Backward doubly stochastic differential equations and SPDEs with quadratic growth *

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Backward doubly stochastic differential equations and SPDEs with quadratic growth*

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Abstract. This paper shows the nonlinear stochastic Feynman-Kac formula holds under quadratic growth. For this, we initiate the study of backward doubly stochastic differential equations (BDSDEs, for short) with quadratic growth. The existence, uniqueness, and comparison theorem for one-dimensional BDSDEs are proved when the generator $f(t, Y, Z)$ grows in Z quadratically and the terminal value is bounded, by introducing innovative approaches. Furthermore, in this framework, we utilize BDSDEs to provide a probabilistic representation of solutions to semilinear stochastic partial differential equations (SPDEs, for short) in Sobolev spaces, and use it to prove the existence and uniqueness of such SPDEs, thereby extending the nonlinear stochastic Feynman-Kac formula for linear growth introduced by Pardoux–Peng (*Probab. Theory Related Fields* 98 (1994) 209–227).

Key words: backward doubly stochastic differential equation, stochastic partial differential equation, quadratic growth, Feynman-Kac formula, Sobolev solution.

AMS subject classifications. 60H10, 60H15, 60H30.

1 Introduction

The nonlinear Feynman-Kac formula was introduced by Pardoux–Peng [26] after they proved the existence and uniqueness of nonlinear backward stochastic differential equations (BSDEs, for short) in [25]. It provides a probabilistic representation for a large class of semilinear partial differential equations (PDEs, for short). A few years later, in order to give a probabilistic representation for

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semilinear stochastic PDEs (SPDEs, for short), a new class of BSDEs called the backward doubly stochastic differential equations (BDSDEs, for short) was introduced by them too in [27]. In order to present the work more clearly, we describe the problem in detail.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space on which two standard independent Brownian motions $\{W_t; 0 \leqslant t < \infty\}$ and $\{B_t; 0 \leqslant t < \infty\}$, with values in $\mathbb{R}^d$ and $\mathbb{R}^l$, respectively, are defined. Let $T > 0$ be a fixed terminal time and denote by $\mathcal{N}$ the class of $\mathbb{P}$-null sets of $\mathscr{F}$, where

$$\mathscr{F}_t \triangleq \mathscr{F}_t^W \vee \mathscr{F}_{t,T}^B, \quad \forall t \in [0,T].$$

In the above, for any process $\{\eta_t; 0 \leqslant t \leqslant T\}$, $\mathscr{F}_{s,t}^{\eta} = \sigma\{\eta_r - \eta_s; s \leqslant r \leqslant t\} \vee \mathcal{N}$ and $\mathscr{F}_t^{\eta} = \mathscr{F}_{0,t}^{\eta}$. For each $(t,x) \in [0,T] \times \mathbb{R}^n$, let $\{X_s^{t,x}; t \leqslant s \leqslant T\}$ be the solution of the following stochastic differential equations (SDEs, for short):

$$X_s^{t,x} = x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r, \quad t \leqslant s \leqslant T,$$

and consider the following backward doubly stochastic differential equations: for $t \leqslant s \leqslant T$,

$$Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr + \int_s^T g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})d\overleftarrow{B}_r - \int_s^T Z_r^{t,x}dW_r, \quad (1.1)$$

where the dW integral is a forward Itô's integral and the $d\overleftarrow{B}$ integral is a backward one. The coefficient h is called the *terminal value* and the coefficient f is called the *generator*. The solution of (1.1) is the pair (Y,Z) of $\mathscr{F}$-measurable processes. For convenience, hereafter, by a *quadratic* BDSDE, or BDSDE *with quadratic growth*, we mean that in (1.1), the generator f grows in Z quadratically. Meanwhile, we call h the *bounded terminal value* if it is bounded.

Pardoux–Peng [27] introduced BDSDEs (1.1), gave the well-posedness of solutions under global Lipschitz coefficients, and used it to prove the existence and uniqueness of the following semilinear SPDEs:

$$\begin{aligned} u(s,x) =& h(x) + \int_s^T \Big\{\mathcal{L}u(r,x) + f\big(r,x,u(r,x),(\sigma^\top \nabla u)(r,x)\big)\Big\}dr \\ &+ \int_s^T g\big(r,x,u(r,x),(\sigma^\top \nabla u)(r,x)\big)d\overleftarrow{B}_r, \quad t \leqslant s \leqslant T, \end{aligned} \quad (1.2)$$

where $\sigma^\top$ denotes the transpose of σ, and

$$\mathcal{L} = \sum_{i=1}^n b_i \frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^n a_{ij}\frac{\partial^2}{\partial x_i \partial x_j}, \qquad (a_{ij}) = \sigma\sigma^\top.$$

This result is summarized as the nonlinear stochastic Feynman-Kac formula, which permits us to solve SPDE by BDSDE and, conversely, one can use SPDE to solve BDSDE too. Since then, this important theory has attracted a lot of attention. For example, Buckdahn–Ma [10, 11] introduced the stochastic viscosity solution to the nonlinear SPDEs, which connect BDSDEs to extend the nonlinear stochastic Feynman-Kac formula. Bally–Matoussi [3] obtained the connection between

the solution of BDSDEs and the Sobolev solution of related SPDEs. Zhang–Zhao [38] studied infinite horizon BDSDEs and got the stationary solutions of related SPDEs. Xiong [34] solved a long-standing open problem about the SPDE characterization of the super-Brownian motion by making use of the BDSDE. Buckdahn–Li–Xing [9] studied mean-field BDSDEs and the associated nonlocal semilinear SPDEs. Besides, Shi–Gu–Liu [29] obtained the comparison theorem of solutions of BDSDEs. Han–Peng–Wu [16] got the maximum principle for backward doubly stochastic control systems. Hu–Matoussi–Zhang [18] studied the Wong-Zakai approximations of BDSDEs. For more recent developments of BDSDEs, we refer the readers to Boufoussi–Van Casteren–Mrhardy [6], Gomez et al. [15], Shi–Wen–Xiong [30], Wu–Zhang [32], Zhang–Zhao [39], etc.

Moreover, when the coefficient g is absent, BDSDEs (1.1) are essentially reduced to the following backward stochastic differential equations:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad 0 \leqslant t \leqslant T, \tag{1.3}$$

which were first introduced by Pardoux–Peng [25], where they obtained the existence and uniqueness under Lipschitz coefficients. Since then, BSDE has stimulated some significant developments in many fields, such as partial differential equation (see Pardoux–Peng [26] and Yong [35]), mathematical finance (see El Karoui–Peng–Quenez [14]), and stochastic optimal control (see Yong–Zhou [36]), to mention a few. Meanwhile, based on the wide applications and motivated by the open problem proposed by Peng [24], many efforts have been made to relax the conditions on the generator f of BSDE (1.3) for the existence and/or uniqueness of adapted solutions. For example, Lepeltier–San Martin [21] proved the existence of adapted solutions for BSDE when the generator f is continuous and of linear growth in (Y, Z). In 2000, Kobylanski [20] established the well-posedness for one-dimensional BSDE (1.3) with f growing in Z quadratically and with bounded terminal value. When the terminal value is unbounded, Briand–Hu [7, 8] proved the existence and uniqueness of one-dimensional BSDEs with quadratic growth. Delbaen–Hu–Bao [13] studied BSDEs with the generator f growing in Z super-quadratically. For multi-dimensional situations, Hu–Tang [19] proved the existence and uniqueness of BSDEs with diagonally quadratic generators, and Xing–Zitkovic [33] established the existence and uniqueness for a large class of Markovian BSDEs with quadratic growth. Some other recent developments of quadratic BSDEs can be found in Bahlali–Eddahbi–Ouknine [1], Barrieu–El Karoui [5], Cheridito–Nam [12], Hu–Li–Wen [17], Richou [28], Tevzadze [31], and references cited therein.

Note that in quadratic BSDEs, in order to overcome the difficulty of quadratic growth, one always would like to distinguish whether the terminal value is bounded or not. The reason is that when the terminal value is bounded, by involving bounded mean oscillation martingales (BMO martingale, for short), some nice estimates and regularities for the solution (Y, Z) could be obtained. In other words, in this case, one could prove that Y is bounded and Z belongs to the BMO martingale space, which is useful when proving the existence and uniqueness. In addition, when proving the existence, the main idea is to use the exponential transformation, which implies that there is an essential difference between the study of the one-dimensional case and the multi-dimensional case. In summary, when

studying the quadratic BSDEs, it is better to distinguish whether the terminal value is bounded or not and whether the framework is a one-dimensional situation or a multi-dimensional situation.

On the other hand, during the past two decades, stimulated by the broad applications and the open problem of Peng [24], tremendous efforts have been made to relax the conditions on the generator f of BDSDEs and extend the nonlinear stochastic Feynman-Kac formula. However, to the best of our knowledge, there are few works concerning BDSDEs (1.1) when the generator f grows in Z quadratically. Some related studies are the works of Zhang–Zhao [40] and Bahlali et al. [2], where they obtained the existence and uniqueness of BDSDEs when the generator f is of polynomial growth in Y and is of super-linear (or sub-quadratic) in Z, respectively. For quadratic BDSDEs, the main difficulty is due to the fact that the collection $\mathbb{F} = \{\mathscr{F}_t; 0 \leqslant t \leqslant T\}$ is neither increasing nor decreasing, and it does not constitute a filtration. So the main technique of BMO martingale used in quadratic BSDEs is useless here, and one cannot expect to study the regularity of Z in the BMO martingale space. Besides, the backward Itô's integral appearing in BDSDE (1.1) will bring extra troubles when proving the existence and uniqueness of the solutions.

In this paper, we initiate the theory of BDSDEs with quadratic growth. In particular, we consider the one-dimensional framework of quadratic BDSDEs with bounded terminal value. First, we study the existence under a general assumption on the generator f. Borrowed some ideas from Kobylanski [20], we construct an artful exponential transformation (see Example 3.1), which transforms the quadratic BDSDE into a classical one. It should be pointed out that the solution (Y, Z) we considered is in the space $L^{\infty}_{\mathbb{F}}(0,T;\mathbb{R}) \times L^{2}_{\mathbb{F}}(0,T;\mathbb{R}^d)$ (see Section 2 for detailed definitions), since $\mathbb{F} = \{\mathscr{F}_t; 0 \leqslant t \leqslant T\}$ is not a filtration. Moreover, in order to overcome the troubles that come from the backward Itô's integral, we introduce the idea of the comparison theorem of classical BDSDEs, and some restricted condition (see (3.11)) on the coefficient g is made, otherwise, Y cannot be bounded (see Example 3.2). Furthermore, based on a priori estimate (see Proposition 3.7) and the monotone stability obtained (see Proposition 3.9), we prove that BDSDE with quadratic growth admits a solution $(Y, Z) \in L^{\infty}_{\mathbb{F}}(0,T;\mathbb{R}) \times L^{2}_{\mathbb{F}}(0,T;\mathbb{R}^d)$ (see Theorem 3.11). Second, in order to prove the uniqueness, we study a comparison theorem, due to the fact that the one-dimensional framework allows us to provide a comparison theorem that directly implies the uniqueness as a by-product (see Theorem 4.4). However, a stronger assumption on the coefficients f and g than that for the existence result is used to prove the comparison theorem. In other words, we assume that the partial derivatives of the coefficients f and g with respect to Z are linear growth and bounded, respectively. Finally, we use BDSDE (1.1) with quadratic growth to give a probabilistic representation for the solutions of SPDE (1.2) in Sobolev spaces, and use it to prove the existence and uniqueness of SPDE (1.2) when f grows in $\sigma^{\top}\nabla u$ quadratically (see Theorem 5.6), thus extending the nonlinear stochastic Feynman-Kac formula of Pardoux–Peng [27] to quadratic growth situation.

The rest of the paper is organized as follows. In Section 2, some preliminaries are presented. In Section 3, a priori estimate, the monotonicity stability, and the existence of the solutions are proved. In Section 4, we focus on the comparison theorem and thus derive the uniqueness of the solutions. In Section 5, we study the relationship between the solution of BDSDEs with quadratic

growth and the Sobolev solution of related SPDEs. Section 6 concludes the results.

2 Preliminaries

Repeat that the triple $(\Omega, \mathscr{F}, \mathbb{P})$ is a complete probability space, and on which two standard independent Brownian motions $\{W_t; 0 \leqslant t < \infty\}$ and $\{B_t; 0 \leqslant t < \infty\}$ are defined with values in $\mathbb{R}^d$ and $\mathbb{R}^l$, respectively. Let $T > 0$ be a fixed terminal time and denote by $\mathcal{N}$ the class of $\mathbb{P}$-null sets of $\mathscr{F}$, where

$$\mathscr{F}_t \triangleq \mathscr{F}_t^W \vee \mathscr{F}_{t,T}^B, \qquad \forall t \in [0,T].$$

In the above, for any process $\{\eta_t; 0 \leqslant t \leqslant T\}$, $\mathscr{F}_{s,t}^{\eta} = \sigma\{\eta_r - \eta_s; s \leqslant r \leqslant t\} \vee \mathcal{N}$ and $\mathscr{F}_t^{\eta} = \mathscr{F}_{0,t}^{\eta}$. Note that the collection $\mathbb{F} = \{\mathscr{F}_t; 0 \leqslant t \leqslant T\}$ is not a filtration since it is neither increasing nor decreasing.

Let us introduce some notations that will be used below. Denote by $|\cdot|$ and $\langle \cdot, \cdot \rangle$ the Euclidean norm and dot product, respectively, throughout the paper. For any $p \geqslant 1$ and Euclidean space $\mathbb{H}$, denote by $C^p(\mathbb{H})$ the set of functions valued in $\mathbb{H}$ and of class C^p with the partial derivations of order less than or equal to p are bounded. In addition, we introduce the following spaces:

$$L^p_{\mathscr{F}_T}(\Omega;\mathbb{H}) = \Big\{\xi : \Omega \to \mathbb{H} \,\Big|\, \xi \text{ is } \mathscr{F}_T\text{-measurable}, \ \|\xi\|_{L^p} \triangleq \big(\mathbb{E}|\xi|^p\big)^{\frac{1}{p}} < \infty\Big\},$$
$$L^\infty_{\mathscr{F}_T}(\Omega;\mathbb{H}) = \Big\{\xi : \Omega \to \mathbb{H} \,\Big|\, \xi \text{ is } \mathscr{F}_T\text{-measurable}, \ \|\xi\|_{\infty} \triangleq \operatorname*{esssup}_{\omega\in\Omega} |\xi(\omega)| < \infty\Big\},$$

and for any $t \in [0,T)$ and $s \in [t,T]$, define

$$L^p_{\mathbb{F}}(t,T;\mathbb{H}) = \Big\{\varphi : [t,T]\times\Omega \to \mathbb{H} \,\Big|\, \varphi_s \text{ is } \mathscr{F}_s\text{-measurable}, \ \|\varphi\|_{L^p_{\mathbb{F}}(t,T)} \triangleq \Big(\mathbb{E}\int_t^T |\varphi_s|^p ds\Big)^{\frac{1}{p}} < \infty\Big\},$$
$$L^\infty_{\mathbb{F}}(t,T;\mathbb{H}) = \Big\{\varphi : [t,T]\times\Omega \to \mathbb{H} \,\Big|\, \varphi_s \text{ is } \mathscr{F}_s\text{-measurable}, \ \|\varphi\|_{L^\infty_{\mathbb{F}}(t,T)} \triangleq \operatorname*{esssup}_{(s,\omega)\in[t,T]\times\Omega} |\varphi_s(\omega)| < \infty\Big\},$$
$$S^p_{\mathbb{F}}(t,T;\mathbb{H}) = \Big\{\varphi : [t,T]\times\Omega \to \mathbb{H} \,\Big|\, \varphi_s \text{ is } \mathscr{F}_s\text{-measurable, continuous, and}$$
$$\|\varphi\|_{S^p_{\mathbb{F}}(t,T)} \triangleq \Big\{\mathbb{E}\Big[\sup_{s\in[t,T]} |\varphi_s|^p\Big]\Big\}^{\frac{1}{p}} < \infty\Big\}.$$

Consider the following backward doubly stochastic differential equations:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds + \int_t^T g(s, Y_s, Z_s)\, d\overleftarrow{B}_s - \int_t^T Z_s dW_s, \qquad 0 \leqslant t \leqslant T, \tag{2.1}$$

where $\xi : \Omega \to \mathbb{R}^k$ is a random variable, $f : \Omega \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k$ and $g : \Omega \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^{k\times l}$ are two coefficients. Note that the dW integral is a forward Itô's integral and the $d\overleftarrow{B}$ integral is a backward one, and both integrals are particular cases of Itô-Skorohod integral (see Nualart [23]).

Definition 2.1. A pair of measurable processes $(Y, Z) \in S^2_{\mathbb{F}}(0,T;\mathbb{R}^k) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^{k\times d})$ is called a solution of BDSDE (2.1), if $\mathbb{P}$-almost surely, it satisfies (2.1). In addition, if Y is bounded, then we call the pair (Y, Z) a bounded solution.

The following result is the Itô's formula for backward doubly stochastic differential equations, which comes from Pardoux–Peng [27].

Proposition 2.2. Let $\Phi \in C^2(\mathbb{R}^k)$, $\alpha \in S^2_{\mathbb{F}}(0,T;\mathbb{R}^k)$, $\beta \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)$, $\gamma \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{k\times l})$, $\delta \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{k\times d})$ be such that

$$\alpha_t = \alpha_0 + \int_0^t \beta_s ds + \int_0^t \gamma_s d\overleftarrow{B}_s + \int_0^t \delta_s dW_s, \qquad 0 \leqslant t \leqslant T.$$

Then

$$\begin{aligned}\Phi(\alpha_t) = \Phi(\alpha_0) &+ \int_0^t \langle \Phi'(\alpha_s), \beta_s\rangle ds + \int_0^t \langle \Phi'(\alpha_s), \gamma_s d\overleftarrow{B}_s\rangle + \int_0^t \langle \Phi'(\alpha_s), \delta_s dW_s\rangle \\ &- \frac{1}{2}\int_0^t \mathrm{Tr}[\Phi''(\alpha_s)\gamma_s\gamma_s^\top]ds + \frac{1}{2}\int_0^t \mathrm{Tr}[\Phi''(\alpha_s))\delta_s\delta_s^\top]ds.\end{aligned}$$

Assumption 1. The terminal value ξ comes from $L^2_{\mathscr{F}_T}(\Omega;\mathbb{R}^k)$, and for any $y \in \mathbb{R}^k$ and $z \in \mathbb{R}^{k\times d}$,

$$f(\cdot, y, z) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^k) \quad \text{and} \quad g(\cdot, y, z) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{k\times l}).$$

Moreover, there exist two positive constants C and α with $\alpha \in (0,1)$ such that for any $t \in [0,T]$, $y_1, y_2 \in \mathbb{R}^k$, $z_1, z_2 \in \mathbb{R}^{k\times d}$,

$$\begin{aligned}|f(t,y_1,z_1) - f(t,y_2,z_2)|^2 &\leqslant C\big[|y_1-y_2|^2 + |z_1-z_2|^2\big], \\ |g(t,y_1,z_1) - g(t,y_2,z_2)|^2 &\leqslant C|y_1-y_2|^2 + \alpha|z_1-z_2|^2.\end{aligned} \tag{2.2}$$

Note that under Assumption 1, BDSDEs (2.1) admit a unique solution $(Y,Z) \in S^2_{\mathbb{F}}(0,T;\mathbb{R}^k) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^{k\times d})$ (see Pardoux–Peng [27, Theorem 1.1]). In addition, the related comparison theorem holds (see Shi et al. [29, Theorem 3.1]).

3 Quadratic BDSDE

In this section, we study the existence of one-dimensional BDSDEs with quadratic growth, i.e., $k = 1$. Before proving the existence, we study a priori estimate and the monotone stability, which are important for us to prove existence later. In order to justify the assumptions that were used to prove the existence, we give two examples. It should be pointed out that the first example also implies the main idea to prove existence, which is the central technique throughout this section.

Example 3.1. Consider the following backward doubly stochastic differential equation,

$$Y_t = \xi + C\int_t^T |Z_s|^2 ds + \alpha\int_t^T Z_s d\overleftarrow{B}_s - \int_t^T Z_s dW_s, \qquad 0 \leqslant t \leqslant T, \tag{3.1}$$

where $C > 0$ and $\alpha \in (-1,1)$ are two constants. Let β be a positive constant which will be given later. Applying the exponential change of variable to $y = \exp(\beta Y)$ implies that

$$y_T = y_t - C\int_t^T \beta y_s |Z_s|^2 ds - \alpha\int_t^T \beta y_s Z_s d\overleftarrow{B}_s + \int_t^T \beta y_s Z_s dW_s$$

$$
\begin{aligned}
&-\frac{\alpha^2}{2}\int_t^T \beta^2 y_s|Z_s|^2ds+\frac{1}{2}\int_t^T \beta^2 y_s|Z_s|^2ds\\
&=y_t-\Big[C-\frac{1-\alpha^2}{2}\beta\Big]\int_t^T \beta y_s|Z_s|^2ds-\alpha\int_t^T \beta y_s Z_s d\overleftarrow{B}_s+\int_t^T \beta y_s Z_s dW_s.
\end{aligned}
$$

Let $\beta=\frac{2C}{1-\alpha^2}$ and denote by $z=\beta yZ$, then the above equation can be rewritten as

$$
y_t=\exp(\beta\xi)+\int_t^T \alpha z_s d\overleftarrow{B}_s-\int_t^T z_s dW_s,\quad 0\leqslant t\leqslant T, \tag{3.2}
$$

which essentially is a linear backward doubly stochastic differential equation. Then the classical existence and uniqueness (see Pardoux–Peng [27, Theorem 1.1]) implies that when

$$
\exp(\beta\xi)\in L^2_{\mathscr{F}_T}(\Omega;\mathbb{R}), \tag{3.3}
$$

there exists a unique solution $(y,z)\in S^2_{\mathbb{F}}(0,T;\mathbb{R})\times L^2_{\mathbb{F}}(0,T;\mathbb{R}^d)$ for BDSDE (3.2). Note that, β is a positive constant, so taking $\xi\in L^\infty_{\mathscr{F}_T}(\Omega;\mathbb{R})$ is a sufficient condition to insure that ξ satisfies (3.3). In fact, if $\xi\in L^\infty_{\mathscr{F}_T}(\Omega;\mathbb{R})$, then $\exp(\xi)\in L^\infty_{\mathscr{F}_T}(\Omega;\mathbb{R})$ and (3.3) holds, thus Eq. (3.2) admits a unique solution $(y,z)\in S^2_{\mathbb{F}}(0,T;\mathbb{R})\times L^2_{\mathbb{F}}(0,T;\mathbb{R}^d)$. Moreover, for BDSDE (3.2), by the comparison theorem (see Shi et al. [29, Theorem 3.1]), one has

$$
\exp(-\beta\|\xi\|_\infty)\leqslant y_t\leqslant\exp(\beta\|\xi\|_\infty),\quad \forall t\in[0,T],
$$

which implies that y_t is positive and bounded. Now, we can define (Y,Z) by

$$
Y_t=\frac{\ln(y_t)}{\beta},\quad Z_t=\frac{z_t}{\beta y_t},\quad \forall t\in[0,T].
$$

Then, it is easy to check that the pair $(Y,Z)\in L^\infty_{\mathbb{F}}(0,T;\mathbb{R})\times L^2_{\mathbb{F}}(0,T;\mathbb{R}^d)$ defined above is a solution of BDSDE (3.1). Finally, the uniqueness of (3.1) in $L^\infty_{\mathbb{F}}(0,T;\mathbb{R})\times L^2_{\mathbb{F}}(0,T;\mathbb{R}^d)$ follows from the uniqueness of (3.2) and the fact that the exponential change of variable is no longer formal.

Example 3.2. Consider the following linear backward doubly stochastic differential equation,

$$
Y_t=\xi+\int_t^T (H+d_sY_s)d\overleftarrow{B}_s-\int_t^T Z_s dW_s,\quad 0\leqslant t\leqslant T, \tag{3.4}
$$

where H is a constant and $d_\cdot$ is a bounded deterministic function. From Pardoux–Peng [27, Theorem 1.1], BDSDE (3.4) admits a unique solution (Y,Z). Moreover, Pardoux–Peng [27, Remark 3.4] implies that the solution Y is given explicitly by

$$
Y_t=\mathbb{E}\left[\Psi(t,T)\xi+H\int_t^T d_s\Psi(t,s)ds\Big|\mathscr{F}_t\right],\quad 0\leqslant t\leqslant T,
$$

where

$$
\Psi(t,s)=\exp\left(\int_t^s d_r d\overleftarrow{B}_r-\frac{1}{2}\int_t^s |d_r|^2dr\right),\quad t\leqslant s\leqslant T,
$$

which is not bounded even if the terminal value ξ is bounded.

Remark 3.3. On the one hand, Example 3.1 tells us why we require the boundedness of the terminal value ξ. On the other hand, Example 3.2 implies that even in the classical situation with bounded terminal value ξ, the normal Lipschitz condition of the coefficient g with respect to y (see (2.2)) could not guarantee that Y is bounded.

To enrich the content, we would like to give some more examples.

Example 3.4. Consider the following BDSDE: for $0 \leqslant t \leqslant T$,

$$Y_t = \xi + 1 + \int_t^T \frac{1}{2}|Z_s|^2 ds + \int_t^T \left\{ Y_s - \ln \mathbb{E}\left[e^{\xi} \middle| \mathscr{F}_s^W \right] \right\} d\overleftarrow{B}_s - \int_t^T Z_s dW_s, \tag{3.5}$$

where $\xi \in L^{\infty}_{\mathscr{F}_T}(\Omega; \mathbb{R})$. In order to find a solution of (3.5), we introduce the following two equations:

$$\tilde{Y}_t = \xi + \int_t^T \frac{1}{2}|\tilde{Z}_s|^2 ds - \int_t^T \tilde{Z}_s dW_s, \quad 0 \leqslant t \leqslant T, \tag{3.6}$$

and

$$\bar{Y}_t = 1 + \int_t^T \bar{Y}_s d\overleftarrow{B}_s, \quad 0 \leqslant t \leqslant T. \tag{3.7}$$

On the one hand, for Eq. (3.6), Kobylanski [20] implies that

$$\tilde{Y}_t = \ln \mathbb{E}\left[e^{\xi} \middle| \mathscr{F}_t^W \right].$$

On the other hand, for Eq. (3.7), the same argument as in Example 3.2 implies that

$$\bar{Y}_t = \exp\left\{ B_T - B_t - \frac{1}{2}(T-t) \right\},$$

which is not bounded. Finally, it is easy to verify that the pair

$$Y = \tilde{Y} + \bar{Y} \quad \text{and} \quad Z = \tilde{Z} \tag{3.8}$$

is a solution of quadratic BDSDE (3.5) with Y being not bounded.

Example 3.5. Let $\delta > 0$, $C > 0$, H be some constants, and let $(Y_\cdot, Z_\cdot)$ be the solution of the following BSDE:

$$Y_t = \xi + C\int_t^T |Z_s|^{2+\delta} ds - \int_t^T Z_s dW_s. \tag{3.9}$$

Then the pair $(Y_\cdot + H(B_T - B_\cdot), Z_\cdot)$ is a solution of the following super-quadratic BDSDE:

$$Y_t = \xi + C\int_t^T |Z_s|^{2+\delta} ds + \int_t^T H d\overleftarrow{B}_s - \int_t^T Z_s dW_s. \tag{3.10}$$

Note that Delbaen–Hu–Bao [13] implies that, for some bounded random variable ξ, BSDE (3.9) admits infinity many solutions. Then, the super-quadratic BDSDE (3.10) also admits infinitely many solutions for such bounded random variables.

3.1 A priori estimate

In this subsection, we prove a priori estimate for the solution (Y, Z) of quadratic BDSDEs. First, we introduce the assumption.

Assumption 2. Suppose that the terminal value $\xi \in L^{\infty}_{\mathscr{F}_T}(\Omega;\mathbb{R})$. Let $a : [0,T] \to \mathbb{R}^+$ and $b : [0,T] \to \mathbb{R}^+$ be two functions, and C and α be two positive constants with $\alpha \in (0,1)$ such that for all $(t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$,

$$|f(t,y,z)| \leqslant a_t|y| + b_t + C|z|^2, \quad |g(t,y,z)|^2 \leqslant \alpha|z|^2, \quad \text{a.s.} \tag{3.11}$$

Remark 3.6. We point out that the boundedness of the solution Y requires us to make the condition (3.11) concerning the coefficient g; if not, the solution Y may not be bounded even if the terminal value ξ is bounded (see Example 3.2 and Remark 3.3). Here we give an example that satisfying (3.11):

$$g(t,y,z) = \sin(y)l(z) \quad \text{with} \quad |l(z)|^2 \leqslant \alpha|z|^2, \quad \forall (t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d.$$

Proposition 3.7 (A priori estimate)**.** *Let $(Y,Z) \in L^{\infty}_{\mathbb{F}}(0,T;\mathbb{R}) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^d)$ be a solution of BDSDE (2.1) with parameters (f,g,ξ) that satisfying Assumption 2. Then for every $t \in [0,T]$,*

$$Y_t \leqslant \Big[\sup_{\Omega}(Y_T)\Big]^+ \cdot \exp\Big(\int_t^T a_s ds\Big) + \int_t^T b_s \exp\Big(\int_t^s a_r dr\Big) ds, \quad a.s. \tag{3.12}$$

$$\Big(resp.\ Y_t \geqslant \Big[\inf_{\Omega}(Y_T)\Big]^- \cdot \exp\Big(\int_t^T a_s ds\Big) - \int_t^T b_s \exp\Big(\int_t^s a_r dr\Big) ds, \quad a.s.\Big). \tag{3.13}$$

Moreover, there exists a positive constant K depending only on $\|Y\|_{\infty}$, $\|a\|_{L^1}$, C and α such that

$$\mathbb{E}\int_0^T |Z_s|^2 ds \leqslant K. \tag{3.14}$$

Proof. Consider the following linear ordinary differential equation,

$$\varphi_t = \Big[\sup_{\Omega}(Y_T)\Big]^+ + \int_t^T (a_s\varphi_s + b_s) ds, \quad 0 \leqslant t \leqslant T.$$

Then we have that for $0 \leqslant t \leqslant T$,

$$\varphi_t = \Big[\sup_{\Omega}(Y_T)\Big]^+ \cdot \exp\Big(\int_t^T a_s ds\Big) + \int_t^T b_s \exp\Big(\int_t^s a_r dr\Big) ds.$$

So the result (3.12) holds if we can prove that $Y_t \leqslant \varphi_t$ for all $t \in [0,T]$. Applying Itô's formula (see Proposition 2.2) to the process $Y_t - \varphi_t$ and to an nonnegative and increasing $C^2(\mathbb{R})$ function Φ with $\Phi''(u) \geqslant 0$ for any $u \in \mathbb{R}$, which will be determined later, deduce that

$$\begin{aligned}\Phi(Y_t - \varphi_t) = \Phi(Y_T - \varphi_T) &+ \int_t^T \Phi'(Y_s - \varphi_s)[f(s,Y_s,Z_s) - (a_s\varphi_s + b_s)] ds \\ &+ \int_t^T \Phi'(Y_s - \varphi_s) g(s,Y_s,Z_s) d\overleftarrow{B}_s - \int_t^T \Phi'(Y_s - \varphi_s) Z_s dW_s\end{aligned}$$

$$+\frac{1}{2}\int_t^T \Phi''(Y_s-\varphi_s)|g(s,Y_s,Z_s)|^2ds-\frac{1}{2}\int_t^T \Phi''(Y_s-\varphi_s)|Z_s|^2ds.$$

From Assumption 2, note that φ is non-negative, we have that for every $s\in[t,T]$,

$$f(s,Y_s,Z_s)-(a_s\varphi_s+b_s)\leqslant a_s(|Y_s|-\varphi_s)+C|Z_s|^2\leqslant a_s|Y_s-\varphi_s|+C|Z_s|^2.$$

Moreover, note that Φ is increasing, we obtain

$$\begin{aligned}\Phi(Y_t-\varphi_t)\leqslant \Phi(Y_T-\varphi_T)&+\int_t^T a_s\big|\Phi'(Y_s-\varphi_s)(Y_s-\varphi_s)\big|ds\\&+\int_t^T\Big(C\Phi'-\frac{1-\alpha}{2}\Phi''\Big)(Y_s-\varphi_s)|Z_s|^2ds\\&+\int_t^T\Phi'(Y_s-\varphi_s)g(s,Y_s,Z_s)d\overleftarrow{B}_s-\int_t^T\Phi'(Y_s-\varphi_s)Z_sdW_s.\end{aligned}\tag{3.15}$$

Set $M=\|Y\|_\infty+\|\varphi\|_\infty$ and define the function Φ on the interval $[-M,M]$ by

$$\Phi(u)=\begin{cases}e^{\frac{2C}{1-\alpha}u}-1-\dfrac{2C}{1-\alpha}u-\dfrac{2C^2}{(1-\alpha)^2}u^2, & u\in[0,M],\\ 0, & u\in[-M,0].\end{cases}$$

Then it is easy to check that for all $u\in[-M,M]$,

$$\begin{aligned}\Phi(u)\geqslant 0,\quad\text{and}\quad \Phi(u)=0\ \text{ if and only if }\ u\leqslant 0,\\ \Phi'(u)\geqslant 0,\\ 0\leqslant u\Phi'(u)\leqslant (M+1)\frac{2C}{1-\alpha}\Phi(u),\\ C\Phi'-\frac{1-\alpha}{2}\Phi''\leqslant 0.\end{aligned}$$

Hence, if we set

$$k_t=a_t(M+1)\frac{2C}{1-\alpha},$$

then the function k is positive and deterministic. Note that $Y_T-\varphi_T\leqslant 0$ implies that $\Phi(Y_T-\varphi_T)=0$, and thus for all $0\leqslant t\leqslant T$, one has

$$\begin{aligned}0\leqslant\Phi(Y_t-\varphi_t)\leqslant\int_t^T k_s\Phi(Y_s-\varphi_s)ds\\ +\int_t^T\Phi'(Y_s-\varphi_s)g(s,Y_s,Z_s)d\overleftarrow{B}_s-\int_t^T\Phi'(Y_s-\varphi_s)Z_sdW_s.\end{aligned}\tag{3.16}$$

Taking expectations on both sides of the inequality (3.16) implies that

$$0\leqslant\mathbb{E}\Phi(Y_t-\varphi_t)\leqslant\mathbb{E}\int_t^T k_s\Phi(Y_s-\varphi_s)ds.$$

Then Gronwall's inequality yields that

$$\mathbb{E}\Phi(Y_t - \varphi_t) = 0, \qquad \forall t \in [0,T].$$

Therefore, note that $\Phi(u) \geqslant 0$, for all $t \in [0,T]$,

$$\Phi(Y_t - \varphi_t) = 0, \qquad \text{a.s.}$$

Thus $Y_t - \varphi_t \leqslant 0$, a.s., which implies that the inequality (3.12) holds. Similarly, by using the same computation, one can prove that (3.13) holds too.

Finally, in order to prove the estimate (3.14), we use again (3.15) with $t = 0$, $\varphi = 0$, $M = \|Y\|_\infty$, and define Φ on $[-M, M]$ by

$$\Phi(u) = \frac{1-\alpha}{2C^2}\Big[\exp\Big(\frac{2C}{1-\alpha}(u+M)\Big) - 1 - \frac{2C}{1-\alpha}(u+M)\Big].$$

It is easy to check that, for every $u \in [-M, M]$,

$$\begin{aligned}
&\Phi(u) \geqslant 0, \qquad \Phi'(u) \geqslant 0,\\
&u\Phi'(u) \leqslant \frac{M}{C}\Big[\exp\Big(\frac{4CM}{1-\alpha}\Big) - 1\Big],\\
&\frac{1-\alpha}{2}\Phi'' - C\Phi' = 1.
\end{aligned}$$

Then (3.15) implies that

$$\begin{aligned}
0 \leqslant \Phi(Y_0) \leqslant \Phi(Y_T) &+ \int_0^T a_s \frac{M}{C}\Big[\exp\Big(\frac{4CM}{1-\alpha}\Big) - 1\Big]ds - \int_0^T |Z_s|^2 ds\\
&+ \int_0^T \Phi'(Y_s) g(s, Y_s, Z_s) d\overleftarrow{B}_s - \int_0^T \Phi'(Y_s) Z_s dW_s,
\end{aligned}$$

which leads that

$$\mathbb{E}\int_0^T |Z_s|^2 ds \leqslant \Phi(M) + \frac{M}{C}\Big[\exp\Big(\frac{4CM}{1-\alpha}\Big) - 1\Big]\|a\|_{L^1([0,T];\mathbb{R}^+)}.$$

This completes the proof. □

An immediate consequence of this proposition is the following corollary.

Corollary 3.8. *Let $(Y, Z) \in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^d)$ be a solution of BDSDE with parameters (f, g, ξ), where ξ belongs to $L^\infty_{\mathscr{F}_T}(\Omega;\mathbb{R})$, and f and g satisfy the condition Assumption 2 with a, b, $C > 0$ and $\alpha \in (0,1)$ such that $a, b \in L^1(0,T;\mathbb{R}^+)$, then*

$$\|Y\|_\infty \leqslant \big(\|\xi\|_\infty + \|b\|_{L^1([0,T];\mathbb{R}^+)}\big)\exp\big(\|a\|_{L^1([0,T];\mathbb{R}^+)}\big).$$

3.2 Monotone stability

In this subsection, we would like to study the monotone stability of BDSDEs (2.1) with quadratic growth, which is also important for us to prove the existence later.

Proposition 3.9 (Monotone stability)**.** *Let (f, g, ξ) be a set of parameters and let (f^n, g, ξ^n) be a sequence of parameters such that:*

(i) *There exist two positive constants C and α with $0 < \alpha < 1$ such that for each $n \in \mathbb{N}$,*

$$|f^n(t,y,z)| \leqslant C(1+|y|+|z|^2), \quad |g(t,y,z)|^2 \leqslant \alpha|z|^2, \quad \forall t \in [0,T], y \in \mathbb{R}, z \in \mathbb{R}^d, \tag{3.17}$$
$$|g(t,y_1,z_1) - g(t,y_2,z_2)|^2 \leqslant C|y_1-y_2|^2 + \alpha|z_1-z_2|^2, \quad \forall t \in [0,T], y_1, y_2 \in \mathbb{R}, z_1, z_2 \in \mathbb{R}^d.$$

(ii) *The sequence $\{f^n; n \in \mathbb{N}\}$ converges to f locally uniformly in (y,z) in compact sets, and for each $n \in \mathbb{N}$, the sequence $\{\xi^n; n \in \mathbb{N}\}$ converges to ξ almost-surely.*

(iii) *For each $n \in \mathbb{N}$, the BDSDE with parameters (f^n, g, ξ^n) has a solution*

$$(Y^n, Z^n) \in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^d),$$

and the sequence $\{Y^n; n \in \mathbb{N}\}$ is monotonic such that for every $n \in \mathbb{N}$, $\|Y^n\|_\infty \leqslant M$, where M is a positive constant.

Then there exists a pair of processes $(Y,Z) \in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^d)$ such that

$$\begin{cases} \lim\limits_{n\to\infty} Y^n = Y \text{ uniformly on } [0,T], \\ \{Z^n; n \in \mathbb{N}\} \text{ converges to } Z \text{ in } L^2_{\mathbb{F}}(0,T;\mathbb{R}^d). \end{cases}$$

Moreover, (Y,Z) is a solution of BDSDE with parameters (f,g,ξ).

Proof. From the assumptions, we observe that for all $t \in [0,T]$, the sequence $\{Y^n_t; n \in \mathbb{N}\}$ is monotonic and bounded. Consequently, it possesses a limit, denoted by Y. Without loss of generality, let's consider the case where the sequence $Y^n; n \in \mathbb{N}$ is monotonically increasing towards Y. Additionally, based on Proposition 3.7, we can deduce the existence of a positive constant K such that

$$\mathbb{E}\int_0^T |Z^n|^2 ds \leqslant K, \quad \forall n \in \mathbb{N}.$$

Then there exists a process $Z \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^d)$ such that a subsequence of $\{Z^n; n \in \mathbb{N}\}$ converges to Z weakly in $L^2_{\mathbb{F}}(0,T;\mathbb{R}^d)$. For simplicity presentation, by otherwise choosing a subsequence, we may assume that the whole sequence $\{Z^n; n \in \mathbb{N}\}$ converges to Z weakly in $L^2_{\mathbb{F}}(0,T;\mathbb{R}^d)$.

Next, we would like to divide the proof into two steps, and denote the following notations:

$$\Delta Y^n \triangleq Y - Y^n, \quad \Delta Y^{m,n} \triangleq Y^m - Y^n, \quad \Delta Z^n \triangleq Z - Z^n, \quad \Delta Z^{m,n} \triangleq Z^m - Z^n.$$

Step 1. The sequence $\{Z^n; n \in \mathbb{N}\}$ strongly converges to Z in $L^2_{\mathbb{F}}(0,T;\mathbb{R}^d)$, i.e.,

$$\lim_{n\to\infty} \mathbb{E}\int_0^T |\Delta Z_t^n|^2 dt = 0. \tag{3.18}$$

The main arguments of this step come from Zhang [37] and Kobylanski [20]. It is easy to see that the pair $(\Delta Y^{m,n}, \Delta Z^{m,n})$ satisfies the following equation,

$$\begin{cases} -d\Delta Y_t^{m,n} = [f^m(t,Y_t^m,Z_t^m) - f^n(t,Y_t^n,Z_t^n)]dt \\ \qquad\qquad\quad +[g(t,Y_t^m,Z_t^m) - g(t,Y_t^n,Z_t^n)]d\overleftarrow{B}_t - \Delta Z_t^{m,n} dW_t, \quad t\in[0,T], \\ \Delta Y_T^{m,n} = \xi^m - \xi^n. \end{cases}$$

By assumptions, there exists a constant C_0 depending on C and M, such that for all $m,n \in \mathbb{N}$, $t \in [0,T]$,

$$\begin{aligned} |f^m(t,Y_t^m,Z_t^m) - f^n(t,Y_t^n,Z_t^n)| &\leqslant C_0[1 + |\Delta Z_t^{m,n}|^2 + |\Delta Z_t^n|^2 + |Z_t|^2], \\ |g(t,Y_t^m,Z_t^m) - g(t,Y_t^n,Z_t^n)|^2 &\leqslant C|\Delta Y_t^{m,n}|^2 + \alpha|\Delta Z_t^{m,n}|^2. \end{aligned} \tag{3.19}$$

Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a smooth increasing function which will be specified later. For $m \geqslant n$, using Itô's formula (see Proposition 2.2) to $\Phi(\Delta Y^{m,n})$ we have that

$$\begin{aligned} d\Phi(\Delta Y_t^{m,n}) =& -\Phi'(\Delta Y_t^{m,n})[f^m(t,Y_t^m,Z_t^m) - f^n(t,Y_t^n,Z_t^n)]dt \\ & - \Phi'(\Delta Y_t^{m,n})[g(t,Y_t^m,Z_t^m) - g(t,Y_t^n,Z_t^n)]d\overleftarrow{B}_t + \Phi'(\Delta Y_t^{m,n})\Delta Z_t^{m,n} dW_t \\ & - \frac{1}{2}\Phi''(\Delta Y_t^{m,n})|g(t,Y_t^m,Z_t^m) - g(t,Y_t^n,Z_t^n)|^2 dt + \frac{1}{2}\Phi''(\Delta Y_t^{m,n})|\Delta Z_t^{m,n}|^2 dt \\ \geqslant& - C_0\Phi'(\Delta Y_t^{m,n})[1 + |\Delta Z_t^{m,n}|^2 + |\Delta Z_t^n|^2 + |Z_t|^2]dt \\ & - \Phi'(\Delta Y_t^{m,n})[g(t,Y_t^m,Z_t^m) - g(t,Y_t^n,Z_t^n)]d\overleftarrow{B}_t + \Phi'(\Delta Y_t^{m,n})\Delta Z_t^{m,n} dW_t \\ & - \frac{C}{2}\Phi''(\Delta Y_t^{m,n})|\Delta Y_t^{m,n}|^2 dt - \frac{\alpha}{2}\Phi''(\Delta Y_t^{m,n})|\Delta Z_t^{m,n}|^2 dt + \frac{1}{2}\Phi''(\Delta Y_t^{m,n})|\Delta Z_t^{m,n}|^2 dt, \end{aligned}$$

where we have used the inequality (3.19).

In the following, we expect $\Delta Z^{m,n}$ and ΔZ^n to be closed. For this purpose, define

$$\Phi(u) = \frac{1-\alpha}{8C_0^2}\Big[\exp(\frac{4C_0}{1-\alpha}u) - \frac{4C_0}{1-\alpha}u - 1\Big].$$

It is straightforward to check that for $u \geqslant 0$, we have

$$\begin{aligned} &\Phi(u) \geqslant 0 \quad \text{with} \quad \Phi(0) = 0, \\ &\Phi'(u) \geqslant 0 \quad \text{with} \quad \Phi'(0) = 0, \\ &\frac{1-\alpha}{2}\Phi''(u) = 4C_0\Phi'(u) + 2. \end{aligned}$$

Note that $Y^m \geqslant Y^n$ when $m \geqslant n$. Then

$$\begin{aligned} d\Phi(\Delta Y_t^{m,n}) \geqslant & - C_0\Phi'(\Delta Y_t^{m,n})[1+|\Delta Z_t^{m,n}|^2+|\Delta Z_t^n|^2+|Z_t|^2]dt \\ & - \Phi'(\Delta Y_t^{m,n})[g(t,Y_t^m,Z_t^m)-g(t,Y_t^n,Z_t^n)]d\overleftarrow{B}_t + \Phi'(\Delta Y_t^{m,n})\Delta Z_t^{m,n}dW_t \\ & + [4C_0\Phi'(\Delta Y_t^{m,n})+2]|\Delta Z_t^{m,n}|^2dt - \frac{C}{2}\Phi''(\Delta Y_t^{m,n})|\Delta Y_t^{m,n}|^2dt \\ = & - C_0\Phi'(\Delta Y_t^{m,n})[1+|\Delta Z_t^n|^2+|Z_t|^2]dt - \frac{C}{2}\Phi''(\Delta Y_t^{m,n})|\Delta Y_t^{m,n}|^2dt \\ & - \Phi'(\Delta Y_t^{m,n})[g(t,Y_t^m,Z_t^m)-g(t,Y_t^n,Z_t^n)]d\overleftarrow{B}_t + \Phi'(\Delta Y_t^{m,n})\Delta Z_t^{m,n}dW_t \\ & + [3C_0\Phi'(\Delta Y_t^{m,n})+2]|\Delta Z_t^{m,n}|^2dt, \end{aligned}$$

and thus

$$\begin{aligned} & \mathbb{E}\Phi(\Delta Y_0^{m,n}) + \mathbb{E}\int_0^T [3C_0\Phi'(\Delta Y_t^{m,n})+2]|\Delta Z_t^{m,n}|^2dt \qquad (3.20) \\ \leqslant & \mathbb{E}\Big[\Phi(\Delta Y_T^{m,n}) + C_0\int_0^T \Phi'(\Delta Y_t^{m,n})[1+|\Delta Z_t^n|^2+|Z_t|^2]dt + \frac{C}{2}\int_0^T \Phi''(\Delta Y_t^{m,n})|\Delta Y_t^{m,n}|^2dt\Big]. \end{aligned}$$

Next we would like to fix n and pass to the infimum limit as $m \to \infty$. Note that there is a subsequence of $\{Z^m; m \geqslant 1\}$ which converges weakly to Z in $L^2_{\mathbb{F}}(0,T;\mathbb{R}^d)$ (still indexed by m for simplicity), the convergence of $Y^m \to Y$ is pointwise, and Y^m is uniformly bounded by M. Then Zhang [37, Problem 1.4.11 (ii)] implies that on the one hand,

$$\mathbb{E}\int_0^T [3C_0\Phi'(\Delta Y_t^n)+2]|\Delta Z_t^n|^2dt \leqslant \liminf_{m\to\infty} \mathbb{E}\int_0^T [3C_0\Phi'(\Delta Y_t^{m,n})+2]|\Delta Z_t^{m,n}|^2dt. \qquad (3.21)$$

On the other hand, again, note that the convergence of $Y^m \to Y$ is pointwise and Y^m is uniformly bounded by M, one has

$$\begin{aligned} & \liminf_{m\to\infty} \mathbb{E}\Big[\Phi(\Delta Y_T^{m,n}) + C_0\int_0^T \Phi'(\Delta Y_t^{m,n})[1+|\Delta Z_t^n|^2+|Z_t|^2]dt + \frac{C}{2}\int_0^T \Phi''(\Delta Y_t^{m,n})|\Delta Y_t^{m,n}|^2dt\Big] \\ \leqslant & \mathbb{E}\Big[\Phi(\Delta Y_T^n) + C_0\int_0^T \Phi'(\Delta Y_t^n)[1+|\Delta Z_t^n|^2+|Z_t|^2]dt + \frac{C}{2}\int_0^T \Phi''(\Delta Y_t^n)|\Delta Y_t^n|^2dt\Big]. \qquad (3.22) \end{aligned}$$

Therefore, combining (3.20), (3.21) and (3.22), we have

$$\begin{aligned} & \mathbb{E}\int_0^T [2C_0\Phi'(\Delta Y_t^n)+2]|\Delta Z_t^n|^2dt \\ \leqslant & \mathbb{E}\Big[\Phi(\Delta Y_T^n) + C_0\int_0^T \Phi'(\Delta Y_t^n)[1+|Z_t|^2]dt + \frac{C}{2}\int_0^T \Phi''(\Delta Y_t^n)|\Delta Y_t^n|^2dt\Big]. \end{aligned} \qquad (3.23)$$

Now, note that $\Phi(0) = \Phi'(0) = 0$, we have that as $n \to \infty$, the claim (3.18) follows from the Lebesgue's dominated convergence theorem.

Step 2. The uniform convergence of a subsequence of $\{Y^n; n \in \mathbb{N}\}$ to Y.

At this stage we know that for all $t \in [0,T]$, $\lim_{n\to\infty} Y_t^n = Y_t$, and the sequence $\{Z^n; n \in \mathbb{N}\}$ converges to Z in $L^2_{\mathbb{F}}(0,T;\mathbb{R}^d)$. Then from Kobylanski [20, Lemma 2.5], there exists a subsequence $\{Z^{n_j}; n_j \in \mathbb{N}\}$ of $\{Z^n; n \in \mathbb{N}\}$ such that Z^{n_j} converges almost surely to Z and such that

$$\widetilde{Z} \triangleq \sup_j |Z^{n_j}| \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^d).$$

For simplicity presentation, by otherwise choosing a subsequence, we would like to still assume that the whole sequence $\{Z^n; n \in \mathbb{N}\}$ converges almost surely to Z in $L^2_{\mathbb{F}}(0,T;\mathbb{R}^d)$, and thus we have

$$Z^n \to Z \ \text{ a.s. } dt \otimes d\mathbb{P} \quad \text{and} \quad \widetilde{Z} = \sup_n |Z^n| \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^d).$$

Recall that the sequence $\{f^n; n \in \mathbb{N}\}$ converges to f locally uniformly in (y,z), we obtain that

$$\lim_{n\to\infty} f^n(t, Y_t^n, Z_t^n) = f(t, Y_t, Z_t), \quad t \in [0,T], \ a.s.$$

In addition, due to that f^n satisfies the condition (3.17), one has

$$|f^n(t, Y_t^n, Z_t^n)| \leqslant C(1 + M + \sup_n |Z_t^n|^2) \leqslant C_0(1 + \widetilde{Z}_t^2).$$

Thus, Lebesgue's dominated convergence theorem gives

$$\lim_{n\to\infty} \int_0^T f^n(t, Y_t^n, Z_t^n)dt = \int_0^T f(t, Y_t, Z_t)dt, \quad a.s.$$

Similarly, we have

$$\lim_{n\to\infty} \mathbb{E}\int_0^T |g(t, Y_t^n, Z_t^n) - g(t, Y_t, Z_t)|^2 dt \leqslant \lim_{n\to\infty} \mathbb{E}\int_0^T [C|\Delta Y_t^n|^2 + \alpha|\Delta Z_t^n|^2]dt = 0,$$

and therefore

$$g(t, Y^n, Z^n) \to g(t, Y, Z) \ \text{ a.s. } dt \otimes d\mathbb{P} \quad \text{and} \quad \sup_n |g(t, Y^n, Z^n)| \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^d).$$

Extracting a subsequence again if necessary, we may assume that the above convergence is $\mathbb{P}$-a.s. Finally, we have

$$\begin{aligned} |\Delta Y_t^{m,n}| \leqslant |\Delta Y_T^{m,n}| &+ \int_t^T |f^m(s, Y_s^m, Z_s^m) - f^n(s, Y_s^n, Z_s^n)|ds \\ &+ \Big| \int_t^T [g(s, Y_s^m, Z_s^m) - g(s, Y_s^n, Z_s^n)] d\overleftarrow{B}_s \Big| + \Big| \int_t^T \Delta Z_s^{m,n} dW_s \Big|. \end{aligned}$$

Now fix n, and taking limits on m and supremum over $t \in [0,T]$, we obtain that for almost all $\omega \in \Omega$,

$$\sup_{0\leqslant t\leqslant T} |\Delta Y_t^n| \leqslant |\Delta Y_T^n| + \int_t^T |f(s, Y_s, Z_s) - f^n(s, Y_s^n, Z_s^n)|ds$$

$$+\sup_{0\leqslant t\leqslant T}\Big|\int_t^T [g(s,Y_s,Z_s)-g(s,Y_s^n,Z_s^n)]d\overleftarrow{B}_s\Big|+\sup_{0\leqslant t\leqslant T}\Big|\int_t^T \Delta Z_s^n dW_s\Big|,$$

from which we deduce that the sequence $\{Y^n; n\in\mathbb{N}\}$ converges to Y uniformly for $t\in[0,T]$. In particular, Y is continuous.

Finally, we could pass to the limit in

$$Y_t^n = Y_T^n + \int_t^T f^n(s,Y_s^n,Z_s^n)ds + \int_t^T g(s,Y_s^n,Z_s^n)d\overleftarrow{B}_s - \int_t^T Z_s^n dW_s, \quad 0\leqslant t\leqslant T,$$

deducing that the pair (Y,Z) is a solution of BDSDE with parameters (f,g,ξ). □

3.3 Existence

Based on the results of a priori estimate and the monotone stability, now we can prove the existence of solutions of one-dimensional BDSDEs (2.1) with quadratic growth.

Assumption 3. Suppose that the terminal value ξ belongs to $L^\infty_{\mathscr{F}_T}(\Omega;\mathbb{R})$, and there exist some positive constants a, b, C and α with $0<\alpha<1$ such that for all $t\in[0,T], y,\bar{y}\in\mathbb{R}, z,\bar{z}\in\mathbb{R}^d$,

$$|f(t,y,z)|\leqslant b+a|y|+C|z|^2, \quad |g(t,y,z)|^2\leqslant\alpha|z|^2, \quad \text{a.s.}$$
$$|g(t,y,z)-g(t,\bar{y},\bar{z})|^2\leqslant C|y-\bar{y}|^2+\alpha|z-\bar{z}|^2, \quad \text{a.s.}$$

Remark 3.10. Note that Assumption 3 implies $g(t,y,0)=0$ for all $(t,y)\in[0,T]\times\mathbb{R}$. Here we present some examples that satisfying Assumption 3:

$$\text{(i)}\quad g(t,y,z)=\sqrt{\alpha}\cos(y)\sin(z); \qquad \text{(ii)}\quad g(t,y,z)=\frac{\sqrt{\alpha}}{2}[\cos(y)\sin(z)+z];$$
$$\text{(iii)}\quad g(t,y,z)=l(z)\ \text{ with }\ |l(z)|^2\leqslant\alpha|z|^2\ \text{ and }\ |l(z)-l(\bar{z})|^2\leqslant\alpha|z-\bar{z}|^2.$$

The following theorem is the main result of this section.

Theorem 3.11 (Existence)**.** *Suppose that the parameters (f,g,ξ) satisfy Assumption 3, then BDSDE (2.1) has at least one solution $(Y,Z)\in L^\infty_{\mathbb{F}}(0,T;\mathbb{R})\times L^2_{\mathbb{F}}(0,T;\mathbb{R}^d)$.*

Proof. The main idea here we used is the *double approximation* (see Briand–Hu [7] and Morlais [22]). First, for the case of the generator f being non-negative, we construct a sequence f^n which is globally Lipschitz continuous, non-decreasing, and converges pointwise to f. Second, we can construct similarly a sequence $f^{n,m}$ for the general situation of the generator f.

Step 1. The case of non-negative generator f

In this step, we would like to assume that the generator f is non-negative for all $(t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d$. Then, we proceed by defining the sequence $\{f^n; n\geqslant r\}$ by inf-convolution, i.e.,

$$f^n(t,y,z)=\inf\Big\{f(t,p,q)+n|p-y|+n|q-z| : (p,q)\in\mathbb{Q}^{1+d}\Big\}, \tag{3.24}$$

where r is an integer. It is easy to see that each f^n is well defined and it is globally Lipschitz continuous in y and z with the constant n. In addition, the sequence $\{f^n; n \geqslant r\}$ is non-decreasing and converges pointwise to the non-negative generator f. In addition, Dini's theorem implies that the convergence is uniformly in (y, z) in compact sets.

Then, the classical results of the existence and uniqueness of BDSDEs (see Pardoux–Peng [27, Theorem 1.1]) and the comparison theorem (see Shi et al. [29, Theorem 3.1]) for Lipschitz continuous coefficients give us that, for every n, there exists a unique solution (Y^n, Z^n) of BDSDE (f^n, g, ξ) such that Y^n is non-decreasing, i.e.,

$$Y^n \leqslant Y^{n+1}. \tag{3.25}$$

In addition, the sequence $\{Y^n; n \geqslant r\}$ is uniformly bounded. In fact, note that f^n is non-negative and

$$f^n(t, y, z) \leqslant f(t, y, z) \leqslant b + a|y| + C|z|^2, \tag{3.26}$$

which implies that f^n satisfies Assumption 2. Hence from Proposition 3.7, the unique solution (Y^n, Z^n) of BDSDE (f^n, g, ξ) is bounded by

$$Y^n \leqslant \big(\|\xi\|_\infty + bT\big)e^{aT}, \quad \text{for any} \quad n \geqslant r. \tag{3.27}$$

Now, from Proposition 3.9, we have that there exists a pair of processes $(Y, Z) \in L^\infty_{\mathbb{F}}(0, T; \mathbb{R}) \times L^2_{\mathbb{F}}(0, T; \mathbb{R}^d)$ such that, the sequence $\{Y^n; n \geqslant r\}$ converges uniformly to Y, the sequence $\{Z^n; n \geqslant r\}$ converges to Z in $L^2_{\mathbb{F}}(0, T; \mathbb{R}^d)$, and (Y, Z) is a solution of BDSDE (f, g, ξ).

Step 2. The general case

Let us explain quickly how to extend the above construction to the general case. The idea consists in using two successive approximations, for which we refer once again to Briand–Hu [8]. Denote by f^+ and f^- the non-negative part and the non-positive part of f, respectively. Define

$$\begin{aligned} f^{n,m}(t, y, z) =& \inf\Big\{f^+(t, p, q) + n|p - y| + n|q - z| : (p, q) \in \mathbb{Q}^{1+d}\Big\} \\ &- \inf\Big\{f^-(t, p, q) + m|p - y| + m|q - z| : (p, q) \in \mathbb{Q}^{1+d}\Big\}, \end{aligned} \tag{3.28}$$

which is non-decreasing with respect to n and non-increasing with respect to m. The entire proof can be rewritten by passing to the limit as n goes to ∞ (m being fixed) and then as m goes to ∞. In fact, when n goes to ∞ (m being fixed), the related solution $Y^{n,m}$ converges to Y^m, which is the solution of $f^+ - f^-_m$, where f^-_m is defined similarly by (3.24). In addition, similar to (3.27), Y^m has also a uniformly lower bound by Proposition 3.7. Then as m goes to ∞ we have that Y^m converges to Y, which is the solution of BDSDE with parameters (f, g, ξ). □

4 Comparison Theorem

As we all know that the one-dimensional circumstance allows us to establish a comparison theorem for the solutions of BDSDEs, which implies the uniqueness of solutions of BDSDEs as a by-product.

So in this section, we would like to prove a comparison theorem for the solutions of BDSDEs with quadratic growth. For $i = 1, 2$, we consider the following equation:

$$Y_t^i = \xi^i + \int_t^T f^i\left(s, Y_s^i, Z_s^i\right) ds + \int_t^T g\left(s, Y_s^i, Z_s^i\right) d\overleftarrow{B}_s - \int_t^T Z_s^i dW_s, \quad 0 \leqslant t \leqslant T. \tag{4.1}$$

Then, under Assumption 3, there exists a pair of measurable processes $(Y^i, Z^i) \in L_{\mathbb{F}}^{\infty}(0, T; \mathbb{R}) \times L_{\mathbb{F}}^2(0, T; \mathbb{R}^d)$ that satisfies the corresponding BDSDE (4.1), where $i = 1, 2$. Moreover, Proposition 3.7 implies that there exists a positive constant K such that

$$\|Y^i\|_{\infty} + \mathbb{E}\int_0^t |Z_s^i|^2 ds \leqslant K, \quad \forall t \in [0, T].$$

In order to prove the comparison theorem, we assume that the terminal value ξ is bounded, and the generator f is locally Lipschitz continuous and is of quadratic growth in Z in a strong sense.

Assumption 4. *There exist three functions l, l_ε, $k : [0, T] \to \mathbb{R}$ and three positive constants C, ε, and $\alpha \in (0, 1)$ such that for all $t \in [0, T]$, $y \in [-M, M]$ and $z \in \mathbb{R}^d$, the coefficients f and g satisfy the following conditions:*

$$\begin{aligned} &|f(t, y, z)| \leqslant l(t) + C|z|^2, \qquad |g(t, y, z)|^2 \leqslant \alpha|z|^2, \qquad a.s., \\ &\frac{\partial f}{\partial y}(t, y, z) \leqslant l_\varepsilon(t) + \varepsilon|z|^2, \qquad \left|\frac{\partial g}{\partial y}(t, y, z)\right|^2 \leqslant C, \qquad a.s., \\ &\left|\frac{\partial f}{\partial z}(t, y, z)\right| \leqslant k(t) + C|z|, \qquad \left|\frac{\partial g}{\partial z}(t, y, z)\right|^2 \leqslant \alpha. \qquad a.s. \end{aligned} \tag{4.2}$$

Similar to Kobylanski [20], following the method used by Barles–Murat [4] for PDEs, we would like to divide the proof of the comparison theorem into two steps. First, we consider the comparison theorem under the condition that the generator f satisfies a structure condition (STR). Then, we show the proof by giving a change of variable that transforms a BDSDE with the generator f satisfying Assumption 4 into a BDSDE with the generator $\widetilde{f}$ satisfying this structure condition (STR).

Definition 4.1. We say that a generator f satisfies the structure condition (STR), if there exists a constant $a > 0$ and a function $b : [0, T] \to \mathbb{R}$ such that

$$\frac{\partial f}{\partial y}(t, y, z) + a\left|\frac{\partial f}{\partial z}\right|^2 (t, y, z) \leqslant b(t), \quad (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d. \tag{4.3}$$

We have the following comparison theorem under the structure condition (STR).

Proposition 4.2. *Assume that both ξ^1 and ξ^2 are bounded, the coefficient g satisfies Assumption 4, and either f^1 or f^2 satisfies the structure condition (STR) with the constant $a > 0$ and the function $b \in L^1([0, T]; \mathbb{R})$. For $i = 1, 2$, denote by (Y^i, Z^i) a solution of BDSDE (4.1) with parameters (f^i, g, ξ^i), respectively. Moreover, suppose that for all $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$,*

$$\xi^1 \leqslant \xi^2, \quad f^1(t, y, z) \leqslant f^2(t, y, z), \quad a.s., \tag{4.4}$$

then for any $t \in [0,T]$, we have

$$Y_t^1 \leqslant Y_t^2, \quad a.s.$$

Remark 4.3. (i) It should be point out that in the condition (STR), when $a > 1/(p-1)(1-\alpha)$ and the function b is bounded, Proposition 4.2 holds true in the space $L^p_{\mathbb{F}}(0,T;\mathbb{R}) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^d)$.

(ii) Proposition 4.2 still holds true if $f^1\left(t, Y_t^2, Z_t^2\right) \leqslant f^2\left(t, Y_t^2, Z_t^2\right)$ a.s. for all $t \in [0,T]$ and the generator f^1 satisfies the condition (STR), or if either $f^1\left(t, Y_t^1, Z_t^1\right) \leqslant f^2\left(t, Y_t^1, Z_t^1\right)$ a.s. for all $t \in [0,T]$ and the generator f^2 satisfies the condition (STR).

Proof of Proposition 4.2. For simplicity of presentation, we denote that for $t \in [0,T]$,

$$\widehat{Y}_t = Y_t^1 - Y_t^2 \quad \text{and} \quad \widehat{Z}_t = Z_t^1 - Z_t^2,$$

and for any $s \in [t,T]$,

$$\widehat{f}(s) = f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2) \quad \text{and} \quad \widehat{g}(s) = g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2).$$

Then, note that $\widehat{Y}^+ = \max\{0, \widehat{Y}\}$, using Itô's formula (see Proposition 2.2) to $\left(\widehat{Y}^+\right)^p$ for $p \in \mathbb{N}$ with $p > 2$ implies that

$$\begin{aligned}(\widehat{Y}_t^+)^p = {} & p\int_t^T (\widehat{Y}_s^+)^{p-1}\widehat{f}(s)ds + p\int_t^T (\widehat{Y}_s^+)^{p-1}\widehat{g}(s)d\overleftarrow{B}_s - p\int_t^T (\widehat{Y}_s^+)^{p-1}\widehat{Z}_s dW_s \\ & + \frac{p(p-1)}{2}\int_t^T (\widehat{Y}_s^+)^{p-2}|\widehat{g}(s)|^2 ds - \frac{p(p-1)}{2}\int_t^T (\widehat{Y}_s^+)^{p-2}|\widehat{Z}_s|^2 ds.\end{aligned} \tag{4.5}$$

For the term $\widehat{f}$ in the above, note (4.4), we have that

$$\begin{aligned}\widehat{f}(s) &= f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2) \\ &= f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^1, Z_s^1) + f^2(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2) \\ &\leqslant f^2(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2) \\ &\leqslant \Big(\int_0^1 \frac{\partial f^2}{\partial y}(*)d\lambda\Big)\widehat{Y}_s + \Big(\int_0^1 \frac{\partial f^2}{\partial z}(*)d\lambda\Big)\widehat{Z}_s,\end{aligned}$$

where

$$(*) = (s, \lambda Y_s^1 + (1-\lambda)Y_s^2, \lambda Z_s^1 + (1-\lambda)Z_s^2).$$

Then, for the first term in the right hand side of (4.5), we have that

$$(\widehat{Y}_s^+)^{p-1}\widehat{f}(s) \leqslant (\widehat{Y}_s^+)^p \int_0^1 \left(\frac{\partial f^2}{\partial y} + a\left|\frac{\partial f^2}{\partial z}\right|^2\right)(*)d\lambda + \frac{1}{4a}(\widehat{Y}_s^+)^{p-2}|\widehat{Z}_s|^2 \mathbf{1}_{\{\widehat{Y}^+ \geqslant 0\}}, \tag{4.6}$$

where we have used the well-known inequality $2\beta\gamma \leqslant |\beta|^2 + |\gamma|^2$ with

$$\beta = \sqrt{2a}\frac{\partial f^2}{\partial z}(*)\left(\widehat{Y}_s^+\right)^{p/2} \quad \text{and} \quad \gamma = \frac{1}{\sqrt{2a}}\left(Y_s^+\right)^{(p-2)/2}\widehat{Z}_s \mathbf{1}_{\{Y^+ \geqslant 0\}}.$$

For the term $\widehat{g}$ in (4.5), note Assumption 4 satisfied by the coefficient g, we have that

$$\begin{aligned}
|\widehat{g}(s)|^2 &= \Big\{\Big(\int_0^1 \frac{\partial g}{\partial y}(*)d\lambda\Big)\widehat{Y}_s + \Big(\int_0^1 \frac{\partial g}{\partial z}(*)d\lambda\Big)\widehat{Z}_s\Big\}^2 \\
&\leqslant \Big(\int_0^1 \frac{\partial g}{\partial y}(*)d\lambda\Big)^2|\widehat{Y}_s|^2 + 2\frac{\sqrt{\varepsilon}}{\sqrt{\varepsilon}}\Big(\int_0^1 \frac{\partial g}{\partial y}(*)d\lambda\Big)\Big(\int_0^1 \frac{\partial g}{\partial z}(*)d\lambda\Big)|\widehat{Y}_s\widehat{Z}_s| + \Big(\int_0^1 \frac{\partial g}{\partial z}(*)d\lambda\Big)^2|\widehat{Z}_s|^2 \\
&\leqslant \widetilde{C}|\widehat{Y}_s|^2 + (\alpha+\varepsilon)|\widehat{Z}_s|^2,
\end{aligned}$$

where

$$\widetilde{C} = C + \frac{C\alpha}{\varepsilon} \quad \text{and} \quad \varepsilon = \frac{1-\alpha}{2}.$$

So, for the fourth term on the right hand side of (4.5), we have that

$$(\widehat{Y}_s^+)^{p-2}|\widehat{g}(s)|^2 \leqslant \widetilde{C}(\widehat{Y}_s^+)^p + \frac{1+\alpha}{2}(\widehat{Y}_s^+)^{p-2}|\widehat{Z}_s|^2\mathbf{1}_{\{\widehat{Y}^+\geqslant 0\}}. \tag{4.7}$$

Combining (4.5)-(4.7) and (STR), one has

$$\begin{aligned}
&(\widehat{Y}_t^+)^p + \frac{p}{4}\left((p-1)(1-\alpha) - \frac{1}{a}\right)\int_t^T (\widehat{Y}_s^+)^{p-2}|\widehat{Z}_s|^2\mathbf{1}_{\{\widehat{Y}^+\geqslant 0\}}ds \\
&\leqslant p\int_t^T \big[b(s) + \frac{1}{2}(p-1)\widetilde{C}\big](\widehat{Y}_s^+)^p ds + p\int_t^T (\widehat{Y}_s^+)^{p-1}\widehat{g}(s)d\overleftarrow{B}_s - p\int_t^T (\widehat{Y}_s^+)^{p-1}\widehat{Z}_s dW_s,
\end{aligned} \tag{4.8}$$

where the function b comes from (4.3). Note that $\widehat{Y}$ is bounded, which implies that $(\widehat{Y}_s^+)^{p-1}\widehat{g}(s)$ and $(\widehat{Y}_s^+)^{p-1}\widehat{Z}_s$ belong to the space $L^2_{\mathbb{F}}(0,T;\mathbb{R}^d)$. Take the expectation on both sides of (4.8),

$$\begin{aligned}
&\mathbb{E}(\widehat{Y}_t^+)^p + \frac{p}{4}\left((p-1)(1-\alpha) - \frac{1}{a}\right)\mathbb{E}\int_t^T (\widehat{Y}_s^+)^{p-2}|\widehat{Z}_s|^2\mathbf{1}_{\{\widehat{Y}^+\geqslant 0\}}ds \\
&\leqslant p\int_t^T \big[b(s) + \frac{1}{2}(p-1)\widetilde{C}\big](\widehat{Y}_s^+)^p ds.
\end{aligned}$$

It is easy to choose a large enough p such that

$$\frac{p}{4}\left((p-1)(1-\alpha) - \frac{1}{a}\right) \geqslant 0.$$

Finally, Gronwall's inequality deduces that

$$\mathbb{E}(\widehat{Y}_t^+)^p \leqslant 0, \qquad \forall t \in [0,T].$$

Therefore, for all $t \in [0,T]$ we have that

$$Y_t^1 \leqslant Y_t^2, \qquad a.s.$$

This completes the proof. □

Based on the first step concerning the structure condition (STR), next we are going to prove the comparison theorem under Assumption 4.

Theorem 4.4 (Comparison theorem)**.** *Let (ξ^1, f^1, g) and (ξ^2, f^2, g) be two parameters of BDSDE (4.1) with bounded terminal values, and suppose that*

(i) $\xi^1 \leqslant \xi^2$, *a.s., and* $f^1(t,y,z) \leqslant f^2(t,y,z)$, *a.s., for all* $(t,y,z) \in [0,T] \times [-M,M] \times \mathbb{R}^d$.

(ii) *Either* (f^1, g) *or* (f^2, g) *satisfies Assumption 4.*

Then if (Y^1, Z^1) *and* (Y^2, Z^2) *are the associated solutions of BDSDE (4.1) with parameters* (ξ^1, f^1, g) *and* (ξ^2, f^2, g), *respectively, one has that for any* $t \in [0,T]$,

$$Y_t^1 \leqslant Y_t^2, \quad a.s.$$

Remark 4.5. We point out that Theorem 4.4 still holds true if $f^1\left(t, Y_t^2, Z_t^2\right) \leqslant f^2\left(t, Y_t^2, Z_t^2\right)$ almost surely for all $t \in [0,T]$ and the generator f^1 satisfies Assumption 4. Alternatively, it holds true if either $f^1\left(t, Y_t^1, Z_t^1\right) \leqslant f^2\left(t, Y_t^1, Z_t^1\right)$ almost surely for all $t \in [0,T]$ and the generator f^2 satisfies Assumption 4.

Proof of Theorem 4.4. The main idea of the proof is to propose a change of variables that can transform a coefficient f satisfying Assumption 4 into a coefficient $\widetilde{f}$ satisfying the structure condition (STR). By utilizing Proposition 4.2, we can then achieve our desired goal.

Let $(Y,Z) \in L_{\mathbb{F}}^{\infty}(0,T;\mathbb{R}) \times L_{\mathbb{F}}^{2}(0,T;\mathbb{R}^d)$ be a solution of BDSDE (4.1) with parameters (ξ, f, g), where ξ is a bounded terminal value. Let's choose $M \in \mathbb{R}$ such that $|Y|_{\infty} < M$, and consider the change of variable $y = \phi(\tilde{y})$, where ϕ is a regular increasing function yet to be chosen. Denote

$$Y = \phi(\widetilde{Y}), \quad w(Y) = \phi'(\widetilde{Y}), \quad Z = \phi'(\widetilde{Y})\widetilde{Z} = w(Y)\widetilde{Z}, \tag{4.9}$$

then

$$\widetilde{Y} = \phi^{-1}(Y), \quad \widetilde{Z} = \frac{Z}{\phi'(\widetilde{Y})} = \frac{Z}{w(Y)},$$

where $(\widetilde{Y}, \widetilde{Z})$ is a solution of the following BDSDE with parameters $(\widetilde{\xi}, \widetilde{f}, \widetilde{g})$:

$$\widetilde{Y}_t = \widetilde{\xi} + \int_t^T \widetilde{f}(s, \widetilde{Y}_s, \widetilde{Z}_s)ds + \int_t^T \widetilde{g}(s, \widetilde{Y}_s, \widetilde{Z}_s)d\overleftarrow{B}_s - \int_t^T \widetilde{Z}_s dW_s, \tag{4.10}$$

where

$$\begin{aligned}
\widetilde{\xi} &= \phi^{-1}(\xi), \\
\widetilde{g}(t,\widetilde{y},\widetilde{z}) &= \frac{g(t,y,z)}{\phi'(\widetilde{y})} = \frac{g(t,y,z)}{w(y)} = \frac{g\big(t,\phi(\widetilde{y}),\phi'(\widetilde{y})\widetilde{z}\big)}{\phi'(\widetilde{y})}, \\
\widetilde{f}(t,\widetilde{y},\widetilde{z}) &= \frac{1}{\phi'(\widetilde{y})}\left(f\left(t,\phi(\widetilde{y}),\phi'(\widetilde{y})\widetilde{z}\right) + \frac{1}{2}\phi''(\widetilde{y})\left[\widetilde{z}^2 - \widetilde{g}(t,\widetilde{y},\widetilde{z})^2\right]\right).
\end{aligned} \tag{4.11}$$

In fact, applying Itô's formula to $\phi(\widetilde{Y})$, we get that

$$\begin{aligned}
\phi(\widetilde{Y}_t) = \phi(\widetilde{\xi}) &+ \int_t^T \left[\phi'(\widetilde{Y}_s)\widetilde{f}(s,\widetilde{Y}_s,\widetilde{Z}_s) + \frac{1}{2}\phi''(\widetilde{Y}_s)\widetilde{g}(s,\widetilde{Y}_s,\widetilde{Z}_s)^2 - \frac{1}{2}\phi''(\widetilde{Y}_s)\widetilde{Z}_s^2\right]ds \\
&+ \int_t^T \phi'(\widetilde{Y}_s)\widetilde{g}(s,\widetilde{Y}_s,\widetilde{Z}_s)d\overleftarrow{B}_s - \int_t^T \phi'(\widetilde{Y}_s)\widetilde{Z}_s dW_s,
\end{aligned}$$

which, note that $Y=\phi(\widetilde{Y})$, comparing with BDSDE (4.1) implies (4.11). In other words, BDSDEs (4.1) and (4.10) are equivalent. It is easy to check that the coefficient $\widetilde{g}$ satisfies Assumption 4. Next, we are going to verify that the coefficient $\widetilde{f}$ given by (4.11) satisfies the structure condition (STR). For this, in order to make it more clear, recalling (4.9), we have

$$y=\phi(\widetilde{y}),\qquad w(y)=\phi'(\widetilde{y})=w(\phi(\widetilde{y})),\qquad z=\phi'(\widetilde{y})\widetilde{z}=w(y)\widetilde{z}=w(\phi(\widetilde{y}))\widetilde{z}. \tag{4.12}$$

Thus

$$\begin{aligned}
&\frac{\partial y}{\partial\widetilde{y}}=w,\quad \frac{\partial z}{\partial\widetilde{y}}=w'w\widetilde{z},\quad \frac{\partial z}{\partial\widetilde{z}}=w,\\
&\phi''(\widetilde{y})=\frac{\partial w}{\partial\widetilde{y}}=\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial\widetilde{y}}=w'w,\quad \frac{\partial w'}{\partial\widetilde{y}}=\frac{\partial w'}{\partial y}\cdot\frac{\partial y}{\partial\widetilde{y}}=w''w,\\
&\frac{\partial\widetilde{g}}{\partial\widetilde{y}}(t,\widetilde{y},\widetilde{z})=\frac{\partial\widetilde{g}}{\partial y}(t,\widetilde{y},\widetilde{z})\cdot\frac{\partial y}{\partial\widetilde{y}}=\frac{\partial\big(g(t,y,z)/w\big)}{\partial y}\cdot\frac{\partial y}{\partial\widetilde{y}}=\frac{\partial g}{\partial y}(t,y,z)-\frac{w'}{w}g(t,y,z),\\
&\frac{\partial\widetilde{g}}{\partial\widetilde{z}}(t,\widetilde{y},\widetilde{z})=\frac{\partial\widetilde{g}}{\partial z}(t,\widetilde{y},\widetilde{z})\cdot\frac{\partial z}{\partial\widetilde{z}}=\frac{\partial\big(g(t,y,z)/w\big)}{\partial z}\cdot\frac{\partial z}{\partial\widetilde{z}}=\frac{\partial g}{\partial z}(t,y,z).
\end{aligned}$$

Then, from (4.11), we could compute that

$$\begin{aligned}
\frac{\partial\widetilde{f}}{\partial\widetilde{y}}(t,\widetilde{y},\widetilde{z})&=-\frac{w'}{w}\left(f\,(t,y,z)+\frac{1}{2}w'w[\widetilde{z}^2-\widetilde{g}(t,\widetilde{y},\widetilde{z})^2]\right)\\
&\quad+\frac{1}{w}\bigg(\frac{\partial f}{\partial y}(t,y,z)w+\frac{\partial f}{\partial z}(t,y,z)w'w\widetilde{z}+\frac{1}{2}[w''w^2+(w')^2w]\cdot[\widetilde{z}^2-\widetilde{g}(t,\widetilde{y},\widetilde{z})^2]\\
&\quad+w'w\widetilde{g}(t,\widetilde{y},\widetilde{z})\Big[\frac{w'}{w}g(t,y,z)-\frac{\partial g}{\partial y}(t,y,z)\Big]\bigg)\\
&=-\frac{w'}{w}\left(f\,(t,y,z)+\frac{1}{2}\frac{w'}{w}[z^2-g(t,y,z)^2]\right)\\
&\quad+\frac{\partial f}{\partial y}(t,y,z)+\frac{w'}{w}\frac{\partial f}{\partial z}(t,y,z)z+\frac{1}{2}\Big[\frac{w''}{w}+\Big(\frac{w'}{w}\Big)^2\Big]\cdot[z^2-g(t,y,z)^2]\\
&\quad+\Big(\frac{w'}{w}\Big)^2g(t,y,z)^2-\frac{w'}{w}g(t,y,z)\cdot\frac{\partial g}{\partial y}(t,y,z)\\
&=\frac{1}{2}\frac{w''}{w}z^2+\frac{w'}{w}\left(\frac{\partial f}{\partial z}(t,y,z)z-f(t,y,z)\right)+\frac{\partial f}{\partial y}(t,y,z)\\
&\quad+\Big[\Big(\frac{w'}{w}\Big)^2-\frac{1}{2}\frac{w''}{w}\Big]g(t,y,z)^2-\frac{w'}{w}g(t,y,z)\cdot\frac{\partial g}{\partial y}(t,y,z),\\
\frac{\partial\widetilde{f}}{\partial\widetilde{z}}(t,\widetilde{y},\widetilde{z})&=\frac{\partial f}{\partial z}(t,y,z)+\frac{w'}{w}\Big[z-g(t,y,z)\cdot\frac{\partial g}{\partial z}(t,y,z)\Big].
\end{aligned}$$

We now show that a good choice of ϕ allows $\widetilde{f}$ to satisfy the structure condition (STR). Indeed, if ϕ is such that $w>0$, $w'>0$, and $w''<0$, note that (4.2) with $0<\alpha^2<\alpha<1$, then

$$\left(\frac{\partial\widetilde{f}}{\partial\widetilde{y}}+a\left|\frac{\partial\widetilde{f}}{\partial\widetilde{z}}\right|^2\right)(t,\widetilde{y},\widetilde{z})$$

$$
\begin{aligned}
&= \frac{1}{2}\frac{w''}{w}z^2 + \frac{w'}{w}\Big(\frac{\partial f}{\partial z}z - f\Big) + \Big[\Big(\frac{w'}{w}\Big)^2 - \frac{1}{2}\frac{w''}{w}\Big]g^2 - \frac{w'}{w}g\cdot\frac{\partial g}{\partial y} \\
&\quad + \frac{\partial f}{\partial y} + a\left|\frac{\partial f}{\partial z} + \frac{w'}{w}\Big[z - g\cdot\frac{\partial g}{\partial z}\Big]\right|^2 \\
&\leqslant \Big[\alpha\Big(\frac{w'}{w}\Big)^2 + \frac{1-\alpha}{2}\frac{w''}{w}\Big]|z|^2 + \frac{w'}{w}\Big(k(t)|z| + l(t) + 2C|z|^2\Big) + \Big[\Big(\frac{w'}{w}\Big)^2 - \frac{1}{2}\frac{w''}{w}\Big]l(t) \\
&\quad + \frac{w'}{w}C\alpha|z| + l_\varepsilon(t) + \varepsilon|z|^2 + a\left(k(t) + \Big(C + \frac{w'}{w}(1+\alpha)\Big)|z|\right)^2 \\
&\leqslant |z|^2\left[\alpha\Big(\frac{w'}{w}\Big)^2 + \frac{1-\alpha}{2}\frac{w''}{w} + \frac{w'}{w}2C + \varepsilon + a\Big(C + \frac{w'}{w}(1+\alpha)\Big)^2\right] \\
&\quad + |z|\left[\frac{w'}{w}\Big[k(t) + C\alpha\Big] + 2ak(t)\Big(C + \frac{w'}{w}(1+\alpha)\Big)\right] \\
&\quad + \Big[\Big(\frac{w'}{w}\Big)^2 - \frac{1}{2}\frac{w''}{w}\Big]l(t) + \frac{w'}{w}l(t) + l_\varepsilon(t) + ak(t)^2 \\
&\leqslant |z|^2\left[\frac{1-\alpha}{2}\frac{w''}{w} + \frac{w'}{w}2C + \Big(\frac{w'}{w}\Big)^2 + \varepsilon + 2a\left(C + \frac{w'}{w}(1+\alpha)\right)^2\right] \\
&\quad + \Big[\Big(\frac{w'}{w}\Big)^2 - \frac{1}{2}\frac{w''}{w}\Big]l(t) + \frac{w'}{w}l(t) + l_\varepsilon(t) + ak(t)^2 + \frac{1}{4(1-\alpha)}\Big[k(t) + C\alpha\Big]^2 + ak(t)^2,
\end{aligned}
$$

where in the last inequality we have used the inequality $2\beta_i\gamma_i \leqslant |\beta_i|^2 + |\gamma_i|^2$ for $i = 1, 2$, with

$$
\begin{aligned}
\beta_1 &= \sqrt{1-\alpha}\frac{w'}{w}|z|, \qquad \gamma_1 = \frac{1}{2\sqrt{1-\alpha}}\Big[k(t) + C\alpha\Big]; \\
\beta_2 &= |z|\sqrt{a}\left(C + \frac{w'}{w}(1+\alpha)\right), \qquad \gamma_2 = \sqrt{a}k(t).
\end{aligned}
$$

Thus, if we find ϕ satisfying all the required assumptions and such that on $[-M, M]$,

$$
\frac{1-\alpha}{2}\frac{w''}{w} + \frac{w'}{w}2C + \Big(\frac{w'}{w}\Big)^2 < -\delta < 0,
$$

then choosing a and ε small enough, the coefficient before $|z|^2$ is non-positive for all $y \in [-M, M]$. Therefore (STR) is satisfied. In fact, by setting

$$
\phi(\tilde{y}) = \frac{1}{\lambda}\ln\left(\frac{e^{\lambda A\tilde{y}} + 1}{A}\right) - M,
$$

then, recalling (4.12), a straightforward yet tedious computation gives us that

$$
w(y) = A - \exp\{-\lambda(y+M)\}, \qquad w'(y) = \lambda\exp\{-\lambda(y+M)\}, \qquad w''(y) = -\lambda^2\exp\{-\lambda(y+M)\},
$$

which implies that $w'' < 0$ for all $y \in [-M, M]$ and $\lambda > 0$. Moreover, when $A > 1$ and $\lambda > 0$, it is easy to see that $w > 0$ and $w' > 0$ on the interval $[-M, M]$. Furthermore, one could choose a

proper A and λ such that

$$\begin{aligned}&\frac{1-\alpha}{2}\frac{w''}{w}+\frac{w'}{w}2C+\left(\frac{w'}{w}\right)^2\\&=\Big[\lambda^2\Big(-(1-\alpha)A+(3-\alpha)\exp(-\lambda(u+M))\Big)+\lambda 4C(A-\exp(-\lambda(u+M)))\Big]\\&\quad\times\frac{1}{2}\frac{\exp(-\lambda(u+M))}{(A-\exp(-\lambda(u+M)))^2}\leqslant-\delta<0.\end{aligned}$$

The proof is complete. □

5 Application to SPDE

When the generator f is of linear growth with respect to y and z, Pardoux–Peng [27] used BDSDEs to give a probabilistic representation for the classical solution of semilinear SPDEs; and Bally–Matoussi [3] and Zhang–Zhao [38] obtained the relationship between the solution of BDSDEs and the Sobolev solution of SPDEs. Then, Wu–Zhang [32] got the Sobolev solution of related SPDEs when f is continuous and locally monotone in y. Zhang–Zhao [40] and Bahlali et al. [2] used BDSDEs to prove the existence and uniqueness of related SPDEs when the generator f is of polynomial growth in Y and grows in Z super-linearly (or sub-quadratically), respectively. In this section, when the generator f is of quadratic growth in z, we use BDSDEs to give a probabilistic representation for the solutions of related semilinear SPDEs in Sobolev spaces, and use it to prove the existence and uniqueness of Sobolev solutions of the SPDEs, thus extending the nonlinear stochastic Feynman-Kac formula.

First, we recall some notations. For Euclidean spaces $\mathbb{H}$ and $\mathbb{G}$, denote by $C^k_{l,b}(\mathbb{H};\mathbb{G})$ the set of functions of class C^k from $\mathbb{H}$ to $\mathbb{G}$, whose partial derivatives of order less than or equal to k are bounded. Denote by $\mathcal{C}_c^{1,\infty}([0,T]\times\mathbb{H})$ the set of compactly supported functions $\varphi(t,x)$ which are continuously derivable in the t-variable and infinitely continuously derivable in the x-variable.

Consider the following forward-backward doubly stochastic differential equation:

$$X_s^{t,x}=x+\int_t^s b(X_r^{t,x})dr+\int_t^s\sigma(X_r^{t,x})dW_r,\quad t\leqslant s\leqslant T,\tag{5.1}$$

$$Y_s^{t,x}=h(X_T^{t,x})+\int_s^T f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})dr+\int_s^T g(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})d\overleftarrow{B}_r-\int_s^T Z_r^{t,x}dW_r,\tag{5.2}$$

where the coefficients b and σ come from $C^2_{l,b}(\mathbb{R}^n;\mathbb{R}^n)$ and $C^3_{l,b}(\mathbb{R}^n;\mathbb{R}^{n\times d})$, respectively. Then it is well known that the forward equation (5.1) admits a unique adapted solution, denoted by $\{X_s^{t,x};t\leqslant s\leqslant T\}$, which satisfying

$$\mathbb{E}\left[\sup_{t\leqslant s\leqslant T}|X_s^{t,x}|^p\right]<\infty,\quad\forall p>1.$$

Now, we would like to connect the forward-backward system (5.1) and (5.2) with quadratic growth to the following semilinear stochastic partial differential equation:

$$\begin{aligned} u(s,x) =& h(x) + \int_s^T \Big\{ \mathcal{L}u(r,x) + f\big(r,x,u(r,x),(\sigma^\top \nabla u)(r,x)\big) \Big\} dr \\ &+ \int_s^T g\big(r,x,u(r,x),(\sigma^\top \nabla u)(r,x)\big) d\overleftarrow{B}_r, \quad t \leqslant s \leqslant T, \end{aligned} \tag{5.3}$$

where $\sigma^\top$ denotes the transpose of σ, and

$$\mathcal{L} = \sum_{i=1}^n b_i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}, \qquad (a_{ij}) = \sigma\sigma^\top. \tag{5.4}$$

Before introducing the definition of Sobolev solutions of SPDE (5.3), we let $\rho : \mathbb{R}^n \to \mathbb{R}^+$ be an integrable continuous non-negative function, and $L^2(\mathbb{R}^n; \rho^{-1}(x)\mathrm{d}x)$ be the weighted L^2 space endowed with the norm

$$\|u\|_\rho^2 \triangleq \int_{\mathbb{R}^n} |u(x)|^2 \rho^{-1}(x) dx.$$

Let us take the weight $\rho(x) = \exp\{F(x)\}$, where $F : \mathbb{R}^n \to \mathbb{R}$ is a continuous function and there is a positive constant $R > 0$ such that $F \in C^2_{l,b}(\mathbb{R}^n, \mathbb{R})$ when $|x| > R$. For example, one can take $\rho(x) = \exp\{\delta|x|\}$ with $\delta \in \mathbb{R}^+$ or $\rho(x) = (1+|x|)^q$ with $q > n+2$.

Let $\mathscr{H}$ be the set of random fields $\{u(t,x); 0 \leqslant t \leqslant T, x \in \mathbb{R}^n\}$ such that $u(t,x)$ is $\mathscr{F}^B_{t,T}$-measurable, and both u and $\sigma^\top \nabla u$ belong to $L^2\big(\Omega \times (0,T) \times \mathbb{R}^n; \mathrm{d}\mathbb{P} \otimes \mathrm{d}t \otimes \rho^{-1}(x)\mathrm{d}x\big)$. Then $\mathscr{H}$ is a Banach space endowed with the following norm:

$$\|u\|_{\mathscr{H}}^2 \triangleq \mathbb{E}\left[\int_{\mathbb{R}^n} \int_0^T \left(|u(t,x)|^2 + \Big|(\sigma^\top \nabla u)(t,x)\Big|^2 \right) dt \rho^{-1}(x) dx \right].$$

Now we present the definition of Sobolev solution of SPDE (5.3).

Definition 5.1. We say that u is a Sobolev solution of SPDE (5.3), if $u \in \mathscr{H}$ and for any $\varphi \in \mathcal{C}_c^{1,\infty}([0,T] \times \mathbb{R}^n)$,

$$\begin{aligned} &\int_{\mathbb{R}^n} \int_t^T u(s,x) \partial_s \varphi(s,x) ds dx + \int_{\mathbb{R}^n} u(t,x)\varphi(t,x) dx - \int_{\mathbb{R}^n} h(x)\varphi(T,x) dx \\ &\quad - \frac{1}{2} \int_{\mathbb{R}^n} \int_t^T (\sigma^\top \nabla u)(s,x) \cdot (\sigma^\top \nabla \varphi)(s,x) ds dx - \int_{\mathbb{R}^n} \int_t^T u \mathrm{div}[(b - \widetilde{A})\varphi](s,x) ds dx \\ &= \int_{\mathbb{R}^n} \int_t^T f\big(s,x,u(s,x),(\sigma^\top \nabla u)(s,x)\big)\varphi(s,x) ds dx \\ &\quad + \int_{\mathbb{R}^n} \int_t^T g\big(s,x,u(s,x),(\sigma^\top \nabla u)(s,x)\big)\varphi(s,x) dB_s dx, \end{aligned} \tag{5.5}$$

where A is a n-vector whose coordinates are given by $\widetilde{A}_j = \frac{1}{2}\sum_{i=1}^n \frac{\partial a_{ij}}{\partial x_i}$ with $1 \leqslant j \leqslant n$.

It should be pointed out that, for the classical solution, indeed, if one supposes that u is a solution of SPDE (5.3) of class C^2, then, similar to the method of Pardoux–Peng [27], applying Proposition 2.2 to $u(s, X_s^{t,x})$ implies that the pair of processes $\{Y_s^{t,x}, Z_s^{t,x}; t \leqslant s \leqslant T\}$ defined by

$$Y_s^{t,x} = u(s, X_s^{t,x}) \quad \text{and} \quad Z_s^{t,x} = (\sigma^\top \nabla u)(s, X_s^{t,x})$$

is a solution of the backward doubly stochastic differential equation (5.2).

In the following, we discuss this relationship between the Sobolev solution of SPDEs and the solution of BDSDEs with quadratic growth. In particular, we first focus on a simple situation to better reflect the idea of solving the problem.

5.1 Simple situation

Consider the following type of backward doubly stochastic differential equations: for $t \leqslant s \leqslant T$,

$$Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T \left[f(r, X_r^{t,x}, Y_r^{t,x}) + C(Z_r^{t,x})^2\right] dr + \int_s^T \alpha Z_r^{t,x} d\overleftarrow{B}_r - \int_s^T Z_r^{t,x} dW_r, \tag{5.6}$$

where $X^{t,x}$ is the solution of (5.1), $C > 0$ and $\alpha \in (-1, 1)$ are two constants, and the functions $f : [0, T] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$ satisfy the following assumption.

Assumption 5. The function h is bounded, and there exist bounded functions $l : [0, T] \to \mathbb{R}^+$ and $k : [0, T] \to \mathbb{R}^+$ such that for all $t \in [0, T]$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$, the generator f satisfies the following condition:

$$|f(t, x, y)| \leqslant l(t), \quad |\frac{\partial f}{\partial y}(t, x, y)| \leqslant k(t), \quad \text{a.s.}$$

Proposition 5.2. *Under Assumption 5, the following stochastic partial differential equation*

$$\begin{aligned} u(s, x) =& h(x) + \int_s^T \left\{\mathcal{L}u(r, x) + f\big(r, x, u(r, x)\big) + C\big((\sigma^\top \nabla u)(r, x)\big)^2\right\} dr \\ &+ \alpha \int_s^T (\sigma^\top \nabla u)(r, x) d\overleftarrow{B}_r, \quad t \leqslant s \leqslant T, \end{aligned} \tag{5.7}$$

admits a unique Sobolev solution $u \in \mathscr{H}$, and for every $t \in [0, T]$,

$$u(s, X_s^{t,x}) = Y_s^{t,x} \quad \text{and} \quad (\sigma^\top \nabla u)(s, X_s^{t,x}) = Z_s^{t,x}, \quad a.s., a.e.\ s \in [t, T],\ x \in \mathbb{R}^n,$$

where $\{(Y_s^{t,x}, Z_s^{t,x}); t \leqslant s \leqslant T\}$ is the unique solution of quadratic BDSDE (5.6).

Proof. Applying the exponential transformation of variable to $\bar{Y} = \exp\{\beta Y\}$ transforms formally BDSDE (5.6) into the following BDSDE:

$$\begin{aligned} \bar{Y}_s^{t,x} = \bar{Y}_T^{t,x} &+ \int_s^T \beta \bar{Y}_r^{t,x} f(r, X_r^{t,x}, Y_r^{t,x}) dr + \Big[C - \frac{1 - \alpha^2}{2}\beta\Big] \int_s^T \beta \bar{Y}_r^{t,x} |Z_r^{t,x}|^2 dr \\ &+ \alpha \int_s^T \beta \bar{Y}_r^{t,x} Z_r^{t,x} d\overleftarrow{B}_r - \int_s^T \beta \bar{Y}_r^{t,x} Z_r^{t,x} dW_r, \quad t \leqslant s \leqslant T. \end{aligned}$$

By taking $\beta = \frac{2C}{1-\alpha^2}$, the above equation becomes

$$\bar{Y}_s^{t,x} = \bar{h}(X_T^{t,x}) + \int_s^T \bar{f}(r, X_r^{t,x}, \bar{Y}_r^{t,x})dr + \alpha \int_s^T \bar{Z}_r^{t,x} d\overleftarrow{B}_r - \int_s^T \bar{Z}_r^{t,x} dW_r, \quad t \leqslant s \leqslant T, \tag{5.8}$$

where

$$\begin{aligned} &\bar{Z}_r^{t,x} = \beta \bar{Y}^{t,x} Z_r^{t,x}, \quad \bar{h}(X_T^{t,x}) = \exp\{\beta h(X_T^{t,x})\}, \\ &\bar{f}(r, X_r^{t,x}, \bar{Y}_r^{t,x}) = \beta \bar{Y}_r^{t,x} f\Big(r, X_r^{t,x}, \frac{1}{\beta} \ln \bar{Y}_r^{t,x}\Big). \end{aligned} \tag{5.9}$$

Under Assumption 5, Eq. (5.8) is a classical BDSDE with a globally Lipschitz generator. Then Bally–Matoussi [3] (see also Zhang–Zhao [39] and Wu–Zhang [32]) implies that the following SPDE

$$\bar{u}(s,x) = \bar{h}(x) + \int_s^T \big\{\mathcal{L}\bar{u}(r,x) + \bar{f}\big(r, x, \bar{u}(r,x)\big)\big\} dr + \alpha \int_s^T (\sigma^\top \nabla \bar{u})(r,x) d\overleftarrow{B}_r, \quad t \leqslant s \leqslant T, \tag{5.10}$$

admits a unique Sobolev solution $\bar{u} \in \mathscr{H}$, and for every $t \in [0,T]$,

$$\bar{u}(s, X_s^{t,x}) = \bar{Y}_s^{t,x} \quad \text{and} \quad \sigma^\top \nabla \bar{u}(s, X_s^{t,x}) = \bar{Z}_s^{t,x}, \quad a.s., a.e.\ s \in [t,T],\ x \in \mathbb{R}^n.$$

On the other hand, we see that

$$\bar{Y}_s^{t,x} = \exp(\beta Y_s^{t,x}) \quad \text{and} \quad \bar{Z}_s^{t,x} = \beta \bar{Y}_s^{t,x} Z_s^{t,x}, \quad t \leqslant s \leqslant T,$$

where the pair $(Y^{t,x}, Z^{t,x}) \in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^d)$ is the solution of BDSDE (5.6). So we would like to define

$$u(s, X_s^{t,x}) \triangleq \frac{\ln \bar{u}(s, X_s^{t,x})}{\beta} = \frac{\ln \bar{Y}_s^{t,x}}{\beta} = Y_s^{t,x}, \quad t \leqslant s \leqslant T, \tag{5.11}$$

and then one can compute that

$$(\sigma^\top \nabla u)(s, X_s^{t,x}) = \frac{(\sigma^\top \nabla \bar{u})(s, X_s^{t,x})}{\beta \bar{u}(s, X_s^{t,x})} = \frac{\bar{Z}_s^{t,x}}{\beta \bar{Y}_s^{t,x}} = Z_s^{t,x}, \quad t \leqslant s \leqslant T. \tag{5.12}$$

Hence, by applying Itô's formula (see Proposition 2.2) to $u = \ln(\bar{u}/\beta)$, one can formally transform SPDE (5.10) into SPDE (5.7). In fact, by using Itô's formula to $u = \ln \bar{u}/\beta$, one has that

$$\begin{aligned} du(s,x) = {} & \frac{-1}{\beta \bar{u}(s,x)} \big\{\mathcal{L}\bar{u}(s,x) + \bar{f}\big(s, x, \bar{u}(s,x)\big)\big\} ds \\ & - \frac{\alpha (\sigma^\top \nabla \bar{u})(s,x)}{\beta \bar{u}(s,x)} d\overleftarrow{B}_s + \frac{\alpha^2}{2} \frac{\big((\sigma^\top \nabla \bar{u})(s,x)\big)^2}{\beta \bar{u}(s,x)^2} ds. \end{aligned} \tag{5.13}$$

From the definitions (5.4) and (5.11), it is easy to compute that

$$\mathcal{L}\bar{u}(s,x) = \beta \bar{u}(s,x) \Big[\mathcal{L}u(s,x) + \frac{\beta}{2}\big((\sigma^\top \nabla u)(s,x)\big)^2\Big]. \tag{5.14}$$

Then, combining (5.9), (5.11), (5.12), (5.13), (5.14), and note that $\beta = \frac{2C}{1-\alpha^2}$, we deduce that

$$\begin{cases} du(s,x) = -\left\{\mathcal{L}u(s,x) + f\big(s,x,u(s,x)\big) + C\big((\sigma^\top \nabla u)(s,x)\big)^2\right\} ds \\ \qquad\qquad - \alpha(\sigma^\top \nabla u)(s,x) d\overleftarrow{B}_s, \quad t \leqslant s \leqslant T, \\ u(T,x) = h(x). \end{cases} \tag{5.15}$$

In other words, u is a Sobolev solution of SPDE (5.7), and for every $t \in [0,T]$,

$$u(s, X_s^{t,x}) = Y_s^{t,x} \quad \text{and} \quad (\sigma^\top \nabla u)(s, X_s^{t,x}) = Z_s^{t,x}, \quad a.s., a.e.\ s \in [t,T],\ x \in \mathbb{R}^n,$$

where $(Y^{t,x}, Z^{t,x})$ is the unique solution of BDSDE (5.6).

Finally, the uniqueness of solutions of SPDE (5.7) comes from the uniqueness of BDSDE (5.6). This completes the proof. □

Remark 5.3. The concept of the above proof is first to transform a simple quadratic BDSDE into a classical BDSDE. Then, by leveraging the relationship between classical BDSDEs and SPDEs, one can obtain the corresponding relationship between the BDSDE with quadratic growth and its associated SPDE.

5.2 General situation

In this subsection, we utilize BDSDE (5.2) to provide a probabilistic representation for the solutions of SPDE (5.3) in the Sobolev space. This representation is then used to establish the existence and uniqueness of Sobolev solutions for SPDE (5.3), resulting in the derivation of the nonlinear stochastic Feynman-Kac formula within this framework. The following lemma extends the equivalence of norm principle given in Bally–Matoussi [3] and plays a crucial role in subsequent analysis.

Lemma 5.4. There exist two positive constants k_1 and K_1 which depend on T, ρ, b and σ, such that for any $t \leqslant s \leqslant T$ and $\phi \in L^1\left(\Omega \times \mathbb{R}^n;\ \mathrm{d}\mathbb{P} \otimes \rho^{-1}(x)\mathrm{d}x\right)$ which is independent of $\mathscr{F}_{t,s}^W$,

$$k_1 \mathbb{E}\int_{\mathbb{R}^d} |\phi(x)|\rho^{-1}(x)\mathrm{d}x \leqslant \mathbb{E}\int_{\mathbb{R}^d} \left|\phi\left(X_s^{t,x}\right)\right| \rho^{-1}(x)\mathrm{d}x \leqslant K_1 \mathbb{E}\int_{\mathbb{R}^d} |\phi(x)|\rho^{-1}(x)\mathrm{d}x.$$

Moreover, for any $\Phi \in L^1\left(\Omega \times [0,T] \times \mathbb{R}^d;\ \mathrm{d}\mathbb{P} \otimes \mathrm{d}t \otimes \rho^{-1}(x)\mathrm{d}x\right)$ such that $\Phi(s,\cdot)$ is independent of $\mathscr{F}_{t,s}^W$, one has

$$\begin{aligned} k_1 \mathbb{E}\int_{\mathbb{R}^d}\int_t^T |\Phi(s,x)|\mathrm{d}s\rho^{-1}(x)\mathrm{d}x &\leqslant \mathbb{E}\int_{\mathbb{R}^d}\int_t^T \left|\Phi\left(s, X_s^{t,x}\right)\right| \mathrm{d}s\rho^{-1}(x)\mathrm{d}x \\ &\leqslant K_1 \mathbb{E}\int_{\mathbb{R}^d}\int_t^T |\Phi(s,x)|\mathrm{d}s\rho^{-1}(x)\mathrm{d}x. \end{aligned}$$

For the coefficients of BDSDE (5.2), we present the following assumption.

Assumption 6. For all $t \in [0,T]$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}$ and $z \in \mathbb{R}^d$, the function $h : \mathbb{R}^n \to \mathbb{R}$ is bounded, and the functions $f : [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and $g : [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^l$ satisfy Assumption 4.

Remark 5.5. Note that under Assumption 6, BDSDE (5.2) has a unique solution $(Y^{t,x}, Z^{t,x}) \in L^{\infty}_{\mathbb{F}}(0,T;\mathbb{R}) \times L^{2}_{\mathbb{F}}(0,T;\mathbb{R}^d)$. Moreover, the coefficients f, g and h satisfy that

$$\int_{\mathbb{R}^n} \int_0^T \Big(|h(x)|^2 + |f(t,x,0,0)|^2 + |g(t,x,0,0)|^2 \Big) dt \rho^{-1}(x) dx < \infty.$$

Now we are in a position to present the main result of this section.

Theorem 5.6 (Feynman-Kac formula)**.** *Under Assumption 6, SPDE (5.3) admits a unique Sobolev solution $u \in \mathscr{H}$, and for every $t \in [0,T]$,*

$$u(s, X_s^{t,x}) = Y_s^{t,x} \quad \text{and} \quad (\sigma^{\top} \nabla u)(s, X_s^{t,x}) = Z_s^{t,x}, \quad a.s., a.e.\ s \in [t,T],\ x \in \mathbb{R}^n,$$

where $\{(Y_s^{t,x}, Z_s^{t,x}); t \leqslant s \leqslant T\}$ is the unique solution of BDSDE (5.2) with quadratic growth.

Proof. Uniqueness. The uniqueness of SPDE (5.3) follows from the uniqueness of BDSDE (5.2). In fact, if u^1 and u^2 are two Sobolev solutions of SPDE (5.3), then both the following two pairs

$$\big(u^1(s, X_s^{t,x}), (\sigma^{\top} \nabla u^1)(s, X_s^{t,x})\big) \quad \text{and} \quad \big(u^2(s, X_s^{t,x}), (\sigma^{\top} \nabla u^2)(s, X_s^{t,x})\big)$$

solve BDSDE (5.2). So the uniqueness of BDSDE (5.2) gives us that

$$u^1(s, X_s^{t,x}) = u^2(s, X_s^{t,x}), \quad a.s., a.e.\ s \in [t,T],\ x \in \mathbb{R}^n,$$

and in particular

$$u^1(t,x) = u^2(t,x), \quad a.s., a.e.,\ x \in \mathbb{R}^n.$$

Next, we prove the existence. Compared with the proof of the simple situation (see Proposition 5.2), the main idea here we used comes from the proof of Theorem 3.11. However, the techniques here are more complex than the proof of Theorem 3.11. For this, first we still assume that the generator f is non-negative for all $(t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$, and for which the proof will be divided into the following steps.

Step 1. Approximation of SPDE.

Let the sequence $\{f^n; n \in \mathbb{N}\}$ be defined as in (3.24). Then, the same arguments we developed as in the proof of Theorem 3.11 deduce that f^n is globally Lipschitz continuous with the constant n and non-decreasing converges pointwise to the generator f. Now, for each $(t,x) \in [0,T] \times \mathbb{R}^n$ and $n \in \mathbb{N}$, we define the processes u^n and v^n by

$$u^n(t,x) = Y_t^{t,x,n} \quad \text{and} \quad v^n(t,x) = Z_t^{t,x,n}, \tag{5.16}$$

where $(Y^{t,x,n}, Z^{t,x,n})$ is the unique solution of BDSDE (5.2) with parameters (f^n, g, h). Then Bally–Matoussi [3] (see also Zhang–Zhao [39] and Wu–Zhang [32]) deduce that

$$v^n(s,x) = (\sigma^{\top} \nabla u^n)(s,x), \tag{5.17}$$

and u^n is the unique Sobolev solution to the following SPDE:

$$\begin{aligned} u^n(s,x) =& h(x) + \int_s^T \Big\{ \mathcal{L}u^n(r,x) + f^n\big(r,x,u^n(r,x),(\sigma^\top \nabla u^n)(r,x)\big) \Big\} dr \\ &+ \int_s^T g\big(r,x,u^n(r,x),(\sigma^\top \nabla u^n)(r,x)\big) d\overleftarrow{B}_r, \quad s \in [t,T], \end{aligned} \tag{5.18}$$

with

$$u^n\left(s, X_s^{t,x}\right) = Y_s^{t,x,n} \quad \text{and} \quad \left(\sigma^\top \nabla u^n\right)\left(s, X_s^{t,x}\right) = Z_s^{t,x,n}, \quad a.s., a.e.\ s \in [t,T],\ x \in \mathbb{R}^n. \tag{5.19}$$

On the other hand, similar to the discussion of (3.28) and (3.27), the sequence $\{Y^{t,x,n}; n \in \mathbb{N}\}$ is non-decreasing and uniformly bounded. In addition, by using the same arguments we developed in the proofs of Proposition 3.7 and Proposition 3.9, there exists a positive constant K such that

$$\sup_{n\in\mathbb{N}} \mathbb{E} \int_t^T |Z_s^{t,x,n}|^2 ds \leqslant K, \tag{5.20}$$

and

$$\lim_{n\to\infty} Y_s^{t,x,n} = Y_s^{t,x}, \ \text{ a.s. } s \in [t,T]; \qquad \lim_{n\to\infty} \mathbb{E} \int_t^T |Z_s^{t,x,n} - Z_s^{t,x}|^2 ds = 0, \tag{5.21}$$

where $(Y^{t,x}, Z^{t,x}) \in L^\infty_{\mathbb{F}}(0,T;\mathbb{R}) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^d)$ is the solution of BDSDE (5.2) with (f,g,h).

Step 2. Convergence of SPDE (5.18)

We point out that the limits which we consider below hold along with a subsequence, however for simplicity, the subsequence will still be indexed by n. From Lemma 5.4, note (5.17), (5.19), (5.20), and $\{Y^{t,x,n}; n \in \mathbb{N}\}$ is uniformly bounded, we have that

$$\begin{aligned} &\mathbb{E} \int_{\mathbb{R}^n} \int_t^T \left(|u^n(s,x)|^2 + |v^n(s,x)|^2 \right) ds \rho^{-1}(x) dx \\ &\leqslant K_1 \mathbb{E} \int_{\mathbb{R}^n} \Big(T \|Y^{t,x,n}\|^2_{L^\infty_{\mathbb{F}}(t,T)} + \int_t^T |Z_s^{t,x,n}|^2 ds \Big) \rho^{-1}(x) dx \\ &\leqslant K_1 \int_{\mathbb{R}^n} \big[T \exp(2\beta \|h\|_\infty) + K \big] \rho^{-1}(x) dx < \infty. \end{aligned} \tag{5.22}$$

Moreover, note that f^n satisfies the inequality (3.26) for any $n \in \mathbb{N}$, and

$$|g\big(s,x,u^n(s,x),v^n(s,x)\big)|^2 \leqslant \alpha |v^n(s,x)|^2, \quad a.s., a.e.\ s \in [t,T],\ x \in \mathbb{R}^n.$$

Then we deduce from (5.22) that

$$\mathbb{E} \int_{\mathbb{R}^n} \int_t^T \Big[|f^n\big(s,x,u^n(s,x),v^n(s,x)\big)| + |g\big(s,x,u^n(s,x),v^n(s,x)\big)|^2 \Big] ds \rho^{-1}(x) dx \leqslant \widetilde{K},$$

where $\widetilde{K}$ is some constant depending on $\|h\|_\infty$, K, K_1 and T. Now, from Lemma 5.4, (5.16), (5.21) and Lebesgue dominated convergence theorem, we have that

$$\lim_{n,m}\int_{\mathbb{R}^n}\int_t^T |u^n(s,x)-u^m(s,x)|^2 ds\rho^{-1}(x)dx = 0,$$
$$\lim_{n,m}\int_{\mathbb{R}^n}\int_t^T |(\sigma^\top\nabla u^n)(s,x)-(\sigma^\top\nabla u^m)(s,x)|^2 ds\rho^{-1}(x)dx = 0.$$

Then, using Lemma 5.4 again, and note that the space $\mathscr{H}$ is complete, it follows that there exists $u\in\mathscr{H}$ such that

$$\lim_{n}\int_{\mathbb{R}^n}\int_t^T |u^n(s,x)-u(s,x)|^2 ds\rho^{-1}(x)dx = 0, \tag{5.23}$$
$$\lim_{n}\int_{\mathbb{R}^n}\int_t^T |(\sigma^\top\nabla u^n)(s,x)-(\sigma^\top\nabla u)(s,x)|^2 ds\rho^{-1}(x)dx = 0, \tag{5.24}$$
$$\int_{\mathbb{R}^n}\int_t^T \Big(|u(s,x)|^2+(\sigma^\top\nabla u)(s,x)|^2\Big) ds\rho^{-1}(x)dx < \infty.$$

Step 3. We show that for any $t\in[0,T]$,

$$u(s,X_s^{t,x}) = Y_s^{t,x} \quad \textit{and} \quad (\sigma^\top\nabla u)(s,X_s^{t,x}) = Z_s^{t,x}, \quad a.s., a.e.\ s\in[t,T],\ x\in\mathbb{R}^n. \tag{5.25}$$

By triangular inequality, it is easy to see that

$$\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}^n}\int_t^T |u(s,X_s^{t,x})-Y_s^{t,x}|^2 ds\rho^{-1}(x)dx\\
&\leqslant \mathbb{E}\int_{\mathbb{R}^n}\int_t^T |u(s,X_s^{t,x})-u^n(s,X_s^{t,x})|^2 ds\rho^{-1}(x)dx \qquad (5.26)\\
&\quad + \mathbb{E}\int_{\mathbb{R}^n}\int_t^T |u^n(s,X_s^{t,x})-Y_s^{t,x}|^2 ds\rho^{-1}(x)dx. \qquad (5.27)
\end{aligned}$$

From Lemma 5.4 and (5.23), one has that the term (5.26) tends to 0 as n tends to ∞. In addition, Lemma 5.4, (5.19) and (5.21) deduce that the term (5.27) tends to 0 as n tends to ∞. So the first result in (5.25) holds. We next prove the second result in (5.25), i.e.,

$$(\sigma^\top\nabla u)(s,X_s^{t,x}) = Z_s^{t,x}, \quad a.s., a.e.\ s\in[t,T],\ x\in\mathbb{R}^n. \tag{5.28}$$

It is sufficient to show that the following term

$$\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}^n}\int_t^T |(\sigma^\top\nabla u)(s,X_s^{t,x})-Z_s^{t,x}|^2 ds\rho^{-1}(x)dx\\
&\leqslant \mathbb{E}\int_{\mathbb{R}^n}\int_t^T |(\sigma^\top\nabla u)(s,X_s^{t,x})-(\sigma^\top\nabla u^n)(s,X_s^{t,x})|^2 ds\rho^{-1}(x)dx\\
&\quad + \mathbb{E}\int_{\mathbb{R}^n}\int_t^T |(\sigma^\top\nabla u^n)(s,X_s^{t,x})-Z_s^{t,x}|^2 ds\rho^{-1}(x)dx\\
&\longrightarrow 0 \quad \text{as} \quad n\to\infty.
\end{aligned} \tag{5.29}$$

In fact, similarly, combining Lemma 5.4, (5.19), (5.21), and (5.24), it is easy to see that (5.29) holds, which implies that (5.28) holds too.

Step 4. We show that u is a Sobolev solution to the following SPDE:

$$\begin{aligned} u(s,x) =& h(x) + \int_s^T \Big\{ \mathcal{L}u(r,x) + f\big(r,x,u(r,x),(\sigma^\top \nabla u)(r,x)\big) \Big\} dr \\ &+ \int_s^T g\big(r,x,u(r,x),(\sigma^\top \nabla u)(r,x)\big) d\overleftarrow{B}_r, \quad s \in [t,T]. \end{aligned} \tag{5.30}$$

First, it was shown in *Step 2* that u belongs to the space $\mathscr{H}$, so by Definition 5.1, we only need to verify that u satisfies Eq. (5.5) with parameters (f,g,h). Note that for every $n \in \mathbb{N}$, since u^n is the Sobolev solution of SPDE (5.18), so for any $\varphi \in \mathcal{C}_c^{1,\infty}([0,T]\times\mathbb{R}^n)$, u^n satisfies

$$\begin{aligned} &\int_{\mathbb{R}^n}\int_t^T u^n(s,x)\partial_s\varphi(s,x)dsdx + \int_{\mathbb{R}^n} u^n(t,x)\varphi(t,x)dx - \int_{\mathbb{R}^n} h(x)\varphi(T,x)dx \\ &\quad - \frac{1}{2}\int_{\mathbb{R}^n}\int_t^T (\sigma^\top\nabla u^n)(s,x)\cdot(\sigma^\top\nabla\varphi)(s,x)dsdx - \int_{\mathbb{R}^n}\int_t^T u^n \mathrm{div}[(b-\widetilde{A})\varphi](s,x)dsdx \\ &= \int_{\mathbb{R}^n}\int_t^T f^n\big(s,x,u^n(s,x),(\sigma^\top\nabla u^n)(s,x)\big)\varphi(s,x)dsdx \\ &\quad + \int_{\mathbb{R}^n}\int_t^T g\big(s,x,u^n(s,x),(\sigma^\top\nabla u^n)(s,x)\big)\varphi(s,x)dB_s dx. \end{aligned} \tag{5.31}$$

Hence, if we can prove that along a subsequence (5.31) converges to (5.5) with parameters (f,g,h), then u satisfies (5.5) with parameters (f,g,h).

Due to that $\varphi \in \mathcal{C}_c^{1,\infty}([0,T]\times\mathbb{R}^n)$, $b \in C_{l,b}^2(\mathbb{R}^n;\mathbb{R}^n)$ and $\sigma \in C_{l,b}^3(\mathbb{R}^n;\mathbb{R}^{n\times n})$, then as $n\to\infty$, clearly the left hand side of (5.31) tends to the following term:

$$\begin{aligned} &\int_{\mathbb{R}^n}\int_t^T u(s,x)\partial_s\varphi(s,x)dsdx + \int_{\mathbb{R}^n} u(t,x)\varphi(t,x)dx - \int_{\mathbb{R}^n} h(x)\varphi(T,x)dx \\ &\quad - \frac{1}{2}\int_{\mathbb{R}^n}\int_t^T (\sigma^\top\nabla u)(s,x)\cdot(\sigma^\top\nabla\varphi)(s,x)dsdx - \int_{\mathbb{R}^n}\int_t^T u\mathrm{div}[(b-\widetilde{A})\varphi](s,x)dsdx. \end{aligned}$$

Next, we compute the limit on the right hand side of (5.31). We show that along a sequence

$$\begin{aligned} &\lim_{n\to\infty}\int_{\mathbb{R}^n}\int_t^T f^n\big(s,x,u^n(s,x),(\sigma^\top\nabla u^n)(s,x)\big)\varphi(s,x)dsdx \\ &= \int_{\mathbb{R}^n}\int_t^T f\big(s,x,u(s,x),(\sigma^\top\nabla u)(s,x)\big)\varphi(s,x)dsdx. \end{aligned} \tag{5.32}$$

In fact, note that the sequence $\{f^n; n\in\mathbb{N}\}$ is globally Lipschitz continuous and non-decreasing converges pointwise to the function f, and for every $t\in[0,T]$,

$$\big(u^n(s,X_s^{t,x}),(\sigma^\top\nabla u^n)(s,X_s^{t,x})\big) = \big(Y_s^{t,x,n},Z_s^{t,x,n}\big), \quad a.s., a.e.\ s\in[t,T],\ x\in\mathbb{R}^n,$$

$$\big(u(s,X_s^{t,x}),(\sigma^\top\nabla u)(s,X_s^{t,x})\big)=\big(Y_s^{t,x},Z_s^{t,x}\big),\quad a.s.,a.e.\ s\in[t,T],\ x\in\mathbb{R}^n,$$

with

$$\lim_{n\to\infty}Y_s^{t,x,n}=Y_s^{t,x},\quad \text{a.s. } s\in[t,T];\qquad \lim_{n\to\infty}\mathbb{E}\int_t^T|Z_s^{t,x,n}-Z_s^{t,x}|^2ds=0.$$

Then, similar to the argument as in the proof of Proposition 3.9, we get that

$$\lim_{n\to\infty}\mathbb{E}\int_t^T|f^n(s,X_s^{t,x},u^n(s,X_s^{t,x}),(\sigma^\top\nabla u^n)(s,X_s^{t,x}))-f(s,X_s^{t,x},u(s,X_s^{t,x}),(\sigma^\top\nabla u)(s,X_s^{t,x}))|ds=0.$$

So combining Lemma 5.4 and the Lebesgue dominated convergence theorem we have that

$$\lim_{n\to\infty}\mathbb{E}\int_{\mathbb{R}^n}\int_t^T|f^n(s,x,u^n(s,x),(\sigma^\top\nabla u^n)(s,x))-f(s,x,u(s,x),(\sigma^\top\nabla u)(s,x))|ds\rho^{-1}(x)dx=0,$$

which implies that (5.32) holds. Finally, for the second term on the right hand side of (5.31), similar to the previous discussion, one has that

$$\begin{aligned}\lim_{n\to\infty}\Big|&\int_{\mathbb{R}^n}\int_t^T g\big(s,x,u^n(s,x),(\sigma^\top\nabla u^n)(s,x)\big)\varphi(s,x)dB_sdx\\&-\int_{\mathbb{R}^n}\int_t^T g\big(s,x,u(s,x),(\sigma^\top\nabla u)(s,x)\big)\varphi(s,x)dB_sdx\Big|=0,\quad\text{in probability.}\end{aligned}$$

Therefore u satisfies Eq. (5.5), and thus u is a Sobolev solution to SPDE (5.30). This completes the proof of the case of the non-negative generator f.

Finally, as for the general situation of the generator f, one can also introduce a sequence $f^{n,m}$ defined as in (3.28), which is non-decreasing with respect to n and non-increasing with respect to m. The entire proof can be rewritten by passing to the limit as n goes to ∞ (m being fixed) and then as m goes to ∞. This completes the proof. □

6 Conclusion Remarks

We initiate the study of quadratic BDSDEs, where we establish the existence, uniqueness, and comparison theorem for one-dimensional BDSDEs with quadratic growth and bounded terminal value. It is worth noting that, while proving the a priori estimate, we introduce the condition (3.11) regarding the coefficient g to ensure the boundedness of Y, which we consider necessary in the case of bounded terminal value. However, if someone discusses the solution of quadratic BDSDEs with an unbounded terminal value, the condition (3.11) can be relaxed since the boundedness of Y is not required in such contexts. We intend to pursue further investigations in cases where the terminal value ξ is unbounded and/or Y is multi-dimensional. Moreover, within this framework, we employ BDSDEs to prove the existence and uniqueness of Sobolev solutions for semilinear SPDEs, thus extending the nonlinear Feynman-Kac formula proposed by Pardoux and Peng [27]. Additionally, it would be interesting and important to conduct further studies on the existence and uniqueness of viscosity solutions for related SPDEs in the near future.

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