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Abstract

We consider the system modeling the motion of a rigid body into a viscous incompressible fluid. Such a system couples the Navier-Stokes system for the fluid with the Newton laws for the rigid body and has a free boundary due to the motion of the rigid body. We work here in the case where the fluid domain and the structure domain are confined into a bounded domain. We show in this article that a weak solution for this system such that the fluid velocity satisfies a Prodi-Serrin condition is smooth in time and space. As for the proof in the case of the standard Navier-Stokes system, here we consider a particular linearization of our system around our weak solution. The corresponding linear system written in a moving spatial domain is then studied with the help of the Prodi-Serrin condition and to show some uniqueness result that allows us to identify the solutions of the linear system and of the nonlinear system, we also study the adjoint of this system.

Key words. Navier-Stokes system, fluid-structure interaction, weak solution
AMS subject classifications. 35Q35, 76D03, 76D05, 74F10.

1 Introduction

We study the weak solutions of a fluid-structure interaction system composed by a rigid body immersed into a viscous incompressible fluid. The complete system is confined into a bounded smooth domain $\Omega \subset \mathbb{R}^3$. The domain of the rigid body is completely characterized by the position of its center of mass $h \in \mathbb{R}^3$, its orientation $Q \in SO(3)$ (where $SO(3)$ is the group of rotations of $\mathbb{R}^3$) and its shape $\mathcal{S}$ that is assumed to be a bounded smooth domain of $\mathbb{R}^3$. More precisely, the structure and the fluid domains are given respectively by

$$\mathcal{S}_{(h,Q)} := h + Q\mathcal{S}, \quad \mathcal{F}_{(h,Q)} := \Omega \setminus \mathcal{S}_{(h,Q)}.$$

We assume that $\overline{\mathcal{S}} \subset \Omega$ and that $\mathcal{F} := \Omega \setminus \mathcal{S}$ is connected. The equations for the rigid body are the Newton laws whereas we use the standard Navier-Stokes system for the fluid dynamics. We denote by $\ell = \ell(t) \in \mathbb{R}^3$ and $\omega = \omega(t) \in \mathbb{R}^3$ the linear and angular velocities of the rigid body and by $u = u(t,x) \in \mathbb{R}^3$ and $p = p(t,x) \in \mathbb{R}$ the velocity and pressure fields in the fluid. The system of equations for the fluid-structure interaction system writes as follows

$$\left\{\begin{array}{rl} \partial_t u + (u\cdot\nabla)u - \Delta u + \nabla p = 0 & (t>0,\ x \in \mathcal{F}_{(h(t),Q(t))}), \\ \operatorname{div} u = 0 & (t>0,\ x \in \mathcal{F}_{(h(t),Q(t))}), \\ u(t,x) = \ell(t) + \omega(t)\times(x-h(t)) & (t>0,\ x \in \partial\mathcal{S}_{(h(t),Q(t))}), \\ u(t,x) = 0 & (t>0,\ x \in \partial\Omega), \end{array}\right. \tag{1.1}$$

$$\begin{cases} m\ell'(t) = -\displaystyle\int_{\partial\mathcal{S}_{(h(t),Q(t))}} \mathbb{T}(u,p)\nu \, \mathrm{d}\gamma_x \quad (t>0), \\ \left(J_Q\omega\right)'(t) = -\displaystyle\int_{\partial\mathcal{S}_{(h(t),Q(t))}} (x-h(t)) \times \mathbb{T}(u,p)\nu \, \mathrm{d}\gamma_x \quad (t>0), \end{cases} \tag{1.2}$$

$$h' = \ell, \quad Q' = \mathbb{S}(\omega)Q, \quad (t>0), \tag{1.3}$$

$$u(0,\cdot) = a \quad \text{in } \mathcal{F}, \quad h(0) = 0, \quad Q(0) = I_3, \quad \ell(0) = \ell_a, \quad \omega(0) = \omega_a. \tag{1.4}$$

In the above system, we have assumed that the viscosity and the density of the fluid are constant and, to simplify the presentation, we have taken them equal to 1. We have also assumed that the density $\rho_{\mathcal{S}} > 0$ of the rigid body is a positive constant and thus its mass and its inertia tensor are given by the standard formula:

$$m = \rho_{\mathcal{S}} \int_{\mathcal{S}} dy, \quad J_Q = QJQ^\top, \quad J := \rho_{\mathcal{S}} \int_{\mathcal{S}} \left(|y|^2 I_3 - y \otimes y\right) \, dy,$$

where $|\cdot|$ is Euclidean norm of $\mathbb{R}^3$. We have denoted by $\nu = \nu_{(h,Q)}$ the unit vector field normal to $\partial\mathcal{S}_{(h,Q)}$ and directed towards the interior of $\mathcal{S}_{(h,Q)}$. The Cauchy stress tensor field in the fluid is given by the constitutive law

$$\mathbb{T}(u,p) = -p\mathbb{I}_3 + 2\mathbb{D}u, \quad \mathbb{D}u = \frac{1}{2}(\nabla u + \nabla u^\top),$$

and where $M^\top$ denotes the transpose of a matrix M. We have also used the notation

$$\mathbb{S}(\omega) := \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}. \tag{1.5}$$

Note that we have assumed, without any loss of generality that at $t = 0$, the rigid body occupies the domain $\mathcal{S}$.

In what follows, we extend the fluid velocity by the rigid velocity in $\mathcal{S}_{(h,Q)}$, that is

$$u(t,x) := \ell(t) + \omega(t) \times (x - h(t)) \quad (t > 0, \ x \in \mathcal{S}_{(h(t),Q(t))})$$

and we do the same for the initial velocity a. We recall that (see, for instance, [33, Lemma 1.1]), $\mathbb{D}\varphi = 0$ in $\mathcal{S}_{(h,Q)}$ if and only if there exist $\ell_\varphi, \omega_\varphi \in \mathbb{R}^3$ such that

$$\varphi(x) := \ell_\varphi + \omega_\varphi \times (x - h) \quad (x \in \mathcal{S}_{(h,Q)}).$$

With this notation, we have in (1.1)–(1.4), $\ell(t) = \ell_u(t)$ and $\omega(t) = \omega_u(t)$ for all $t \geqslant 0$. We also define a global density for any $\left(\widetilde{h}, \widetilde{Q}\right) \in \mathbb{R}^3 \times SO(3)$ by the following formula

$$\rho_{\left(\widetilde{h},\widetilde{Q}\right)} := 1_{\mathcal{F}_{(\widetilde{h},\widetilde{Q})}} + \rho_{\mathcal{S}} 1_{\mathcal{S}_{(\widetilde{h},\widetilde{Q})}}. \tag{1.6}$$

In the particular case, where $\left(\widetilde{h}, \widetilde{Q}\right) = (0, I_3)$, we simply note by ρ the corresponding global density:

$$\rho := \rho_{(0,I_3)} = 1_{\mathcal{F}} + \rho_{\mathcal{S}} 1_{\mathcal{S}}. \tag{1.7}$$

We now introduce several function spaces in order to define the concept of weak solution for the system (1.1)–(1.4) and state our main results. Let G be a domain in $\mathbb{R}^3$. For any $q \in [1,\infty]$, and $s \geqslant 0$, we denote by $L^q(G)$ and $W^{s,q}(G)$ the usual Lebesgue and Sobolev-Slobodeckij spaces. We use the notation $H^s(G) = W^{s,2}(G)$. Further, $W_0^{s,q}(G)$ is defined as the completion of the set $C_0^\infty(G)$ of smooth functions with compact supports in G with respect to the $W^{s,q}(G)$ norm. We also introduce the following spaces, associated with our problem: for any $h \in \mathbb{R}^3$, $Q \in SO(3)$ and for any $q \in (1,\infty)$

$$\mathcal{L}^q_{(h,Q)} := \left\{ \varphi \in L^q(\Omega)^3 \ : \ \operatorname{div}\varphi = 0, \ \varphi \cdot \nu = 0 \text{ on } \partial\Omega, \ \mathbb{D}(\varphi) = 0 \text{ in } \mathcal{S}_{(h,Q)} \right\}, \tag{1.8}$$

$$\mathcal{H}^1_{(h,Q)} := \left\{\varphi \in H^1_0(\Omega)^3 \ : \ \operatorname{div}\varphi = 0,\ \mathbb{D}(\varphi) = 0 \text{ in } \mathcal{S}_{(h,Q)}\right\}, \tag{1.9}$$

$$\mathcal{W}^{1,q}_{(h,Q)} = \left\{\varphi \in W^{1,q}_0(\Omega)^3 \ : \ \operatorname{div}\varphi = 0,\ \mathbb{D}(\varphi) = 0 \text{ in } \mathcal{S}_{(h,Q)}\right\}. \tag{1.10}$$

If $(h,Q) = (0, I_3)$, then we denote the above spaces by $\mathcal{L}^q, \mathcal{H}^1$ and $\mathcal{W}^{1,q}$ respectively. In what follows, we also need the notation

$$L^2\left(0,T;\mathcal{H}^1_{(h,Q)}\right),\ H^1\left(0,T;\mathcal{L}^2_{(h,Q)}\right),\ \text{etc.}$$

for $(h,Q) \in W^{1,\infty}\left(0,T;\mathbb{R}^3 \times SO(3)\right)$. They are all defined in the same way: for instance $H^1\left(0,T;\mathcal{L}^2_{(h,Q)}\right)$ is the subspace of $\varphi \in H^1\left(0,T;L^2(\Omega)^3\right)$ such that for a.e. $t \in (0,T)$, $\varphi(t,\cdot) \in \mathcal{L}^2_{(h(t),Q(t))}$. Note that if h and Q are constant functions, we recover the usual definition.

Definition 1.1. *Assume $a \in \mathcal{L}^2$ and $T > 0$. We say that (h,Q,u) is a weak solution of (1.1)–(1.4) in $(0,T)$ if*

$$(h,Q) \in W^{1,\infty}\left(0,T;\mathbb{R}^3 \times SO(3)\right), \quad u \in L^\infty\left(0,T;\mathcal{L}^2_{(h,Q)}\right) \cap L^2\left(0,T;\mathcal{H}^1_{(h,Q)}\right), \tag{1.11}$$

satisfy

$$h' = \ell_u, \quad Q' = \mathbb{S}(\omega_u)Q \quad (t \in (0,T)), \tag{1.12}$$

$$\overline{\mathcal{S}_{(h(t),Q(t))}} \subset \Omega \quad (t \in [0,T]), \tag{1.13}$$

and for any

$$\varphi \in H^1\left(0,T;\mathcal{L}^2_{(h,Q)}\right) \cap L^\infty\left(0,T;\mathcal{H}^1_{(h,Q)}\right) \quad \textit{such that } \varphi(T,\cdot) = 0,$$

we have the weak formulation

$$-\iint_{(0,T)\times\Omega} \rho_{(h,Q)} u \cdot \left(\partial_t\varphi + (u\cdot\nabla\varphi)\right)\,\mathrm{d}x\,\mathrm{d}t + \iint_{(0,T)\times\Omega} 2\mathbb{D}(u) : \mathbb{D}(\varphi)\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \rho a\cdot\varphi(0,\cdot)\,\mathrm{d}x. \tag{1.14}$$

We recall the following result regarding the existence of weak solutions of the system (1.1)–(1.4) (see, for instance, [3], [5], [15]):

Theorem 1.2. *There exists a weak solution (h,Q,u) in the sense of Definition 1.1, and it satisfies the following energy inequality*

$$\frac{1}{2}\int_\Omega \rho_{(h(t),Q(t))}(x)\,|u(t,x)|^2\,\mathrm{d}x + 2\int_0^t \int_{\mathcal{F}_{(h(t),Q(t))}} |\mathbb{D}u|^2\,\mathrm{d}x\,\mathrm{d}\tau \leqslant \frac{1}{2}\int_\Omega \rho(x)\,|a(x)|^2\,\mathrm{d}x \quad (\textit{a.e. } t \in (0,T)).$$

Our aim is to show that if a weak solution of (1.1)–(1.4) satisfies a Prodi–Serrin condition for the fluid velocity, then the solution is smooth in time and space. More precisely, let us define, for $(h,Q) \in W^{1,\infty}\left(0,T;\mathbb{R}^3 \times SO(3)\right)$ the subset

$$\mathcal{E}_{(h,Q)} := \left\{(t,x) \in (0,T]\times\Omega\ ;\ x \in \overline{\mathcal{F}_{(h(t),Q(t))}}\right\}.$$

Then, our first main result states as follows:

Theorem 1.3. *Let (h,Q,u) be a weak solution of (1.1)–(1.4) associated with an initial condition $a \in \mathcal{L}^2$ in the sense of Definition 1.1. Assume*

$$u \in L^r(0,T;L^q(\mathcal{F}_{(h,Q)}))^3 \quad \textit{with} \quad \frac{2}{r} + \frac{3}{q} = 1, \quad r \in [2,\infty),\ q \in (3,\infty]. \tag{1.15}$$

Then

$$h, Q \in C^\infty((0,T];\mathbb{R}^3 \times SO(3)), \quad u \in C^\infty(\mathcal{E}_{(h,Q)})^3.$$

We have a similar result in the "critical" case ($r = \infty$, $q = 3$), but with a smallness assumption:

Theorem 1.4. *There exists $\varepsilon > 0$ such that if (h, Q, u) is a weak solution of (1.1)–(1.4) associated with an initial condition $a \in \mathcal{L}^2_{(0,I_3)}$ in the sense of Definition 1.1 and if*

$$u \in L^\infty\left(0,T;L^3\left(\mathcal{F}_{(h,Q)}\right)\right)^3, \quad \text{with} \quad \|u\|_{L^\infty\left(0,T;L^3\left(\mathcal{F}_{(h,Q)}\right)\right)^3} \leqslant \varepsilon, \tag{1.16}$$

then

$$h, Q \in C^\infty((0,T];\mathbb{R}^3 \times SO(3)), \quad u \in C^\infty(\mathcal{E}_{(h,Q)})^3.$$

Remark 1.5. *In Theorem 1.3 and Theorem 1.4, we have use the notation $u \in L^r(0,T;L^q(\mathcal{F}_{(h,Q)}))^3$ where $(h,Q) \in W^{1,\infty}\left(0,T;\mathbb{R}^3 \times SO(3)\right)$. One can define rigorously this condition by extending u by zero in $\mathcal{S}_{(h,Q)}$ and by saying that the corresponding extension belongs to $L^r(0,T;L^q(\Omega))^3$. We can also define this condition by using the change of variables defined in the next section.*

Remark 1.6. *We have obtained in [20] similar results in the case where $\Omega = \mathbb{R}^3$ and where $\mathcal{S}$ is a ball. The main difference between this work and [20] comes from the change of variables used to transforms the system (1.1)–(1.4) into a system written in a cylindrical domain. In [20], due to the particular geometry, it consists in a translation whereas here*

Note also that the above theorems improve the results obtained in [23] where the authors show a similar regularity property with the additional hypothesis that

$$\ell', \omega' \in L^\infty(0,T)^3.$$

Let us also add that the Prodi-Serrin conditions have been used to obtain uniqueness results for weak solutions of (1.1)–(1.4): [24] in the case where Ω is a bounded domain of $\mathbb{R}^3$ and [20] in the case where $\Omega = \mathbb{R}^3$ and where $\mathcal{S}$ is a ball. In [20], we have been able to deal with the critical case (that is if $u \in L^\infty\left(L^3\right)^3$).

Remark 1.7. *Concerning the smallness condition in Theorem 1.4, note that for the Navier-Stokes system without any structure and written in the whole space $\mathbb{R}^3$, the corresponding result without the smallness condition is obtained in [9]. The proof is completely different from the one proposed here and relies on Carleman estimates for the heat equation and on backward uniqueness (see also [8]). In [26], the author extends this result to the half-space and some local results near a boundary are obtained in [22, 27], but up to our knowledge, the cases of an exterior domain or of a bounded domain remain open.*

Remark 1.8. *The results in Theorem 1.3 and Theorem 1.4 can be extended to the case of several rigid bodies, the main modifications are on the definition of change of variables where we need to localize the the rigid velocities around each rigid body. We can also extend the results to the case where $\Omega = \mathbb{R}^3$ and a rigid body of arbitrary (smooth) shape. In that case, we need to consider the change of variables used in [4] (in the same spirit as the one used presented here for a bounded domain) instead of the "simpler" change of variables consisting of a translation and a rotation. With such a "simpler" change of variables, we obtain some additional terms in the Navier-Stokes system with coefficients that can be unbounded in the space variables. We refer, for instance, to [10], where the authors show the existence of strong solutions for the corresponding system.*

The mathematical analysis of the system (1.1)–(1.4) has been considered by many authors and there are nowadays a large amount of articles devoted to this subject in the literature. We can quote some of the earlier works, mainly devoted to the well-posedness of the above system: [4], [10], [28], [29], [31], etc. in the case where $\Omega = \mathbb{R}^d$, $d = 2, 3$ and [3], [5], [11], [13], [14], [15], [25], [30], etc. In the last years, many other problems on this system have been tackled: the possible contact between rigid bodies ([12], [16], [17], [25], etc.), the control of such a system ([1], [2], etc.), some asymptotic behavior ([6], [7], [18], etc.). Concerning the use of the Prodi-Serrin condition to obtain regularity properties or uniqueness results, we have already given the corresponding result in Remark 1.6.

The outline of this article is as follows: in Section 2, we recall a possible construction for the change of variables in order to transform (1.1)–(1.4) into a system written in a cylindrical domain. The new system involves operators with variable coefficients and we also give in this section some estimates of these operators.

We introduce in Section 3 some functional spaces and the semigroup associated with a linearization of the system. In this section, we give in particular some estimates of this linear system after a change of variables. The corresponding result is important in order to analyze another linearization of (1.1)–(1.4) studied in Section 4. For this linear system, we define several notions of solutions: strong solutions, weak solutions and very weak solutions and using the Prodi-Serrin condition of the weak solution of the nonlinear system, we show the existence and uniqueness of strong solutions for this system for initial conditions and right-hand side regular enough. We can also show a similar result for the adjoint of this linear system and this allows us to obtain a uniqueness result for the very weak solution of the direct linear system. Using these results, we can then show in Section 5 the main results.

Notation 1.9. *In the whole paper, we use C as a generic positive constant that does not depend on the other terms of the inequality. The value of the constant C may change from one appearance to another. We also use the notation $X \lesssim_p Y$ if there exists a constant $C > 0$ depending only on the parameter p such that we have the inequality $X \leqslant CY$.*

2 Change of variables

In this section, we define a change of variables that allows us to transform systems written in a moving spatial domain into systems written in a fixed spatial domain. The construction of this change of variables is the same as in [30] (see [4] for the case where $\Omega = \mathbb{R}^3$ as in Remark 1.8), but we recall it here for sake of completeness.

Let us consider

$$(h, Q) \in W^{1,\infty}(0, T; \mathbb{R}^3 \times SO(3)) \tag{2.1}$$

such that

$$h(0) = 0, \quad Q(0) = I_3, \quad \text{and} \quad \overline{\mathcal{S}_{(h(t),Q(t))}} \subset \Omega \quad (t \in [0, T]). \tag{2.2}$$

Let us define $(\ell, \omega) \in L^\infty(0, T; \mathbb{R}^3 \times \mathbb{R}^3)$ by

$$h'(t) = \ell(t), \quad Q'(t) = \mathbb{S}(\omega(t))Q(t) \quad (t \in (0, T)), \tag{2.3}$$

where $\mathbb{S}$ is defined by (1.5). We take $\alpha > 0$ such that

$$\operatorname{dist}\left(\mathcal{S}_{(h(t),Q(t))}, \partial\Omega\right) \geqslant \alpha \quad (t \in [0, T]), \tag{2.4}$$

and we consider a cut-off function $\psi \in C_0^\infty(\mathbb{R}^3)$ such that

$$\operatorname{supp}\psi \subset \{x \in \Omega \ : \ \operatorname{dist}(x, \partial\Omega) > \alpha/2\} \quad \text{and} \quad \psi \equiv 1 \quad \text{in } \{x \in \Omega \ : \ \operatorname{dist}(x, \partial\Omega) \geqslant \alpha\}.$$

Then, we introduce the function λ defined in $[0, T] \times \mathbb{R}^3$ by

$$\lambda(t, x) := \frac{1}{2} \operatorname{curl}\left(\psi(x)\left(\ell(t) \times (x - h(t)) - |x - h(t)|^2 \, \omega(t)\right)\right). \tag{2.5}$$

We have the following properties of λ:

Lemma 2.1. *The function λ defined by (2.5) satisfies $\lambda \in L^\infty(0, T; C_0^k(\mathbb{R}^3)^3)$ for any $k \geqslant 0$, with*

$$\operatorname{supp}\lambda \subset \{x \in \Omega \ : \ \operatorname{dist}(x, \partial\Omega) > \alpha/2\}.$$

Moreover,

$$\begin{gathered} \operatorname{div}\lambda = 0 \quad \text{in } (0, T) \times \mathbb{R}^3, \\ \lambda(t, x) = \ell(t) + \omega(t) \times (x - h(t)) \quad (t \in (0, T), \ x \in \mathcal{S}_{(h(t),Q(t))}). \end{gathered} \tag{2.6}$$

Let X be the flow associated with the vector field λ:

$$\begin{cases} \partial_t X(t,y) = \lambda(t, X(t,y)) \quad (t>0) \\ X(0,y) = y \in \mathbb{R}^3. \end{cases} \tag{2.7}$$

The following results were proved in [4] and [30].

Lemma 2.2. *For any $y \in \mathbb{R}^3$, the initial value problem (2.7) admits a unique solution $X(\cdot, y) \in W^{1,\infty}(0,T;\mathbb{R}^3)$. For any $t \in [0,T]$, $X(t,\cdot)$ is a C^∞-diffeomorphism from Ω onto itself such that*

$$X(t,\mathcal{F}) = \mathcal{F}_{(h(t),Q(t))}, \quad X(t,y) = h(t) + Q(t)y \quad (t \in [0,T],\ y \in \mathcal{S}). \tag{2.8}$$

Moreover,

$$\det(\nabla X) = 1 \quad \text{in } [0,T]\times\Omega.$$

Lemma 2.3. *For any $t \in [0,T]$, we denote by $Y(t,\cdot) = [X(t,\cdot)]^{-1}$ the inverse of $X(t,\cdot)$. For any $k \geqslant 1$, there exists a positive constant $C = C\left(\|\ell\|_{L^\infty(0,T)^3}, \|\omega\|_{L^\infty(0,T)^3}\right)$ such that*

$$\left\|\nabla^k X\right\|_{L^\infty((0,T)\times\Omega)^{3^{k+1}}} + \left\|\nabla^k Y(X)\right\|_{L^\infty((0,T)\times\Omega)^{3^{k+1}}} \leqslant C, \tag{2.9}$$

$$\left\|\partial_t\nabla^k X\right\|_{L^\infty(0,T)\times\Omega)^{3^{k+1}}} + \left\|\partial_t\nabla^k Y(X)\right\|_{L^\infty((0,T)\times\Omega)^{3^{k+1}}} \leqslant C, \tag{2.10}$$

Moreover, for any $k \geqslant 1$, there exists a constant $C>0$ such that for any $t_1, t_2 \in [0,T]$, we have the following estimate

$$\left\|\nabla^k X(t_1,\cdot) - \nabla^k X(t_2,\cdot)\right\|_{L^\infty(\Omega)^{3^{k+1}}} + \left\|\nabla^k Y(t_1, X(t_1,\cdot)) - \nabla^k Y(t_2, X(t_2,\cdot))\right\|_{L^\infty(\Omega)^{3^{k+1}}} \leqslant C\left|t_2 - t_1\right|. \tag{2.11}$$

Let us set

$$\mathcal{G} := (\nabla X)^\top (\nabla X), \tag{2.12}$$

and for a smooth function $W : \Omega \to \mathbb{R}^3$, we also define

$$\mathcal{D}W := \frac{1}{2}\left(\nabla\left[(\nabla X)\, W\right]\nabla Y(X) + (\nabla Y(X))^\top \left(\nabla\left[(\nabla X)\, W\right]\right)^\top\right), \tag{2.13}$$

$$\mathcal{K}W := (\nabla X)^\top \operatorname{div}\left(2\mathcal{D}W\,(\nabla Y(X))^\top\right). \tag{2.14}$$

Finally, if (h,Q,u) is a weak solution of (1.1)–(1.4), then we set

$$U(t,y) := \nabla Y(t,X(t,y))u(t,X(t,y)), \quad \Lambda(t,y) := \nabla Y(t,X(t,y))\lambda(t,X(t,y)) \tag{2.15}$$

and

$$\mathcal{M}W := (\nabla X)^\top \left((U-\Lambda)\cdot\nabla\right)\left[(\nabla X)\, W\right]. \tag{2.16}$$

These operators $\mathcal{G}, \mathcal{D}, \mathcal{K}$ and $\mathcal{M}$ appear in the following change of variables: if $w : (0,T)\times\Omega \to \mathbb{R}^3$ and $\pi : (0,T)\times\Omega \to \mathbb{R}$, then we set

$$W(t,y) := \nabla Y(t,X(t,y))w(t,X(t,y)), \quad \Pi(t,y) := \pi(t,X(t,y)). \tag{2.17}$$

More precisely, after standard computations, we have the following result:

Lemma 2.4. *Let W and Π defined in (2.17). Then the following relations hold true:*

$$(\operatorname{div} w)(X) = \operatorname{div} W, \quad (\nabla X)^\top (\nabla\pi)(X) = \nabla\Pi, \tag{2.18}$$

$$(2\mathbb{D}w)(X) = \mathcal{D}W, \quad (\nabla X)^\top (\Delta w)(X) = \mathcal{K}W, \tag{2.19}$$

$$(\nabla X)^\top (\partial_t w + (u\cdot\nabla)w)(X) = (\nabla X)^\top \partial_t\left[(\nabla X)\, W\right] + \mathcal{M}W. \tag{2.20}$$

Moreover, if w satisfies

$$w(t,x) = \ell_w(t) + \omega_w(t) \times (x - h(t)) \quad (x \in \mathcal{S}_{(h(t),Q(t))}),$$

then W satisfies

$$W(t,y) = \ell_W(t) + \omega_W(t) \times y \quad (y \in \mathcal{S})$$

with

$$\ell_W(t) = Q(t)^\top \ell_w(t), \quad \omega_W(t) = Q(t)^\top \omega_w(t). \tag{2.21}$$

As a corollary of Lemma 2.3, we have the following result:

Corollary 2.5. *Assume W and w satisfy the relation (2.17). Then, for any $p, s \in (1,\infty)$,*

$$w \in L^p\left(0,T;\mathcal{L}^s_{(h,Q)}\right) \iff W \in L^p\left(0,T;\mathcal{L}^s\right),$$

$$w \in L^p\left(0,T;\mathcal{W}^{1,s}_{(h,Q)}\right) \iff W \in L^p\left(0,T;\mathcal{W}^{1,s}\right),$$

$$w \in W^{1,p}\left(0,T;\mathcal{L}^s_{(h,Q)}\right) \iff W \in W^{1,p}\left(0,T;\mathcal{L}^s\right).$$

We can also write the Prodi-Serrin conditions (1.15) or (1.16) in a different way (recall Remark 1.5):

Corollary 2.6. *Assume u and U satisfy the relation (2.17). Then, for any $q, r \in [1,\infty]$,*

$$u \in L^r(0,T;L^q(\mathcal{F}_{(h,Q)}))^3 \iff U \in L^r(0,T;L^q(\mathcal{F}))^3.$$

The end of this section is devoted to some estimates on $\mathcal{G}, \mathcal{D}, \mathcal{K}$ and $\mathcal{M}$ defined above. To this aim, we introduce the following spaces: for $p \in (1,\infty)$ and for $t^1, t^2 \in \mathbb{R}$, with $t^1 < t^2$, we consider the spaces

$$\mathcal{X}^p(t^1,t^2) := \left\{ w \in L^p(t^1,t^2;\mathcal{W}^{1,p}) \cap W^{1,p}(t^1,t^2;\mathcal{L}^p) \ : \ w|_{\mathcal{F}} \in L^p(t^1,t^2;W^{2,p}(\mathcal{F}))^3 \right\}, \tag{2.22}$$

equipped with the norm

$$\|w\|_{\mathcal{X}^p(t^1,t^2)} := \|w\|_{L^p(t^1,t^2;\mathcal{W}^{1,p})} + \|\partial_t w\|_{L^p(t^1,t^2;\mathcal{L}^p)} + \|w|_{\mathcal{F}}\|_{L^p(t^1,t^2;W^{2,p}(\mathcal{F}))^3}.$$

We also define

$$\mathcal{X}^p_0(t^1,t^2) := \left\{ w \in \mathcal{X}^p(t^1,t^2) \ : \ w(t^1) = 0 \right\}. \tag{2.23}$$

Note that, for any $w \in \mathcal{X}^p_0(t^1,t^2)$, we have

$$\|w(t)\|_{\mathcal{L}^p} \leqslant (t^2 - t^1)^{1/p'} \|w\|_{\mathcal{X}^p(t^1,t^2)} \quad (t \in (t^1,t^2)), \tag{2.24}$$

where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. For any $t^1, t^2 \in [0,T]$ and let us write, to simplify,

$$X^j := X(t^j,\cdot), \quad Y^j := Y(t^j,\cdot), \quad \mathcal{G}^j := \mathcal{G}(t^j,\cdot) = \left(\nabla X^j\right)^\top \left(\nabla X^j\right), \quad \mathcal{K}^j := \mathcal{K}(t^j,\cdot). \tag{2.25}$$

From Lemma 2.3, we have the following results

Lemma 2.7. *There exists a positive constant $C = C\left(\|\ell\|_{L^\infty(0,T)^3}, \|\omega\|_{L^\infty(0,T)^3}\right)$ such that for any $t^1, t^2 \in [0,T]$, for any $p \in [1,\infty]$, and for any $W \in \mathcal{X}^p(0,T)$,*

$$\left\|\mathcal{G}^1 \partial_t W - \mathcal{G}^2 \partial_t W\right\|_{L^p(0,T;L^p(\mathcal{F})^3)} \leqslant C\left|t^1 - t^2\right| \|W\|_{\mathcal{X}^p(0,T)}, \tag{2.26}$$

$$\left\|\mathcal{K}^1 W - \mathcal{K}^2 W\right\|_{L^p(0,T;L^p(\mathcal{F})^3)} \leqslant C\left|t^1 - t^2\right| \|W\|_{\mathcal{X}^p(0,T)}. \tag{2.27}$$

Lemma 2.8. *There exists a positive constant* $C = C\left(\|\ell\|_{L^\infty(0,T)^3}, \|\omega\|_{L^\infty(0,T)^3}\right)$ *such that for any* $t^1, t^2 \in [0,T]$, *with* $t^1 < t^2$, *for any* $p \in [1,\infty]$, *and for any* $W \in \mathcal{X}_0^p(t^1,t^2)$

$$\left\|(\nabla X)^\top \partial_t \left[(\nabla X) W\right] - \mathcal{G}^1 \partial_t W\right\|_{L^p(t^1,t^2;L^p(\mathcal{F})^3)} \leqslant C \max\left\{(t^2-t^1), (t^2-t^1)^{1/p'}\right\} \|W\|_{\mathcal{X}^p(t^1,t^2)}, \tag{2.28}$$

$$\left\|\mathcal{K}W - \mathcal{K}^1 W\right\|_{L^p(t^1,t^2;L^p(\mathcal{F})^3)} \leqslant C(t^2-t^1) \|W\|_{\mathcal{X}^p(t^1,t^2)}. \tag{2.29}$$

Proof. We only show (2.28), the proof of (2.29) can be done similarly. Note that,

$$(\nabla X)^\top \partial_t \left[(\nabla X) W\right] = \mathcal{G}\partial_t W + (\nabla X)^\top (\partial_t \nabla X) W.$$

Combining this with Lemma 2.3 and (2.24), we obtain

$$\left\|(\nabla X)^\top \partial_t \left[(\nabla X) W\right] - \mathcal{G}^1 \partial_t W\right\|_{L^p(t^1,t^2;L^p(\mathcal{F})^3)} \lesssim (t^2-t^1) \|W\|_{\mathcal{X}^p(t^1,t^2)} + (t^2-t^1)^{1/p'} \|W\|_{\mathcal{X}^p(t^1,t^2)}.$$

□

Lemma 2.9. *Assume*

$$U \in L^r(0,T;L^q(\mathcal{F}))^3 \quad \text{with } \frac{2}{r} + \frac{3}{q} = 1,\ r \in [2,\infty],\ q \in [3,\infty]. \tag{2.30}$$

Then there exists a positive constant $C = C\left(\|\ell\|_{L^\infty(0,T)^3}, \|\omega\|_{L^\infty(0,T)^3}\right)$ *such that for any* $0 \leqslant t^1 < t^2 \leqslant T$, *and for any* $W \in \mathcal{X}^2(t^1,t^2)$

$$\|\mathcal{M}W\|_{L^2(t^1,t^2;L^2(\mathcal{F})^3)} \leqslant C\left(\|U\|_{L^r(t^1,t^2;L^q(\mathcal{F}))^3} + \|\ell\|_{L^2(t^1,t^2)^3} + \|\omega\|_{L^2(t^1,t^2)^3}\right) \|W\|_{\mathcal{X}^2(t^1,t^2)}. \tag{2.31}$$

Proof. First we note that from Lemma 2.3,

$$\left|(\nabla X)^\top (\Lambda \cdot \nabla) \left[(\nabla X) W\right]\right| \lesssim (|W| + |\nabla W|)|\Lambda|, \quad \left|(\nabla X)^\top (U \cdot \nabla) \left[(\nabla X) W\right]\right| \lesssim (|W| + |\nabla W|)|U|. \tag{2.32}$$

From (2.5) and (2.15), we have $|\Lambda| \lesssim |\lambda| \lesssim |\ell| + |\omega|$. Thus the above estimates imply

$$\left\|(\nabla X)^\top (\Lambda \cdot \nabla) \left[(\nabla X) W\right]\right\|_{L^2(t^1,t^2;L^2(\mathcal{F})^3)} \lesssim \left(\|\ell\|_{L^2(t^1,t^2)^3} + \|\omega\|_{L^2(t^1,t^2)^3}\right) \|W\|_{L^\infty(t^1,t^2;H^1(\mathcal{F})^3)}. \tag{2.33}$$

For the other estimate, we note that if $W \in \mathcal{X}^2(t^1,t^2)$, then a Sobolev embedding yields

$$\nabla W \in L^2\left(t^1,t^2;L^6(\mathcal{F})\right)^9 \cap L^\infty\left(t^1,t^2;L^2(\mathcal{F})\right)^9. \tag{2.34}$$

In particular, if $U \in L^\infty(\left(t^1,t^2;L^3(\mathcal{F})\right)^3$ or $U \in L^2\left(t^1,t^2;L^3(\mathcal{F})\right)^3$ (that is the cases $(r,q) = (\infty,3)$ and $(r,q) = (2,\infty)$), we can estimate the second relation of (2.32) and conclude the proof. Else, by an interpolation result, we deduce from (2.34) and from (2.30) that

$$\|\nabla W\|_{L^{2r/(r-2)}(t^1,t^2;L^{2q/(q-2)}(\mathcal{F}))^9} \leqslant C \|W\|_{\mathcal{X}^2(t^1,t^2)}.$$

Applying the Hölder inequality to the second relation of (2.32), we deduce

$$\left\|(\nabla X)^\top (U \cdot \nabla) \left[(\nabla X) W\right]\right\|_{L^2(t^1,t^2;L^2(\mathcal{F})^3)} \lesssim \|U\|_{L^r(t^1,t^2;L^q(\mathcal{F}))^3} \|W\|_{\mathcal{X}^2(t^1,t^2)}.$$

Combining the above relation with (2.33) and with (2.16), we conclude the proof of the lemma. □

3 Functional framework and preliminary estimates

As in the previous section, we consider (h, Q) satisfying (2.1) and (2.2). For any $t^1 \in [0, T]$, we set

$$h^1 := h(t^1), \quad Q^1 := Q(t^1) \tag{3.1}$$

In this section, we first study a linear fluid structure system posed in the fixed spatial domains $\mathcal{F}_{(h^1,Q^1)}$ and $\mathcal{S}_{(h^1,Q^1)}$ for (h^1, Q^1) defined as above. In fact, the results we present are valid for any $(h^1, Q^1) \in \mathbb{R}^3 \times SO(3)$ such that $\mathcal{F}_{(h^1,Q^1)} \subset \Omega$. The corresponding linear system was studied in several articles, for instance [21], [30]. We introduce the functional framework to analyze such a problem, and introduce the fluid-structure operator associated with the linear problem. We will use this fluid-structure operator to define very weak solutions for another linear system in Section 4. In Section 3.2, we reformulate the linear problem in the spatial domains $\mathcal{F}, \mathcal{S}$, and we obtain estimates that are uniform with respect to the choice of t^1. Such estimates will be used in Section 4. We also study a stationary Stokes system in moving domain and its reformulation in the reference configuration at the end of this section.

3.1 A linear fluid-structure system in $\mathcal{F}_{(h^1,Q^1)}$, $\mathcal{S}_{(h^1,Q^1)}$.

We consider the following linear problem

$$\begin{cases} \partial_t w - \Delta w + \nabla \pi = f & (t \in (0,T),\ x \in \mathcal{F}_{(h^1,Q^1)}), \\ \operatorname{div} w = 0 & (t \in (0,T),\ x \in \mathcal{F}_{(h^1,Q^1)}), \\ w(t,x) = \ell_w(t) + \omega_w(t) \times (x - h^1) & (t \in (0,T),\ x \in \partial\mathcal{S}_{(h^1,Q^1)}), \\ w(t,x) = 0 & (t \in (0,T),\ x \in \partial\Omega), \end{cases} \tag{3.2}$$

$$m\ell_w'(t) = -\int_{\partial\mathcal{S}_{(h^1,Q^1)}} \mathbb{T}(w,\pi)\nu \, \mathrm{d}\gamma_x + \int_{\mathcal{S}_{(h^1,Q^1)}} \rho_{\mathcal{S}} f \, dx \quad (t \in (0,T)), \tag{3.3}$$

$$J_{Q^1}\omega_w'(t) = -\int_{\partial\mathcal{S}_{(h^1,Q^1)}} (x - h^1) \times \mathbb{T}(w,\pi)\nu \, \mathrm{d}\gamma_x + \int_{\mathcal{S}_{(h^1,Q^1)}} \rho_{\mathcal{S}}(x - h^1) \times f \, dx \quad (t \in (0,T)), \tag{3.4}$$

$$w(0,\cdot) = b \quad \text{in } \mathcal{F}_{(h^1,Q^1)}, \quad \ell_w(0) = \ell_b, \quad \omega_w(0) = \omega_b. \tag{3.5}$$

The above system is a linearization of (1.1)–(1.4) and was studied by several authors (see, for instance, [21], [30], etc.). One can use a semigroup approach and write the above system as

$$w' = \mathbb{A}_{(h^1,Q^1)} w + \mathbb{P}_{(h^1,Q^1)} f \quad \text{in } (0,T), \quad w(0) = b. \tag{3.6}$$

In order to do that, we need to introduce several spaces and operators. We start by the following Helmholtz type decomposition: from [34], we have

$$L^q(\Omega)^3 = \mathcal{L}^q_{(h^1,Q^1)} \oplus \mathcal{G}^q_{(h^1,Q^1)} \tag{3.7}$$

where $\mathcal{L}^q_{(h^1,Q^1)}$ is defined in (1.8) and where

$$\begin{aligned} \mathcal{G}^q_{(h^1,Q^1)} := \Big\{ \varphi \in L^q(\Omega)^3 \ :\ & \exists \pi \in L^1_{loc}(\mathcal{F}_{(h^1,Q^1)}), \quad \varphi = \nabla\pi \quad \text{in } \mathcal{F}_{(h^1,Q^1)}, \\ & \int_{\mathcal{S}_{(h^1,Q^1)}} \rho_{\mathcal{S}}\varphi \ \mathrm{d}y + \int_{\partial\mathcal{S}_{(h^1,Q^1)}} \pi\nu \ \mathrm{d}\gamma_x = 0, \\ & \int_{\mathcal{S}_{(h^1,Q^1)}} \rho_{\mathcal{S}}\varphi \times (x - h^1) \ \mathrm{d}x + \int_{\partial\mathcal{S}_{(h^1,Q^1)}} \pi\nu \times (x - h^1) \ \mathrm{d}\gamma_x = 0 \Big\}. \end{aligned}$$

In the particular case $q=2$, note that $\mathcal{L}^2_{(h^1,Q^1)}$ and $\mathcal{G}^2_{(h^1,Q^1)}$ are orthogonal for the scalar product in $\langle\cdot,\cdot\rangle_{\mathcal{L}^2_{(h^1,Q^1)}}$ defined by

$$\langle f,g\rangle_{\mathcal{L}^2_{(h^1,Q^1)}}:=\int_\Omega \rho_{(h^1,Q^1)} f\cdot g\ \mathrm{d}y. \tag{3.8}$$

We recall that $\rho_{(h^1,Q^1)}$ is defined by (1.6). Note also that $\varphi\in\mathcal{G}^q_{(h^1,Q^1)}$ and $\mathbb{D}\varphi=0$ in $\mathcal{S}_{(h^1,Q^1)}$ if and only if there exists $\pi\in L^1_{loc}(\mathcal{F})$, $\nabla\pi\in L^q(\mathcal{F}_{(h^1,Q^1)})^3$ such that

$$\varphi=\nabla\pi\quad\text{in }\mathcal{F}_{(h^1,Q^1)},\quad \ell_\varphi=-m^{-1}\int_{\partial\mathcal{S}_{(h^1,Q^1)}}\pi\nu\ \mathrm{d}\gamma_x,\quad \omega_\varphi=-J_{Q^1}^{-1}\int_{\partial\mathcal{S}_{(h^1,Q^1)}}\pi\nu\times x\ \mathrm{d}\gamma_x. \tag{3.9}$$

We denote by $\mathbb{P}^{(q)}_{(h^1,Q^1)}:L^q(\Omega)^3\to\mathcal{L}^q_{(h^1,Q^1)}$ the projection onto $\mathcal{L}^q_{(h^1,Q^1)}$ along $\mathcal{G}^q_{(h^1,Q^1)}$. Then, we can define the operator $\mathbb{A}^{(q)}_{(h^1,Q^1)}:\mathcal{D}(\mathbb{A}^{(q)}_{(h^1,Q^1)})\to\mathcal{L}^q_{(h^1,Q^1)}$ as follows

$$\mathcal{D}\left(\mathbb{A}^{(q)}_{(h^1,Q^1)}\right):=\left\{\varphi\in\mathcal{W}^{1,q}_{(h^1,Q^1)}\ :\ \varphi_{|\mathcal{F}_{(h^1,Q^1)}}\in W^{2,q}(\mathcal{F}_{(h^1,Q^1)})^3\right\},\quad \mathbb{A}^{(q)}_{(h^1,Q^1)}\varphi:=\mathbb{P}^{(q)}_{(h^1,Q^1)}\widetilde{\mathbb{A}}^{(q)}_{(h^1,Q^1)}\varphi,$$

where

$$\widetilde{\mathbb{A}}^{(q)}_{(h^1,Q^1)}\varphi:=\Delta\varphi\quad\text{in }\mathcal{F}_{(h^1,Q^1)},$$

and

$$\left(\widetilde{\mathbb{A}}^{(q)}_{(h^1,Q^1)}\varphi\right)(x):=-m^{-1}\left(\int_{\partial\mathcal{S}_{(h^1,Q^1)}}2\mathbb{D}(\varphi)\nu\ d\gamma_x\right)$$
$$-J_{Q^1}^{-1}\left(\int_{\partial\mathcal{S}_{(h^1,Q^1)}}(y-h^1)\times 2\mathbb{D}(\varphi)\nu\ d\gamma_y\right)\times(x-h^1)\quad(x\in\mathcal{S}_{(h^1,Q^1)}).$$

As usual, if $(h^1,Q^1)=(0,I_3)$, then we simply write $\mathcal{G}^q$, $\mathbb{P}^{(q)}$, $\mathbb{A}^{(q)}$ instead of $\mathcal{G}^q_{(h^1,Q^1)}$, $\mathbb{P}^{(q)}_{(h^1,Q^1)}$, $\mathbb{A}^{(q)}_{(h^1,Q^1)}$. With the above notation, we have the following result obtained in [21, Theorem 5.1] and [30, Corollary 5.4]:

Proposition 3.1. *Assume $T>0$, $p\in(1,\infty)$ and $t^1\in[0,T]$. Then there exists a constant $C>0$ such that for any $b\in\mathcal{W}^{2(1-1/p),p}_{(h^1,Q^1)}$ and $f\in L^p(0,T;L^p(\Omega)^3)$, the system (3.6) admits a unique strong solution*

$$w\in L^p\left(0,T;\mathcal{D}\left(\mathbb{A}^{(p)}_{(h^1,Q^1)}\right)\right)\cap W^{1,p}\left(0,T;\mathcal{L}^p_{(h^1,Q^1)}\right).$$

Moreover, we have the estimate

$$\|w\|_{L^p\left(0,T;\mathcal{D}\left(\mathbb{A}^{(p)}_{(h^1,Q^1)}\right)\right)\cap W^{1,p}\left(0,T;\mathcal{L}^p_{(h^1,Q^1)}\right)}\leqslant C\left(\|b\|_{\mathcal{W}^{2(1-1/p),p}_{(h^1,Q^1)}}+\|f\|_{L^p\left(0,T;\mathcal{L}^p_{(h^1,Q^1)}\right)}\right).$$

Remark 3.2. *Note that if $(h^1,Q^1)=(0,I_3)$, then*

$$L^p\left(0,T;\mathcal{D}\left(\mathbb{A}^{(p)}\right)\right)\cap W^{1,p}\left(0,T;\mathcal{L}^p\right)=\mathcal{X}^p(0,T).$$

where $\mathcal{X}^p$ is defined by (2.22).

3.2 Uniform estimates after the change of variables

Let (h,Q) satisfying (2.1)-(2.2) and let us consider the change of variables constructed in Section 2. We remind the notation (2.25) to define $X^1,Y^1,\mathcal{G}^1$ and $\mathcal{K}^1$. Then if we set

$$W(t,y):=\nabla Y^1(X^1(y))w(t,X^1(y)),\quad \Pi(t,y):=\pi(t,X^1(y)), \tag{3.10}$$

$$F^1(t,y) := \nabla Y^1(X^1(y)) f(t, X^1(y)), \quad b^1(y) := \nabla Y^1(X^1(y)) b(X^1(y)), \tag{3.11}$$

we can see from (2.18)–(2.19) that the system (3.2)–(3.5) is transformed into

$$\begin{cases} \mathcal{G}^1 \partial_t W - \mathcal{K}^1 W + \nabla \Pi = \mathcal{G}^1 F^1 & \text{in } (0,T) \times \mathcal{F}, \\ \operatorname{div} W = 0 & \text{in } (0,T) \times \mathcal{F}, \\ W = \ell_W + \omega_W \times y & \text{on } (0,T) \times \partial\mathcal{S}, \\ W = 0 & \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{3.12}$$

$$m\ell'_W = -\int_{\partial\mathcal{S}} \mathbb{T}(W,\Pi)\nu \, \mathrm{d}\gamma_y + \int_{\mathcal{S}} \rho_{\mathcal{S}} F^1 \, dy \quad (t \in (0,T)), \tag{3.13}$$

$$J\omega'_W = -\int_{\partial\mathcal{S}} y \times \mathbb{T}(W,\Pi)\nu \, \mathrm{d}\gamma_y + \int_{\mathcal{S}} \rho_{\mathcal{S}} y \times F^1 \, dy \quad (t \in (0,T)), \tag{3.14}$$

$$W(0,\cdot) = b^1 \quad \text{in } \mathcal{F}, \quad \ell_W(0) = \ell_{b^1}, \quad \omega_W(0) = \omega_{b^1}. \tag{3.15}$$

Using the above transformation and Proposition 3.1, we can obtain the following result:

Proposition 3.3. *Let (h,Q) satisfying (2.1)-(2.2) and $p \in (1,\infty)$. Then, there exists a constant $C > 0$ such that for any $t^1 \in [0,T]$, $b^1 \in \mathcal{W}^{2(1-1/p),p}$ and $F^1 \in L^p(0,T;L^p(\Omega)^3)$, the system (3.12)–(3.15) admits a unique strong solution $W \in L^p(0,T;\mathcal{D}(\mathbb{A}^{(p)})) \cap W^{1,p}(0,T;\mathcal{L}^p)$ together with the estimate*

$$\|W\|_{L^p(0,T;\mathcal{D}(\mathbb{A}^{(p)}))\cap W^{1,p}(0,T;\mathcal{L}^p)} \leqslant C\left(\|b^1\|_{\mathcal{W}^{2(1-1/p),p}} + \|F^1\|_{L^p(0,T;L^p(\Omega)^3)}\right).$$

Note that, with respect with Proposition 3.1, here the constant C is independent of $t^1 \in [0,T]$.

Proof. Using the inverse transformation to (3.10)-(3.11), we deduce that for any $t^1 \in [0,T]$, the system (3.12)–(3.15) admits a unique strong solution $W \in L^p(0,T;\mathcal{D}(\mathbb{A}^{(p)})) \cap W^{1,p}(0,T;\mathcal{L}^p)$ together with the estimate

$$\|W\|_{L^p(0,T;\mathcal{D}(\mathbb{A}^{(p)}))\cap W^{1,p}(0,T;\mathcal{L}^p)} \leqslant C(t^1)\left(\|b^1\|_{\mathcal{W}^{2(1-1/p),p}} + \|F^1\|_{L^p(0,T;L^p(\Omega)^3)}\right) \tag{3.16}$$

with a positive constant $C(t^1)$. In order to obtain the above estimate with a constant independent of t^1, we proceed as follows. For any $t^2 \in [0,T]$, we write the corresponding system (3.12)–(3.15) in the form

$$\begin{cases} \mathcal{G}^1 \partial_t W - \mathcal{K}^1 W + \nabla \Pi = \left(\mathcal{G}^1 - \mathcal{G}^2\right) \partial_t W + \left(\mathcal{K}^2 - \mathcal{K}^1\right) W + \mathcal{G}^2 F^2 & \text{in } (0,T) \times \mathcal{F}, \\ \operatorname{div} W = 0 & \text{in } (0,T) \times \mathcal{F}, \\ W = \ell_W + \omega_W \times y & \text{on } (0,T) \times \partial\mathcal{S}, \\ W = 0 & \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{3.17}$$

$$m\ell'_W = -\int_{\partial\mathcal{S}} \mathbb{T}(W,\Pi)\nu \, \mathrm{d}\gamma_y + \int_{\mathcal{S}} \rho_{\mathcal{S}} F^2 \, dy \quad (t \in (0,T)), \tag{3.18}$$

$$J\omega'_W = -\int_{\partial\mathcal{S}} y \times \mathbb{T}(W,\Pi)\nu \, \mathrm{d}\gamma_y + \int_{\mathcal{S}} \rho_{\mathcal{S}} y \times F^2 \, dy \quad (t \in (0,T)), \tag{3.19}$$

$$W(0,\cdot) = b^2 \quad \text{in } \mathcal{F}, \quad \ell_W(0) = \ell_{b^2}, \quad \omega_W(0) = \omega_{b^2}. \tag{3.20}$$

Using (3.16) and (2.26)-(2.27), we deduce that for $\left|t^1 - t^2\right|$ small, we have the following estimate for the solution of (3.12)–(3.15) for t^2:

$$\|W\|_{L^p(0,T;\mathcal{D}(\mathbb{A}^{(p)}))\cap W^{1,p}(0,T;\mathcal{L}^p)} \leqslant 2C(t^1)\left(\|b^2\|_{\mathcal{W}^{2(1-1/p),p}} + \|F^2\|_{L^p(0,T;L^p(\Omega)^3)}\right). \tag{3.21}$$

Then using the compactness of $[0,T]$, we deduce that $[0,T]$ has a finite subcover of open intervals on which the above property holds. Taking the maximum of the corresponding constant, we conclude the proof of the proposition. □

3.3 Uniform estimates on a Stokes type system

Finally, let us consider the following elliptic system (depending on the parameter $t \in [0,T]$):

$$\begin{cases} -\mathcal{K}W + \nabla \Pi = \mathcal{G}F & \text{in } (0,T)\times\mathcal{F}, \\ \operatorname{div} W = 0 & \text{in } (0,T)\times\mathcal{F}, \\ W = \ell_W + \omega_W \times y & \text{on } (0,T)\times\partial\mathcal{S}, \\ W = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{3.22}$$

Proposition 3.4. *Let (h,Q) satisfying (2.1)-(2.2) and $k \geqslant 1$. Then for any*

$$F \in L^2(0,T;H^k(\mathcal{F}))^3, \quad (\ell_W,\omega_W) \in \mathbb{R}^3\times\mathbb{R}^3,$$

the system (3.22) admits a unique strong solution $(W,\Pi) \in L^2(0,T;H^{2+k}(\mathcal{F})^3 \times H^{1+k}(\mathcal{F})/\mathbb{R})$.

Proof. Assume $t \in [0,T]$ and let us define (w,π) by the inverse of the formula (2.17) and similarly

$$f(t,x) := \nabla X(t,Y(t,x))F(t,Y(t,x)).$$

This allows us to transform (3.22) into

$$\begin{cases} -\Delta w + \nabla \pi = f & (t\in[0,T],\ x \in \mathcal{F}_{(h(t),Q(t))}), \\ \operatorname{div} w = 0 & (t\in[0,T],\ x \in \mathcal{F}_{(h(t),Q(t))}), \\ w(t,x) = \ell_w(t) + \omega_w(t)\times(x-h(t)) & (t\in[0,T],\ x \in \partial\mathcal{S}_{(h(t),Q(t))}), \\ w(t,x) = 0 & (t\in[0,T],\ x \in \partial\Omega). \end{cases} \tag{3.23}$$

By using [32, Proposition 2.2, p.33] and a similar proof to that of Proposition 3.3, we deduce the existence of a constant $C>0$ such that for any $t\in[0,T]$

$$\|(w(t,\cdot),\pi(t,\cdot))\|_{H^{2+k}(\mathcal{F}_{(h(t),Q(t))})^3\times H^{1+k}(\mathcal{F}_{(h(t),Q(t))})/\mathbb{R}} \leqslant C\left(\|(\ell_w(t),\omega_w(t))\|_{\mathbb{R}^3\times\mathbb{R}^3} + \|f(t,\cdot)\|_{H^k(\mathcal{F}_{(h(t),Q(t))})^3}\right).$$

Going back the variables W,Π,F and using Lemma 2.3, we obtain from the above estimates

$$\|(W(t,\cdot),\Pi(t,\cdot))\|_{H^{2+k}(\mathcal{F})^3\times H^{1+k}(\mathcal{F})/\mathbb{R}} \leqslant C\left(\|(\ell_W(t),\omega_W(t))\|_{\mathbb{R}^3\times\mathbb{R}^3} + \|F(t,\cdot)\|_{H^k(\mathcal{F})^3}\right),$$

where C is independent of t. By integrating in time the above result, we conclude the proof of the proposition. □

4 Study of a linear system

In this section, we study a linear fluid-structure in a given moving domain. More precisely, assume (h,Q,u) is a weak solution of (1.1)–(1.4) in the sense of Definition 1.1, and satisfies (1.15) or (1.16). We consider the following linear problem

$$\begin{cases} \partial_t w - \Delta w + \nabla\pi + (u\cdot\nabla)w = f & (t\in(0,T),\ x \in \mathcal{F}_{(h(t),Q(t))}), \\ \operatorname{div} w = 0 & (t\in(0,T),\ x \in \mathcal{F}_{(h(t),Q(t))}), \\ w(t,x) = \ell_w(t) + \omega_w(t)\times(x-h(t)) & (t\in(0,T),\ x \in \partial\mathcal{S}_{(h(t),Q(t))}), \end{cases} \tag{4.1}$$

$$m\ell_w'(t) = -\int_{\partial\mathcal{S}_{(h(t),Q(t))}} \mathbb{T}(w,\pi)\nu \, \mathrm{d}\gamma_y + \int_{\mathcal{S}_{(h(t),Q(t))}} \rho_{\mathcal{S}} f \, dx \quad (t\in(0,T)), \tag{4.2}$$

$$(J_Q\omega_w)'(t) = -\int_{\partial\mathcal{S}_{(h(t),Q(t))}} (x-h(t))\times\mathbb{T}(w,\pi)\nu \, \mathrm{d}\gamma_y + \int_{\mathcal{S}_{(h(t),Q(t))}} \rho_{\mathcal{S}}(x-h)\times f \, dx \quad (t\in(0,T)), \tag{4.3}$$

$$w(0,\cdot) = b \quad \text{in } \mathcal{F}, \quad \ell_w(0) = \ell_b, \quad \omega_w(0) = \omega_b. \tag{4.4}$$

The difference between the above linear system and the linear system (3.2)–(3.5) is that the spatial domains $\mathcal{F}_{(h,Q)}, \mathcal{S}_{(h,Q)}$ are varying according to (h, Q) and that we have added the drift term $(u \cdot \nabla)w$. In what follows, we are going to show that if w is a weak solution of the above system with $f = 0$ and $b = a$, then $v = w$. We will then study the regularity of the solutions of the above linear system for $t > 0$ and deduce the corresponding regularity for v.

We also need to introduce the adjoint system of (4.1)–(4.4):

$$\begin{cases} -\partial_t \varphi - \Delta \varphi + \nabla \pi - (u \cdot \nabla)\varphi = f & (t \in (0,T),\ x \in \mathcal{F}_{(h(t),Q(t))}), \\ \operatorname{div} \varphi = 0 & (t \in (0,T),\ x \in \mathcal{F}_{(h(t),Q(t))}), \\ \varphi(t,x) = \ell_\varphi(t) + \omega_\varphi(t) \times (x - h(t)) & (t \in (0,T),\ x \in \partial\mathcal{S}_{(h(t),Q(t))}), \end{cases} \tag{4.5}$$

$$-m\ell_\varphi' = -\int_{\partial\mathcal{S}_{(h(t),Q(t))}} \mathbb{T}(\varphi,\pi)\nu \, \mathrm{d}\gamma_x + \int_{\mathcal{S}_{(h(t),Q(t))}} \rho_{\mathcal{S}} f \, dx \quad (t \in (0,T)), \tag{4.6}$$

$$-J_Q\omega_\varphi' = -\int_{\partial\mathcal{S}_{(h(t),Q(t))}} (x-h) \times \mathbb{T}(\varphi,\pi)\nu \, \mathrm{d}\gamma_x + \int_{\mathcal{S}_{(h(t),Q(t))}} \rho_{\mathcal{S}}(x-h) \times f \, dx \quad (t \in (0,T)), \tag{4.7}$$

$$\varphi(T,\cdot) = b \quad \text{in } \mathcal{F}, \quad \ell_\varphi(T) = \ell_b, \quad \omega_\varphi(T) = \omega_b. \tag{4.8}$$

In the next subsections, we will see how this adjoint system allows us to define a notion of weak solutions for the system of (4.1)–(4.4).

We consider the change of variables defined in Section 2 and in particular the transformations (2.17), (2.15). We also set for $(t, y) \in [0, T] \times \Omega$,

$$\Phi(t,y) := \nabla Y(t, X(t,y))\varphi(t, X(t,y)), \quad \Pi(t,y) := \pi(t, X(t,y)), \tag{4.9}$$

$$F(t,y) := \nabla Y(t, X(t,y))f(t, X(t,y)), \quad B(y) := \nabla Y(T, X(T,y))b(X(T,y)). \tag{4.10}$$

Then, we can transform the system (4.1)–(4.4) into

$$\begin{cases} (\nabla X)^\top \partial_t \left[(\nabla X)\, W\right] + \mathcal{M}W - \mathcal{K}W + \nabla\Pi = \mathcal{G}F & \text{in } (0,T) \times \mathcal{F}, \\ \operatorname{div} W = 0 & \text{in } (0,T) \times \mathcal{F}, \\ W = \ell_W + \omega_W \times y & \text{on } (0,T) \times \partial\mathcal{S}, \\ W = 0 & \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{4.11}$$

$$m\ell_W' + m\omega_U \times \ell_W = -\int_{\partial\mathcal{S}} \mathbb{T}(W,\Pi)\nu \, \mathrm{d}\gamma_y + \int_{\mathcal{S}} \rho_{\mathcal{S}} F \, dy \quad (t \in (0,T)), \tag{4.12}$$

$$J\omega_W' + \omega_U \times J\omega_W = -\int_{\partial\mathcal{S}} y \times \mathbb{T}(W,\Pi)\nu \, \mathrm{d}\gamma_y + \int_{\mathcal{S}} \rho_{\mathcal{S}} y \times F \, dy \quad (t \in (0,T)), \tag{4.13}$$

$$W(0,\cdot) = b \quad \text{in } \mathcal{F}, \quad \ell_W(0) = \ell_b, \quad \omega_W(0) = \omega_b \tag{4.14}$$

and the system (4.5)–(4.8) into

$$\begin{cases} -(\nabla X)^\top \partial_t \left[(\nabla X)\, \Phi\right] - \mathcal{M}\Phi - \mathcal{K}\Phi + \nabla\Pi = \mathcal{G}F & \text{in } (0,T) \times \mathcal{F}, \\ \operatorname{div} \Phi = 0 & \text{in } (0,T) \times \mathcal{F}, \\ \Phi = \ell_\Phi + \omega_\Phi \times y & \text{on } (0,T) \times \partial\mathcal{S}, \\ \Phi = 0 & \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{4.15}$$

$$-m\ell_\Phi' - m\omega_U \times \ell_\Phi = -\int_{\partial\mathcal{S}} \mathbb{T}(\Phi,\Pi)\nu \, \mathrm{d}\gamma_y + \int_{\mathcal{S}} \rho_{\mathcal{S}} F \, \mathrm{d}y \quad (t \in (0,T)), \tag{4.16}$$

$$-J\omega_\Phi'(t) - J\left(\omega_U \times \omega_\Phi\right) = -\int_{\partial\mathcal{S}} y \times \mathbb{T}(\Phi,\Pi)\nu \, \mathrm{d}\gamma_y + \int_{\mathcal{S}} \rho_{\mathcal{S}} y \times F \, dy \quad (t \in (0,T)), \tag{4.17}$$

$$\Phi(T,\cdot) = B \quad \text{in } \mathcal{F}, \quad \ell_\Phi(T) = \ell_B, \quad \omega_\Phi(T) = \omega_B. \tag{4.18}$$

4.1 Strong solutions for the linear system

We now construct strong solutions to the system (4.1)–(4.4) and to the adjoint system (4.5)–(4.8). We say that

$$(W,\Pi)\in\mathcal{X}^q(0,T)\times L^q(0,T;W^{1,q}(\mathcal{F})/\mathbb{R})$$

is a strong solution of the system (4.11)–(4.14) if it satisfies the system (4.11)–(4.14) a.e. in time and space. We define similarly the strong solutions of (4.15)–(4.18). In the case $q=2$, we recall that

$$\mathcal{X}^2(0,T)\subset C^0([0,T];\mathcal{H}^1),$$

and if $W\in\mathcal{X}_0^2(0,T)$, then there exists a constant $C>0$ independent of T such that

$$\|W\|_{C^0([0,T];\mathcal{H}^1)}\leqslant C\,\|W\|_{\mathcal{X}^2(0,T)}\,.$$

The above result Lemma 2.8 allows us to show the following result:

Proposition 4.1. *There exists $\varepsilon>0$ small enough such that if (h,Q,u) is a weak solution of (1.1)–(1.4) satisfying (1.15) or (1.16), then for any $F\in L^2(0,T;L^2(\Omega)^3)$ and for any $b\in\mathcal{H}^1$, there exists a unique strong solution*

$$(W,\Pi)\in\mathcal{X}^2(0,T)\times L^2(0,T;H^1(\mathcal{F})/\mathbb{R})$$

of (4.11)–(4.14).

Proof. Let us consider $t^1,t^2\in[0,T]$ and $t^2>t^1$. Let us show the existence and uniqueness of strong solutions for the system

$$\begin{cases}(\nabla X)^{\top}\,\partial_t\left[(\nabla X)\,W\right]+\mathcal{M}W-\mathcal{K}W+\nabla\Pi=\mathcal{G}F & \text{in } (t^1,t^2)\times\mathcal{F},\\ \operatorname{div}W=0 & \text{in } (t^1,t^2)\times\mathcal{F},\\ W=\ell_W+\omega_W\times y & \text{on } (t^1,t^2)\times\partial\mathcal{S},\\ W=0 & \text{on } (t^1,t^2)\times\partial\Omega,\end{cases}\tag{4.19}$$

$$m\ell_W'+m\omega_U\times\ell_W=-\int_{\partial\mathcal{S}}\mathbb{T}(W,\Pi)\nu\;\mathrm{d}\gamma_y+\rho_{\mathcal{S}}\int_{\mathcal{S}}F\;dy\quad(t\in(t^1,t^2)),\tag{4.20}$$

$$J\omega_W'+\omega_U\times J\omega_W=-\int_{\partial\mathcal{S}}y\times\mathbb{T}(W,\Pi)\nu\;\mathrm{d}\gamma_y+\rho_{\mathcal{S}}\int_{\mathcal{S}}y\times F\;dy\quad(t\in(t^1,t^2)),\tag{4.21}$$

$$W(t^1,\cdot)=\widetilde{b}\quad\text{in }\mathcal{F},\quad\ell_W(t^1)=\ell_{\widetilde{b}},\quad\omega_W(t^1)=\omega_{\widetilde{b}}.\tag{4.22}$$

We can write the above system in the form (3.12)–(3.15), where X^1, Y^1, $\mathcal{G}^1$, $\mathcal{K}^1$ are given by (2.25) and where $F^1:=F+\mathbb{B}_X(W)$ with

$$\mathbb{B}_X(W):=\left(\mathcal{G}^1\right)^{-1}\Big(-\left(\nabla X\right)^{\top}\partial_t\left[(\nabla X)\,W\right]+\mathcal{G}^1\partial_tW-\mathcal{M}W+\left(\mathcal{K}W-\mathcal{K}^1W\right)\Big)\quad\text{in }\mathcal{F},$$

and

$$\mathbb{B}_X(W)=-\omega_U\times\ell_W-\left[J^{-1}\left(\omega_U\times J\omega_W\right)\right]\times y\quad\text{in }\mathcal{S}.$$

The last relation allows us to have

$$\rho_{\mathcal{S}}\int_{\mathcal{S}}\mathbb{B}_X(W)\;dy=-m\omega_U\times\ell_W,\quad\rho_{\mathcal{S}}\int_{\mathcal{S}}y\times\mathbb{B}_X(W)\;dy=-\omega_U\times J\omega_W.$$

Therefore, in order to show the existence and the uniqueness of a solution of (4.19)–(4.22), it is sufficient to show the existence and the uniqueness of a fixed-point of the mapping

$$\mathfrak{N}:L^2(t^1,t^2;L^2\left(\Omega\right)^3)\to L^2(t^1,t^2;L^2\left(\Omega\right)^3),\quad g\mapsto\mathbb{B}_X(W),$$

where $W \in \mathcal{X}^2(t^1,t^2)$ is the solution of (3.12)–(3.15) with $F^1 := F + g$ and $b^1 = \widetilde{b}$.

Let us consider $g^\sharp, g^\flat \in L^2(t^1,t^2;L^2(\Omega)^3)$ and $W^\sharp, W^\flat \in \mathcal{X}^2(t^1,t^2)$ the corresponding solutions of (3.12)–(3.15) with respectively $F^1 := F + g^\sharp$, $F^1 := F + g^\flat$ and $b^1 = \widetilde{b}$. We set

$$g := g^\flat - g^\sharp, \quad W := W^\flat - W^\sharp,$$

and we see that $W \in \mathcal{X}^2(t^1,t^2)$ is the solution of (3.12)–(3.15) with $F^1 := g$ and $b^1 = 0$. Applying Proposition 3.3, we deduce that

$$\|W\|_{\mathcal{X}^2(t^1,t^2)} \lesssim \|g\|_{L^2(t^1,t^2;L^2(\Omega)^3)}. \tag{4.23}$$

Combining the above estimate with Lemma 2.8 and with Lemma 2.9, we obtain the existence of a constant $C > 0$ independent of t_1 such that

$$\begin{aligned}&\left\|\mathfrak{N}g^\flat - \mathfrak{N}g^\sharp\right\|_{L^2(t^1,t^2;L^2(\Omega))^3}\\ &\quad\leqslant C\left((t^2-t^1) + (t^2-t^1)^{1/2} + \|U\|_{L^r(t^1,t^2;L^q(\mathcal{F}))^3} + \|\ell\|_{L^2(t^1,t^2)^3} + \|\omega\|_{L^2(t^1,t^2)^3}\right)\|g\|_{L^2(t^1,t^2;L^2(\Omega))^3}.\end{aligned} \tag{4.24}$$

If $r < \infty$, we deduce that for $t^2 - t^1$ small, we can apply the Banach fixed-point theorem and obtain the existence and uniqueness of a strong solution $W \in \mathcal{X}^2(t^1,t^2)$ of (4.19)–(4.22). If $r = \infty$, we use the smallness condition (1.16) to deduce the same result with $\varepsilon > 0$ small enough. In both case, we can thus consider an increasing sequence τ_k such that $\tau_0 = 0$, $\tau_N = T$ and

$$C\left((\tau_{k+1}-\tau_k) + (\tau_{k+1}-\tau_k)^{1/2} + \|U\|_{L^r(\tau_k,\tau_{k+1};L^q(\mathcal{F}))^3} + \|\ell\|_{L^2(\tau_k,\tau_{k+1})^3} + \|\omega\|_{L^2(\tau_k,\tau_{k+1})^3}\right) \leqslant \frac{1}{2} \quad (k = 0,\dots,N-1) \tag{4.25}$$

and the above construction allows us to prove the existence and the uniqueness of a strong solution for (4.11)–(4.14) in $(0,T)$. □

In a similar manner we also have the following result.

Proposition 4.2. *There exists $\varepsilon > 0$ small enough such that if (h,Q,u) is a weak solution of (1.1)–(1.4) satisfying (1.15) or (1.16), then for any $F \in L^2(0,T;L^2(\Omega)^3)$ and $B \in \mathcal{H}^1$, there exists a unique strong solution*

$$(\Phi,\Pi) \in \mathcal{X}^2(0,T) \times L^2(0,T;H^1(\mathcal{F})/\mathbb{R})$$

of (4.15)–(4.18).

In what follows, we fix $\varepsilon > 0$ small enough so that the conclusions of Proposition 4.1 and Proposition 4.2 hold.

We can also study the following linear system that is similar to (4.11)–(4.14), but where we have removed the term $\mathcal{M}W$ and some terms in the differential equations:

$$\begin{cases} (\nabla X)^\top \partial_t\left[(\nabla X)\, W\right] - \mathcal{K}W + \nabla\Pi = \mathcal{G}F & \text{in } (0,T)\times\mathcal{F},\\ \operatorname{div} W = 0 & \text{in } (0,T)\times\mathcal{F},\\ W = \ell_W + \omega_W \times y & \text{on } (0,T)\times\partial\mathcal{S},\\ W = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{4.26}$$

$$m\ell_W' = -\int_{\partial\mathcal{S}} \mathbb{T}(W,\Pi)\nu \,\mathrm{d}\gamma_y + \int_{\mathcal{S}} \rho_{\mathcal{S}} F \,dy \quad (t \in (0,T)), \tag{4.27}$$

$$J\omega_W' = -\int_{\partial\mathcal{S}} y \times \mathbb{T}(W,\Pi)\nu \,\mathrm{d}\gamma_y + \int_{\mathcal{S}} \rho_{\mathcal{S}} y \times F \,dy \quad (t \in (0,T)), \tag{4.28}$$

$$W(0,\cdot) = 0 \quad \text{in } \mathcal{F}, \quad \ell_W(0) = 0, \quad \omega_W(0) = 0. \tag{4.29}$$

With a proof similar to the proof of Proposition 4.1, one can show the following result.

Proposition 4.3. *Assume (h, Q, u) is a weak solution of (1.1)–(1.4) and $p > 1$. For any $F \in L^p(0,T;L^p(\Omega)^3)$, there exists a unique strong solution*

$$(W, \Pi) \in \mathcal{X}^p(0,T) \times L^p(0,T;W^{1,p}(\mathcal{F})/\mathbb{R})$$

of (4.26)–(4.29).

4.2 Weak solutions for the linear system

Let us introduce two different notions of weak solutions for the system (4.1)–(4.4):

Definition 4.4. *Assume (h, Q, u) is a weak solution of (1.1)–(1.4) in $(0,T)$. Assume $f \in L^2(0,T;L^2(\Omega)^3)$ and $a \in \mathcal{L}^2$.*

- *We say that $w \in L^\infty\left(0,T;\mathcal{L}^2_{(h,Q)}\right) \cap L^2\left(0,T;\mathcal{H}^1_{(h,Q)}\right)$ is a weak solution of (4.1)–(4.4) if for any*

$$\varphi \in H^1\left(0,T;\mathcal{L}^2_{(h,Q)}\right) \cap L^\infty\left(0,T;\mathcal{H}^1_{(h,Q)}\right) \quad \text{such that } \varphi(T,\cdot) = 0,$$

we have

$$-\iint_{(0,T)\times\Omega} \rho_{(h,Q)} w \cdot (\partial_t \varphi + (u\cdot\nabla)\varphi)\,\mathrm{d}x\,\mathrm{d}t + \iint_{(0,T)\times\Omega} 2\mathbb{D}(w) : \mathbb{D}(\varphi)\,\mathrm{d}x\,\mathrm{d}t = \iint_{(0,T)\times\Omega} \rho_{(h,Q)} f\cdot\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \rho b\cdot\varphi(0,\cdot)\,\mathrm{d}x. \quad (4.30)$$

- *We say that $w \in L^2(0,T;\mathcal{L}^2_{(h,Q)})$ is a very weak solution of (4.1)–(4.4) if for any*

$$\varphi \in H^1(0,T;\mathcal{L}^2_{(h,Q)}) \cap L^\infty(0,T;\mathcal{H}^1_{(h,Q)}) \cap L^2(0,T;H^2(\mathcal{F}_{(h,Q)}))^3 \quad \text{such that } \varphi(T,\cdot) = 0,$$

we have

$$\iint_{(0,T)\times\Omega} \rho_{(h,Q)} w \cdot \left(-\partial_t\varphi - (u\cdot\nabla)\varphi + \mathbb{A}_{(h,Q)}\varphi\right)\,\mathrm{d}x\,\mathrm{d}t = \iint_{(0,T)\times\Omega} \rho_{(h,Q)} f\cdot\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \rho b\cdot\varphi(0,\cdot)\,\mathrm{d}x. \quad (4.31)$$

In the above definition, we have used the notation $L^2(0,T;H^2(\mathcal{F}_{(h,Q)}))^3$ for (h,Q) satisfying (2.1)–(2.2). We define this space as follows:

$$\varphi \in L^2(0,T;H^2(\mathcal{F}_{(h,Q)}))^3 \iff \Phi \in L^2(0,T;H^2(\mathcal{F}))^3$$

where φ and Φ satisfy the first relation of (4.9).

We can also give similar notions of weak solutions for the system (4.11)–(4.14):

Definition 4.5. *Assume (h, Q, u) is a weak solution of (1.1)–(1.4) in $(0,T)$. Assume $F \in L^2(0,T;L^2(\Omega)^3)$ and $a \in \mathcal{L}^2$.*

- *We say that $W \in L^\infty(0,T;\mathcal{L}^2) \cap L^2(0,T;\mathcal{H}^1)$ is a weak solution of (4.11)–(4.14) if for any*

$$\Phi \in H^1(0,T;\mathcal{L}^2) \cap L^\infty(0,T;\mathcal{H}^1) \quad \text{such that } \Phi(T,\cdot) = 0,$$

we have

$$-\iint_{(0,T)\times\Omega} \rho W \cdot \left((\nabla X)^\top \partial_t\left[(\nabla X)\,\Phi\right] + \mathcal{M}\Phi\right)\,\mathrm{d}y\,\mathrm{d}t + \iint_{(0,T)\times\Omega} 2\mathcal{D}(w) : \mathcal{D}(\varphi)\,\mathrm{d}y\,\mathrm{d}t = \iint_{(0,T)\times\Omega} \rho\mathcal{G}F\cdot\Phi\,\mathrm{d}y\,\mathrm{d}t + \int_\Omega \rho b\cdot\Phi(0,\cdot)\,\mathrm{d}y. \quad (4.32)$$

- *We say that* $W \in L^2(0,T;\mathcal{L}^2)$ *is a very weak solution of* (4.11)–(4.14) *if for any*

$$\Phi \in H^1(0,T;\mathcal{L}^2) \cap L^\infty(0,T;\mathcal{H}^1) \cap L^2(0,T;H^2(\mathcal{F})^3) \quad \textit{such that } \Phi(T,\cdot) = 0,$$

we have

$$\begin{aligned} -\iint_{(0,T)\times\mathcal{F}} W \cdot \Big[(\nabla X)^\top \partial_t \left[(\nabla X)\,\Phi\right] + \mathcal{M}\Phi + \mathcal{K}\Phi\Big]\,\mathrm{d}y\,\mathrm{d}t \\ + \int_0^T m\ell_W \cdot \left(-\ell_\Phi' - \omega_U \times \ell_\Phi + m^{-1}\int_{\partial\mathcal{S}} 2\mathbb{D}(\Phi)\nu\,\mathrm{d}\gamma_y\right)\mathrm{d}t \\ + \int_0^T J\omega_W \cdot \left(-\omega_\Phi'(t) - \omega_U \times \omega_\Phi + J^{-1}\int_{\partial\mathcal{S}} y \times 2\mathbb{D}(\Phi)\nu\,\mathrm{d}\gamma_y\right)\mathrm{d}t \\ = \iint_{(0,T)\times\Omega} \rho F \cdot \mathcal{G}\Phi\,\mathrm{d}y\,\mathrm{d}t + \int_\Omega \rho b \cdot \Phi(0,\cdot)\,\mathrm{d}y. \end{aligned} \tag{4.33}$$

Since the system (4.11)–(4.14) is obtained from (4.1)–(4.4) via a change of coordinates, the notions of weak solutions and very weak solutions for each system are equivalent. More precisely, we have the following result:

Lemma 4.6. *Assume* (h,Q,u) *is a weak solution of* (1.1)–(1.4) *in* $(0,T)$. *Assume* $f \in L^2(0,T;L^2(\Omega)^3)$ *and* $a \in \mathcal{L}^2$. *We define* F *by* (4.10).

- *The function* $w \in L^\infty(0,T;\mathcal{L}^2_{(h,Q)}) \cap L^2(0,T;\mathcal{H}^1_{(h,Q)})$ *is a weak solution of* (4.1)–(4.4) *if and only if* $W \in L^\infty(0,T;\mathcal{L}^2) \cap L^2(0,T;\mathcal{H}^1)$ *defined by* (2.17) *is a weak solution of* (4.11)–(4.14).
- *The function* $w \in L^2(0,T;\mathcal{L}^2_{(h,Q)})$ *is a very weak solution of* (4.1)–(4.4) *if and only if* $W \in L^2(0,T;\mathcal{L}^2)$ *defined by* (2.17) *is a very weak solution of* (4.11)–(4.14).

Proof. The first point is a direct consequence of (2.20) and (2.19). For the second point we write

$$\begin{aligned} \iint_{(0,T)\times\Omega} \rho_{(h,Q)} w \cdot \left(-\partial_t\varphi - (u\cdot\nabla)\varphi + \mathbb{A}_{(h,Q)}\varphi\right)\mathrm{d}x\,\mathrm{d}t \\ = \iint_{(0,T)\times\mathcal{F}_{(h,Q)}} w\cdot(-\partial_t\varphi - (u\cdot\nabla)\varphi - \Delta\varphi)\,\mathrm{d}x\,\mathrm{d}t \\ + m\ell_w \cdot \left(-\ell_\varphi'(t) + m^{-1}\int_{\partial\mathcal{S}_{(h,Q)}} 2\mathbb{D}(\varphi)\nu\,\mathrm{d}\gamma_x\right) \\ + J_Q\omega_w \cdot \left(-\omega_\varphi'(t) + J_Q^{-1}\int_{\partial\mathcal{S}_{(h,Q)}} (x-h(t))\times 2\mathbb{D}(\varphi)\nu\,\mathrm{d}\gamma_x\right). \end{aligned} \tag{4.34}$$

Then, using (2.20) and (2.19),

$$\begin{aligned} \iint_{(0,T)\times\mathcal{F}_{(h,Q)}} w\cdot(-\partial_t\varphi - (u\cdot\nabla)\varphi - \Delta\varphi)\,\mathrm{d}x\,\mathrm{d}t \\ = -\iint_{(0,T)\times\mathcal{F}} W\cdot\Big[(\nabla X)^\top\partial_t\left[(\nabla X)\,\Phi\right] + \mathcal{M}\Phi + \mathcal{K}\Phi\Big]\,\mathrm{d}y\,\mathrm{d}t, \end{aligned} \tag{4.35}$$

$$m\ell_w \cdot \left(-\ell_\varphi'(t) + m^{-1}\int_{\partial\mathcal{S}_{(h,Q)}} 2\mathbb{D}(\varphi)\nu\,\mathrm{d}\gamma_x\right) = m\ell_W \cdot \left(-\ell_\Phi' - \omega_U\times\ell_\Phi + m^{-1}\int_{\partial\mathcal{S}} 2\mathbb{D}(\Phi)\nu\,\mathrm{d}\gamma_y\right), \tag{4.36}$$

and

$$J_Q\omega_w \cdot \left(-\omega_\varphi'(t) + J_Q^{-1} \int_{\partial\mathcal{S}_{(h,Q)}} (x - h(t)) \times 2\mathbb{D}(\varphi)\nu \, \mathrm{d}\gamma_x \right)$$
$$= J\omega_W \cdot \left(-\omega_\Phi'(t) - \omega_U \times \omega_\Phi + J^{-1} \int_{\partial\mathcal{S}} y \times 2\mathbb{D}(\Phi)\nu \, \mathrm{d}\gamma_y \right). \quad (4.37)$$

□

From the above definitions and the above lemma, one can check that a strong solution of (4.1)–(4.4) is a weak solution of (4.1)–(4.4), and that a weak solution of (4.1)–(4.4) is a very weak solution of (4.1)–(4.4). Using Proposition 4.2, we deduce the following result:

Lemma 4.7. *Assume (h, Q, u) is a weak solution of (1.1)–(1.4) satisfying (1.15) or (1.16) ε given by Proposition 4.1. Then there exists at most one very weak solution of (4.1)–(4.4).*

Proof. The proof is similar to [20, Lemma 6.4]. □

Remark 4.8. *Assume (h, Q, u) is a weak solution of (1.1)–(1.4) satisfying (1.15) or (1.16) ε given by Proposition 4.1. Then any very weak solution w of (4.1)–(4.4) with $f = 0$ and $b = a$ satisfies*

$$w = u.$$

We use this property in what follows to deduce properties of u from the same properties for w.

5 Proof of the main result

In this section, we prove the main results namely Theorem 1.3 and Theorem 1.4. We are going to follow the same approach as [20]. Throughout this section we assume that (h, Q, u) is a weak solution of (1.1)–(1.4) satisfying (1.15) or (1.16) with ε given by Proposition 4.1 and Proposition 4.2. We recall the definition of $\mathcal{X}^p(0,T)$ from (2.22).

Lemma 5.1. *Assume (h, Q, u) is a weak solution of (1.1)–(1.4) satisfying (1.15) or (1.16) with ε given by Proposition 4.1. Then $tU \in \mathcal{X}^2(0,T)$.*

Proof. Let us set

$$w(t,\cdot) := tu(t,\cdot) \quad (t \in (0,T)). \quad (5.1)$$

Then we deduce from (1.14) that w is a weak solution of (4.1)–(4.4) with $f = u$ and $b = 0$. Since $u \in L^2(0,T;\mathcal{L}^2_{(h,Q)})$, we deduce from Proposition 4.1 and Lemma 4.7 that $W \in \mathcal{X}^2(0,T)$. □

Lemma 5.2. *Assume (h, Q, u) is a weak solution of (1.1)–(1.4) satisfying (1.15) or (1.16) with ε given by Proposition 4.1. Then $t^4U \in \mathcal{X}^4(0,T)$.*

Proof. Let us set

$$W(t,\cdot) = t^2U(t,\cdot) \quad (t \in (0,T)). \quad (5.2)$$

Then we deduce from Lemma 5.1 that W is the strong solution of (4.26)–(4.29) with

$$F := -t^2\mathcal{G}^{-1}\mathcal{M}U + 2tU = (\nabla X)^{-1}\left((tU - t\Lambda)\cdot\nabla\right)\left[(\nabla X)\,(tU)\right] + 2tU.$$

On the other hand, from Sobolev embeddings (see, for instance, [19, Lemma 3.3, p.80], [11, Proposition 4.3]), we have

$$L^2(0,T;H^2(\mathcal{F})) \cap H^1(0,T;L^2(\mathcal{F})) \hookrightarrow L^{10}(0,T;L^{10}(\mathcal{F})) \cap L^{10/3}(0,T;W^{1,10/3}(\mathcal{F})).$$

We thus deduce that $F \in L^{5/2}(0,T;\mathcal{L}^{5/2})$ and applying Proposition 4.3, we obtain $W = t^2U \in \mathcal{X}^{5/2}(0,T)$.

We can now repeat the argument: we set

$$W(t,\cdot) = t^4 U(t,\cdot) \quad (t \in (0,T)) \tag{5.3}$$

which is the strong solution of (4.26)–(4.29) with

$$F := -t^4 \mathcal{G}^{-1}\mathcal{M}U + 4t^3 U = (\nabla X)^{-1}\left(\left(t^2 U - t^2\Lambda\right)\cdot\nabla\right)\left[(\nabla X)\,(t^2 U)\right] + 4t^3 U.$$

From Sobolev embeddings (see, for instance, [19, Lemma 3.3, p.80], [11, Proposition 4.3]), we have

$$L^{5/2}(0,T;W^{2,5/2}(\mathcal{F}))\cap W^{1,5/2}(0,T;L^{5/2}(\mathcal{F})) \hookrightarrow L^r(0,T;L^r(\mathcal{F}))\cap L^5(0,T;W^{1,5}(\mathcal{F}))$$

for any $r > 5/2$. We thus deduce that $F \in L^4(0,T;\mathcal{L}^4)$ and applying Proposition 4.3, we obtain $W = t^4 U \in \mathcal{X}^4(0,T)$. □

Proposition 5.3. *Assume (h,Q,u) is a weak solution of (1.1)–(1.4) satisfying (1.15) or (1.16) with ε given by Proposition 4.1. Then*

$$t^{16}U \in H^2(0,T;\mathcal{L}^2)\cap H^1(0,T;\mathcal{D}(\mathbb{A}))\cap L^2(0,T;H^4(\mathcal{F})^3).$$

Proof. Let us set

$$W(t,\cdot) = t^8\partial_t U(t,\cdot) \quad (t\in(0,T)). \tag{5.4}$$

Then we deduce from Lemma 5.1 that W is a very weak solution of (4.11)–(4.14) with

$$F := \mathcal{G}^{-1}\left[-t^8\left(\partial_t\nabla X\right)^\top\partial_t\left[(\nabla X)\,U\right] - t^8\left(\nabla X\right)^\top\partial_t\left[(\partial_t\nabla X)\,U\right]\ -t^8\left(\partial_t\mathcal{M}\right)U + t^8\left(\partial_t\mathcal{K}\right)U\right] + 8t^7\partial_t U. \tag{5.5}$$

In order to estimate F, we first note that since $t^4U\in\mathcal{X}^4(0,T)$ and $t^2U\in L^4(0,T;\mathcal{L}^4)$, and by using (2.21), we have

$$t^4\ell',\ t^4\omega' \in L^4(0,T;\mathbb{R}^3).$$

Thus from (2.5) and Lemma 2.3, we deduce that for any $k\geqslant 0$,

$$t^4\partial_t\lambda\in L^4(0,T;C_0^k(\mathbb{R}^3)^3),\quad t^4\partial_t\Lambda\in L^4(0,T;C_0^k(\mathbb{R}^3)^3).$$

In particular, from (2.7) and Lemma 2.3,

$$t^4\partial_t^2\nabla X\in L^4(0,T;L^\infty(\Omega)^9).$$

The above relations and Lemma 5.1 yield that

$$\mathcal{G}^{-1}\left[-t^8\left(\partial_t\nabla X\right)^\top\partial_t\left[(\nabla X)\,U\right] - t^8\left(\nabla X\right)^\top\partial_t\left[(\partial_t\nabla X)\,U\right]\right] + 8t^7\partial_t U\in L^2(0,T;\mathcal{L}^2). \tag{5.6}$$

Now, from (2.16), we have

$$\begin{aligned}\left(\partial_t\mathcal{M}\right)U = {}&\left(\nabla\partial_t X\right)^\top\left((U-\Lambda)\cdot\nabla\right)\left[(\nabla X)\,U\right] + \left(\nabla X\right)^\top\left(\left(\partial_t U-\partial_t\Lambda\right)\cdot\nabla\right)\left[(\nabla X)\,U\right]\\ &+\left(\nabla X\right)^\top\left((U-\Lambda)\cdot\nabla\right)\left[(\nabla\partial_t X)\,U\right]\end{aligned} \tag{5.7}$$

and from $t^4U\in\mathcal{X}^4(0,T)$, we deduce that $t^8\left(\partial_t\mathcal{M}\right)U\in L^2(0,T;\mathcal{L}^2)$. From (2.13) and (2.14), we deduce similarly that $t^8\left(\partial_t\mathcal{K}\right)U\in L^2(0,T;\mathcal{L}^2)$. Combining these relations with (5.5) and (5.6), we conclude $F\in L^2(0,T;\mathcal{L}^2)$. Combining Proposition 4.1 and Lemma 4.7, we deduce that $W = t^8\partial_t U\in\mathcal{X}^2(0,T)$. Note in particular that

$$t^8U\in H^1(0,T;H^2(\mathcal{F})^3)\hookrightarrow L^\infty(0,T;L^\infty(\mathcal{F})^3). \tag{5.8}$$

Now we use the fact that $\left(t^8U, t^8P\right)$ satisfies the elliptic system

$$\begin{cases} -\mathcal{K}(t^8U) + \nabla(t^8P) = -\left(\nabla X\right)^\top t^8\partial_t\left[\left(\nabla X\right)U\right] - t^8\mathcal{M}U & \text{in } (0,T)\times\mathcal{F}, \\ \operatorname{div}(t^8U) = 0 & \text{in } (0,T)\times\mathcal{F}, \\ (t^8U) = t^8\ell_U + t^8\omega_U \times y & \text{on } (0,T)\times\partial\mathcal{S}, \\ (t^8U) = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{5.9}$$

We have already obtained that

$$-\left(\nabla X\right)^\top t^8\partial_t\left[\left(\nabla X\right)U\right] \in L^2(0,T;H^2(\mathcal{F}))^3, \quad t^8\mathcal{M}U \in L^2(0,T;L^2(\mathcal{F}))^3.$$

Then, from (2.16), $t^8\partial_{y_j}\left(\mathcal{M}U\right)$ involves terms that can be estimated as before and

$$\left(\nabla X\right)^\top\left(\left(\partial_{y_j}\left(t^4U\right)\right)\cdot\nabla\right)\left[\left(\nabla X\right)\left(t^4U\right)\right], \quad \left(\nabla X\right)^\top\left(\left(t^4U\right)\cdot\nabla\right)\left[\left(\nabla X\right)\partial_{y_j}\left(t^4U\right)\right].$$

Using Lemma 2.3 and that $t^4U \in L^4(0,T;W^{2,4}(\mathcal{F}))^3$, we deduce that $t^8\mathcal{M}U \in L^2(0,T;H^1(\mathcal{F}))^3$. Thus from Proposition 3.4, we deduce $t^8U \in L^2(0,T;H^3(\mathcal{F}))^3$.

On the other hand, $t^8\partial_{y_i}\partial_{y_j}\left(\mathcal{M}U\right)$ involves terms that can be estimated as before and terms of the form

$$\left(\nabla X\right)^\top\left(\left(\partial_{y_i}\partial_{y_j}\left(t^4U\right)\right)\cdot\nabla\right)\left[\left(\nabla X\right)\left(t^4U\right)\right], \quad \left(\nabla X\right)^\top\left(\left(\partial_{y_j}\left(t^4U\right)\right)\cdot\nabla\right)\left[\left(\nabla X\right)\partial_{y_i}\left(t^4U\right)\right],$$
$$\left(\nabla X\right)^\top\left(\left(t^4U\right)\cdot\nabla\right)\left[\left(\nabla X\right)\partial_{y_i}\partial_{y_j}\left(t^4U\right)\right].$$

Using Lemma 2.3, $t^4U \in L^4(0,T;W^{2,4}(\mathcal{F}))^3$, $t^8\mathcal{M}U \in L^2(0,T;H^1(\mathcal{F}))^3$ and (5.8), we deduce that $t^{16}\mathcal{M}U \in L^2(0,T;H^2(\mathcal{F}))^3$. Applying Proposition 3.4 allows us to obtain $t^{16}U \in L^2(0,T;H^4(\mathcal{F}))^3$. □

We are now in a position to prove Theorem 1.3 and Theorem 1.4.

Proof of Theorem 1.3. We show by induction that for any $k \geqslant 2$, there exists $m \geqslant 1$ such that

$$t^m\partial_t^jU \in L^2(0,T;H^{2(k-j)}(\mathcal{F})^3) \quad (0\leqslant j\leqslant k), \quad t^m\ell^{(j)},\ t^m\omega^{(j)} \in L^2(0,T)^3 \quad (0\leqslant j\leqslant k). \tag{5.10}$$

We have already obtained the above relation for $k=2$ in the previous proposition. From the above relation, we deduce that, for $M \geqslant m$, $W := t^M\partial_t^kU$ is the very weak solution of (4.11)–(4.14) with F defined by

$$\begin{aligned} \mathcal{G}F := -t^M \sum_{j_1+j_2+j_3=k, j_3\neq k} \binom{k}{j_1,j_2,j_3}\left[\partial_t^{j_1}\nabla X\right]^\top\partial_t\left(\left[\partial_t^{j_2}\nabla X\right]\left[\partial_t^{j_3}U\right]\right) \\ - t^M\sum_{j=1}^k\binom{k}{j}\left(\partial_t^j\mathcal{M}\right)\partial_t^{k-j}U + t^M\sum_{j=1}^k\binom{k}{j}\left(\partial_t^j\mathcal{K}\right)\partial_t^{k-j}U + Mt^{M-1}\mathcal{G}\partial_t^kU. \end{aligned} \tag{5.11}$$

From (5.10), we deduce that for $M \geqslant 3m$ and $\alpha \geqslant 0$, $t^M\partial_t^k\lambda \in L^2(0,T;C_0^\alpha(\mathbb{R}^3))^3$ and thus from the Faà di Bruno formula and from (2.7), we conclude that for $M \geqslant C(m)$ and for any $\alpha \geqslant 0$, $t^M\partial_t^{k+1}X \in L^2(0,T;C_0^\alpha(\mathbb{R}^3))^3$. We also obtain that for for $M \geqslant C(m)$ and for any $\alpha \geqslant 0$, $t^M\partial_t^{k+1}\nabla Y(X) \in L^2(0,T;C_0^\alpha(\mathbb{R}^3))^3$. We deduce that, with $M \geqslant C(m)$, for $j_1, j_2 \leqslant k$ and $j_3+1 \leqslant k$,

$$t^M\left[\partial_t^{j_1}\nabla X\right]^\top\left[\partial_t^{j_2}\nabla X\right]\left[\partial_t^{j_3+1}U\right] \in L^2(0,T;L^2(\mathcal{F}))^3$$

and for $j_1, j_3 \leqslant k$, and $j_2+1 \leqslant k$,

$$t^M\left[\partial_t^{j_1}\nabla X\right]^\top\left[\partial_t^{j_2+1}\nabla X\right]\left[\partial_t^{j_3}U\right] \in L^2(0,T;L^2(\mathcal{F}))^3.$$

For $j_2 = k$, we use that $U \in L^\infty(0,T;L^\infty(\mathcal{F}))^3$ to deduce

$$t^M\left[\nabla X\right]^\top\left[\partial_t^{k+1}\nabla X\right]U \in L^2(0,T;L^2(\mathcal{F}))^3.$$

Next, combining $t^m\partial_t^{k-j}U \in L^2(0,T;H^2(\Omega))^3$ for $j \geqslant 1$ with (2.13) and (2.14), we deduce that for M large enough,

$$t^M \sum_{j=1}^{k} \binom{k}{j} \left(\partial_t^j \mathcal{K}\right) \partial_t^{k-j} U \in L^2(0,T;L^2(\mathcal{F}))^3.$$

From (2.16), we have

$$\left(\partial_t^j \mathcal{M}\right) \partial_t^{k-j} U = \sum_{n_1+n_2+n_3=j} \binom{j}{n_1,n_2,n_3} \left(\partial_t^{n_1}\nabla X\right)^\top \left((\partial_t^{n_2}U - \partial_t^{n_2}\Lambda)\cdot\nabla\right) \left[\left(\partial_t^{n_3}\nabla X\right)\left(\partial_t^{k-j}U\right)\right]. \tag{5.12}$$

Since $n_1, n_3 \leqslant j \leqslant k$, we have for M large enough, $t^M\partial_t^{n_1}\nabla X, \partial_t^{n_3}\nabla X \in L^\infty(0,T;L^\infty(\mathcal{F}))^9$. By using (2.15), we have for M large enough and $\alpha \geqslant 0$, $t^M\partial_t^k\Lambda \in L^2(0,T;C_0^\alpha(\mathbb{R}^3))^3$ and thus for $n_2 \leqslant j \leqslant k$,

$$t^M \left(\partial_t^{n_2}U - \partial_t^{n_2}\Lambda\right) \in L^2(0,T;L^2(\mathcal{F}))^3. \tag{5.13}$$

On the other hand, by the hypothesis of induction, if $j \in \{1,\dots,k\}$,

$$t^{m+1}\partial_t^{k-j}U \in L^2(0,T;H^{2j}(\mathcal{F}))^3 \cap H^1(0,T;H^{2j-2}(\mathcal{F}))^3$$

and thus

$$\nabla\left(t^{m+1}\partial_t^{k-j}U\right) \in L^\infty(0,T;H^{2j-2}(\mathcal{F}))^9. \tag{5.14}$$

If $j \geqslant 2$, a Sobolev embedding and the above relation lead to

$$\nabla\left(t^{m+1}\partial_t^{k-j}U\right) \in L^\infty(0,T;L^\infty(\mathcal{F}))^9,$$

and combining it with (5.13), we deduce

$$t^M \left(\partial_t^{n_1}\nabla X\right)^\top \left((\partial_t^{n_2}U - \partial_t^{n_2}\Lambda)\cdot\nabla\right) \left[\left(\partial_t^{n_3}\nabla X\right)\left(\partial_t^{k-j}U\right)\right] \in L^2(0,T;L^2(\mathcal{F}))^3. \tag{5.15}$$

For $j = 1$, we only get from (5.14) that

$$\nabla\left(t^{m+1}\partial_t^{k-j}U\right) \in L^\infty(0,T;L^2(\mathcal{F}))^9 \tag{5.16}$$

but in that case, $n_2 \leqslant 1$ and by the hypothesis of induction,

$$t^m\partial_t^{n_2}U \in L^2(0,T;H^{2(k-1)}(\mathcal{F}))^3 \subset L^2(0,T;L^\infty(\mathcal{F}))^3$$

since $k \geqslant 2$. Combining the two above relations implies (5.15) in that case and thus for M large enough, $t^M\left(\partial_t^j\mathcal{M}\right)\partial_t^{k-j}U \in L^2(0,T;L^2(\mathcal{F}))^3$. We conclude that for M large enough, F defined by (5.11) satisfies $F \in L^2(0,T;L^2(\mathcal{F}))^3$. Combining Proposition 4.1 and Lemma 4.7, we deduce that $W = t^M\partial_t^k U \in \mathcal{X}^2(0,T)$.

It remains to show that for $j = 0,\dots,k-1$, there exists $M_j \geqslant 0$ such that

$$t^{M_j}\partial_t^j U \in L^2(0,T;H^{2(k-j+1)}(\mathcal{F})^3). \tag{5.17}$$

We have already obtained the above relation for $j = k+1$ and $j = k$. Let us assume that it holds for $j_0+1 \leqslant j \leqslant k+1$, with $j_0 \in \{0,\dots,k-1\}$ and let us show it for j_0. We first see that for M large enough, $(W,\Pi) := \left(t^M\partial_t^{j_0}U, t^M\partial_t^{j_0}P\right)$ are the solution of (3.22) with

$$\begin{aligned}\mathcal{G}F := -t^M \sum_{j_1+j_2+j_3=j_0} \binom{j_0}{j_1,j_2,j_3} \left[\partial_t^{j_1}\nabla X\right]^\top \partial_t\left(\left[\partial_t^{j_2}\nabla X\right]\left[\partial_t^{j_3}U\right]\right) \\ - t^M\sum_{j=0}^{j_0}\binom{j_0}{j}\left(\partial_t^j\mathcal{M}\right)\partial_t^{j_0-j}U + t^M\sum_{j=1}^{j_0}\binom{j_0}{j}\left(\partial_t^j\mathcal{K}\right)\partial_t^{j_0-j}U.\end{aligned} \tag{5.18}$$

Let us first analyze the first sum in the right-hand side of (5.18). For $j_1 \leqslant j_0 \leqslant k-1$ and $j_2 \leqslant j_0 \leqslant k-1$, we have for any $\alpha \geqslant 0$,

$$t^M \partial_t^{j_1} \nabla X \in L^\infty(0,T; C_0^\alpha(\mathbb{R}^3)^3), \quad t^M \partial_t^{j_2+1} \nabla X \in L^\infty(0,T; C_0^\alpha(\mathbb{R}^3)^3).$$

Now, from (5.10), we have for $j_3 \leqslant j_0$,

$$t^m \partial_t^{j_3} U \in L^2(0,T; H^{2(k-j_3)}(\Omega))^3 \subset L^2(0,T; H^{2(k-j_0)}(\Omega))^3,$$

and the same result holds for $t^m \partial_t^{j_3+1} U$ with $j_3 + 1 \leqslant j_0$. The only term that remains to be tackled in the first sum in the right-hand side of (5.18) is

$$-t^M \mathcal{G} \partial_t^{j_0+1} U.$$

Using (5.17) for $j = j_0 + 1$, we deduce that the above term is in $L^2(0,T; H^{2(k-j_0)}(\Omega))^3$ for $M \geqslant M_{j_0+1}$ and we have thus shown that for M large enough

$$-t^M \sum_{j_1+j_2+j_3=j_0} \binom{j_0}{j_1, j_2, j_3} \left[\partial_t^{j_1} \nabla X\right]^\top \partial_t \left(\left[\partial_t^{j_2} \nabla X\right] \left[\partial_t^{j_3} U\right]\right) \in L^2(0,T; H^{2(k-j_0)}(\Omega))^3. \tag{5.19}$$

For the second sum in the right-hand side of (5.18), we use (2.16) to write it as

$$\begin{aligned} &- t^M \sum_{j=0}^{j_0} \binom{j_0}{j} \left(\partial_t^j \mathcal{M}\right) \partial_t^{j_0-j} U \\ &\qquad = -t^M \sum_{j_1+j_2+j_3+j_4=j_0} \binom{j_0}{j_1, j_2, j_3, j_4} \left(\partial_t^{j_1} \nabla X\right)^\top \left(\left(\partial_t^{j_2} U - \partial_t^{j_2} \Lambda\right) \cdot \nabla\right) \left[\left(\partial_t^{j_3} \nabla X\right) \left(\partial_t^{j_4} U\right)\right]. \end{aligned} \tag{5.20}$$

We consider 3 cases:
Case 1: if $j_2 \in \{0, \dots, j_0 - 1\}$ and if $j_4 \in \{0, \dots, j_0 - 1\}$, then we deduce from (5.10) that

$$t^M \left(\partial_t^{j_2} U - \partial_t^{j_2} \Lambda\right) \in L^\infty(0,T; H^{2(k-j_2)-1}(\Omega))^3 \subset L^\infty(0,T; H^{2(k-j_0)+1}(\Omega))^3$$

and we have

$$\nabla \left(t^m \partial_t^{j_4} U\right) \in L^2(0,T; H^{2(k-j_4)-1}(\Omega))^9 \subset L^2(0,T; H^{2(k-j_0)+1}(\Omega))^9$$

so that, since $2(k-j_0)+1 \geqslant 3$,

$$t^M \left(\partial_t^{j_1} \nabla X\right)^\top \left(\left(\partial_t^{j_2} U - \partial_t^{j_2} \Lambda\right) \cdot \nabla\right) \left[\left(\partial_t^{j_3} \nabla X\right) \left(\partial_t^{j_4} U\right)\right] \in L^2(0,T; H^{2(k-j_0)+1}(\Omega))^3.$$

Case 2: if $j_2 = j_0$, then $j_1 = j_3 = j_4 = 0$, and we deduce from (5.10) that

$$t^M \left(\partial_t^{j_0} U - \partial_t^{j_0} \Lambda\right) \in L^\infty(0,T; H^{2(k-j_0)-1}(\Omega))^3$$

and

$$\nabla (t^m U) \in L^2(0,T; H^{2k-1}(\Omega))^9.$$

If $j_0 = 0$, then $2(k-j_0) - 1 = 2k - 1 \geqslant 3$ and we deduce that

$$t^M (\nabla X)^\top \left(\left(\partial_t^{j_0} U - \partial_t^{j_0} \Lambda\right) \cdot \nabla\right) [(\nabla X) U] \in L^2(0,T; H^{2(k-j_0)-1}(\Omega))^3. \tag{5.21}$$

Else, $j_0 \geqslant 1$ and

$$\nabla (t^m U) \in L^2(0,T; W^{2(k-j_0)-1,\infty}(\Omega))^9$$

and we obtain again (5.21).

Case 3: if $j_4 = j_0$, then $j_1 = j_2 = j_3 = 0$, and we deduce from (5.10) that

$$t^M (U - \Lambda) \in L^\infty(0, T; H^{2k-1}(\Omega))^3$$

and

$$\nabla \left(t^m \partial_t^{j_0} U \right) \in L^2(0, T; H^{2(k-j_0)-1}(\Omega))^9.$$

With the same proof as in Case 2, we obtain

$$t^M (\nabla X)^\top ((U - \Lambda) \cdot \nabla) \left[(\nabla X) \partial_t^{j_0} U \right] \in L^2(0, T; H^{2(k-j_0)-1}(\Omega))^3. \tag{5.22}$$

We can conclude that

$$-t^M \sum_{j=0}^{j_0} \binom{j_0}{j} \left(\partial_t^j \mathcal{M} \right) \partial_t^{j_0 - j} U \in L^2(0, T; H^{2(k-j_0)-1}(\Omega))^3. \tag{5.23}$$

For the second sum in the right-hand side of (5.18), we use (2.13), (2.14) and (5.10) to obtain that

$$t^M \sum_{j=1}^{j_0} \binom{j_0}{j} \left(\partial_t^j \mathcal{K} \right) \partial_t^{j_0 - j} U \in L^2(0, T; H^{2(k-j_0)}(\Omega))^3. \tag{5.24}$$

We thus deduce from (5.20), (5.19), (5.23), (5.24) and Proposition 3.4 that for some $M \geqslant 0$,

$$t^M \partial_t^{j_0} U \in L^2(0, T; H^{2(k-j_0)+1}(\Omega))^3. \tag{5.25}$$

We can use this relation to improve (5.23) and more precisely (5.21) and (5.22). Indeed, for $j_2 = j_0$ and $j_1 = j_3 = j_4 = 0$, we have now

$$t^M \left(\partial_t^{j_0} U - \partial_t^{j_0} \Lambda \right) \in L^\infty(0, T; H^{2(k-j_0)}(\Omega))^3$$

and

$$\nabla (t^m U) \in L^2(0, T; H^{2k-1}(\Omega))^9.$$

If $j_0 = 0$, we use (5.25) to replace the last relation by

$$\nabla (t^m U) \in L^2(0, T; H^{2k}(\Omega))^9$$

and, using that $2(k - j_0) = 2k \geqslant 4$, we deduce

$$t^M (\nabla X)^\top \left(\left(\partial_t^{j_0} U - \partial_t^{j_0} \Lambda \right) \cdot \nabla \right) [(\nabla X) U] \in L^2(0, T; H^{2(k-j_0)}(\Omega))^3. \tag{5.26}$$

Else, if $j_0 \geqslant 1$, then

$$\nabla (t^m U) \in L^2(0, T; H^{2(k-j_0)}(\Omega))^9$$

and since $2(k - j_0) \geqslant 2$, we deduce (5.26). For the other case, $j_4 = j_0$ and $j_1 = j_2 = j_3 = 0$, we have now

$$t^M (U - \Lambda) \in L^\infty(0, T; H^{2k-1}(\Omega))^3$$

and

$$\nabla \left(t^M \partial_t^{j_0} U \right) \in L^2(0, T; H^{2(k-j_0)}(\Omega))^9.$$

With the same manipulations as for the previous case, we deduce

$$t^M (\nabla X)^\top ((U - \Lambda) \cdot \nabla) \left[(\nabla X) \partial_t^{j_0} U \right] \in L^2(0, T; H^{2(k-j_0)}(\Omega))^3. \tag{5.27}$$

The above relations allow us to conclude that

$$-t^M \sum_{j=0}^{j_0} \binom{j_0}{j} \left(\partial_t^j \mathcal{M}\right) \partial_t^{j_0-j} U \in L^2(0,T;H^{2(k-j_0)}(\Omega))^3. \tag{5.28}$$

We thus deduce from (5.20), (5.19), (5.28), (5.24) and Proposition 3.4 that for some $M \geqslant 0$,

$$t^M \partial_t^{j_0} U \in L^2(0,T;H^{2(k-j_0+1)}(\Omega))^3. \tag{5.29}$$

This concludes the induction (5.17) and then the induction (5.10), and this ends the proof of the theorem. □

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