

Supplemental Material

Direct measurement of the viscopillary lift force near a liquid interface

Appendix A: Derivation of the equations

Here, we describe the calculation of the lift force. The problem is that of a fixed sphere, immersed in a fluid with viscosity η at a distance d from an interface with a fluid of large viscosity $\eta_0 \gg \eta$. The lower fluid is in contact with a plate. We use an orthonormal cartesian frame $(\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z)$, where $\hat{\mathbf{e}}_i$ is the unit vector in the direction i , with $\hat{\mathbf{e}}_z$ normal to the interface when it is not deformed. The plate moves at velocity $Vf(\omega t)\hat{\mathbf{e}}_x$. We chose the origin of coordinates $x = 0, y = 0$ at the center of the sphere, whereas $z = 0$ corresponds to the plane containing the interface when it is not deformed. The altitude of the surface of the sphere is

$$z = h_0(x, y) = d + r^2/(2R), \quad (\text{A1})$$

with R the radius of the sphere, $r = (x^2 + y^2)^{1/2}$ and d is the minimal distance between the sphere and the (undeformed) interface. The altitude of the interface is $z = -\delta(x, y)$, so that the local distance between the sphere and the interface is $h = \delta + h_0$. In the limit $\varepsilon = d/R \ll 1$, we analyse this problem using the standard scalings of lubrication theory. We define $\ell = \sqrt{Rd}$ the characteristic length in the lateral direction, and we use rescaled coordinates

$$X = x/\ell, \quad Y = y/\ell, \quad Z = z/d, \quad (\text{A2})$$

so that the equation for the upper surface is

$$Z = 1 + \frac{1}{2}(X^2 + Y^2) \equiv H_0(X, Y), \quad (\text{A3})$$

and the lower surface is located at

$$Z = -\Delta \equiv -\delta/d. \quad (\text{A4})$$

We also define

$$H = H_0 + \Delta. \quad (\text{A5})$$

Let us first describe the flow (u_x, u_y, u_z) , associated with the pressure field p , generated by the motion of the wall at velocity V . In lubrication theory we have the scaling

$$u_x = VU_X(X, Y, Z), \quad u_y = VU_Y(X, Y, Z), \quad (\text{A6})$$

$$u_z = V\varepsilon^{1/2}U_Z(X, Y, Z), \quad p = \frac{\eta V \ell}{d^2}P(X, Y). \quad (\text{A7})$$

In terms of rescaled quantities, Stokes equation becomes

$$-\nabla_{\parallel} P + \partial_Z^2 \mathbf{U}_{\parallel} = 0, \quad \nabla_{\parallel} \mathbf{U}_{\parallel} + \partial_Z U_Z = 0, \quad (\text{A8})$$

where

$$\mathbf{U}_{\parallel} = U_X \hat{\mathbf{e}}_x + U_Y \hat{\mathbf{e}}_y, \quad \nabla_{\parallel} = \hat{\mathbf{e}}_x \partial_X + \hat{\mathbf{e}}_y \partial_Y. \quad (\text{A9})$$

Denoting $\mathbf{U}_{\parallel}^* = \mathbf{U}_{\parallel}(Z = -\Delta)$ the value of the dimensionless longitudinal velocity at the interface, and noting that $\mathbf{U}_{\parallel} = \mathbf{0}$ at the sphere's surface $Z = H_0$, we obtain:

$$\mathbf{U}_{\parallel} = \frac{H_0 - Z}{H} \mathbf{U}_{\parallel}^* + (\nabla_{\parallel} P) \frac{(Z + \Delta)(Z + \Delta - H)}{2}. \quad (\text{A10})$$

Inserting the above expression (A10) into the continuity equation (A8) and integrating over $Z \in [-\Delta; H_0]$ leads to

$$\int_{-\Delta}^{H_0} dZ \nabla_{\parallel} \left[\frac{H_0 - Z}{H} \mathbf{U}_{\parallel}^* + (\nabla_{\parallel} P) \frac{(Z + \Delta)(Z - H_0)}{2} \right] + U_Z(H_0) - U_Z(-\Delta) = 0. \quad (\text{A11})$$

We can also write this expression as

$$\nabla_{\parallel} \int_{-\Delta}^{H_0} dZ \left[\frac{H_0 - Z}{H} \mathbf{U}_{\parallel}^* + (\nabla_{\parallel} P) \frac{(Z + \Delta)(Z - H_0)}{2} \right] - (\nabla_{\parallel} \Delta) \mathbf{U}_{\parallel}^* + U_Z(H_0) - U_Z(-\Delta) = 0, \quad (\text{A12})$$

where we have compensated the fact of placing ∇_{\parallel} in front of the integral by the first term on the second line of the above equation. Noting that $U_Z(H_0) = 0$, and performing the integrals over Z leads to

$$\nabla_{\parallel} \left[\frac{H}{2} \mathbf{U}_{\parallel}^* - (\nabla_{\parallel} P) \frac{H^3}{12} \right] - (\nabla_{\parallel} \Delta) \mathbf{U}_{\parallel}^* = U_Z(-\Delta). \quad (\text{A13})$$

Now let us find a kinematic condition for the vertical flow at the interface $U_Z(-\Delta)$. The apparent vertical velocity of the interface is

$$\mathbf{v} = -\partial_t \delta \hat{\mathbf{e}}_z. \quad (\text{A14})$$

Let us write the normal vector \mathbf{n} to the interface, pointing upwards

$$\hat{\mathbf{n}} = \hat{\mathbf{e}}_z + \sqrt{\varepsilon} \nabla_{\parallel} \Delta + \mathcal{O}(\varepsilon). \quad (\text{A15})$$

The natural condition for the flow is the continuity of the component of the velocity that is normal to the interface, so

$$\mathbf{v} \cdot \hat{\mathbf{n}} = (V \mathbf{U}_{\parallel}^* + \sqrt{\varepsilon} V U_Z(-\Delta) \hat{\mathbf{e}}_z) \cdot \hat{\mathbf{n}}. \quad (\text{A16})$$

The kinematic condition that sets the value of $U_Z(-\Delta)$ then reads

$$U_Z(-\Delta) + \mathbf{U}_{\parallel}^* \cdot \nabla_{\parallel} \Delta = -\Omega \partial_{\tau} \Delta, \quad (\text{A17})$$

where we use the dimensionless time

$$\tau = \omega t, \quad (\text{A18})$$

and we have defined the dimensionless excitation frequency,

$$\Omega = \frac{\omega d}{V\sqrt{\varepsilon}} = \frac{\omega\sqrt{dR}}{V}. \quad (\text{A19})$$

Using the condition (A17), equation (A13) becomes

$$\Omega\partial_\tau\Delta = \nabla_{\parallel}\left[\frac{H}{2}\mathbf{U}_{\parallel}^* + (\nabla_{\parallel}P)\frac{H^3}{12}\right]. \quad (\text{A20})$$

Next, we consider the Young-Laplace equation

$$P = -\frac{\sigma d^3}{\eta V \ell^3}\nabla_{\parallel}^2\Delta = -\frac{\sigma d^{3/2}}{\eta V R^{3/2}}\nabla_{\parallel}^2\Delta. \quad (\text{A21})$$

Note that here we have neglected the contribution of the pressure variations in the lower fluid, this approximation will be discussed in Section B.

Next, Eq. (A21) suggests the definition of the dimensionless softness parameter

$$\kappa = \frac{\eta V R^{3/2}}{\sigma d^{3/2}}, \quad (\text{A22})$$

so that the dimensionless equilibrium equation for the lower interface reads

$$\kappa P + \nabla_{\parallel}^2\Delta = 0. \quad (\text{A23})$$

For simplicity, we will assume that the viscosity of the lower fluid is high enough so that the velocity at the interface is also the excitation velocity, so that

$$\mathbf{U}_{\parallel}^* = f(\tau)\hat{\mathbf{e}}_x. \quad (\text{A24})$$

This leads to

$$\Omega\partial_\tau\Delta = \nabla_{\parallel}\left[\frac{H}{2}f(\tau)\hat{\mathbf{e}}_x + (\nabla_{\parallel}P)\frac{H^3}{12}\right]. \quad (\text{A25})$$

Finally, our goal is to calculate the lift force

$$F = \iint dx dy p = F^* \iint dX dY P(X, Y), \quad (\text{A26})$$

where

$$F^* = Rd\frac{\eta V \ell}{d^2} = R^{3/2}\frac{\eta V}{d^{1/2}}. \quad (\text{A27})$$

In summary, the lift force can be calculated by integrating Eq. (A26), after one has solved the partial differential equations (A23) and (A25) for the pressure and deformation fields. To compare with the experimental data, we chose to introduce another dimensionless parameter s as

$$\begin{aligned} s &= \Omega\kappa^{1/3} = \frac{\omega\sqrt{dR}}{V} \times \left(\frac{\eta V R^{3/2}}{\sigma d^{3/2}}\right)^{1/3} \\ &= \frac{\omega R}{V^{2/3}} \times \left(\frac{\eta}{\sigma}\right)^{1/3} \end{aligned} \quad (\text{A28})$$

which does not depend on d . Thus, the curves obtained for F/F^* as a function of κ obtained by varying the gap d correspond to constant values of s .

Appendix B: Discussion on the validity of approximations

Let us discuss here the approximations. First, the equality between the velocity of the fluid at the interface and the excitation velocity is not exact, since the upper fluid exerts non-uniform constraints on the lower one. We estimate the characteristic stress as $\sigma_{xz} \sim \eta V/d$, leading to a characteristic force scale $\eta V \ell^2/d$. Hence the characteristic variations of fluid velocities at the surface are of the order of $(\eta V \ell^2/d)/(\eta_0 \ell)$, which are small compared to V at the condition

$$\frac{\eta_0}{\eta} \sqrt{\frac{d}{R}} \gg 1. \quad (\text{B1})$$

In an experiment where d varies, the inequality (B1) is no longer satisfied when κ becomes of the order of κ^* defined by

$$\kappa^* = \frac{\eta V}{\sigma} \left(\frac{\eta_0}{\eta}\right)^3. \quad (\text{B2})$$

For small values of d , or for $\kappa > \kappa^*$, we may therefore expect that the theory fails, possibly explaining why dynamic effects are not quantitatively captured by the theory for finite values of κ .

Next, we have also neglected the pressure variation in the lower fluid to write Eq. (A21). Considering that in the lower fluid the only relevant length scale is ℓ , we estimate that this pressure variation is of the order $\Delta p \sim \eta_0 V^*/\ell$, with V^* the characteristic deviation of velocity at the interface (with respect to V). If the condition (B1) is satisfied, then $V^* \sim \eta V \ell / (d \eta_0)$, leading to a pressure variation in the lower fluid of the order $\Delta p \sim \eta V/d$, which is small compared with $p \sim \eta V \ell / d^2$ in the upper fluid. In the opposite case where $\frac{\eta_0}{\eta} \sqrt{\frac{d}{R}} \ll 1$, we estimate $V^* \sim V$, so that $\Delta p \sim \eta_0 V/\ell \ll \eta V/d$ is again small compared to the pressure variations $p \sim \eta V \ell / d^2$ in the upper fluid. In both cases, neglecting the pressure variations in the lower fluid is therefore a valid approximation.

In conclusion, we expect our approximations to be valid when the condition (B1) is realized, corresponding to not-too-small distances d , or equivalently values of κ that are small compared to κ^* .

Appendix C: Regime of small deformations in the static case ($\Omega = 0$)

Here we consider the static case, with $\Omega = 0$ and $f = 1$ (constant velocity), in the regime $\kappa \ll 1$ (strong surface tension). We can analyse the problem by using a series expansions

$$P = P_0 + \kappa P_1 + \mathcal{O}(\kappa^2), \quad \Delta = \kappa \Delta_1 + \mathcal{O}(\kappa^2). \quad (\text{C1})$$

At leading order, we obtain

$$\nabla_{\parallel} \left(\frac{H_0^3}{12} (\nabla_{\parallel} P_0) \right) = -\frac{1}{2} (\nabla_{\parallel} H_0) \hat{\mathbf{e}}_x, \quad (\text{C2})$$

whose solution is just the pressure for the problem of a sphere moving near a rigid plane [1]

$$P_0 = 6\rho \cos \theta / (5H_0^2), \quad (\text{C3})$$

where (ρ, θ) are polar coordinates, with $X = \rho \cos \theta$ and $Y = \rho \sin \theta$. Next,

$$P_0 + \nabla_{\parallel}^2 \Delta_1 = 0, \quad (\text{C4})$$

which admits the solution

$$\Delta_1 = \frac{6}{5\rho} \ln(1 + \rho^2/2) \cos \theta. \quad (\text{C5})$$

Next, the equation for the pressure at next-to-leading order reads

$$0 = \hat{\mathbf{e}}_x \cdot \nabla_{\parallel} \frac{\Delta_1}{2} + \nabla_{\parallel} \left[\frac{H_0^3}{12} \nabla_{\parallel} P_1 + 3 \frac{H_0^2 \Delta_1}{12} \nabla_{\parallel} P_0 \right]. \quad (\text{C6})$$

Using polar coordinates (ρ, θ) this equation can be written as

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \frac{\rho H_0^3}{12} \partial_{\rho} P_1 + \frac{1}{\rho^2} \frac{\partial}{\partial \theta} \frac{H_0^3}{12} \partial_{\theta} P_1 = G, \quad (\text{C7})$$

with

$$G = -(\cos \theta \partial_{\rho} - \frac{\sin \theta}{\rho} \partial_{\theta}) \frac{\Delta_1}{2} - \frac{1}{4\rho} \frac{\partial}{\partial \rho} \rho H_0^2 \Delta_1 \partial_{\rho} P_0 - \frac{1}{4\rho^2} \frac{\partial}{\partial \theta} H_0^2 \Delta_1 \partial_{\theta} P_0. \quad (\text{C8})$$

We can write

$$G(\rho, \theta) = g_0(\rho) + g_2(\rho) \cos(2\theta). \quad (\text{C9})$$

We may therefore look for solutions under the form

$$P_1(\rho, \theta) = \tilde{P}_1(\rho) + \tilde{P}_2(\rho) \cos(2\theta). \quad (\text{C10})$$

The term \tilde{P}_1 will be the only one to contribute to the lift force, and it satisfies

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{H_0^3}{12} \partial_{\rho} \tilde{P}_1 = g_0. \quad (\text{C11})$$

This equation can be solved by requiring that \tilde{P}_1 remains finite at small ρ . We find

$$\partial_{\rho} \tilde{P}_1 = \frac{576(\rho^2 - 4)}{25 \rho (\rho^2 + 2)^4} \log \frac{\rho^2 + 2}{2}. \quad (\text{C12})$$

The lift force then reads

$$\frac{F}{F^*} = 2\pi\kappa \int_0^{\infty} d\rho \rho \tilde{P}_1 = -\pi\kappa \int_0^{\infty} d\rho \rho^2 \partial_{\rho} \tilde{P}_1. \quad (\text{C13})$$

The result of this integration is

$$\frac{F}{F^*} \underset{\kappa \rightarrow 0}{\simeq} \frac{6\pi\kappa}{25}, \quad (\text{C14})$$

as announced in the main text, see Eq. (5).

[1] M.E. O'neill, K. Stewartson, On the slow motion of a sphere parallel to a nearby plane wall, *J. Fluid Mech.* **27**, 705 (1967).