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# Dedekind sums and mean square value of $L(1, \chi)$ over subgroups

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## Abstract

An explicit formula for the quadratic mean value at  $s = 1$  of the Dirichlet  $L$ -functions associated with the odd Dirichlet characters modulo  $f > 2$  is known. Here we present a situation where we could prove an explicit formula for the quadratic mean value at  $s = 1$  of the Dirichlet  $L$ -functions associated with the odd Dirichlet characters modulo not necessarily prime moduli  $f > 2$  that are trivial on a subgroup  $H$  of the multiplicative group  $(\mathbb{Z}/f\mathbb{Z})^*$ . This explicit formula involves summation  $S(H, f)$  of Dedekind sums  $s(h, f)$  over the  $h \in H$ . A result on some cancelation of the denominators of the  $s(h, f)$ 's when computing  $S(H, f)$  is known. Here, we prove that for some explicit families of  $f$ 's and  $H$ 's this known result on cancelation of denominators is the best result one can expect. Finally, we surprisingly prove that for  $p$  a prime,  $m \geq 2$  and  $1 \leq n \leq m/2$ , the values of the Dedekind sums  $s(h, p^m)$  do not depend on  $h$  as  $h$  runs over the elements of order  $p^n$  of the multiplicative cyclic group  $(\mathbb{Z}/p^m\mathbb{Z})^*$ .

## 1 A general mean square value formula in terms of Dedekind sums

For  $c \in \mathbb{Z}$  and  $d > 1$  such that  $\gcd(c, d) = 1$ , the *Dedekind sum* is defined by

$$s(c, d) := \frac{1}{4d} \sum_{a=1}^{d-1} \cot\left(\frac{\pi a}{d}\right) \cot\left(\frac{\pi ac}{d}\right) \quad (1)$$

(see [Apo, Chapter 3, Exercise 11] or [RG, (26)]). It depends only on  $c$  modulo  $d$ . We also set  $s(c, 1) = 0$  for  $c \in \mathbb{Z}$ . Notice that  $s(c^*, d) = s(c, d)$  whenever  $cc^* \equiv 1 \pmod{d}$  (make the change of variables  $n \mapsto nc$  in  $s(c^*, d)$ ). We have a reciprocity law for Dedekind sums (see e.g. [Apo, Theorem 3.7], [RG, (4)] or [Lou15, (7) and (9)])

$$s(c, d) + s(d, c) = \frac{c^2 + d^2 - 3cd + 1}{12cd} \quad (c > 1, d > 1, \gcd(c, d) = 1).$$

We deduce (by induction) that  $s(c, d) \in \mathbb{Q}$  and that (see also [Lou94, Lemma (a)(i)])

$$s(1, d) = \frac{(d-1)(d-2)}{12d} \quad (d \geq 1). \quad (2)$$

For  $d > 1$  and  $\gcd(c, d) = 1$ , we set

$$\tilde{s}(c, d) := \frac{1}{4d} \sum_{\substack{n=1 \\ \gcd(n, d)=1}}^{d-1} \cot\left(\frac{\pi n}{d}\right) \cot\left(\frac{\pi nc}{d}\right). \quad (3)$$

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Using (1) we have

$$\tilde{s}(c, d) = \sum_{\delta|d} \frac{\mu(\delta)}{\delta} s(c, d/\delta). \quad (4)$$

In particular, using (2) we obtain

$$\tilde{s}(1, d) = \frac{\phi(d)}{12} \left( \prod_{p|d} \left( 1 + \frac{1}{p} \right) - \frac{3}{d} \right) \quad (d > 1). \quad (5)$$

For  $f > 2$ , let  $X_f$  be the group of order  $\phi(f)$  of the Dirichlet characters modulo  $f$ . Let  $X_f^- := \{\chi \in X_f \text{ and } \chi(-1) = -1\}$  be the set of the  $\phi(f)/2$  odd Dirichlet characters modulo  $f$ . If  $H$  is a subgroup of order  $n$  of the multiplicative group  $(\mathbb{Z}/f\mathbb{Z})^*$  which does not contain  $-1$ , we set

$$X_f^-(H) = \{\chi \in X_f^-; \chi|_H = 1\}.$$

Hence,  $\#X_f^-(H) = \phi(f)/(2n)$ . Let  $L(s, \chi)$  be the Dirichlet  $L$ -function associated with  $\chi \in X_f$ .

**Theorem 1.** *Let  $H$  be a subgroup of order  $n$  of the multiplicative group  $(\mathbb{Z}/f\mathbb{Z})^*$ , with  $f > 2$ . Assume that  $-1 \notin H$ , which is the case if  $n$  is odd. We have the mean square value formula*

$$M(f, H) := \frac{1}{\#X_f^-(H)} \sum_{\chi \in X_f^-(H)} |L(1, \chi)|^2 = \frac{2\pi^2}{f} \tilde{S}(H, f), \text{ where } \tilde{S}(H, f) := \sum_{h \in H} \tilde{s}(h, f). \quad (6)$$

In particular, by (5), we have the mean square value formula (see also [Lou16, Theorem 2])

$$M(f, \{1\}) := \frac{2}{\phi(f)} \sum_{\chi \in X_f^-} |L(1, \chi)|^2 = \tilde{s}(1, f) = \frac{\pi^2 \phi(f)}{6f} \left( \prod_{p|f} \left( 1 + \frac{1}{p} \right) - \frac{3}{f} \right). \quad (7)$$

*Proof.* For (6), see [Lou16, Proof of Theorem 2]. □

**Corollary 2.** *Let  $n \geq 1$  be an odd divisor of  $p-1$ , where  $p \geq 3$  is an odd prime number. Let  $H_n$  be the only subgroup of order  $n$  of the multiplicative cyclic group  $(\mathbb{Z}/p\mathbb{Z})^*$ . We have the mean square value formula*

$$M(p, H_n) := \frac{1}{\#X_p^-(H_n)} \sum_{\chi \in X_p^-(H_n)} |L(1, \chi)|^2 = \frac{2\pi^2}{p} S(H_n, p) = \frac{\pi^2}{6} \left( 1 + \frac{N(H_n, p)}{p} \right),$$

where

$$S(H_n, p) := \sum_{h \in H_n} s(h, p) \text{ and } N(H_n, p) := 12S(H_n, p) - p.$$

Moreover, by [Lou19, Theorem 6], for  $n > 1$  the rational number  $2S(H_n, p)$  is an integer of the same parity as  $(p-1)/2$  and  $N(H_n, p) = 12S(H_n, p) - p$  is an odd rational integer.

By [LM21, Theorem 1.1], we have

$$N(p, H_n) = o(p) \text{ and } M(p, H_n) = \frac{\pi^2}{6} + o(1)$$

as  $p$  tends to infinity and  $H_n$  runs over the subgroup of  $(\mathbb{Z}/p\mathbb{Z})^*$  of odd orders  $n \leq \frac{\log p}{3 \log \log p}$ . By [MS, Theorem 2.1 and Remark 2.2] we can relax this constraint on  $n$  down to  $\phi(n) = o(\log p)$ , which is optimal. Indeed, if  $p$  runs over the Mersenne primes  $p = 2^n - 1$ ,  $n \geq 3$  odd and prime, and  $H_n$  is the subgroup of order  $n$  of  $(\mathbb{Z}/p\mathbb{Z})^*$  generated by 2, then  $N(p, H_n) = 2p - (6n - 3)$ , by [LM23, Theorem 5.4].

## 2 A conjecture for the case of prime moduli

According to our numerical computations it seems reasonable to conjecture the following:

**Conjecture 3.** (i). (See [Lou19, Section 2.2] for some numerical evidence). Let  $p$  range over the odd prime integers and  $H$  over the subgroups of odd order of the multiplicative cyclic group  $(\mathbb{Z}/p\mathbb{Z})^*$ . Then  $N(H, p) = 12S(H, p) - p \leq 0$  and hence  $M(p, H) \leq \pi^2/6$  with a probability greater than or equal to  $1/2$ , i.e.

$$\liminf_{B \rightarrow \infty} \rho(B) \geq \frac{1}{2}, \text{ where } \rho(B) := \frac{\#\{(p, n); n \geq 1 \text{ odd divides } p-1, N(H_n, p) \leq 0 \text{ and } p \leq B\}}{\#\{(p, n); n \geq 1 \text{ odd divides } p-1 \text{ and } p \leq B\}}.$$

(ii). For a given odd integer  $n \geq 3$ , let  $p$  range over the odd prime integers  $p \equiv 1 \pmod{2n}$ . Let  $H_n$  be the only subgroup of order  $n$  of the multiplicative cyclic group  $(\mathbb{Z}/p\mathbb{Z})^*$ . Then  $N(H_n, p) = 12S(H_n, p) - p \leq 0$  and hence  $M(p, H_n) \leq \pi^2/6$  with a probability greater than or equal to  $1/2$ , i.e.

$$\liminf_{B \rightarrow \infty} \rho_n(B) \geq \frac{1}{2}, \text{ where } \rho_n(B) := \frac{\#\{p; p \equiv 1 \pmod{2n}, N(H_n, p) \leq 0 \text{ and } p \leq B\}}{\#\{p; p \equiv 1 \pmod{2n} \text{ and } p \leq B\}}.$$

If  $p \equiv 1 \pmod{6}$  then  $N(H_3, p) = -1$  (Corollary 8), and point (ii) of Conjecture 3 holds true for  $n = 3$ . As an example of our computation, for  $n = 9$  we have the following numerical datas:

$B$	$\#\{p \leq B; p \equiv 1 \pmod{18}\}$	$\#\{p; p \leq B \equiv 1 \pmod{18} \text{ and } N(H_9, p) \leq 0\}$	$\rho_9(B)$
$10^5$	1592	838	0.52638...
$10^6$	13063	6820	0.52208...
$10^7$	110772	56779	0.51257...
$10^8$	959959	490984	0.51146...
$10^9$	8474566	4317341	0.50944...
$10^{10}$	75841588	38573928	0.50861...
$10^{11}$	686345266	348497259	0.50775...

and

$A$	$B$	$c_{\text{prime}}(A, B)$	$c_{\leq 0}(A, B)$	$\rho_9(A, B)$
$10^{10}$	$10^6$	7226	3695	0.51134...
$10^{10}$	$10^7$	72505	36731	0.50659...
$10^{10}$	$10^8$	724408	368910	0.50925...
$10^{10}$	$10^9$	7224235	3672183	0.50831...
$10^{10}$	$10^{10}$	71191018	36166905	0.50802...
$10^{11}$	$10^6$	6558	3301	0.50335...
$10^{11}$	$10^7$	65747	33253	0.50577...
$10^{11}$	$10^8$	658053	333640	0.50701...
$10^{11}$	$10^9$	6579598	3337952	0.50731...
$10^{11}$	$10^{10}$	65673261	33320115	0.50736...
$10^{12}$	$10^6$	6076	3145	0.51761...
$10^{12}$	$10^7$	60361	30850	0.51109...
$10^{12}$	$10^8$	602908	305658	0.50697...
$10^{12}$	$10^9$	6031209	3056473	0.50677...
$10^{12}$	$10^{10}$	60305132	30562355	0.50679...
$10^{13}$	$10^6$	5564	2782	0.50000...
$10^{13}$	$10^7$	55572	28003	0.50390...
$10^{13}$	$10^8$	557166	282186	0.50646...
$10^{13}$	$10^9$	5566301	2817547	0.50617...
$10^{13}$	$10^{10}$	55673215	28179022	0.50615...

where

$$\begin{aligned} c_{\text{prime}}(A, B) &:= \#\{p \equiv 1 \pmod{18}; A \leq p \leq A+B\}, \\ c_{\leq 0}(A, B) &:= \#\{p \equiv 1 \pmod{18}; A \leq p \leq A+B \text{ and } N(H_9, p) \leq 0\} \\ \text{and } \rho_9(A, B) &:= c_{\leq 0}(A, B)/c_{\text{prime}}(A, B). \end{aligned}$$

For  $n = 5, 7, 11, 13$  and  $15$  we have the following numerical datas:

$B$	$\#\{p \leq B; p \equiv 1 \pmod{10}\}$	$\#\{p; p \leq B \equiv 1 \pmod{10} \text{ and } N(H_5, p) \leq 0\}$	$\rho_5(B)$
$10^5$	2387	1335	0.55927...
$10^6$	19617	10403	0.53030...
$10^7$	166104	86814	0.52264...
$10^8$	1440298	744791	0.51710...
$10^9$	12711386	6540511	0.51453...
$10^{10}$	113761519	58352843	0.51294...
$B$	$\#\{p \leq B; p \equiv 1 \pmod{14}\}$	$\#\{p; p \leq B \equiv 1 \pmod{14} \text{ and } N(H_7, p) \leq 0\}$	$\rho_7(B)$
$10^5$	1593	823	0.51663...
$10^6$	13063	6770	0.51825...
$10^7$	110653	56848	0.51375...
$10^8$	960023	490970	0.51141...
$10^9$	8474221	4322243	0.51004...
$10^{10}$	75840762	38584999	0.50876...
$B$	$\#\{p \leq B; p \equiv 1 \pmod{22}\}$	$\#\{p; p \leq B \equiv 1 \pmod{22} \text{ and } N(H_{11}, p) \leq 0\}$	$\rho_{11}(B)$
$10^5$	945	506	0.53544...
$10^6$	7858	4099	0.52163...
$10^7$	66386	34669	0.52223...
$10^8$	576103	300012	0.52076...
$10^9$	5084435	2634688	0.51818...
$10^{10}$	45504543	23481241	0.51601...
$B$	$\#\{p \leq B; p \equiv 1 \pmod{26}\}$	$\#\{p; p \leq B \equiv 1 \pmod{18} \text{ and } N(H_{13}, p) \leq 0\}$	$\rho_{13}(B)$
$10^5$	798	397	0.49749...
$10^6$	6539	3307	0.50573...
$10^7$	55376	28071	0.50691...
$10^8$	480132	242633	0.50534...
$10^9$	4237228	2139817	0.50500...
$10^{10}$	37919477	19125424	0.50436...
$B$	$\#\{p \leq B; p \equiv 1 \pmod{30}\}$	$\#\{p; p \leq B \equiv 1 \pmod{30} \text{ and } N(H_{15}, p) \leq 0\}$	$\rho_{15}(B)$
$10^5$	1189	648	0.54499...
$10^6$	9807	5129	0.52299...
$10^7$	83003	42787	0.51548...
$10^8$	719984	368612	0.51197...
$10^9$	6355189	3240295	0.50986...
$10^{10}$	56878661	28940619	0.50881...

### 3 Mean square values of $L(1, \chi)$ over subgroups and bounds on relative class numbers of imaginary abelian number fields

We refer the reader to [Was, Chapters 3, 4 and 11] for more background details. Let  $K$  be an imaginary abelian number field of degree  $m = 2n > 1$  and conductor  $f_K > 1$ . Let  $f > 1$  be any integer divisible by  $f_K$ , i.e. let  $K$  be an imaginary subfield of a cyclotomic number field  $\mathbb{Q}(\zeta_f)$  (Kronecker-Weber's theorem). Let  $w_K$  be its number of complex roots of unity. Let  $Q_K \in \{1, 2\}$  be its Hasse unit index. Hence,  $Q_K = 1$  if  $K/\mathbb{Q}$  is cyclic (see e.g. [Lem, Example 5, page 352]). In particular, for any imaginary subfield  $K$  of  $\mathbb{Q}(\zeta_p)$  we have  $Q_K = 1$  and  $w_K = 2p$  if  $K = \mathbb{Q}(\zeta_p)$  but  $w_K = 2$  if  $K \subsetneq \mathbb{Q}(\zeta_p)$  (see [Was, Exercise 2.3]). Let  $K^+$  be the maximal real subfield of  $K$  of degree  $n$  fixed by the complex conjugation. The class number  $h_{K^+}$  of  $K^+$  divides the class number  $h_K$  of  $K$ . The *relative class number* of  $K$  is defined by  $h_{\bar{K}} = h_K/h_{K^+}$ . Let  $d_K$  and  $d_{K^+}$  be the absolute values of the discriminants of  $K$  and  $K^+$ . For  $\gcd(t, f) = 1$ , let  $\sigma_t$  be the  $\mathbb{Q}$ -automorphism of  $\mathbb{Q}(\zeta_f)$  defined by  $\sigma_t(\zeta_f) = \zeta_f^t$ . Then  $t \mapsto \sigma_t$  is a canonical isomorphism from the multiplicative group  $(\mathbb{Z}/f\mathbb{Z})^*$  to the Galois group  $\text{Gal}(\mathbb{Q}(\zeta_f)/\mathbb{Q})$ . Set

$$H := \text{Gal}(\mathbb{Q}(\zeta_f)/K) = \{t \in (\mathbb{Z}/f\mathbb{Z})^*; \alpha \in K \Rightarrow \sigma_t(\alpha) = \alpha\},$$

a subgroup of  $(\mathbb{Z}/f\mathbb{Z})^*$  of index  $m$  and order  $\phi(f)/m$ . Notice also that  $\#X_f^-(H) = n$ . Now,  $-1 \notin H$  (notice that  $\sigma_{-1}$  is the complex conjugation restricted to  $\mathbb{Q}(\zeta_f)$ ). Any  $\chi \in X_f$  is induced by a unique primitive Dirichlet character  $\chi^*$  of conductor  $f_{\chi^*}$  dividing  $f$ . We have the relative class number formula

$$h_K^- = \frac{Q_K w_K}{(2\pi)^n} \sqrt{\frac{d_K}{d_{K^+}}} \prod_{\chi \in X_f^-(H)} L(1, \chi^*). \quad (8)$$

By [Lou99], dealing with primitive characters is not going to give explicit formulas. However, noticing that

$$L(1, \chi^*) = L(1, \chi) \prod_{q|f} \left(1 - \frac{\chi^*(q)}{q}\right)^{-1} \quad (\chi \in X_f)$$

and using (8) and the arithmetic-geometric mean inequality, we obtain

$$h_K^- \leq \frac{Q_K w_K}{\Pi(f, H)} \sqrt{\frac{d_K}{d_{K^+}}} \left(\frac{M(f, H)}{4\pi^2}\right)^{n/2}, \quad (9)$$

where

$$\Pi(f, H) := \prod_{q|f} \prod_{\chi \in X_f^-(H)} \left(1 - \frac{\chi^*(q)}{q}\right) \quad (q \text{ runs over the prime divisors of } f).$$

Notice that  $\Pi(f, H) = 1$  whenever  $f = p^m$  is power of a prime.

For example, let  $p \geq 3$  be an odd prime. Let  $K$  be an imaginary subfield of degree  $(K : \mathbb{Q}) = m$  of the cyclotomic field  $\mathbb{Q}(\zeta_p)$ . Set  $H = \text{Gal}(\mathbb{Q}(\zeta_p)/K)$ , a subgroup of order  $(p-1)/m$  of the multiplicative group  $(\mathbb{Z}/p\mathbb{Z})^*$ . Then  $d_K = p^{m-1}$  and  $d_{K^+} = p^{m/2-1}$ , by the conductor-discriminant formula. Therefore, by (9) we have

$$h_K^- \leq w_K \left(\frac{pM(p, H)}{4\pi^2}\right)^{m/4}, \quad \text{where } m = (K : \mathbb{Q}) \text{ and } w_K = \begin{cases} 2 & \text{if } K \subsetneq \mathbb{Q}(\zeta_p), \\ 2p & \text{if } K = \mathbb{Q}(\zeta_p). \end{cases} \quad (10)$$

In particular, for  $H = \{1\}$  and using (7) and (10) we recover [Wal]:

$$M(p, \{1\}) := \frac{2}{p-1} \sum_{\chi \in X_p^-} |L(1, \chi)|^2 = \frac{\pi^2}{6} \left(1 - \frac{1}{p}\right) \left(1 - \frac{2}{p}\right) \leq \frac{\pi^2}{6} \quad (p \geq 3) \quad (11)$$

and obtain the following upper bound

$$h_{\mathbb{Q}(\zeta_p)}^- \leq 2p \left(\frac{pM(p, \{1\})}{4\pi^2}\right)^{(p-1)/4} \leq 2p \left(\frac{p}{24}\right)^{(p-1)/4}. \quad (12)$$

The mean square value of  $L(1, \chi)$ ,  $\chi \in X_p^-$  being asymptotic to  $\pi^2/6$ , by (11), for  $K \subsetneq \mathbb{Q}(\zeta_p)$  we might expect to have bounds close to

$$M(p, H) \leq \pi^2/6 \text{ and } h_K^- \leq 2(p/24)^{n/2}, \quad (13)$$

by (10). At least, according to Corollary 2 and Conjecture 3 these bounds should hold true with probability greater than or equal to  $1/2$ . However, it is hopeless to expect such a universal mean square upper bound. Indeed (e.g. see [CK]), it is likely that there are infinitely many imaginary abelian number fields of a given degree  $m = 2n$  and prime conductors  $p$  for which

$$M(p, H) = \frac{1}{n} \sum_{\chi \in X_p^-(H)} |L(1, \chi)|^2 \geq \left(\prod_{\chi \in X_p^-(H)} L(1, \chi)\right)^{2/n} \gg (\log \log p)^2.$$

## 4 An explicit formula for some mean square values of $L(1, \chi)$ over subgroups

Formula (7) gives an explicit formula for  $M(f, \{1\})$  for  $f > 2$ . We now present in Theorem 5 the only situation where we could get an explicit formula for  $M(f, H)$  for non trivial subgroups  $H$  of the multiplicative group  $(\mathbb{Z}/f\mathbb{Z})^*$  where  $f$  is not necessarily prime.

**Lemma 4.** *Let  $f = \prod_{k=1}^t p_k^{e_k}$  be an integer such that all its  $t \geq 1$  distinct prime divisors  $p_k$  are equal to 1 modulo 3. Then the set*

$$E_f := \{a/b \in (\mathbb{Z}/f\mathbb{Z})^*; f = a^2 + ab + b^2 \text{ and } \gcd(a, b) = 1\}$$

is of cardinal  $2^t$  and its elements are of order 3 in the multiplicative group  $(\mathbb{Z}/f\mathbb{Z})^*$ .

Moreover, if  $\delta \geq 1$  divides  $f$ , then there exist  $a'$  and  $b'$  such that  $\delta = a'^2 + a'b' + b'^2$ ,  $\gcd(a', b') = 1$  and  $a/b = a'/b'$  in  $(\mathbb{Z}/\delta\mathbb{Z})^*$ .

*Proof.* Let  $p \equiv 1 \pmod{3}$  be prime. Then  $p = \pi\bar{\pi}$  splits in the principal ideal domain  $\mathbb{Z}[\zeta_6]$  of  $\mathbb{Z}$ -basis  $\{1, \zeta_6\}$ , where  $\zeta_6$  is a primitive complex sixth root of unity and  $\pi \in \mathbb{Z}[\zeta_6]$  is irreducible in  $\mathbb{Z}[\zeta_6]$ . For each  $k$ , fix a factorisation of  $p_k$  into a product of 2 complex conjugate irreducible elements of  $\mathbb{Z}[\zeta_6]$ . By taking  $\alpha = a + b\zeta_6 = \prod_{k=1}^t \pi_k^{e_k}$ , where  $\pi_k$  is any of the 2 given complex conjugate irreducible factors of  $p_k$ , we get  $2^t$  ways to write  $f = \alpha\bar{\alpha} = a^2 + ab + b^2$  with  $\gcd(a, b) = 1$ . Moreover, given two distinct such  $\alpha$ 's, say  $\alpha_1 = a_1 + b_1\zeta_6$  and  $\alpha_2 = a_2 + b_2\zeta_6$ , there exists some index  $k$  for which  $\pi_k$  divides  $\alpha_1$  but does not divide  $\alpha_2$ . Since  $b_2\alpha_1 - b_1\alpha_2 = a_1b_2 - a_2b_1$  and  $\gcd(b_1, f) = 1$ , it follows that  $\pi_k$  does not divide  $b_2\alpha_1 - b_1\alpha_2$ , which implies that  $p_k$  does not divide  $a_1b_2 - a_2b_1$ . Hence,  $a_1/b_1 \neq a_2/b_2$  in  $(\mathbb{Z}/f\mathbb{Z})^*$ . Conversely, if  $f = a^2 + ab + b^2$  with  $\gcd(a, b) = 1$  then  $f = \alpha\bar{\alpha}$  where  $\alpha = a + b\zeta_6$  with  $\gcd(a, b) = 1$ . Therefore,  $\alpha = \eta\alpha_f$  for one of the 6 invertible elements  $\eta \in \{\zeta_6^k; 0 \leq k \leq 5\}$  of  $\mathbb{Z}[\zeta_6]$ , where  $\alpha_f := \prod_{k=1}^t \pi_k^{e_k} = a_f + b_f\zeta_6$  is also such that  $f = a_f^2 + a_fb_f + b_f^2$ . However,  $a/b = a_f/b_f$  in  $(\mathbb{Z}/p\mathbb{Z})^*$ . Indeed,  $\zeta_6\alpha_f = -b_f + (a_f + b_f)\zeta_6$  and  $-b_f/(a_f + b_f) = a_f/b_f$  in  $(\mathbb{Z}/p\mathbb{Z})^*$ .

Finally, let  $\delta = \prod_{k=1}^t p_k^{e'_k}$  be a divisor of  $f$ . Set  $\beta := \prod_{k=1}^t \pi_k^{e'_k} = a' + b'\zeta_6$ , that divides  $\alpha$  in  $\mathbb{Z}[\zeta_6]$ , say  $\alpha = \beta\gamma$  with  $\gamma = a'' + b''\zeta_6$ . Therefore,  $a + b\zeta_6 = (a' + b'\zeta_6)(a'' + b''\zeta_6) = (a'a'' - b'b'') + (a'b'' + a''b' + b'b'')\zeta_6$ , hence  $a = a'a'' - b'b''$ ,  $b = a'b'' + a''b' + b'b''$ ,  $\gcd(a', b') = 1$  and noticing that  $\delta = \beta\bar{\beta} = a'^2 + a'b' + b'^2$  we obtain

$$a/b = (a'a'' - b'b'')/(a'b'' + a''b' + b'b'') = a'/b'$$

in  $(\mathbb{Z}/\delta\mathbb{Z})^*$ . □

We can use  $E_f$  to explicitly construct  $2^{t-1}$  subgroups  $\{1, a/b, b/a\}$  of order 3 of the multiplicative group  $(\mathbb{Z}/f\mathbb{Z})^*$ . Notice that by the Chinese remainder theorem there are  $(3^t - 1)/2$  subgroups of order 3 in the multiplicative group  $(\mathbb{Z}/f\mathbb{Z})^*$ . We will now prove the following new result:

**Theorem 5.** *Let  $f > 1$  be an integer such that all its  $t \geq 1$  distinct prime divisors are equal to 1 modulo 3. Then, for the  $2^{t-1}$  subgroups  $H_3 = \{1, a/b, b/a\}$  of the multiplicative group  $(\mathbb{Z}/f\mathbb{Z})^*$  generated by the  $2^t$  elements  $a/b \in E_f$  we have*

$$\tilde{S}(f, H_3) := \sum_{h \in H_3} \tilde{s}(h, f) = \frac{\phi(f)}{12} \left( \prod_{p|f} \left( 1 + \frac{1}{p} \right) - \frac{1}{f} \right)$$

and (with the notation in (6), and compare with (7))

$$M(f, H_3) = \frac{\pi^2 \phi(f)}{6 f} \left( \prod_{p|f} \left( 1 + \frac{1}{p} \right) - \frac{1}{f} \right).$$

*Proof.* We have  $\tilde{S}(f, H_3) = \tilde{s}(1, f) + 2\tilde{s}(a/b, f)$  and we use (5) and Lemma 6.  $\square$

**Lemma 6.** *Assume that  $f = a^2 + ab + b^2 > 3$ , where  $a, b \in \mathbb{Z}$  and  $\gcd(a, b) = 1$  (i.e. assume that all the prime divisors of  $f$  are equal to 1 mod 3). Set  $h_3 = a/b$ , of order 3 in the multiplicative group  $(\mathbb{Z}/f\mathbb{Z})^*$ . Then for any divisor  $\delta \geq 1$  of  $f$  we have  $s(h_3, \delta) = \frac{\delta-1}{12\delta}$ . By (4), it follows that*

$$\tilde{s}(h_3, f) = \sum_{\delta|f} \frac{\mu(\delta)}{\delta} s(h_3, f/\delta) = \sum_{\delta|f} \mu(\delta) \frac{f/\delta - 1}{12f} = \frac{\phi(f)}{12f}.$$

In particular,  $\tilde{s}(h_3, f)$  does not depend on the choice of  $h_3$  in  $E_f$ .

*Proof.* By Lemma 4 we have  $s(h_3, \delta) = s(h'_3, \delta)$  where  $h'_3 = a'/b'$  with  $\delta = a'^2 + a'b' + b'^2$  and  $\gcd(a', b') = 1$ . By [Lou16, Lemma 4] (see also [LM23, Lemma 6.1]), we have  $s(h'_3, \delta) = \frac{\delta-1}{12\delta}$ .  $\square$

**Remarks 7.** (i). *Let  $f$  be a product of  $t > 1$  prime numbers equal to 1 modulo 3. There are  $3^t - 1$  elements  $h$  of order 3 in the multiplicative group  $(\mathbb{Z}/f\mathbb{Z})^*$  and  $\tilde{s}(h, f)$  may depend on the choice of  $h$ . For example, take  $f = 91 = 7 \cdot 13$ . Then  $h = 29$  and  $h' = 53$  are of order 3 in the multiplicative group  $(\mathbb{Z}/f\mathbb{Z})^*$  but such that  $-\frac{22}{91} = \tilde{s}(h, f) \neq \tilde{s}(h', f) = -\frac{46}{91}$  are not equal and both different from  $\frac{\phi(f)}{12f} = \frac{6}{91}$ . Moreover,  $\frac{610}{91} = \tilde{S}(f, \{1, h, h^2\}) \neq S(f, \{1, h', h'^2\}) = \frac{562}{91}$  are not equal and both not given by the formula in Theorem 5 that gives  $\tilde{S}(f, H_3) = \frac{666}{91}$ .*

(ii). *It seems difficult to find other situations where results similar to those in Theorem 5 would hold true. For example, take the moduli of the form  $f = (a^5 - 1)/(a - 1)$  with  $|a| \geq 2$  and  $a \not\equiv 1 \pmod{5}$ . Then the prime divisors of  $f$  are equal to 1 modulo 5 and  $H_5 = \{1, a, a^2, a^3, a^4\}$  is a subgroup of order 5 of  $(\mathbb{Z}/f\mathbb{Z})^*$ . An explicit formula for  $M(f, H_5)$  is known in the case that  $f$  is prime (see [Lou16, Theorem 5]). But we have not been able to obtain an explicit formula for  $M(f, H_5)$  for non prime moduli  $f$ .*

In particular, we have (13) for some non-cyclotomic numbers fields:

**Corollary 8.** *(See [Lou16, Theorem 1] or [LM23, Theorem 6.6]). Let  $p \equiv 1 \pmod{6}$  be a prime integer. Let  $K$  be the imaginary subfield of degree  $(p-1)/3$  of the cyclotomic number field  $\mathbb{Q}(\zeta_p)$ . Let  $H_3$  be the only subgroup of order 3 of the multiplicative cyclic group  $(\mathbb{Z}/p\mathbb{Z})^*$ . Then (compare with (11))*

$$M(p, H_3) := \frac{6}{p-1} \sum_{\chi \in X_{\bar{p}}(H_3)} |L(1, \chi)|^2 = \frac{\pi^2}{6} \left(1 - \frac{1}{p}\right) \leq \frac{\pi^2}{6}.$$

Hence by (10) we have (compare with (12))

$$h_{\bar{K}} \leq 2 \left( \frac{pM(p, H_3)}{4\pi^2} \right)^{\frac{p-1}{12}} \leq 2 \left( \frac{p}{24} \right)^{(p-1)/12} \quad (14)$$

(note the misprint in the exponent in [Lou16, (8)]), i.e. the expected bounds (13) hold true.

## 5 On the denominator of Dedekind sums

**Proposition 9.** *(See [RG, Theorem 2 page 27]). For  $\gcd(c, d) = 1$  we have  $2d \gcd(3, d) s(c, d) \in \mathbb{Z}$ .*

If  $p \equiv 7 \pmod{12}$  then  $2p \gcd(3, p) s(1, p) = (p-1)(p-2)/6$  is an odd integer coprime with  $p$  and the information on the denominator of the rational number  $s(c, d)$  given in Proposition 9 is optimal in this case. Hence,  $2f \gcd(3, f) S(H_n, f) \in \mathbb{Z}$ , where  $H_n$  is a subgroup of order  $n$  of the multiplicative group  $(\mathbb{Z}/f\mathbb{Z})^*$  and

$$S(H_n, f) := \sum_{h \in H_n} s(h, f) \in \mathbb{Q}.$$

We always have some cancelation on the common denominator  $2f \gcd(3, f)$  of the  $s(h, f)$ 's when we sum over all the elements  $h$  of a subgroup of order  $n > 1$  of the multiplicative group  $(\mathbb{Z}/f\mathbb{Z})^*$ :



**Theorem 10.** Let  $H_n$  be a subgroup of order  $n$  of the multiplicative group  $(\mathbb{Z}/f\mathbb{Z})^*$ . Set

$$T(H_n, f) := \sum_{h \in H_n} h \in \mathbb{Z}/f\mathbb{Z}.$$

(i). (See [Lou19, Lemma 5]). If  $n > 1$ , then  $\gcd(f, T(H_n, f)) > 1$ .

(ii). (See [Lou19, Theorem 10]). If  $f$  is odd then the rational number

$$2 \gcd(3, f) \frac{f}{\gcd(f, T(H_n, f))} S(H_n, f)$$

is an integer of the same parity as  $n \frac{f-1}{2}$ .

Here is a general example which shows that the formulation of Theorem 10 is optimal:

**Theorem 11.** Let  $p \geq 3$  be prime,  $f' > 1$  odd and divisible by  $p$  and  $n \geq 1$ . Set  $f = p^n f'$  and

$$H_{p^n} = \{1 + kf'; 0 \leq k \leq p^n - 1\} = \ker((\mathbb{Z}/f\mathbb{Z})^* \rightarrow (\mathbb{Z}/f'\mathbb{Z})^*),$$

a subgroup of order  $p^n$  of the multiplicative group  $(\mathbb{Z}/f\mathbb{Z})^*$ .

Then  $T(H_{p^n}, f) = p^n + \frac{p^n-1}{2}f$ , hence  $\gcd(f, T(H_{p^n}, f)) = p^n$ , and

$$S(H_{p^n}, f) := \sum_{h \in H_n} s(h, f) = \sum_{k=0}^{p^n-1} s(1 + kf', f) = \frac{p^{n+1} + p^n - 1}{12p^{n+1}} f - \frac{p^n}{4} + \frac{p^n}{6f}. \quad (15)$$

Consequently (compare with Proposition 9), the rational number  $2 \gcd(3, f) \frac{f}{p^n} S(H_{p^n}, f)$  is a rational integer not divisible by  $p$  and it is odd if and only if  $f \equiv 3 \pmod{4}$ .

*Proof.* Take  $d \geq 2$ . Set  $\zeta_d = \exp(2i\pi/d)$ . Taking the logarithmic derivative of  $\prod_{k=0}^{d-1} (x - \zeta_d^k) = x^d - 1$  at  $x = 1/\lambda$  we obtain

$$\sum_{k=0}^{d-1} \frac{1}{\zeta_d^k \lambda - 1} = \frac{d}{\lambda^d - 1} \text{ whenever } \lambda^d \neq 1. \quad (16)$$

Taking the logarithmic derivative of  $\prod_{k=1}^{d-1} (x - \zeta_d^k) = (x^d - 1)/(x - 1) = x^{d-1} + \dots + x + 1$  at  $x = 1$  we obtain

$$\sum_{k=1}^{d-1} \frac{1}{\zeta_d^k - 1} = -\frac{d-1}{2}. \quad (17)$$

Noticing that  $\cot x = i + 2i/(\exp(2ix) - 1)$  and using (1) and (17), we have

$$s(c, d) = \frac{d-1}{4d} - \frac{1}{d} \sum_{a=1}^{d-1} \frac{1}{(\zeta_d^a - 1)(\zeta_d^{ac} - 1)} \quad (\text{for } \gcd(c, d) = 1). \quad (18)$$

In particular,

$$s(1 + kf', f) = \frac{f-1}{4f} - \frac{1}{f} \sum_{a=1}^{f-1} \frac{1}{(\zeta_f^a - 1)(\zeta_f^{a c a^k} - 1)} \quad (\text{for } f = p^n f' \text{ and } p \mid f'), \quad (19)$$

and for  $c = 1$  and in using (2) we obtain

$$\sum_{a=1}^{d-1} \frac{1}{(\zeta_d^a - 1)^2} = -\frac{(d-1)(d-5)}{12} \quad (\text{for } d \geq 2). \quad (20)$$

Using (19) and (20), we deduce that

$$\begin{aligned}
\sum_{k=0}^{p^n-1} s(1+kf', f) &= p^n \frac{f-1}{4f} - \frac{1}{f} \sum_{a=1}^{f-1} \sum_{k=0}^{p^n-1} \frac{1}{(\zeta_f^a - 1)(\zeta_f^a \zeta_{p^n}^{ak} - 1)} \\
&= p^n \frac{f-1}{4f} - \frac{p^n}{f} \sum_{\substack{a=1 \\ p^n | a}}^{f-1} \frac{1}{(\zeta_f^a - 1)^2} - \frac{1}{f} \sum_{\substack{a=1 \\ p^n \nmid a}}^{f-1} \sum_{k=0}^{p^n-1} \frac{1}{(\zeta_f^a - 1)(\zeta_f^a \zeta_{p^n}^{ak} - 1)} \\
&= p^n \frac{f-1}{4f} + \frac{(f-p^n)(f-5p^n)}{12p^n f} - \frac{1}{f} \sum_{l=0}^{n-1} \sum_{\substack{a=1 \\ p^l | a}}^{f-1} \sum_{k=0}^{p^n-1} \frac{1}{(\zeta_f^a - 1)(\zeta_f^a \zeta_{p^n}^{ak} - 1)}. \tag{21}
\end{aligned}$$

Now, using (16) we obtain

$$\begin{aligned}
\sum_{\substack{a=1 \\ p^l | a}}^{f-1} \sum_{k=0}^{p^n-1} \frac{1}{(\zeta_f^a - 1)(\zeta_f^a \zeta_{p^n}^{ak} - 1)} &= \sum_{\substack{a=1 \\ p^l \nmid a}}^{f/p^l-1} \sum_{k=0}^{p^n-1} \frac{1}{(\zeta_{f/p^l}^a - 1)(\zeta_{f/p^l}^a \zeta_{p^{n-l}}^{ak} - 1)} \\
&= \sum_{\substack{a=1 \\ p^l \nmid a}}^{f/p^l-1} \sum_{k=0}^{p^{n-l}-1} \frac{p^l}{(\zeta_{f/p^l}^a - 1)(\zeta_{f/p^l}^a \zeta_{p^{n-l}}^{ak} - 1)} \\
&= \sum_{\substack{a=1 \\ p^l \nmid a}}^{f/p^l-1} \frac{p^n}{(\zeta_{f/p^l}^a - 1)(\zeta_{f/p^n}^a - 1)}. \tag{22}
\end{aligned}$$

Since  $\{a; 1 \leq a \leq f/p^l - 1 \text{ and } p \nmid a\} = \{Af/p^n + B; 0 \leq A \leq p^{n-l} - 1 \text{ and } 1 \leq B \leq f/p^n - 1\}$   
 $\setminus \{Af/p^n + pB; 0 \leq A \leq p^{n-l} - 1 \text{ and } 1 \leq B \leq f/p^{n+1} - 1\}$ , we have

$$\begin{aligned}
\sum_{\substack{a=1 \\ p^l \nmid a}}^{f/p^l-1} \frac{1}{(\zeta_{f/p^l}^a - 1)(\zeta_{f/p^n}^a - 1)} &= \sum_{B=1}^{f/p^n-1} \frac{1}{\zeta_{f/p^n}^B - 1} \sum_{A=0}^{p^{n-l}-1} \frac{1}{\zeta_{p^{n-l}}^A \zeta_{f/p^l}^B - 1} \\
&\quad - \sum_{B=1}^{f/p^{n+1}-1} \frac{1}{\zeta_{f/p^{n+1}}^B - 1} \sum_{A=0}^{p^{n-l}-1} \frac{1}{\zeta_{p^{n-l}}^A \zeta_{f/p^{l+1}}^B - 1} \\
&= \sum_{B=1}^{f/p^n-1} \frac{p^{n-l}}{(\zeta_{f/p^n}^B - 1)^2} - \sum_{B=1}^{f/p^{n+1}-1} \frac{p^{n-l}}{(\zeta_{f/p^{n+1}}^B - 1)^2} \\
&= -p^{-l} \frac{(f-p^n)(f-5p^n)}{12p^n} + p^{-l} \frac{(f-p^{n+1})(f-5p^{n+1})}{12p^{n+2}}, \tag{23}
\end{aligned}$$

by (16) and (20). Finally, using (21), (22) and (23) we obtain

$$S(H_{p^n}, f) = p^n \frac{f-1}{4f} + \frac{(f-p^n)(f-5p^n)}{12p^n f} + \frac{1-p^{-n}}{1-p^{-1}} \frac{(f-p^n)(f-5p^n)}{12f} - \frac{1-p^{-n}}{1-p^{-1}} \frac{(f-p^{n+1})(f-5p^{n+1})}{12p^2 f}$$

and the desired formula.  $\square$

**Corollary 12.** *Let  $p \geq 3$  be prime and  $f = p^m$  with  $m \geq 2$ . Assume that  $1 \leq n \leq m-1$ . Then  $E_{p^n} = \{1+kf/p^n; 1 \leq k \leq p^n-1 \text{ and } \gcd(p, k) = 1\}$  is the set of the  $\phi(p^n) = p^{n-1}(p-1)$  elements of order  $p^n$  of the multiplicative cyclic group  $(\mathbb{Z}/p^m\mathbb{Z})^*$  and we have the following mean value formula*

$$\frac{1}{\#E_{p^n}} \sum_{h \in E_{p^n}} s(h, f) = \frac{f}{12p^{2n}} - \frac{1}{4} + \frac{1}{6f} \text{ for } 1 \leq n \leq m-1.$$

*Proof.* We have

$$\frac{1}{\#E_{p^n}} \sum_{h \in E_{p^n}} s(h, f) = \frac{1}{p^{n-1}(p-1)} \left( \sum_{k=1}^{p^n-1} s(1 + kf/p^n, f) - \sum_{k=1}^{p^{n-1}-1} s(1 + kf/p^{n-1}, f) \right).$$

Using (15) for  $n$  and  $n-1$  the desired result follows.  $\square$

**Remarks 13.** Assume that  $1 \leq n \leq m-1$ . A Dirichlet character  $\chi$  modulo  $f = p^m$  is trivial on the subgroup  $H_{p^n}$  if and only if it is induced by a Dirichlet character  $\chi'$  modulo  $f' = f/p^n$  and in that situation we have  $L(1, \chi) = L(1, \chi')$ . Therefore, noticing that  $p^n/\phi(f) = 1/\phi(f')$ , we have  $M(f, H_{p^n}) = M(f', \{1\})$ . Now, on the one hand by (7) we have

$$M(f', \{1\}) = \frac{\pi^2}{6} \left(1 - \frac{1}{p}\right) \left(1 + \frac{1}{p} - \frac{3}{f'}\right).$$

On the other hand by (6) and (4) we have

$$M(f, H_{p^n}) = \frac{2\pi^2}{f} \left( \sum_{h \in H_{p^n}} s(h, f) - \frac{1}{p} \sum_{h \in H_{p^n}} s(h, f/p) \right) = \frac{2\pi^2}{f} (S(H_{p^n}, f) - S(H_{p^{n-1}}, f/p)).$$

Using Theorem 11, we do recover that  $M(f, H_{p^n}) = M(f', \{1\})$ .

While checking the statements of Theorem 11 and Corollary 12 on various values of  $p, f', n$  and  $m$ , we came across the following surprising Theorem which in the range  $1 \leq n \leq m/2$  is much more precise than Corollary 12 and implies Corollary 12:

**Theorem 14.** Assume that  $f \geq 1$  divides  $f'^2$  and that  $f'$  divides  $f$ . Then for  $k \in \mathbb{Z}$  we have  $\gcd(1 + kf', f) = 1$  and

$$s(1 + kf', f) = \frac{f'^2}{12f} - \frac{1}{4} + \frac{1}{6f} \text{ for } \gcd(k, f) = 1. \quad (24)$$

In particular, for  $p \geq 3$  is prime,  $f = p^m$  and  $1 \leq n \leq m/2$ , we have

$$s(1 + kf/p^n, f) = \frac{f}{12p^{2n}} - \frac{1}{4} + \frac{1}{6f} \text{ for } 1 \leq k \leq p^n - 1 \text{ and } \gcd(k, p) = 1. \quad (25)$$

Therefore, for  $m \geq 2$  and  $1 \leq n \leq m/2$ , the Dedekind sums  $s(h, p^m)$  do not depend on  $h$  as  $h$  runs over the  $\phi(p^n) = p^{n-1}(p-1)$  elements of order  $p^n$  of the multiplicative cyclic group  $(\mathbb{Z}/p^m\mathbb{Z})^*$ .

*Proof.* Set  $q = f/f'$ . Notice that  $q$  divides  $f'$ . By (18), proving (24) is equivalent to proving that:

$$\sum_{a=1}^{f-1} \frac{1}{(\zeta_f^a - 1)(\zeta_f^a \zeta_q^{ak} - 1)} = \frac{6f - f'^2 - 5}{12}.$$

Write  $a = Aq + B$  with  $0 \leq A \leq f' - 1$ ,  $0 \leq B \leq q - 1$  and  $(A, B) \neq (0, 0)$ . Then  $(\zeta_f^a - 1)(\zeta_f^a \zeta_q^{ak} - 1) = (\lambda \zeta_{f'}^A - 1)(\mu \zeta_{f'}^A - 1)$ , where  $\lambda = \lambda_B = \zeta_f^B$  and  $\mu = \mu_{B,k} = \zeta_f^B \zeta_q^{Bk}$ . Since  $\gcd(k, f) = 1$  and  $1 \leq B \leq f' - 1$ , we have  $\lambda \neq \mu$  and

$$\frac{1}{(\lambda \zeta_{f'}^A - 1)(\mu \zeta_{f'}^A - 1)} = -\frac{1}{\lambda - \mu} \left( \frac{\lambda}{\lambda \zeta_{f'}^A - 1} - \frac{\mu}{\mu \zeta_{f'}^A - 1} \right).$$

Noticing that  $\lambda^{f'} = \mu^{f'} = \zeta_q^B$ , as  $q \mid f'$ , and using (16) we get

$$\sum_{A=0}^{f'-1} \frac{1}{(\lambda_B \zeta_{f'}^A - 1)(\mu_{B,k} \zeta_{f'}^A - 1)} = -\frac{f'}{\zeta_q^B - 1} \text{ for } 1 \leq B \leq f' - 1.$$

Therefore, we do have

$$\begin{aligned} \sum_{a=1}^{f'-1} \frac{1}{(\zeta_f^a - 1)(\zeta_f^a \zeta_q^{ak} - 1)} &= \sum_{A=1}^{f'-1} \frac{1}{(\zeta_{f'}^A - 1)^2} + \sum_{B=1}^{q-1} \sum_{A=0}^{f'-1} \frac{1}{(\lambda_B \zeta_{f'}^A - 1)(\mu_{B,k} \zeta_{f'}^A - 1)} \\ &= -\frac{(f'-1)(f'-5)}{12} - f' \sum_{B=1}^{q-1} \frac{1}{\zeta_q^B - 1} = -\frac{(f'-1)(f'-5)}{12} + \frac{f'(q-1)}{2} = \frac{6f - f'^2 - 5}{12}, \end{aligned}$$

by (20) and (17).  $\square$

**Remarks 15.** *The restriction  $1 \leq n \leq m/2$  is of paramount importance: for  $m/2 < n \leq m-1$  the Dedekind sum  $s(1 + kp^{m-n}, p^m)$  may depend on  $k$  with  $1 \leq k \leq p^n - 1$  and  $\gcd(k, p) = 1$ .*

## References

- [Apo] M. T. Apostol. *Modular functions and Dirichlet series in number theory*. Graduate Texts in Mathematics **41**. Springer-Verlag, New York, 1976.
- [CK] P. J. Cho and A. H. H. Kim. Dihedral and cyclic extensions with large class numbers. *J. Théor. Nombres Bordeaux* **24** (2012), 583–603.
- [Lem] F. Lemmermeyer. Ideal class groups of cyclotomic number fields. I. *Acta Arith.* **72** (1995), 347–359.
- [LM21] S. Louboutin and M. Munsch. Second moment of Dirichlet  $L$ -functions, character sums over subgroups and upper bounds on relative class numbers. *Quart. J. Math.* **72** (2021), 1379–1399.
- [LM23] S. Louboutin and M. Munsch. Mean square values of  $L$ -functions over subgroups for non primitive characters, Dedekind sums and bounds on relative class numbers. *Canad. J. Math.* **75** (2023), 1711–1743 .
- [Lou94] S. Louboutin. Quelques formules exactes pour des moyennes de fonctions  $L$  de Dirichlet. *Canad. Math. Bull.* **36** (1993), 190–196. Addendum. *Canad. Math. Bull.* **37** (1994), p. 89.
- [Lou99] S. Louboutin. On the mean value of  $|L(1, \chi)|^2$  for odd primitive Dirichlet characters. *Proc. Japan Acad. Ser. A Math. Sci.* **75** (1999), 143–145.
- [Lou15] S. Louboutin. Twisted quadratic moments for Dirichlet  $L$ -functions. *Bull. Korean Math. Soc.* **52** (2015), 2095–2105.
- [Lou16] S. Louboutin. Dedekind sums, mean square value of  $L$ -functions at  $s = 1$  and upper bounds on relative class numbers. *Bull. Pol. Acad. Sci. Math.* **64** (2016), 165–174.
- [Lou19] S. Louboutin. On the denominator of Dedekind sums. *Bull. Korean Math. Soc.* **56** (2019), 815–827.
- [MS] M. Munsch and I. E. Shparlinski. Moments and non-vanishing of  $L$ -functions over thin subgroups. *Preprint*, submitted. <https://arxiv.org/abs/2309.10207>.
- [RG] H. Rademacher and E. Grosswald. Dedekind sums. *The Carus Mathematical Monographs*, **16**. The Mathematical Association of America, Washington, D.C., 1972.
- [Wal] H. Walum. An exact formula for an average of  $L$ -series. *Illinois J. Math.* **26** (1982), 1–3.
- [Was] L. C. Washington. *Introduction to Cyclotomic Fields*. Second Edition. Graduate Texts in Mathematics **83**. Springer-Verlag, New York, 1997.