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# Displacement within velocity effect in gravitational wave memory

P.-M. Zhang<sup>1\*</sup>, P. A. Horvathy<sup>2,3†</sup>,

<sup>1</sup> *School of Physics and Astronomy,*

*Sun Yat-sen University, Zhuhai, China*

<sup>2</sup> *Laboratoire de Mathématiques et de Physique Théorique,*

*Université de Tours, (France)*

<sup>3</sup> *Erwin Schrödinger Institute,*

*Vienna (Austria)*

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## Abstract

Sandwich gravitational waves exhibit the *velocity memory effect* (VM) which however can become, for specific values of the wave parameters, pure *displacement* (DM) as suggested by Zel'dovich and Polnarev. Fixing such a “miraculous” value, the particle trajectory is an (approximate) standing wave characterized by a unique integer  $m$ , for which the particle does not absorb any energy from the passing wave. Our statements are illustrated by a simple Gaussian and by the Pöschl-Teller potential as profiles.

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\* <mailto:zhangpm5@mail.sysu.edu.cn>

† <mailto:horvathy@lmpt.univ-tours.fr>

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## I. INTRODUCTION

The *Memory Effect*, proposed to detect gravitational waves (GW), has two versions. The *displacement effect* (DM) put forward by Zel'dovich and Polnarev in the context of flyby says that *although the distance between free bodies will change, their relative velocity will become vanishingly small as the flyby concludes* [1–3]. Theoretical confirmation could then be obtained by taking into account non-linear effects [4–7].

Earlier studies [8–14] argued instead in favor of a *velocity effect* (VM): a burst of gravitational wave particles initially at rest would be scattered apart with *constant velocity*.

Our previous investigations [15–19] confirmed VM however did not corroborate the claim of Zel'dovich and Polnarev [1].

Which one of the proposed effects is the physical one ? The difficulty to answer the question comes from that the geodesic equations lead, for a sandwich wave with non-constant profile [13, 14], to a Sturm-Liouville problem [16].

In this note we demonstrate, by using two simple examples whose profile is (i) either a Gaussian, (III.1), or (ii) the Pöschl-Teller potential, (IV.1) [20] that DM arises for exceptional values of the parameters, which measure the *area below the wave profile* and correspond to having an *integer number of half-wave trajectories*.

## II. MEMORY EFFECT

Henceforth we restrict our attention at a gravitational wave with  $D = 1$  transverse dimension whose metric is, in Brinkmann coordinates  $(X^\mu) = (X, U, V)$  [21],

$$g_{\mu\nu}X^\mu X^\nu = (dX)^2 + 2dUdV - \frac{1}{2}\mathcal{A}(U)X^2dU^2, \quad (\text{II.1})$$

where  $X$  is space-like and  $U, V$  are light-cone coordinates. The wave is assumed to be a sandwich wave i.e., one whose profile  $\mathcal{A}$  is zero in both the Beforezone  $U < U_b$  and in the Afterzone  $U > U_a$  and is non-vanishing only in a short Wavezone  $U_b < U < U_a$  (see [13, 14, 19] for the terminology). Our test particles, taken lightlike for simplicity, move along geodesics determined by,

$$\frac{d^2X}{dU^2} + \frac{1}{2}\mathcal{A}X = 0, \quad (\text{II.2a})$$

$$\frac{d^2V}{dU^2} - \frac{1}{4}\frac{d\mathcal{A}}{dU}(X)^2 - \frac{1}{2}\mathcal{A}\frac{d(X^2)}{dU} = 0. \quad (\text{II.2b})$$

The coordinate  $X$  is decoupled from  $V$  from which the projection to transverse space of the lightlike worldlines is independent. Conversely, the geodesic is the horizontal lift of the trajectory  $X(U)$  determined by eqn. (II.2a) with  $U$  viewed as Newtonian time [22, 23]. The  $V$ -equation will not be considered any further.

We note for further reference that the equation (II.2a) describes free motion both in the Before and in the Afterzone where  $\mathcal{A} = 0$ , but *not* in the Wavezone, where  $\mathcal{A} \neq 0$  [14, 19]. For a non-trivial profile  $\mathcal{A}$ , (II.2a) is a Sturm-Liouville problem which, apart of particular cases, has only numerical solutions, for which our previous investigations [17–19] indicate that the particles undergo, generically, VM with non-zero velocity.

## III. GRAVITATIONAL WAVE WITH GAUSSIAN PROFILE

Our first example has a Gaussian profile. By rescaling the lightlike coordinate  $U$  we can achieve that the wave has fixed (unit) width,

$$\mathcal{A}^G(U) = \frac{k}{\sqrt{\pi}}e^{-U^2}, \quad (\text{III.1})$$

normalized as

$$\int \mathcal{A}^G(U)dU = k. \quad (\text{III.2})$$

The amplitude  $k$  is thus the area below the profile. The particles undergo generically (as said before), VM with non-zero velocity [17–19]. However below we argue that *fine-tuning the parameter  $k$*  can lead to (approximate) DM. For  $k = k_{crit} = 9.51455$  we get, for example, a “half-jump solution” shown in FIG. 1.

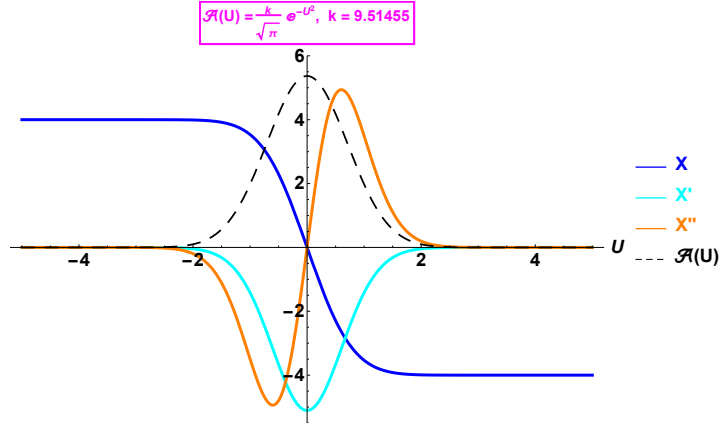


FIG. 1: *Fine-tuning the amplitude to  $k = k_{crit}$  provides us with the “half-wave displacement memory effect” with  $m = 1$  standing half-wave.  $X$  : trajectory,  $dX/dU$  : velocity,  $d^2X/dU^2$  : force.*

This “miracle” is explained by that at the (approximate) boundaries of the Wavezone  $U_b < U < U_a$  both the *velocity and the force* vanish, — and outside the Wavezone it move as required by Newton’s laws.

What happens can also be understood by looking at the plots in FIG.2 which confirm VM for  $k \neq k_{crit}$ , but show no DM : the velocity does not vanish (even approximately) in the Afterzone : for  $k < k_{crit}$  the force falls off before the oscillator reaches its return point, whereas for  $k > k_{crit}$  it pulls the particle back after reaching the bottom.

For  $k = k_{crit}$  we found precisely one half-wave trajectory with wave number  $m = 1$  and one may wonder if DM with several half-waves can also be accommodated. The answer is : fine-tuning the amplitude yields indeed DM with higher wave number, as shown for example, FIG.3. Emboldened by this success, further fine-tuning yields DM for other “magic amplitudes”,

$$k_1 \approx 9.5, m = 1, \quad k_2 \approx 30.7, m = 2, \quad k_3 \approx 63.1, m = 3, \quad k_4 \approx 106.7, m = 4, \dots, \quad (\text{III.3})$$

The trajectories consist of  $m$  half-waves, cf. FIG.1 and 3. Notice that the outgoing position depends on the parity of  $m$  :

$$X_{out} = (-1)^m X_{in}. \quad (\text{III.4})$$

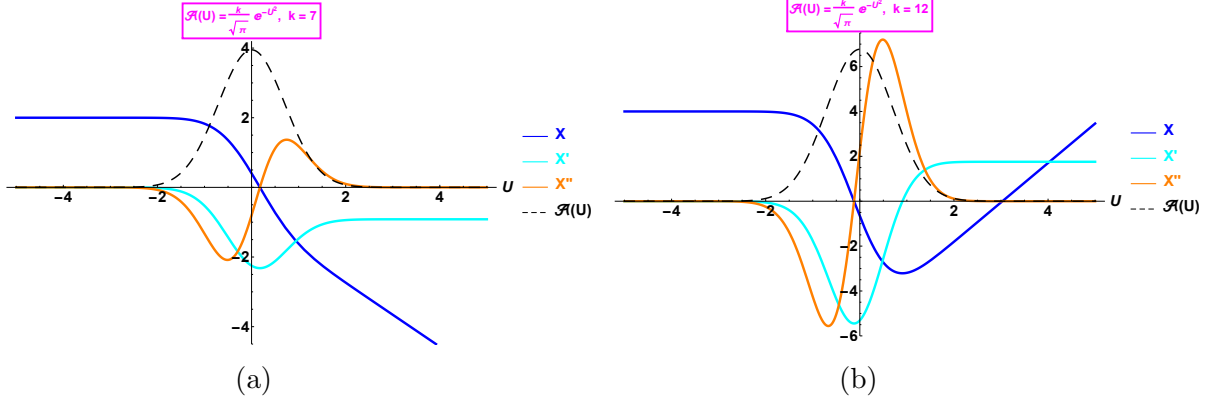


FIG. 2: (a) For  $k < k_{crit}$  the trajectory undershoots and (b) for  $k > k_{crit}$  it overshoots before being straightened out.

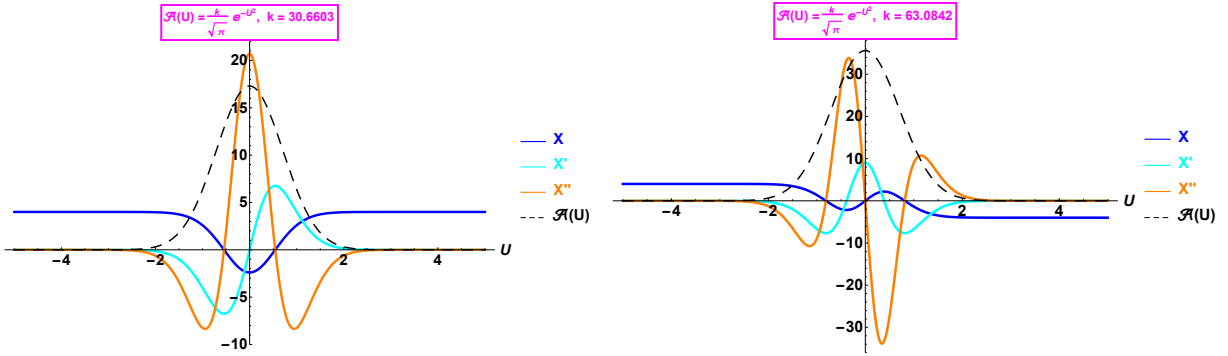


FIG. 3: Fine-tuning the amplitude yields DM with  $m = 2$  resp.  $m = 3$  half-waves as trajectories.

The relation between the [square root of the] amplitude (area) and the wave number,  $\sqrt{k}$  and  $m$ , depicted in FIG.4, is approximately linear,

$$\sqrt{k_m} \approx 0.78 + 2.38m. \quad (\text{III.5})$$

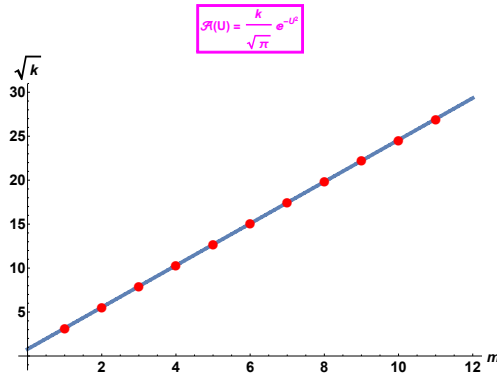


FIG. 4: The relation between the number of half-wave trajectories in the Wavezone,  $m$ , and  $\sqrt{k_{crit}}$  for DM is approximately linear.

#### IV. GRAVITATIONAL WAVE WITH PÖSCHL-TELLER PROFILE

For Gaussian profile the Sturm-Liouville equation (II.2a) has no analytic solution. However its shape is reminiscent of the [symmetric part of the] Pöschl-Teller (PT) potential [20, 24],

$$\mathcal{A}^{PT}(U) = \frac{k}{2 \cosh^2 U}, \quad (\text{IV.1})$$

depicted in FIG.5. For the sake of comparison with the Gaussian  $\mathcal{A}^G$  in (III.1),  $\mathcal{A}^{PT}$  is again normalized,  $\int \mathcal{A}^{PT}(U) dU = k$ , cf. (III.2). The advantage of the Pöschl-Teller profile is that

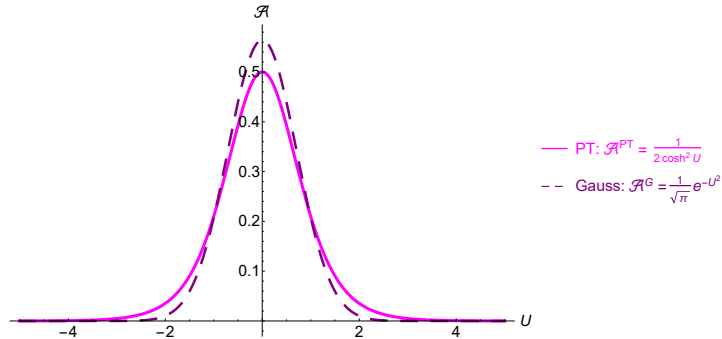


FIG. 5: The *Gaussian bell* (dashed) can be approximated by the *Pöschl-Teller* potential (IV.1) (solid line), which admits analytic solutions. The parameters were chosen so that the area below both profiles be identical and equal to  $k$ .

the trajectories can be found analytically. Writing

$$k_m = 4m(m+1), \quad (\text{IV.2})$$

where  $m$  is a priori a real number, eqn. (II.2a) becomes,

$$\frac{d^2 X}{dU^2} + \frac{m(m+1)}{\cosh^2 U} X = 0. \quad (\text{IV.3})$$

The initial conditions for a particle at rest before the burst arrives are,

$$X(U = -\infty) = X_0, \quad \text{and} \quad \dot{X}(U = -\infty) = 0. \quad (\text{IV.4})$$

Putting  $t = \tanh(U)$  into (IV.3) then yields the Legendre equation,

$$(1-t^2) \frac{d^2 X}{dt^2} - 2t \frac{dX}{dt} + m(m+1) X = 0. \quad (\text{IV.5})$$

The clue is that DM requires that  $X(U)$  tend to a constant for  $U \rightarrow \infty$  which amounts to requiring that the solution of (IV.5) should extend to  $t = \pm 1$  which implies, as it is well-known from QM, that  $m$  must be a positive integer so that the Legendre function becomes a Legendre polynomial,

$$X(U) \equiv X_m(U) = P_m(\tanh U), \quad m = 1, 2, \dots, \quad (\text{IV.6})$$

shown in FIGs. 6-7 should be compared with (III.3) and with FIGs.1 and 3.

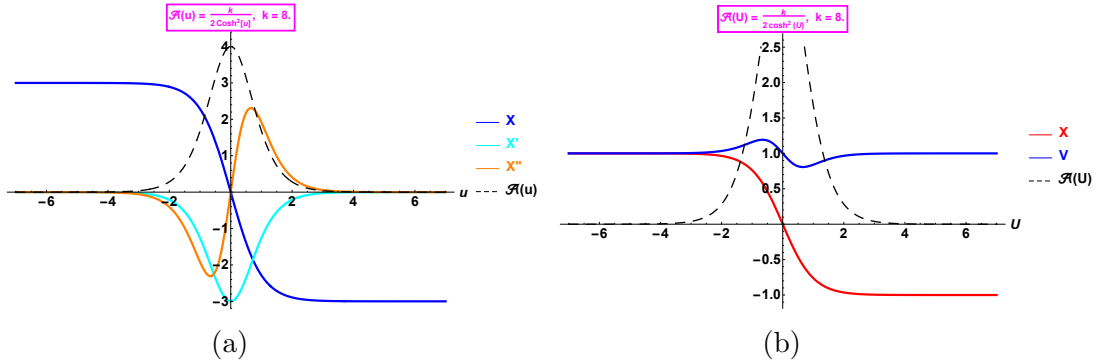


FIG. 6: For the Pöschl-Teller profile with  $k_{crit} = 8$  i.e.  $m = 1$ , both the transverse and the  $V$  trajectories are consistent with DM, as in the Gaussian case.

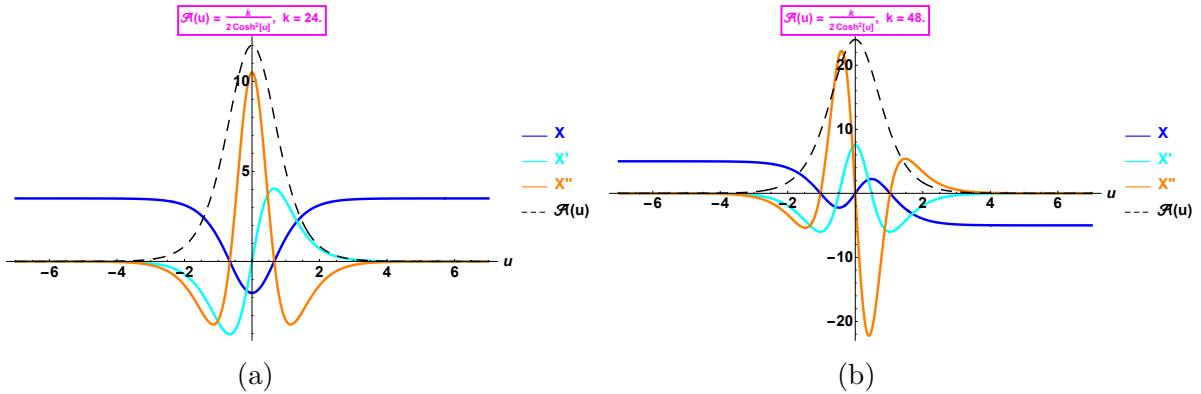


FIG. 7: Trajectory (IV.6) for Pöschl-Teller profile with  $k_{crit} =$  (a) 24 and (b)  $k_{crit} = 48$  have standing waves with  $m = 2$  resp.  $m = 3$  wave numbers, cf. FIG.3.

The trajectories (IV.6) consist again of  $m$  half-waves, as for the Gaussian profile. The Pöschl-Teller counterpart of the  $k_{crit} \Leftrightarrow m$  relation,

$$\sqrt{k_{crit}} = 1.33 + 2.83 m \quad (\text{IV.7})$$

is similar to (III.5) shown in FIG.4. For  $k \neq k_{crit}$  we get again VM but no DM, as it could be illustrated by plots similar to FIG.2.



It is instructive to plot also the velocities (jointly with trajectories, collected from FIGs. 6 and 7), FIG.8. Similar plots could be obtained for the Gaussian profile.

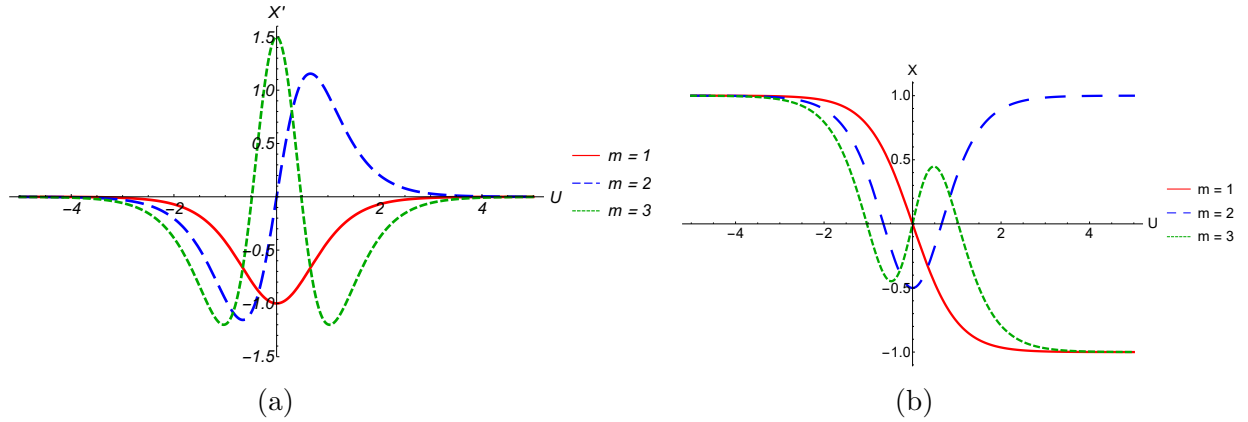


FIG. 8: *The transverse (a) velocities and (b) trajectories (IV.6) for displacement amplitudes  $k = k_m$  in (IV.2), for the Pöschl-Teller profile (IV.1)-(IV.2), shown for  $m=1, 2, 3$ .*

More generally, assuming that the coefficient of  $dU^2$  in (II.1) is that of an oscillator, the particle initially at rest at  $X_0 \neq 0$  is pulled towards the origin, picking up some speed. However passing  $X = 0$  the force changes direction and starts to reduce the velocity. If the profile was  $U$ -independent, then the particle would oscillate between  $X_0$  and  $-X_0$  forever. However for bell-shaped potentials the pull progressively falls off and after a while the motion becomes free.

If the residual velocity is non-zero then we have VM, as in FIG. 2. However if the velocity drops to zero, which happens at the turning point after an integer number of half-oscillations, then the particle stops, and we have DM. Higher amplitude  $k$  means stronger force which then requires a larger  $m$  to stop. These investigations are largely independent of the concrete profile as long as it is roughly bell-shaped.

Interestingly, DM is obtained precisely when the (*transverse*) *energy* cumulated by the particle during the process vanishes and this happens precisely for those “magic” DM values  $k_{crit}$ , as illustrated in FIG.9 for the Gaussian; almost identical pictures could be produced for the Pöschl-Teller profile. The reason is simple : the potential energy is (approximately) zero both in the Before and in the Afterzone, whereas the kinetic energy vanishes when the wave arrives, while DM means zero velocity also in the Afterzone.

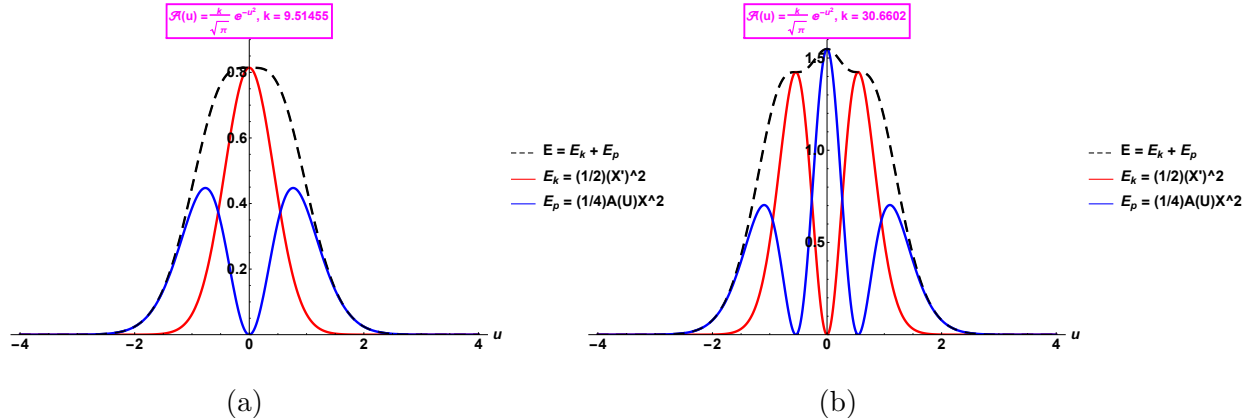


FIG. 9: For DM parameter  $k_{crit}$  the incoming wave first transfers some energy to the particle until reaching a maximal value, but the accumulated energy is then radiated again away, as shown for (a)  $m = 1$  and (b)  $m = 2$ .

## V. CONCLUSION

Particles at rest before the arrival of a sandwich gravitational wave show, generically, VM : the particles fly apart with diverging constant but non-zero velocity [8, 9, 11, 15–19]. Zel’dovich and Polnarev suggested instead that flyby would generate a pure displacement (DM) [1]. In this paper we have shown and illustrated by a numerical and by an analytical example that a judicious choice of the wave parameters, for which the (approximate) Wavezone contains an integer number of standing half-waves *does* produce pure displacement, i.e., one with asymptotically vanishing velocity. Thus the cumulative energy balance is zero : the particle does not absorb any energy from the wave.

The DM values of the parameters were found by fine-tuning and is to some extent analogous to spontaneous symmetry breaking in soliton theory [25, 26].

A seemingly different approach could be to follow [27], which, spelled out for  $D = 1$ , says that for the chosen initial conditions the trajectory is

$$X(U) = P(U)X_0, \quad (\text{V.1})$$

where  $P$  is a solution of the Sturm-Liouville equation

$$\ddot{P} = \mathcal{A}P. \quad (\text{V.2})$$

This is however equivalent to what we did above, because  $P(U)$  is just the geodesic in (II.2a) for initial position  $X(0) = 1$ . Then finding DM would amount to find the parameter for

which  $X(U) \rightarrow \text{const.}$  when  $U \rightarrow \infty$  — as we did above.

Our results can be understood from yet another point of view [28]. In the linear approximation,

$$X(U) = X(0) + U\dot{X}_0 \quad \text{and} \quad \dot{X}(U) = \dot{X}_0, \quad (\text{V.3})$$

where  $\dot{X}_0$  is the initial velocity. The displacement and the velocity after the wave had passed depend on three moments,

$$M_0 = \int \mathcal{A}(U)dU, \quad M_1 = \int U\mathcal{A}(U)dU, \quad M_2 = \int U^2\mathcal{A}(U)dU. \quad (\text{V.4})$$

Then the motion is

$$\dot{X}_{sol}(U) = M_0\dot{X}_0 + \dot{X}_0 + M_1\ddot{X}_0, \quad (\text{V.5a})$$

$$X_{sol}(U) = X_0 + U\dot{X}_{sol}(U) - M_1X_0 - M_2\ddot{X}_0, \quad (\text{V.5b})$$

where  $X_0 = X(U_a)$  and  $\dot{X}_0 = \dot{X}(U_a)$  are the initial position and velocity.

Applied to the Afterzone  $U > U_a$  where  $\mathcal{A}(U) \equiv 0$  implying that all three moments vanish,  $M_0 = M_1 = M_2 = 0$ . Thus the motion is along straight lines with constant velocity. The velocity will vanish only if the particle enters into the Afterzone with zero velocity,

$$\dot{X}_0 = 0. \quad (\text{V.6})$$

The critical values of the amplitude  $k_{crit} = k_m$  for this are precisely those we had found above, which correspond to having  $m$  half-waves in the Wavezone. Then we have also

$$X_{sol}(U) = X_0 = X(U_a), \quad (\text{V.7})$$

the position of the particle when it enters into the Afterzone with vanishing velocity.

We conclude with some speculation. In the VM regime the particles pick up some energy. Tiny as it is, it must be taken off from the wave. Since the distances are of the order of billion lightyears and the wave has to cross interstellar dust, etc, one could figure that for non-zero outgoing velocity the wave would be ultimately dissipated. What can reach us *must be* DM.

Our note aims to answer a question of the (unknown) referee of our previous paper [19]. The miraculous cancellation mentioned above will be studied further in a more comprehensive paper [29] and confirm that reducing VM to DM is may indeed possible for specific choices of the parameters for more general plane gravitation waves which include those generated by flyby, gravitational collapse, etc as proposed in [13].

Our next remark concerns the generalization to higher dimensions [29]. Both numerical and analytical evidence show that for a Brinkmann metric with 2 transverse dimensions,

$$g_{\mu\nu}X^\mu X^\nu = \delta_{ij}dX^i dX^j + 2dUdV + \frac{1}{2}\mathcal{A}(U)\left((X^1)^2 - (X^2)^2\right), \quad (\text{V.8})$$

where  $(X^i)$  are transverse and  $U, V$  light-cone coordinates. The relative minus is mandatory for a vacuum gravitational wave, therefore one of the components necessarily diverges as shown by FIG.10 for the Gaussian profile (III.1) with  $k = 1$ . The diverging coordinate

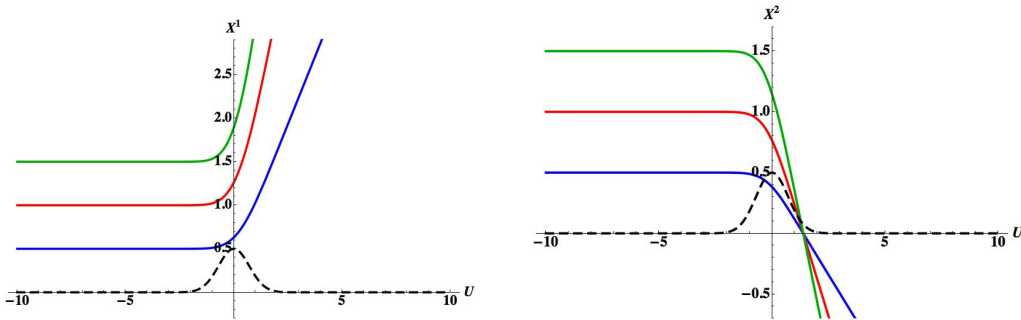


FIG. 10: *The geodesics for the Gaussian burst (III.1) in 2 transverse dimensions for various (blue/red/green) initial positions in the Beforezone. The potential is attractive in the  $X^2$ , but repulsive in the  $X^1$  sector implying diverging trajectories.*

should be discarded by putting it to identically zero when looking for DM allowing for a “half DM” in the best case. This is indeed the general behavior: *even-order* derivatives of the Gaussian exhibit half DM and *odd-order* derivatives exhibit DM in both transverse directions, as it will be explained in [29]. See refs. [30–32] for other recent developments.

In conclusion, the condition for DM is to find a moment  $U_0$  when *both the force and the velocity vanish* — then Newton’ first law take care of it. The difficulty is, precisely, to find such a moment.

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- [1] Ya. B. Zel'dovich and A. G. Polnarev, "Radiation of gravitational waves by a cluster of superdense stars," *Astron. Zh.* **51**, 30 (1974) [*Sov. Astron.* **18** 17 (1974)].
- [2] V B Braginsky and L P Grishchuk, "Kinematic resonance and the memory effect in free mass gravitational antennas," *Zh. Eksp. Teor. Fiz.* **89** 744 (1985) [*Sov. Phys. JETP* 62, 427 (1985)]
- [3] V B Braginsky and K. S. Thorne, "Gravitational-wave bursts with memory experiments and experimental prospects", *Nature* **327**, 123 (1987)
- [4] R. K. Sachs, "Gravitational waves in general relativity. 6. The outgoing radiation condition," *Proc. Roy. Soc. Lond. A* **264** (1961), 309-338 doi:10.1098/rspa.1961.0202
- [5] R. K. Sachs, "Gravitational waves in general relativity. 8. Waves in asymptotically flat spacetimes," *Proc. Roy. Soc. Lond. A* **270** (1962), 103-126 doi:10.1098/rspa.1962.0206
- [6] D. Christodoulou, "Nonlinear nature of gravitation and gravitational wave experiments," *Phys. Rev. Lett.* **67** (1991), 1486-1489 doi:10.1103/PhysRevLett.67.1486
- [7] L. Blanchet and T. Damour, "Hereditary effects in gravitational radiation," *Phys. Rev. D* **46** (1992) 4304.
- [8] J. Ehlers and W. Kundt, "Exact solutions of the gravitational field equations," in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, London, 1962).
- [9] J-M. Souriau, "Ondes et radiations gravitationnelles," *Colloques Internationaux du CNRS* No 220, pp. 243-256. Paris (1973).
- [10] P.C. Aichelburg, H. Balasin, "Generalized Symmetries of Impulsive Gravitational Waves," *Class.Quant.Grav.*14:A31-A42, (1997).
- [11] L. P. Grishchuk and A. G. Polnarev, "Gravitational wave pulses with 'velocity coded memory'," *Sov. Phys. JETP* **69** (1989) 653 [*Zh. Eksp. Teor. Fiz.* **96** (1989) 1153].
- [12] S. Hawking, "Gravitational radiation in an expanding universe," *J. Math. Phys.* **9** (1968), 598-604 doi:10.1063/1.1664615
- [13] G. W. Gibbons and S. W. Hawking, "Theory of the detection of short bursts of gravitational radiation," *Phys. Rev. D* **4** (1971), 2191-2197 doi:10.1103/PhysRevD.4.2191
- [14] H. Bondi and F. A. E. Pirani, "Gravitational Waves in General Relativity. 13: Caustic Prop-

- erty of Plane Waves,” Proc. Roy. Soc. Lond. A **421** (1989) 395.
- [15] P. M. Zhang, C. Duval, G. W. Gibbons and P. A. Horvathy, “Velocity Memory Effect for Polarized Gravitational Waves,” JCAP **05** (2018), 030 doi:10.1088/1475-7516/2018/05/030 [arXiv:1802.09061 [gr-qc]].
- [16] P. M. Zhang, M. Elbistan, G. W. Gibbons and P. A. Horvathy, “Sturm–Liouville and Carroll: at the heart of the memory effect,” Gen. Rel. Grav. **50** (2018) no.9, 107 doi:10.1007/s10714-018-2430-0 [arXiv:1803.09640 [gr-qc]].
- [17] P.-M. Zhang, C. Duval, G. W. Gibbons and P. A. Horvathy, “The Memory Effect for Plane Gravitational Waves,” Phys. Lett. B **772** (2017), 743-746 [arXiv:1704.05997 [gr-qc]].
- [18] P. M. Zhang, C. Duval, G. W. Gibbons and P. A. Horvathy, “Soft gravitons and the memory effect for plane gravitational waves,” Phys. Rev. D **96** (2017) no.6, 064013
- [19] M. Elbistan, P. M. Zhang and P. A. Horvathy, “Memory effect & Carroll symmetry, 50 years later,” Annals Phys. **459** (2023), 169535 doi:10.1016/j.aop.2023.169535
- [20] G. Pöschl and E. Teller, “Bemerkungen zur Quantenmechanik des anharmonischen Oszillators,” Z. Phys. **83** (1933), 143-151 doi:10.1007/BF01331132
- [21] M. W. Brinkmann, “On Riemann spaces conformal to Euclidean spaces,” Proc. Natl. Acad. Sci. U.S. **9** (1923) 1–3; “Einstein spaces which are mapped conformally on each other,” Math. Ann. **94** (1925) 119–145.
- [22] C. Duval, G. Burdet, H. P. Kunzle and M. Perrin, “Bargmann Structures and Newton-cartan Theory,” Phys. Rev. D **31** (1985), 1841-1853 doi:10.1103/PhysRevD.31.1841
- [23] C. Duval, G.W. Gibbons, P. Horvathy, “Celestial mechanics, conformal structures and gravitational waves,” Phys. Rev. **D43** (1991) 3907 [hep-th/0512188].
- [24] I. Chakraborty and S. Kar, “Geodesic congruences in exact plane wave spacetimes and the memory effect,” Phys. Rev. D **101** (2020) no.6, 064022 doi:10.1103/PhysRevD.101.064022 [arXiv:1901.11236 [gr-qc]].
- [25] S. Coleman, “Classical lumps and their quantum descendants,” Erice Lecture (1975), in *Aspects of Symmetry: Selected Erice Lectures*, p.185 Cambridge University Press, (1985),
- [26] P.G. Drazin and R. S. Johnson: “Solitons : an introduction.” Cambridge texts in applied mathematics. Cambridge University Press, (1989).
- [27] C. Duval, G. W. Gibbons, P. A. Horvathy and P. M. Zhang, “Carroll symmetry of plane gravitational waves,” Class. Quant. Grav. **34** (2017) no.17, 175003 doi:10.1088/1361-6382/aa7f62

- [arXiv:1702.08284 [gr-qc]].
- [28] T. Damour, private correspondence, (2024).
- [29] P. M. Zhang and P. A. Horvathy, “Gravitational wave memory: displacement or velocity ?” (in preparation).
- [30] I. Chakraborty and S. Kar, “A simple analytic example of the gravitational wave memory effect,” *Eur. Phys. J. Plus* **137** (2022) no.4, 418 doi:10.1140/epjp/s13360-022-02593-y [arXiv:2202.10661 [gr-qc]].
- [31] K. Mitman, M. Boyle, L. C. Stein, N. Deppe, L. E. Kidder, J. Moxon, H. P. Pfeiffer, M. A. Scheel, S. A. Teukolsky and W. Throwe, *et al.* “A Review of Gravitational Memory and BMS Frame Fixing in Numerical Relativity,” [arXiv:2405.08868 [gr-qc]].
- [32] L. Bieri and A. Polnarev, “Gravitational Wave Displacement and Velocity Memory Effects,” [arXiv:2402.02594 [gr-qc]].