t+2} \leq \beta^{T+1}\sum_{t=0}^{\infty}\beta^t \ln \tilde{Y}_{T+t+2} \\
&< \beta^{T+1}\sum_{t=0}^{\infty}\beta^t \ln\left(\left[a\left(1+\varepsilon\right)^{\gamma}\right]^{\frac{\gamma^{t+2}-1}{\gamma-1}}\tilde{Y}_T^{\gamma^{t+2}}\right) \\
&\leq \beta^{T+1}\sum_{t=0}^{\infty}\beta^t\left(\gamma^{t+2}\ln\tilde{Y}_T + \frac{\gamma^{t+2}-1}{\gamma-1}\ln\left[a\left(1+\varepsilon\right)^{\gamma}\right]\right) \\
&\leq \beta^{T+1}\sum_{t=0}^{\infty}\beta^t\left[\gamma^{t+2}\left(\ln\tilde{Y}_T\right)^{+} + \frac{\gamma^{t+2}-1}{\gamma-1}\left(\ln\left[a\left(1+\varepsilon\right)^{\gamma}\right]\right)^{+}\right] \\
&< \infty
\end{aligned}$$

because $\beta\gamma < 1$. Then, $\sum_{t=0}^{\infty}\beta^t \ln c_t = \sum_{t=0}^{T}\beta^t \ln c_t + \sum_{t=T+1}^{\infty}\beta^t \ln c_t < \infty$.

Finally, we want to prove that the limit of the utility function along any feasible path converges to a finite real number or diverges to $-\infty$.

From the initial part of this lemma, we know that, under Assumption 2, $\sum_{t=0}^{\infty}\beta^t \ln c_t < \infty$ for any $X_0 \geq 0$.

Let $\ln^{-} \equiv \min\{0, \ln\}$ and $\ln^{+} \equiv \max\{0, \ln\}$.

The sequence $\sum_{t=0}^{T}\beta^t\left(\ln\left[\mathcal{F}\left(X_t\right) - X_{t+1}\right]\right)^{-}$ is decreasing in T, so that it converges to a finite non-positive number or diverges to $-\infty$.

We know that, for any $\mu > 0$, there is T such that

$$\sum_{t=T}^{\infty}\beta^t\left(\ln\left[\mathcal{F}\left(X_t\right) - X_{t+1}\right]\right)^{+} < \mu$$

This implies $\sum_{t=0}^{\infty}\beta^t\left(\ln\left[\mathcal{F}\left(X_t\right) - X_{t+1}\right]\right)^{+} < \infty$. Indeed, otherwise,

$$\sum_{t=T}^{\infty}\beta^t\left(\ln\left[\mathcal{F}\left(X_t\right) - X_{t+1}\right]\right)^{+} = \infty$$

for any T. Since $\sum_{t=0}^{\infty}\beta^t\left(\ln\left[\mathcal{F}\left(X_t\right) - X_{t+1}\right]\right)^{+}$ is increasing, it converges to a finite non-negative number.

Notice that

$$\begin{aligned}
&\sum_{t=0}^{T}\beta^t \ln\left[\mathcal{F}\left(X_t\right) - X_{t+1}\right] \\
= &\sum_{t=0}^{T}\beta^t\left(\ln\left[\mathcal{F}\left(X_t\right) - X_{t+1}\right]\right)^{-} + \sum_{t=0}^{T}\beta^t\left(\ln\left[\mathcal{F}\left(X_t\right) - X_{t+1}\right]\right)^{+}
\end{aligned}$$

Then, $\sum_{t=0}^{T}\beta^t \ln\left[\mathcal{F}\left(X_t\right) - X_{t+1}\right]$ converges to a finite number or diverges to $-\infty$. ■

Proof of Proposition 5

The set $\Pi(X_0)$ is compact in the product topology.

We want to show that $U(\mathbf{X}) \equiv \sum_{t=0}^{\infty} \beta^t \ln[\mathcal{F}(X_t) - X_{t+1}]$ is upper semi-continuous in $\Pi(X_0)$.

Let X_0 be given. Consider the maximal sequence $\left(\tilde{X}_t\right)_{t=0}^{\infty}$ with $\tilde{X}_{t+1} = \mathcal{F}\left(\tilde{X}_t\right)$ for any t, and $\tilde{X}_0 = X_0$.

Consider a sufficiently small $\varepsilon \in \left(0, b/\bar{X}\right)$.

There are two cases: either (1) $\tilde{X}_t \leq b/\varepsilon$ for any $t \geq 0$, or (2) there exists T such that $\tilde{X}_T > b/\varepsilon$.

(1) If $\tilde{X}_t \leq b/\varepsilon$ for any $t \geq 0$, then,

$$\begin{aligned}\sum_{t=0}^{\infty} \beta^t \ln[\mathcal{F}(X_t) - X_{t+1}] &\leq \sum_{t=0}^{\infty} \beta^t \ln \mathcal{F}\left(\tilde{X}_t\right) \leq \sum_{t=0}^{\infty} \beta^t \ln \mathcal{F}\left(\frac{b}{\varepsilon}\right) \\ &= \frac{1}{1-\beta} \ln \mathcal{F}\left(\frac{b}{\varepsilon}\right) < \infty\end{aligned}$$

Given $\mu > 0$, there exists T_0 such that, for any $T \geq T_0$ and any $\mathbf{X} \in \Pi(X_0)$, we have

$$\sum_{t=T}^{\infty} \beta^t \left(\ln[\mathcal{F}(X_t) - X_{t+1}]\right)^+ \leq \mu$$

where $\ln^+ \equiv \max\{0, \ln\}$.

Let $\mathbf{X}^n \in \Pi(X_0)$ such that $\mathbf{X}^n \to \mathbf{X} \in \Pi(X_0)$ in the product topology.

Let μ be given. Then, for any $T \geq T_0$ and any n,

$$\begin{aligned}U(\mathbf{X}^n) &\leq \sum_{t=0}^{T} \beta^t \ln\left[\mathcal{F}(X_t^n) - X_{t+1}^n\right] + \sum_{t=T+1}^{\infty} \beta^t \left(\ln\left[\mathcal{F}(X_t^n) - X_{t+1}^n\right]\right)^+ \\ &\leq \sum_{t=0}^{T} \beta^t \ln\left[\mathcal{F}(X_t^n) - X_{t+1}^n\right] + \mu\end{aligned}$$

This implies that

$$\begin{aligned}\limsup_n U(\mathbf{X}^n) &\leq \lim_{n\to\infty} \sum_{t=0}^{T} \beta^t \ln\left[\mathcal{F}(X_t^n) - X_{t+1}^n\right] + \mu \\ &= \sum_{t=0}^{T} \beta^t \ln[\mathcal{F}(X_t) - X_{t+1}] + \mu\end{aligned}$$

Let $T \to \infty$. We have

$$\limsup_n U(\mathbf{X}^n) \leq \sum_{t=0}^{\infty} \beta^t \ln[\mathcal{F}(X_t) - X_{t+1}] + \mu = U(\mathbf{X}) + \mu$$

Since μ is an arbitrary positive number, we get $\limsup_n U(\mathbf{X}^n) \leq U(\mathbf{X})$, that is U is upper semi-continuous.

(2) Now, assume that there exists T such that $\tilde{X}_T > b/\varepsilon$.

As in Lemma 4, we consider the following subcases: (2.1) $\gamma < 1$, (2.2) $\gamma = 1$, (2.3) $1 < \gamma < 1/\beta$.

(2.1) If $\gamma < 1$, according to point (1) in the proof of Lemma 4, we have $\sum_{t=0}^{\infty} \beta^t \ln c_t < \infty$.

(2.2) Let $\gamma = 1$. As above, $\tilde{X}_t > \bar{X}$ for any $t \geq 0$. Then, $\tilde{X}_{t+1} = \mathcal{F}\left(\tilde{X}_t\right) = a\tilde{X}_t + b$ for any $t \geq 0$. If $a < 1$, we follow the point (2.1) in the proof of Lemma 4 to get $\sum_{t=0}^{\infty} \beta^t \ln c_t < \infty$. Similarly, if $a \geq 1$, we follow the point (2.2) in the proof of Lemma 4 to obtain $\sum_{t=0}^{\infty} \beta^t \ln c_t < \infty$.

(2.3) Let $1 < \gamma < 1/\beta$. Since $\varepsilon \in \left(0, b/\bar{X}\right)$, we have $b/\varepsilon > \bar{X}$. Thus, $\tilde{X}_T > b/\varepsilon$ implies $\tilde{X}_T > \bar{X}$. We know that, if $\tilde{X}_T > \bar{X}$ for some T, then, under Assumption 2, $\tilde{X}_t > \bar{X}$ and

$$\begin{aligned} \tilde{X}_{t+1} &= A\left[1 + \frac{\lambda}{q} qL^*\left(\tilde{X}_t\right)\right]^{\theta} \left[\tilde{X}_t - qL^*\left(\tilde{X}_t\right)\right]^{\alpha} \\ &\quad + (1 - \delta_K)\left[\tilde{X}_t - qL^*\left(\tilde{X}_t\right)\right] + (1 - \delta_L)\, qL^*\left(\tilde{X}_t\right) \end{aligned}$$

for any $t \geq T$. Since $\gamma > 1$, we can find $\varepsilon > 0$ such that $a(b/\varepsilon)^{\gamma} > b/\varepsilon$.

Let $\tilde{Y}_{t+1} \equiv a\left(\tilde{Y}_t + b\right)^{\gamma} + c\tilde{Y}_t$ for any $t \geq T$ with $\tilde{Y}_0 = \tilde{X}_0$. It is easy to see that $\tilde{Y}_t > b/\varepsilon$ implies $\tilde{Y}_{t+1} > b/\varepsilon$ and $\tilde{Y}_{t+1} < a(1+\varepsilon)^{\gamma} \tilde{Y}_t^{\gamma}$.

If $\tilde{Y}_t \leq b/\varepsilon$ for any t, since $c_t \leq \mathcal{F}\left(\tilde{X}_t\right) \leq \mathcal{F}\left(\tilde{Y}_t\right) \leq \mathcal{F}(b/\varepsilon)$, we have that the sum $\sum_{t=0}^{\infty} \beta^t \ln c_t$ is finite-valued.

If $\tilde{Y}_T > b/\varepsilon$ for some T, then, $\tilde{Y}_t > b/\varepsilon$ for any $t \geq T$. Hence, $\tilde{X}_{T+1} \leq \tilde{Y}_{T+1} < a(1+\varepsilon)^{\gamma} \tilde{Y}_T^{\gamma}$ and (42) holds. Applying the same argument of the point (3) in the proof of Lemma 4, we get $\sum_{t=0}^{\infty} \beta^t \ln c_t < \infty$.

Therefore, in all the three subcases: (2.1), (2.2), (2.3), given $\mu > 0$, there exists T_0 such that, for any $T \geq T_0$ and any $\mathbf{X} \in \Pi(X_0)$, we have

$$\sum_{t=T}^{\infty} \beta^t \left(\ln\left[\mathcal{F}(X_t) - X_{t+1}\right]\right)^+ \leq \mu$$

Let $\mathbf{X}^n \in \Pi(X_0) \to \mathbf{X} \in \Pi(X_0)$. Then, for any $T \geq T_0$ and any n,

$$\begin{aligned} U(\mathbf{X}^n) &\leq \sum_{t=0}^{T} \beta^t \ln\left[\mathcal{F}(X_t^n) - X_{t+1}^n\right] + \sum_{t=T+1}^{\infty} \beta^t \left(\ln\left[\mathcal{F}(X_t^n) - X_{t+1}^n\right]\right)^+ \\ &\leq \sum_{t=0}^{T} \beta^t \ln\left[\mathcal{F}(X_t^n) - X_{t+1}^n\right] + \mu \end{aligned}$$

This implies that

$$\begin{aligned}\limsup_n U(\mathbf{X}^n) &\leq \lim_{n\to\infty}\sum_{t=0}^{T}\beta^t \ln\left[\mathcal{F}(X_t^n) - X_{t+1}^n\right] + \mu \\ &= \sum_{t=0}^{T}\beta^t \ln\left[\mathcal{F}(X_t) - X_{t+1}\right] + \mu\end{aligned}$$

Let $T \to \infty$. We have

$$\limsup_n U(\mathbf{X}^n) \leq \sum_{t=0}^{\infty}\beta^t \ln\left[\mathcal{F}(X_t) - X_{t+1}\right] + \mu = U(\mathbf{X}) + \mu$$

Since μ is arbitrary, we get $\limsup_n U(\mathbf{X}^n) \leq U(\mathbf{X})$, that is U is upper semi-continuous in $\Pi(X_0)$.

Therefore, there exists $\mathbf{X}$ that maximizes $U(\mathbf{X})$ in $\Pi(X_0)$.

In order to have $U(\mathbf{X}) > -\infty$, we observe the following.

Consider $X_0 \in \left(0, \bar{X}\right]$. According to Assumption 2,

$$X_0 < \hat{X} = \left(\frac{A}{\delta_K}\right)^{\frac{1}{1-\alpha}}$$

In this case, since $X_0 < AX_0^\alpha + (1-\delta_K)X_0$ and $\mathcal{F}(X_0) - X_0 = AX_0^\alpha + (1-\delta_K)X_0 - X_0 > 0$, $\mathbf{X} = (X_0, X_0, \ldots)$ is feasible from X_0. Then,

$$\sum_{t=0}^{\infty}\beta^t \ln\left[\mathcal{F}(X_t) - X_{t+1}\right] = \sum_{t=0}^{\infty}\beta^t \ln\left[\mathcal{F}(X_0) - X_0\right] = \frac{\ln\left[\mathcal{F}(X_0) - X_0\right]}{1-\beta} > -\infty$$

Clearly, $\max U(\mathbf{X}) \geq \sum_{t=0}^{\infty}\beta^t \ln\left[AX_0^\alpha + (1-\delta_K)X_0 - X_0\right] > -\infty$.

If $X_0 > \bar{X}$, consider the sequence $\left(X_0, \bar{X}, \bar{X}, \ldots\right)$. We have $\mathcal{F}(X_0) > \mathcal{F}\left(\bar{X}\right) > \bar{X}$ because, if $\tilde{X}_T > \bar{X}$ for some T, then, under Assumption 2, $\tilde{X}_t > \bar{X}$ for any $t \geq T$. Thus, the sequence $\left(X_0, \bar{X}, \bar{X}, \ldots\right)$ is feasible and

$$\begin{aligned}\sum_{t=0}^{\infty}\beta^t \ln\left[\mathcal{F}(X_t) - X_{t+1}\right] &= \ln\left[\mathcal{F}(X_0) - \bar{X}\right] + \sum_{t=1}^{\infty}\beta^t \ln\left[\mathcal{F}\left(\bar{X}\right) - \bar{X}\right] \\ &= \ln\left[\mathcal{F}(X_0) - \bar{X}\right] + \frac{\beta}{1-\beta}\ln\left[\mathcal{F}\left(\bar{X}\right) - \bar{X}\right] > -\infty\end{aligned}$$

Clearly,

$$\max U(\mathbf{X}) \geq \ln\left[\mathcal{F}(X_0) - \bar{X}\right] + \sum_{t=1}^{\infty}\beta^t \ln\left[A\bar{X}^\alpha + (1-\delta_K)\bar{X} - \bar{X}\right] > -\infty$$

■

Proof of Lemma 6

Since $\mathcal{F}$ is increasing, the optimal sequence $(X_t^*)_{t=0}^{\infty}$ is monotonic (see Amir (1996) and Le Van and Dana (2002)).

It cannot converge to zero, otherwise, for any t large enough, $\mathcal{F}(X_t^*) = AX_t^{*\alpha} + (1-\delta_K) X_t^*$. Since $\mathcal{F}'(0) = \infty$, the Euler equation would lead to a contradiction.

There is a period T such that $X_T^* > \bar{X}$. Otherwise, the optimal path (X_t^*) would satisfy $\mathcal{F}(X_t^*) = AX_t^{*\alpha} + (1-\delta_K) X_t^*$ for any t. Besides, it must converge to the steady state $\tilde{X}$ which is larger than $\bar{X}$. By Assumption 2, we get a contradiction.

The optimal path either converges to a steady state larger than the threshold $\bar{X}$ or diverges to $+\infty$. ■

Proof of Proposition 8

Let $\gamma < 1$.

From the Euler equation, X^* is a steady state if and only if $\mathcal{F}'(X^*) = 1/\beta$. From points (3) and (5) of Lemma 2, $\mathcal{F}$ is strictly concave and differentiable on $(0, +\infty)$.

We recall that $1 - \delta_K + \alpha A \tilde{X}^{\alpha-1} = 1/\beta$ and that $\bar{X} < \tilde{X}$ implies $\mathcal{F}'\left(\bar{X}\right) = 1 - \delta_K + \alpha A \bar{X}^{\alpha-1} > 1/\beta$. We know that the stationary state X^* satisfies $1/\beta = \mathcal{F}'(X^*)$.

$\mathcal{F}'(X^*) = 1/\beta < \mathcal{F}'\left(\bar{X}\right)$ implies that $X^* > \bar{X}$. Indeed, $\mathcal{F}$ is strictly concave. We want to show that $X^* > \tilde{X}$.

We have

$$\mathcal{F}'(X_t) = \alpha A \left[1 + \lambda L^*(X_t)\right]^{\theta} \left[X_t - qL^*(X_t)\right]^{\alpha-1} + 1 - \delta_K$$

We know that $X_t > \bar{X}$ implies $L^*(X_t) > 0$, that is $X_t - qL^*(X_t) < X_t$ and $\left[X_t - qL^*(X_t)\right]^{\alpha-1} > X_t^{\alpha-1}$. In addition, $\left[1 + \lambda L^*(X_t)\right]^{\theta} > 1$.

Then, $\mathcal{F}'(X_t) > \alpha A X_t^{\alpha-1} + 1 - \delta_K$ for any $X_t > \bar{X}$. Since $\tilde{X} > \bar{X}$, we get $\mathcal{F}'\left(\tilde{X}\right) > \alpha A \tilde{X}^{\alpha-1} + 1 - \delta_K = 1/\beta = \mathcal{F}'(X^*)$.

We know also from Lemma 2 that $\mathcal{F}$ is strictly concave. We obtain $\tilde{X} < X^*$.

If $\delta_K = \delta_L$, (28) becomes

$$\lambda\theta(X_t - qL_t) = \alpha q(1 + \lambda L_t) \tag{43}$$

Since

$$X_t = (1 + \zeta_t) qL_t \tag{44}$$

we obtain

$$L_t = \frac{\alpha}{\lambda(\theta\zeta_t - \alpha)}$$

(43) entails $\lambda\theta K_t = \alpha q + \alpha q \lambda L_t$ and we find

$$\zeta_t \equiv \frac{K_t}{qL_t} = \frac{\alpha}{\lambda\theta}\frac{1}{L_t} + \frac{\alpha}{\theta} \tag{45}$$

Thus,

$$\lim_{t\to\infty} \zeta_t = \frac{\alpha}{\theta}\left(1 + \frac{1}{\lambda \lim_{t\to\infty} L_t}\right) = \frac{\alpha}{\theta}\left(1 + \frac{1}{\lambda L^*}\right) > \frac{\alpha}{\theta}$$

■

Proof of Proposition 10

(2) Let $\gamma = 1$ and $\delta_K \geq \delta_L$. Consider only $X_0 \geq \bar{X}$ and $X \geq X_0$. In this case,

$$\mathcal{F}\left(\bar{X}\right) = A\bar{X}^{\alpha} + (1-\delta_K)\bar{X} = \mathcal{F}'\left(\bar{X}\right)\bar{X} + Z$$

where

$$\mathcal{F}'\left(\bar{X}\right) = \alpha A\bar{X}^{\alpha-1} + 1 - \delta_K > \frac{1}{\beta} \tag{46}$$

and $Z \equiv (1-\alpha) A\bar{X}^{\alpha}$. From the first-order condition (32), we know that, for any $X > \bar{X}$,

$$\lambda\theta A\left(\frac{1+\lambda L}{X-qL}\right)^{\theta-1}(X-qL)^{\gamma-1} - \alpha qA\left(\frac{1+\lambda L}{X-qL}\right)^{\theta}(X-qL)^{\gamma-1} + q(\delta_K - \delta_L) = 0$$

and, since $\gamma = 1$,

$$\lambda\theta A\left(\frac{1+\lambda L}{X-qL}\right)^{\theta-1} - \alpha qA\left(\frac{1+\lambda L}{X-qL}\right)^{\theta} + q(\delta_K - \delta_L) = 0 \tag{47}$$

Then,

$$\frac{1+\lambda L^*(X)}{X - qL^*(X)}$$

is constant for any $X \geq \bar{X}$ and equal to $1/\bar{X}$ (since $L^*\left(\bar{X}\right) = 0$). Hence,

$$L_t = L^*(X_t) = \frac{X_t - \bar{X}}{q + \lambda\bar{X}} \tag{48}$$

Moreover,

$$K_t = X_t - qL_t = \bar{X}\frac{q + \lambda X_t}{q + \lambda\bar{X}} \tag{49}$$

and

$$\frac{K_t}{qL_t} = \bar{X}\frac{\frac{\lambda}{q}X_t + 1}{X_t - \bar{X}} \tag{50}$$

For any $X \geq \bar{X}$, we obtain

$$\begin{aligned}
\mathcal{F}'(X) &= \alpha A\left[1+\lambda L^*(X)\right]^{\theta}\left[X - qL^*(X)\right]^{\alpha-1} + 1 - \delta_K \\
&= \alpha A\left[\frac{1+\lambda L^*(X)}{X - qL^*(X)}\right]^{\theta}\left[X - qL^*(X)\right]^{\alpha+\theta-1} + 1 - \delta_K \\
&= \alpha A\left(\frac{1}{\bar{X}}\right)^{1-\alpha} + 1 - \delta_K = \mathcal{F}'\left(\bar{X}\right)
\end{aligned}$$

For any $X_{t+1} \geq \bar{X}$, the Euler equation implies

$$\frac{c_{t+1}}{c_t} = \beta\mathcal{F}'(X_{t+1}) = \beta\mathcal{F}'\left(\bar{X}\right) > 1$$

Optimal consumption grows at a constant rate > 1.

$$c_t = c_0 \left[\beta \mathcal{F}'\left(\bar{X}\right)\right]^t$$

Let us compute $\mathcal{F}(X_t)$ for $X_t > \bar{X}$:

$$\begin{aligned}
\mathcal{F}(X_t) &= A(1+\lambda L_t)^{\theta}(X_t - qL_t)^{\alpha} + (1-\delta_K)(X_t - qL_t) + (1-\delta_L) qL_t \\
&= A(1+\lambda L_t)^{1-\alpha}(X_t - qL_t)^{\alpha} + (1-\delta_K)(X_t - qL_t) + (1-\delta_L) qL_t \\
&= A\left(\frac{1+\lambda L_t}{X_t - qL_t}\right)^{1-\alpha}(X_t - qL_t) + (1-\delta_K)(X_t - qL_t) + (1-\delta_L) qL_t \\
&= \left(A\bar{X}^{\alpha-1} + \delta_L - \delta_K\right)(X_t - qL_t) + (1-\delta_L) X_t
\end{aligned}$$

Using (48) to replace L_t, we get

$$\begin{aligned}
\mathcal{F}(X_t) &= \left(A\bar{X}^{\alpha-1} + \delta_L - \delta_K\right)\left(X_t - q\frac{X_t - \bar{X}}{q+\lambda\bar{X}}\right) + (1-\delta_L) X_t \\
&= \left[1 - \delta_L + \left(A\bar{X}^{\alpha-1} + \delta_L - \delta_K\right)\frac{\lambda\bar{X}}{q+\lambda\bar{X}}\right] X_t \\
&\quad + \left(A\bar{X}^{\alpha-1} + \delta_L - \delta_K\right)\frac{q\bar{X}}{q+\lambda\bar{X}} \\
&= aX_t + b
\end{aligned}$$

which is linear in X_t, where

$$a \equiv 1 - \delta_L + \left(A\bar{X}^{\alpha-1} + \delta_L - \delta_K\right)\frac{\lambda\bar{X}}{q+\lambda\bar{X}} \tag{51}$$

$$b \equiv \left(A\bar{X}^{\alpha-1} + \delta_L - \delta_K\right)\frac{q\bar{X}}{q+\lambda\bar{X}} \tag{52}$$

Notice that $\bar{X}$ is solution to

$$A\bar{X}^{\alpha-1} = \frac{q(\delta_K - \delta_L)}{\alpha q - \lambda\theta\bar{X}} \tag{53}$$

Then, the optimal path (X_t) satisfies for any t:

$$\begin{aligned}
X_{t+1} &= \mathcal{F}(X_t) - c_t = \mathcal{F}(X_t) - c_0\left[\beta\mathcal{F}'\left(\bar{X}\right)\right]^t \\
&= aX_t + b - c_0\left[\beta\mathcal{F}'\left(\bar{X}\right)\right]^t \\
&= aX_t + b - c_0(a\beta)^t
\end{aligned} \tag{54}$$

since $\mathcal{F}'\left(\bar{X}\right) = \mathcal{F}'(X_t) = a$.

Now, we want to prove that the solution to the difference equation:

$$x_{t+1} = ax_t + b - c(a\beta)^t \tag{55}$$

with $x_0 \neq 0$, $0 < \beta < 1$ and

$$a > 1/\beta > 1,\ b > 0,\ c > 0 \tag{56}$$

is given by

$$x_{t+1} = a^{t+1}\left(x_0 - \frac{b}{1-a} - \frac{c}{a}\frac{1-\beta^{t+1}}{1-\beta}\right) + \frac{b}{1-a} \tag{57}$$

that is by

$$x_{t+1} = a^{t+1}\left(x_0 - \frac{b}{1-a} - \frac{c}{a}\frac{1}{1-\beta}\right) + \frac{c}{a}\frac{(a\beta)^{t+1}}{1-\beta} + \frac{b}{1-a} \tag{58}$$

Let us prove (58) by induction.

For $t = 0$, (58) gives

$$x_1 = a\left(x_0 - \frac{b}{1-a} - \frac{c}{a}\right) + \frac{b}{1-a} = ax_0 + b - c$$

that is (55).

Now, let (58) be true at period t. Check it for period $t+1$.

(55) implies

$$x_{t+2} = ax_{t+1} + b - c\,(\beta a)^{t+1}$$

Replacing (58), we have

$$x_{t+2} = a^{t+2}\left(x_0 - \frac{b}{1-a} - \frac{c}{a}\frac{1-\beta^{t+1}}{1-\beta}\right) + \frac{ab}{1-a} + b - c\,(\beta a)^{t+1} \tag{59}$$

Since

$$\frac{ab}{1-a} + b = \frac{b}{1-a}$$

and

$$\begin{aligned} & -a^{t+2}\frac{c}{a}\frac{1-\beta^{t+1}}{1-\beta} - c\,(\beta a)^{t+1} \\ = \ & -a^{t+2}\frac{c}{a}\frac{1}{1-\beta}\left[1 - \beta^{t+1} + \beta^{t+1}(1-\beta)\right] \\ = \ & -a^{t+2}\frac{c}{a}\frac{1}{1-\beta}\left(1-\beta^{t+2}\right) \end{aligned}$$

(59) becomes

$$x_{t+2} = a^{t+2}\left(x_0 - \frac{b}{1-a} - \frac{c}{a}\frac{1-\beta^{t+2}}{1-\beta}\right) + \frac{b}{1-a}$$

Thus, (58) is verified also for $t+1$. Then, it holds for any $t \geq 0$ proving that (57) is the solution to (55).

Equation (54) is exactly equation (57) with $c = c_0$. Notice that all restrictions in (56) are satisfied. Indeed, according to (46),

$$a = \mathcal{F}'\left(\bar{X}\right) > \frac{1}{\beta}$$

and, according to (51) and (52),

$$b = \left[a - (1 - \delta_L)\right] \frac{q}{\lambda} > \left[\frac{1}{\beta} - (1 - \delta_L)\right] \frac{q}{\lambda} > 0 \tag{60}$$

Therefore, using to (58), the solution becomes

$$\begin{aligned} X_{t+1} &= \left[\mathcal{F}'\left(\bar{X}\right)\right]^{t+1} \left(X_0 + \frac{q}{\lambda} \left[1 + \frac{\delta_L}{\mathcal{F}'\left(\bar{X}\right) - 1} \right] - \frac{1}{1-\beta} \frac{c_0}{\mathcal{F}'\left(\bar{X}\right)} \right) \\ &\quad + \frac{\beta}{1-\beta} c_0 \left[\beta \mathcal{F}'\left(\bar{X}\right)\right]^t - \frac{q}{\lambda} \left[1 + \frac{\delta_L}{\mathcal{F}'\left(\bar{X}\right) - 1} \right] \end{aligned} \tag{61}$$

Noticing that $a = \mathcal{F}'\left(\bar{X}\right) > 1$, according to (60),

$$\frac{b}{1-a} = \frac{q}{\lambda} \left(\frac{\delta_L}{1-a} - 1 \right) = -\frac{q}{\lambda} \left[1 + \frac{\delta_L}{\mathcal{F}'\left(\bar{X}\right) - 1} \right] < 0$$

We need $X_t \geq 0$ for any t. If the term multiplying $\left[\mathcal{F}'\left(\bar{X}\right)\right]^{t+1}$ in (61) was negative, i.e. if

$$X_0 + \frac{q}{\lambda} \left[1 + \frac{\delta_L}{\mathcal{F}'\left(\bar{X}\right) - 1} \right] - \frac{1}{1-\beta} \frac{c_0}{\mathcal{F}'\left(\bar{X}\right)} < 0$$

since by (56) $\mathcal{F}'\left(\bar{X}\right) > 1/\beta > 1$, then X_t would become negative sooner or later.

Hence, this term must be non-negative, which implies that

$$\frac{1}{1-\beta} \frac{c_0}{\mathcal{F}'\left(\bar{X}\right)} \leq X_0 + \frac{q}{\lambda} \left[1 + \frac{\delta_L}{\mathcal{F}'\left(\bar{X}\right) - 1} \right] \tag{62}$$

establishing an upper bound on c_0.

The value of the program is $V(c_0) = \sum_{t=0}^{\infty} \beta^t \ln\left(c_0 \left[\beta \mathcal{F}'\left(\bar{X}\right)\right]^t \right)$ and V is increasing in c_0. From (62), the maximum is attained at

$$c_0 = \bar{c}_0 \equiv (1-\beta) \mathcal{F}'\left(\bar{X}\right) \left(X_0 + \frac{q}{\lambda} \left[1 + \frac{\delta_L}{\mathcal{F}'\left(\bar{X}\right) - 1} \right] \right)$$

Note that this implies that the first term in (61) becomes zero.

Therefore, the optimal paths are given by

$$\begin{aligned} c_t &= \bar{c}_0 \left[\beta \mathcal{F}'\left(\bar{X}\right)\right]^t \\ X_{t+1} &= \bar{c}_0 \left[\beta \mathcal{F}'\left(\bar{X}\right)\right]^t \frac{\beta}{1-\beta} - z \end{aligned}$$

for any t, where $z > 0$ is defined by (17). Clearly, when $t \to \infty$,

$$X_{t+1} \sim \bar{c}_0 \left[\beta \mathcal{F}'\left(\bar{X}\right)\right]^t \frac{\beta}{1-\beta}$$

From (48), (49) and (50), L_{t+1}, K_{t+1} and the ratio $K_{t+1}/\left(qL_{t+1}\right)$ obtain

$$\begin{aligned} L_{t+1} &= \frac{X_{t+1} - \bar{X}}{q + \lambda \bar{X}} = \frac{\bar{c}_0 \left[\beta \mathcal{F}'\left(\bar{X}\right)\right]^t \frac{\beta}{1-\beta} - z - \bar{X}}{q + \lambda \bar{X}} \\ K_{t+1} &= \bar{X} \frac{\frac{q}{\lambda} + X_{t+1}}{\frac{q}{\lambda} + \bar{X}} = \bar{X} \frac{\frac{q}{\lambda} + \bar{c}_0 \left[\beta \mathcal{F}'\left(\bar{X}\right)\right]^t \frac{\beta}{1-\beta} - z}{\frac{q}{\lambda} + \bar{X}} \\ \frac{K_{t+1}}{qL_{t+1}} &= \bar{X} \frac{\frac{\lambda}{q} X_{t+1} + 1}{X_{t+1} - \bar{X}} \end{aligned}$$

Thus,

$$\lim_{t \to \infty} K_t = \lim_{t \to \infty} L_t = \infty$$

Taking the limit of the ratio, we have

$$\lim_{t \to \infty} \frac{K_t}{qL_t} = \lim_{t \to \infty} \left(\bar{X} \frac{\frac{\lambda}{q} X_t + 1}{X_t - \bar{X}} \right) = \bar{X} \frac{\lambda}{q} \leq \frac{\alpha}{\theta}$$

where the last inequality follows from (8). Note that this result on the limit is in line with point (3) of Lemma 3.

Finally, we observe that, asymptotically, physical, living and total capital, and consumption grow at the same rate:

$$\frac{L_{t+1}}{L_t} \sim \frac{K_{t+1}}{K_t} \sim \frac{X_{t+1}}{X_t} \sim \frac{c_{t+1}}{c_t} = \beta \mathcal{F}'\left(\bar{X}\right) > 1$$

In this respect, we can say that the unbounded growth is asymptotically balanced. ■

Proof of Proposition 11

Let $1 < \gamma < 1/\beta$**.** In this case, $\mathcal{F}$ is strictly convex for $X_t > \bar{X}$. Hence $\mathcal{F}'$ is increasing and $\mathcal{F}'\left(X_t\right) > \mathcal{F}'\left(\bar{X}\right)$ for any $X_t > \bar{X}$. Since $\mathcal{F}'\left(\bar{X}\right) > 1/\beta$, there exists no steady state. The optimal path $\left(X_t\right)_{t=0}^{\infty}$ diverges to infinity: $\lim_{t \to \infty} X_t = \infty$. Then, according to points (1) and (4) of Lemma 3, $\lim_{t \to \infty} L_t = \lim_{t \to \infty} K_t = \infty$ and

$$\lim_{t \to \infty} \frac{K_t}{qL_t} = \frac{\alpha}{\theta}$$

with $K_t/(qL_t) < \alpha/\theta$ for t large enough.

In the following, we consider the two possible cases according to Assumption 1: $\delta_K = \delta_L$ and $\delta_K > \delta_L$.

(1) Let $\delta_K = \delta_L$. Following the same argument as at the end of the proof of Proposition 8, we obtain again equation (45):

$$\zeta_t = \frac{\alpha}{\lambda\theta}\frac{1}{L_t} + \frac{\alpha}{\theta}$$

However, under the assumptions of Proposition 11, L_t diverges to ∞. Thus, $\zeta_t > \alpha/\theta$ for any t and, since $\lim_{t\to\infty} L_t = \infty$, we have $\lim_{t\to\infty} \zeta_t = \alpha/\theta$.

(2) Let now $\delta_K > \delta_L$.

The first order condition in (28) can be written as:

$$\lambda\theta A K^*(X) - \alpha q A\left[1+\lambda L^*(X)\right] = q\left(\delta_L - \delta_K\right)\left[1+\lambda L^*(X)\right]^{1-\theta} K^*(X)^{1-\alpha}$$

Since $\zeta(X_t) \equiv K^*(X_t)/\left[qL^*(X_t)\right]$, we can divide the above expression to obtain

$$\lambda\theta A\zeta(X_t) = \alpha A\left[\lambda + \frac{1}{L^*(X_t)}\right] + \left(\delta_L - \delta_K\right) q^\theta \zeta(X_t)^\theta K^*(X_t)^{1-\gamma}\left[\lambda + \frac{1}{L^*(X_t)}\right]^{1-\theta}$$

and, taking derivatives on both sides with respect to X_t:

$$\begin{aligned}
&\zeta'(X_t)\left(\lambda\theta A + \theta\left(\delta_K - \delta_L\right) q^\theta \zeta(X_t)^{\theta-1} K^*(X_t)^{1-\gamma}\left[\lambda + \frac{1}{L^*(X_t)}\right]^{1-\theta}\right) \\
= \;& \frac{L^{*\prime}(X_t)}{L^*(X_t)^2}\left(-\alpha A + (1-\theta)\left(\delta_K - \delta_L\right) q^\theta \zeta(X_t)^\theta K^*(X_t)^{1-\gamma}\left[\lambda + \frac{1}{L^*(X_t)}\right]^{-\theta}\right) \\
& + K^{*\prime}(X_t)(\gamma-1) q^\theta \left(\delta_K - \delta_L\right)\zeta(X_t)^\theta K^*(X_t)^{-\gamma}\left[\lambda + \frac{1}{L^*(X_t)}\right]^{1-\theta} \qquad (63)
\end{aligned}$$

We have

$$K^*(X_t)^{-\gamma} K^{*\prime}(X_t) = q^{1-\gamma} L^*(X_t)^{2-\gamma} \frac{K^{*\prime}(X_t)}{qL^{*\prime}(X_t)} \frac{L^{*\prime}(X_t)}{L^*(X_t)^2}\left[\frac{K^*(X_t)}{qL^*(X_t)}\right]^{-\gamma} \qquad (64)$$

(30) and (31) imply

$$\frac{K^{*\prime}(X_t)}{qL^{*\prime}(X_t)} = \frac{\alpha q\lambda\theta + \theta(1-\theta)\lambda^2 \frac{K^*(X_t)}{1+\lambda L^*(X_t)}}{\alpha q\lambda\theta + \alpha(1-\alpha) q^2 \frac{1+\lambda L^*(X_t)}{K^*(X_t)}}$$

since $P(X_t) \equiv 1 + \lambda L^*(X_t)$.

This means that, for any X_t sufficiently large, the term $K^{*\prime}(X_t)/\left[qL^{*\prime}(X_t)\right]$ is uniformly bounded above and below away from zero. We also have that $\zeta(X_t) \equiv K^*(X_t)/\left[qL^*(X_t)\right]$ is uniformly bounded away from zero and from $+\infty$.

(63) can be rewritten as

$$\zeta'(X_t) R(X_t) = S(X_t)$$

where

$$R(X_t) \equiv \lambda\theta A + \theta(\delta_K - \delta_L) q^\theta \zeta(X_t)^{\theta-1} K^*(X_t)^{1-\gamma} \left[\lambda + \frac{1}{L^*(X_t)}\right]^{1-\theta} > 0 \tag{65}$$

and

$$S(X_t) \equiv \sigma(X_t) \frac{L^{*\prime}(X_t)}{L^*(X_t)^2}$$

with

$$\begin{aligned}\sigma(X_t) \equiv\ & -\alpha A + (1-\theta)(\delta_K - \delta_L) q^\theta \zeta(X_t)^\theta K^*(X_t)^{1-\gamma} \left[\lambda + \frac{1}{L^*(X_t)}\right]^{-\theta} \\ & + (\gamma-1)(\delta_K - \delta_L) q^{-\alpha} \zeta(X_t)^{-\alpha} L^*(X_t)^{2-\gamma} \frac{K^{*\prime}(X_t)}{L^{*\prime}(X_t)} \left[\lambda + \frac{1}{L^*(X_t)}\right]^{1-\theta}\end{aligned} \tag{66}$$

because of (64).

Taking the limit of the second term in (66),

$$\begin{aligned}& \lim_{X_t\to\infty} \left((1-\theta)(\delta_K - \delta_L) q^\theta \zeta(X_t)^\theta K^*(X_t)^{1-\gamma} \left[\lambda + \frac{1}{L^*(X_t)}\right]^{-\theta}\right) \\ =\ & (1-\theta)(\delta_K - \delta_L) q^\theta \lambda^{-\theta} \lim_{X_t\to\infty} \left[\zeta(X_t)^\theta K^*(X_t)^{1-\gamma}\right] = 0\end{aligned}$$

Moreover, taking the limit of the third term of (66),

$$\begin{aligned}& \lim_{X_t\to\infty} \left[(\gamma-1)(\delta_K - \delta_L) q^{-\alpha} \zeta(X_t)^{-\alpha} L^*(X_t)^{2-\gamma} \frac{K^{*\prime}(X_t)}{L^{*\prime}(X_t)} \left[\lambda + \frac{1}{L^*(X_t)}\right]^{1-\theta}\right] \\ =\ & (\gamma-1)(\delta_K - \delta_L) q^{-\alpha} \lambda^{1-\theta} \lim_{X_t\to\infty} \left[\zeta(X_t)^{-\alpha} L^*(X)^{2-\gamma} \frac{K^{*\prime}(X)}{L^{*\prime}(X)}\right] = \infty\end{aligned}$$

Hence, for any X_t large enough, $S(X_t) > 0$ and $\zeta'(X_t) > 0$.

Let us prove that $\zeta'(X_t) < 0$ when X_t is sufficiently close to $\bar{X}$ from the right.

We have

$$\zeta'(X_t) = \frac{\sigma(X_t)}{R(X_t)} \frac{L^{*\prime}(X_t)}{L^*(X_t)^2} \tag{67}$$

where $R(X_t) > 0$ is given by (65) and $\sigma(X_t)$ by (66).

Observe that

$$\lim_{X_t\to\bar{X}^+} \sigma(X_t) = -\alpha A + (1-\theta)(\delta_K - \delta_L) \bar{X}^{1-\alpha} \tag{68}$$

because

$$\begin{aligned}
& q^{-\alpha}\zeta\left(X_t\right)^{-\alpha} L^*\left(X_t\right)^{2-\gamma} \frac{K^{*\prime}\left(X_t\right)}{L^{*\prime}\left(X_t\right)} \left[\lambda + \frac{1}{L^*\left(X_t\right)}\right]^{1-\theta} \\
\sim\ & q^{-\alpha}\zeta\left(X_t\right)^{-\alpha} L^*\left(X_t\right)^{2-\gamma} L^*\left(X_t\right)^{\theta-1} \frac{K^{*\prime}\left(X_t\right)}{L^{*\prime}\left(X_t\right)} \\
=\ & q^{-\alpha} q^{1-\theta} \zeta\left(X_t\right)^{-\alpha} K^*\left(X_t\right)^{\theta-1} L^*\left(X_t\right)^{2-\gamma} \frac{K^{*\prime}\left(X_t\right)}{L^{*\prime}\left(X_t\right)} \left[\frac{qL^*\left(X_t\right)}{K^*\left(X_t\right)}\right]^{\theta-1} \\
=\ & q^{1-\gamma}\zeta\left(X_t\right)^{1-\gamma} K^*\left(X_t\right)^{\theta-1} L^*\left(X_t\right)^{2-\gamma} \frac{K^{*\prime}\left(X_t\right)}{L^{*\prime}\left(X_t\right)} \to 0
\end{aligned}$$

since $1-\gamma < 0 < 2-\gamma$, $\lim_{X_t \to \bar{X}^+} L^*\left(X_t\right) = 0$, $\lim_{X_t \to \bar{X}^+} K^*\left(X_t\right) = \bar{X}$, $\lim_{X_t \to \bar{X}^+} \zeta\left(X_t\right)^{1-\gamma} = 0$ and

$$\lim_{X_t \to \bar{X}^+} \frac{K^{*\prime}\left(X_t\right)}{L^{*\prime}\left(X_t\right)} = q \frac{\alpha q \lambda \theta + \theta\left(1-\theta\right)\lambda^2 \bar{X}}{\alpha q \lambda \theta + \alpha\left(1-\alpha\right) q^2 \frac{1}{\bar{X}}}$$

$\bar{X}$ satisfies (7), that is

$$\left(\delta_K - \delta_L\right) \bar{X}^{1-\alpha} = \alpha A - \lambda \theta A \frac{\bar{X}}{q}$$

Replacing in (68), we find

$$\lim_{X_t \to \bar{X}^+} \sigma\left(X_t\right) = -\alpha A + \left(1-\theta\right)\left(\alpha A - \lambda\theta A \frac{\bar{X}}{q}\right) = -\theta A \left[\alpha + \lambda\left(1-\theta\right)\frac{\bar{X}}{q}\right] < 0$$

When X_t is sufficiently close to $\bar{X}$ from the right, we get $\sigma\left(X_t\right)/R\left(X_t\right) < 0$ and, according to (67), $\zeta'\left(X\right) < 0$. ■

Proof of Proposition 13

Under the proposition's assumptions $\gamma > 1$ and $\delta_K = \delta_L$, it is possible to write the dynamics in the variable ζ_t, which, we know, converges to α/θ (see Proposition 11).

The existence of a limit for this reduced dynamics will allow us to simulate the trajectory (ζ_t) and, then, the trajectories (X_t) and (c_t).

We know that

$$\mathcal{F}\left(X_{t+1}\right) = A\left(1+\lambda L_{t+1}\right)^{\theta}\left(X_{t+1} - qL_{t+1}\right)^{\alpha} + \left(1-\delta_K\right)X_{t+1}$$

and that according to (32),

$$\mathcal{F}'\left(X_{t+1}\right) = \alpha A\left(1+\lambda L_{t+1}\right)^{\theta}\left(X_{t+1} - qL_{t+1}\right)^{\alpha-1} + 1 - \delta_K$$

Moreover, $c_t = \mathcal{F}\left(X_t\right) - X_{t+1}$, $L_t = \alpha/\left[\lambda\left(\theta\zeta_t - \alpha\right)\right]$ and

$$X_t = \left(1+\zeta_t\right) qL_t = \frac{q}{\lambda}\frac{\alpha\zeta_t + \alpha}{\theta\zeta_t - \alpha} \equiv X\left(\zeta_t\right) \tag{69}$$

The Euler equation becomes

$$\frac{u'(c_t)}{u'(c_{t+1})} = \frac{c_{t+1}}{c_t} = \beta \mathcal{F}'(X_{t+1})$$

that is (21), where ψ is defined by (22).

Besides, (21) obtains solving the "new" optimization problem:

$$\max \sum_{t=0}^{\infty} \beta^t \ln\left(\psi\left(\zeta_t, \zeta_{t+1}\right)\right)$$

with $\zeta_t \geq \alpha/\theta$ for any t.

The solution will be interior because of ln. Taking the derivative of the intertemporal utility with respect to ζ_{t+1}, we find

$$\frac{\psi_2\left(\zeta_t, \zeta_{t+1}\right)}{\psi\left(\zeta_t, \zeta_{t+1}\right)} + \beta \frac{\psi_1\left(\zeta_{t+1}, \zeta_{t+2}\right)}{\psi\left(\zeta_{t+1}, \zeta_{t+2}\right)} = 0$$

Since $\psi\left(\zeta_t, \zeta_{t+1}\right) \equiv \mathcal{F}(X(\zeta_t)) - X\left(\zeta_{t+1}\right)$, we have

$$\begin{aligned} \psi_1\left(\zeta_{t+1}, \zeta_{t+2}\right) &= \mathcal{F}'\left(X\left(\zeta_{t+1}\right)\right) X'\left(\zeta_{t+1}\right) \\ \psi_2\left(\zeta_t, \zeta_{t+1}\right) &\equiv -X'\left(\zeta_{t+1}\right) \end{aligned}$$

and we recover (21). ■

Proof of Proposition 14

We study now the case $\gamma > 1$ with $\delta_K > \delta_L$. From (28):

$$\lambda\theta A \left(\frac{1+\lambda L_t}{X_t - qL_t}\right)^{\theta-1} = \alpha q A \left(\frac{1+\lambda L_t}{X_t - qL_t}\right)^{\theta} + q\left(\delta_L - \delta_K\right) K_t^{1-\gamma} \tag{70}$$

for any t, where $K_t = X_t - qL_t$. We know that (28) implies that $L_t = L^*(X_t)$ is a strictly increasing function of X_t.

Define $\varphi(X_t)$ according to (24). (70) reduces to

$$\lambda\theta A \varphi(X_t)^{\theta-1} = \alpha q A \varphi(X_t)^{\theta} + q\left(\delta_L - \delta_K\right) K^*(X_t)^{1-\gamma} \tag{71}$$

Taking derivatives in (71) with respect to X_t, we find

$$\theta A \varphi'(X_t) \varphi(X_t)^{\theta-2} \left[\lambda(\theta-1) - \alpha q \varphi(X_t)\right] = q(\gamma-1)\left(\delta_K - \delta_L\right) K^*(X_t)^{-\gamma} K^{*\prime}(X_t) \tag{72}$$

$\theta < 1$, $\gamma > 1$ and $K^{*\prime}(X_t) > 0$ imply $\varphi'(X_t) < 0$. Since φ is strictly monotonic, we can define the inverse function $X_t = \varphi^{-1}(\varphi_t)$ for any $\varphi_t > 0$.

According to (71), we find

$$\lambda\theta A \left[\lim_{X_t \to \infty} \varphi(X_t)\right]^{\theta-1} = \alpha q A \left[\lim_{X_t \to \infty} \varphi(X_t)\right]^{\theta} \tag{73}$$

since $\gamma > 1$ and $\lim_{X_t \to \infty} K^*(X_t)^{1-\gamma} = 0$. Thus, $\lim_{X_t \to \infty} \varphi(X_t) = l > 0$, otherwise, according to (73), $\lim_{X_t \to \infty} \varphi(X_t) = 0$ would imply that $\infty = 0$, which is a contradiction. Then, (73) becomes $\lambda\theta A l^{\theta-1} = \alpha q A l^{\theta}$, that is $l = \lambda\theta/(\alpha q) > 0$, which is not surprising:

$$\begin{aligned}\lim_{X_t \to \infty} \varphi(X_t) &= \lim_{X_t \to \infty} \frac{1 + \lambda L^*(X_t)}{K^*(X_t)} = \lim_{X_t \to \infty} \frac{1}{K^*(X_t)} + \frac{\lambda}{q} \lim_{X_t \to \infty} \frac{qL^*(X_t)}{K^*(X_t)} \\ &= 0 + \frac{\lambda}{q} \lim_{X_t \to \infty} \frac{1}{\zeta(X_t)} = \frac{\lambda}{q}\frac{\theta}{\alpha}\end{aligned}$$

Now, consider the Euler equation:

$$\begin{aligned}\frac{c_{t+1}}{c_t} &= \beta \mathcal{F}'(X_{t+1}) = \beta\left(\alpha A\left[1 + \lambda L^*(X_{t+1})\right]^{\theta}\left[K^*(X_{t+1})\right]^{\alpha-1} + 1 - \delta_K\right) \\ &= \beta\left(\alpha A \varphi(X_{t+1})^{\theta}\left[K^*(X_{t+1})\right]^{\alpha+\theta-1} + 1 - \delta_K\right)\end{aligned}$$

Clearly, $\lim_{t \to \infty}(c_{t+1}/c_t) = \infty$ since $\lim_{X_t \to \infty} \varphi(X_t) = l > 0$ and

$$\lim_{X_t \to \infty} K^*(X_t)^{\alpha+\theta-1} = \infty$$

For the numerical exercises, we will consider the Euler equation $c_{t+1}/c_t = \beta \mathcal{F}'(X_{t+1})$ with $c_t = \mathcal{F}(X_t) - X_{t+1}$:

$$\frac{\mathcal{F}(X_{t+1}) - X_{t+2}}{\mathcal{F}(X_t) - X_{t+1}} = \beta \mathcal{F}'(X_{t+1})$$

Using $X_t = \varphi^{-1}(\varphi_t)$, we obtain

$$\frac{\mathcal{F}\left(\varphi^{-1}\left(\varphi_{t+1}\right)\right) - \varphi^{-1}\left(\varphi_{t+2}\right)}{\mathcal{F}\left(\varphi^{-1}(\varphi_t)\right) - \varphi^{-1}\left(\varphi_{t+1}\right)} = \beta \mathcal{F}'\left(\varphi^{-1}\left(\varphi_{t+1}\right)\right)$$

that is (23), a recursive transition function.

The trajectory is determined by (23) jointly with the transversality condition:

$$\lim_{t \to \infty} \varphi_t = l = \frac{\lambda\theta}{\alpha q}$$

■

Proof of Theorem 16.

First, we prove that (K_t^*) maximizes the firm's profit. Define

$$\Delta^T \equiv \sum_{t=0}^{T} p_t^* \left[F(K_t^*, L_t^*) - K_{t+1}^* - qL_{t+1}^*\right] - \sum_{t=0}^{T} p_t^* \left[F(K_t, L_t^*) - K_{t+1} - qL_{t+1}^*\right]$$

where $K_0^* = K_0$ and $0 \leq K_{t+1} \leq F(K_t, L_t^*) - qL_{t+1}^* = L\left(K_t, L_t^*, L_{t+1}^*\right)$. We have:

$$\begin{aligned}\Delta^T &= \sum_{t=0}^{T-1}\left(-p_t^*\left(K_{t+1}^* - K_{t+1}\right) + p_{t+1}^*\left[F\left(K_{t+1}^*, L_{t+1}^*\right) - F\left(K_{t+1}, L_{t+1}^*\right)\right]\right) \\ &\quad -p_T^*\left(K_{T+1}^* - K_{T+1}\right)\end{aligned}$$

Since $F\left(K_t^*, L_t^*\right) - F\left(K_t, L_t^*\right) \geq F_K\left(K_t^*, L_t^*\right)\left(K_t^* - K_t\right)$, we obtain

$$\Delta^T \geq \sum_{t=0}^{T-1} \left[-p_t^* + p_{t+1}^* F_K\left(K_{t+1}^*, L_{t+1}^*\right)\right]\left(K_{t+1}^* - K_{t+1}\right) - p_T^*\left(K_{T+1}^* - K_{T+1}\right)$$

The Euler equation implies $-p_t^* + p_{t+1}^* F_K\left(K_{t+1}^*, L_{t+1}^*\right) = 0$, that is $\Delta^T \geq -p_T^*\left(K_{T+1}^* - K_{T+1}\right) \geq -p_T^* K_{T+1}^*$.

To finish this proof, we have to show that $\lim_{T\to\infty} p_T^* K_{T+1}^* = 0$. As above, we focus on the three cases: (1) $\gamma < 1$, (2) $\gamma = 1$, and (3) $1 < \gamma < 1/\beta$.

(1) $\gamma < 1$. Since $c_T^* = F\left(K_T^*, L_T^*\right) - K_{T+1}^* - qL_{T+1}^*$, we get

$$p_T^* K_{T+1}^* = \beta^T u'\left(c_T^*\right) K_{T+1}^* = \beta^T \frac{K_{T+1}^*}{c_T^*} = \beta^T \frac{\frac{K_{T+1}^*}{L_{T+1}^*}}{\frac{F\left(K_T^*, L_T^*\right)}{L_{T+1}^*} - \frac{K_{T+1}^*}{L_{T+1}^*} - q} \equiv \beta^T \rho_T^* \tag{74}$$

where the ratio ρ_T^* is uniformly bounded because $\left(K_t^*, L_t^*\right)$ converges to the steady state $\left(K^*, L^*\right)$. According to (74) $\lim_{T\to\infty} \beta^T = 0$ implies that

$$\lim_{T\to\infty} p_T^* K_{T+1}^* = \lim_{T\to\infty} \beta^T \rho_T^* = 0$$

(2) $\gamma = 1$. According to (4), we have

$$\frac{F\left(K_T^*, L_T^*\right)}{L_{T+1}^*} = \left[A\left(\frac{1+\lambda L_T^*}{K_T^*}\right)^{\theta} \frac{K_T^*}{L_T^*} + \left(1-\delta_K\right)\frac{K_T^*}{L_T^*} + q\left(1-\delta_L\right)\right]\frac{L_T^*}{L_{T+1}^*}$$

since $\gamma \equiv \alpha + \theta = 1$. The growth path is asymptotically balanced and, by (19), $\lim_{T\to\infty}\left(L_{T+1}^*/L_T^*\right) = \beta\mathcal{F}'\left(\bar{X}\right)$. Then,

$$\begin{aligned}
\lim_{T\to\infty} \frac{F\left(K_T^*, L_T^*\right)}{L_{T+1}^*} &= \left[A\lambda^{\theta} \lim_{T\to\infty}\left(\frac{K_T^*}{L_T^*}\right)^{1-\theta} + \left(1-\delta_K\right)\lim_{T\to\infty}\frac{K_T^*}{L_T^*} + q\left(1-\delta_L\right)\right]\lim_{T\to\infty}\frac{L_T^*}{L_{T+1}^*} \\
&= \frac{\lambda\bar{X}\left(A\bar{X}^{-\theta} + 1 - \delta_K\right) + q\left(1-\delta_L\right)}{\beta\mathcal{F}'\left(\bar{X}\right)}
\end{aligned} \tag{75}$$

since $\lim_{T\to\infty} K_T^* = \infty$ and $\lim_{T\to\infty}\left(K_T^*/L_T^*\right) = \lambda\bar{X}$.

ρ_T^* in (74) is uniformly bounded because of (75) and $\lim_{T\to\infty}\left(K_T^*/L_T^*\right) = \lambda\bar{X}$. Then, according to (74), $\lim_{T\to\infty}\beta^T = 0$ implies $\lim_{T\to\infty} p_T^* K_{T+1}^* = 0$.

(3) $1 < \gamma < 1/\beta$. We know that

$$\begin{aligned}
c_{T+1}^* &= \beta c_T^* F_K\left(K_{T+1}^*, L_{T+1}^*\right) = \beta c_T^*\left[\alpha A\left(1+\lambda L_{T+1}^*\right)^{\theta} K_{T+1}^{*\alpha-1} + 1 - \delta_K\right] \\
&= \beta\frac{c_T^*}{K_{T+1}^*}\alpha A\left(1+\lambda L_{T+1}^*\right)^{\theta} K_{T+1}^{*\alpha} + \beta c_T^*\left(1-\delta_K\right)
\end{aligned}$$

Hence,

$$\frac{c_{T+1}^*}{K_{T+2}^*} = \beta\frac{c_T^*}{K_{T+1}^*}\left[\alpha A\left(\frac{1+\lambda L_{T+1}^*}{K_{T+1}^*}\right)^{\theta} K_{T+1}^{*\alpha+\theta-1} + 1 - \delta_K\right]\frac{K_{T+1}^*}{K_{T+2}^*} \tag{76}$$

From the budget constraint, at equilibrium, we have:

$$\begin{aligned} K^*_{T+2} + qL^*_{T+2} &\leq A\left(1+\lambda L^*_{T+1}\right)^{\theta} K^{*\alpha}_{T+1} + (1-\delta_K) K^*_{T+1} + q(1-\delta_L) L^*_{T+1} - c^*_{T+1} \\ &\leq A\left(1+\lambda L^*_{T+1}\right)^{\theta} K^{*\alpha}_{T+1} + (1-\delta_K) K^*_{T+1} + q(1-\delta_L) L^*_{T+1} \end{aligned}$$

that is, dividing by K^*_{T+2}, and rearranging

$$\frac{1+\frac{qL^*_{T+2}}{K^*_{T+2}}}{A\left(\frac{1+\lambda L^*_{T+1}}{K^*_{T+1}}\right)^{\theta} K^{*\alpha+\theta-1}_{T+1} + 1 - \delta_K + (1-\delta_L)\frac{qL^*_{T+1}}{K^*_{T+1}}} \leq \frac{K^*_{T+1}}{K^*_{T+2}} \tag{77}$$

Replacing (77) in (76), we get

$$\frac{c^*_{T+1}}{K^*_{T+2}} \geq \beta \frac{c^*_T}{K^*_{T+1}} \frac{\left(1+\frac{qL^*_{T+2}}{K^*_{T+2}}\right)\left[\alpha A\left(\frac{1+\lambda L^*_{T+1}}{K^*_{T+1}}\right)^{\theta} K^{*\alpha+\theta-1}_{T+1} + 1 - \delta_K\right]}{A\left(\frac{1+\lambda L^*_{T+1}}{K^*_{T+1}}\right)^{\theta} K^{*\alpha+\theta-1}_{T+1} + 1 - \delta_K + (1-\delta_L)\frac{qL^*_{T+1}}{K^*_{T+1}}}$$

Since $\gamma \equiv \alpha + \theta > 1$, $\lim_{T\to\infty}(qL^*_T/K^*_T) = \theta/\alpha$ and $\lim_{T\to\infty} K^*_T = \infty$, we find

$$\begin{aligned} &\lim_{T\to\infty} \frac{\left(1+\frac{qL^*_{T+2}}{K^*_{T+2}}\right)\left[\alpha A\left(\frac{1+\lambda L^*_{T+1}}{K^*_{T+1}}\right)^{\theta} K^{*\alpha+\theta-1}_{T+1} + 1 - \delta_K\right]}{A\left(\frac{1+\lambda L^*_{T+1}}{K^*_{T+1}}\right)^{\theta} K^{*\alpha+\theta-1}_{T+1} + 1 - \delta_K + (1-\delta_L)\frac{qL^*_{T+1}}{K^*_{T+1}}} \\ = \quad & \alpha\left(1+\lim_{T\to\infty}\frac{qL^*_{T+2}}{K^*_{T+2}}\right) = \alpha\left(1+\frac{\theta}{\alpha}\right) = \gamma \end{aligned}$$

Fix $\varepsilon \in (0, \gamma-1)$. Then, there exists T_0 such that, for any $T \geq T_0$, we have $c^*_{T+1}/K^*_{T+2} \geq \beta(\gamma-\varepsilon) c^*_T/K^*_{T+1}$ and, by induction, $c^*_{T_0+\tau+1}/K^*_{T_0+\tau+2} \geq [\beta(\gamma-\varepsilon)]^{\tau+1} c^*_{T_0}/K^*_{T_0+1}$ or, equivalently,

$$\frac{K^*_{T_0+\tau+2}}{c^*_{T_0+\tau+1}} \leq \frac{1}{[\beta(\gamma-\varepsilon)]^{\tau+1}} \frac{K^*_{T_0+1}}{c^*_{T_0}} \tag{78}$$

Therefore,

$$\lim_{T\to\infty} p^*_T K^*_{T+1} = \lim_{T\to\infty} \frac{\beta^T K^*_{T+1}}{c^*_T} \leq \frac{\beta^{T_0} K^*_{T_0+1}}{c^*_{T_0}} \lim_{T\to\infty} (\gamma-\varepsilon)^{T_0-T} = 0$$

since $\gamma - \varepsilon > 1$.

Let us prove that the consumer does maximize her overall utility when $0 < \gamma < 1/\beta$, that is, if (c_t) satisfies $\sum_{t=0}^{\infty}\beta^t u(c_t) > \sum_{t=0}^{\infty}\beta^t u(c^*_t)$, then $\sum_{t=0}^{\infty} p^*_t c_t > \pi^* + r^* K_0$.

(1) From (P0), we have $c^*_t + K^*_{t+1} + qL^*_{t+1} = F(K^*_t, L^*_t)$ for any t. Thus, $p^*_t c^*_t = p^*_t F(K^*_t, L^*_t) - p^*_t K^*_{t+1} - q p^*_t L^*_{t+1}$. Let us show that the series $\sum_{t=0}^{\infty} p^*_t c^*_t$,

$\sum_{t=0}^{\infty} p_t^* K_{t+1}^*$, $\sum_{t=0}^{\infty} p_t^* L_{t+1}^*$ and $\sum_{t=0}^{\infty} p_t^* F\left(K_t^*, L_t^*\right)$ are bounded away from ∞ and, thus, $\sum_{t=0}^{\infty} p_t^* c_t^* = \pi^* + r^* K_0$ with $\sum_{t=0}^{\infty} p_t^* c_t^* < \infty$ and $\pi^* + r^* K_0 = \sum_{t=0}^{\infty} \left[p_t^* F\left(K_t^*, L_t^*\right) - p_t^* K_{t+1}^* - q p_t^* L_{t+1}^*\right] < \infty$.

Note that $p_t^* \equiv \beta^t u'\left(c_t^*\right) = \beta^t / c_t^*$. Then, $\sum_{t=0}^{\infty} p_t^* c_t^* = \sum_{t=0}^{\infty} \beta^t = 1/\left(1-\beta\right) < \infty$. Moreover, since $\sum_{t=0}^{\infty} p_t^* c_t^* = \sum_{t=0}^{\infty} p_t^* \left[F\left(K_t^*, L_t^*\right) - K_{t+1}^* - q L_{t+1}^*\right]$ and $\sum_{t=0}^{\infty} p_t^* c_t^* < \infty$, we get $\sum_{t=0}^{\infty} p_t^* F\left(K_t^*, L_t^*\right) = \sum_{t=0}^{\infty} p_t^* c_t^* + \sum_{t=0}^{\infty} p_t^* K_{t+1}^* + q \sum_{t=0}^{\infty} p_t^* L_{t+1}^* < \infty$, provided that $\sum_{t=0}^{\infty} p_t^* K_{t+1}^* < \infty$ and $\sum_{t=0}^{\infty} p_t^* L_{t+1}^* < \infty$.

First, since the sequence $\left(L_t^*/K_t^*\right)$ is bounded away from 0 and from ∞, if $\sum_{t=0}^{\infty} p_t^* K_{t+1}^* < \infty$, then we would have that

$$\sum_{t=0}^{\infty} p_t^* L_{t+1}^* = \sum_{t=0}^{\infty} p_t^* K_{t+1}^* \left(L_{t+1}^*/K_{t+1}^*\right) < \infty$$

In this regard, let us show that $\sum_{t=0}^{\infty} p_t^* K_{t+1}^* < \infty$. We need to distinguish two cases: $\gamma \leq 1$ and $1 < \gamma < 1/\beta$. If $\gamma \leq 1$, we have that ρ_T^* in (74) is uniformly bounded and $\sum_{t=0}^{\infty} p_t^* K_{t+1}^* = \sum_{t=0}^{\infty} \beta^t \rho_T^* < \infty$. In the second case, $1 < \gamma < 1/\beta$ and according to (78), we have

$$\begin{aligned}
\sum_{t=0}^{\infty} p_t^* K_{t+1}^* &= \sum_{t=0}^{\infty} \frac{\beta^t K_{t+1}^*}{c_t^*} = \sum_{t=0}^{T_0} \frac{\beta^t K_{t+1}^*}{c_t^*} + \sum_{\tau=0}^{\infty} \frac{\beta^{T_0+\tau+1} K_{T_0+\tau+2}^*}{c_{T_0+\tau+1}^*} \\
&\leq \sum_{t=0}^{T_0} \frac{\beta^t K_{t+1}^*}{c_t^*} + \sum_{\tau=0}^{\infty} \frac{\beta^{T_0+\tau+1}}{\left[\beta\left(\gamma-\varepsilon\right)\right]^{\tau+1}} \frac{K_{T_0+1}^*}{c_{T_0}^*} \\
&= \sum_{t=0}^{T_0} \frac{\beta^t K_{t+1}^*}{c_t^*} + \frac{\beta^{T_0} K_{T_0+1}^*}{\left(\gamma-\varepsilon\right) c_{T_0}^*} \sum_{\tau=0}^{\infty} \left(\frac{1}{\gamma-\varepsilon}\right)^{\tau} \\
&= \sum_{t=0}^{T_0} \frac{\beta^t K_{t+1}^*}{c_t^*} + \frac{\beta^{T_0} K_{T_0+1}^*}{c_{T_0}^*} \frac{1}{\gamma-\varepsilon-1} < \infty
\end{aligned}$$

Therefore, $\sum_{t=0}^{\infty} p_t^* F\left(K_t^*, L_t^*\right) < \infty$.

Finally, let us show that, if (c_t) satisfies $\sum_{t=0}^{\infty} \beta^t u\left(c_t\right) > \sum_{t=0}^{\infty} \beta^t u\left(c_t^*\right)$, then $\sum_{t=0}^{\infty} p_t^* c_t > \pi^* + r^* K_0$. Indeed, $0 > \sum_{t=0}^{\infty} \beta^t u\left(c_t^*\right) - \sum_{t=0}^{\infty} \beta^t u\left(c_t\right) \geq \sum_{t=0}^{\infty} \beta^t u'\left(c_t^*\right)\left(c_t^* - c_t\right)$. Thus, $0 > \sum_{t=0}^{\infty} p_t^* c_t^* - \sum_{t=0}^{\infty} p_t^* c_t$ and $\sum_{t=0}^{\infty} p_t^* c_t > \sum_{t=0}^{\infty} p_t^* c_t^* = \pi^* + r^* K_0$. To finish the proof, take $r^* = p_0^* F_K\left(K_0, L_0\right)$. ■

Proof of Corollary 12.

(1) When $\gamma = 1$, according to (19), growth factors are asymptotically balanced with

$$\frac{K_{t+1}}{K_t} \sim \frac{L_{t+1}}{L_t} = \beta \mathcal{F}'\left(\bar{X}\right) \tag{79}$$

and the definition for $\bar{X}$ is given by equation (53) in the proof of Proposition 10:

$$A \bar{X}^{\alpha-1} = \frac{q\left(\delta_K - \delta_L\right)}{\alpha q - \lambda \theta \bar{X}} \tag{80}$$

Equation (80) can be written as

$$\alpha - \frac{\lambda\theta}{q}\bar{X} = \frac{\delta_K - \delta_L}{A}\bar{X}^{1-\alpha} \tag{81}$$

It is easy to show that equation (81) has a unique solution. Note first that the LHS strictly decreases in $\bar{X}$, while the RHS strictly increases in $\bar{X}$. Since $LHS\,(0) = \alpha > RHS\,(0) = 0$, the RHS and the LHS cross only once proving the existence and uniqueness of the solution to (81), $\bar{X}^*\,(q)$.

From (81), it is easy to see that $\bar{X}^*$ is strictly increasing in q.

According to (46) in the proof of Proposition 10, $\mathcal{F}'\left(\bar{X}\right) = \alpha A\bar{X}^{\alpha-1} + 1 - \delta_K$. Hence, (79) implies that the balanced growth factor decreases when q increases.

Moreover, (18) is equivalent to $\lim_{t\to\infty}(K_t/L_t) = \lambda\bar{X}$, entailing that, when q increases, the asymptotic capital ratio K_t/L_t increases as well.

(2) Let $1 < \gamma < 1/\beta$. Equation (20) implies $\lim_{t\to\infty}(K_t/L_t) = q\alpha/\theta$, entailing that, when q increases, the asymptotic capital ratio K_t/L_t increases as well. ■

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