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KUNTAL BHANDARI†, RAJIB DUTTA‡, MANISH KUMAR§

ABSTRACT. This article investigates the $\phi(\Delta t)$-null controllability of a time-discrete parabolic system with Kirchhoff boundary conditions, using a single boundary control. We demonstrate this $\phi(\Delta t)$-null controllability by deriving an appropriate Carleman estimate. Furthermore, we establish that the limit of these controls for the discrete system turns out to be a control responsible for the null controllability of the associated continuous system.

CONTENTS

1. INTRODUCTION

Let us consider the following one dimensional parabolic control system with Kirchhoff type boundary condition

$$\begin{cases} \partial_t u_1 - \partial_x(\gamma_1 \partial_x u_1) = 0, & (t,x) \in (0,T)\times(0,1), \\ \partial_t u_2 - \partial_x(\gamma_2 \partial_x u_2) = 0, & (t,x) \in (0,T)\times(0,1), \\ u_1(t,0) = 0, u_2(t,0) = v(t), & t \in (0,T), \\ u_1(t,1) = u_2(t,1), & t \in (0,T), \\ \gamma_1(1)\partial_x u_1(t,1) + \gamma_2(1)\partial_x u_2(t,1) + \alpha u_1(t,1) = 0, & t \in (0,T), \\ u_1(0,x) = u_{1,0}(x), u_2(0,x) = u_{2,0}(x), & x \in (0,1), \end{cases} \tag{1.1}$$

where $\alpha \geqslant 0$, $\gamma_i \in C^1[0,1]$ with $\gamma_{min} = \inf_{x\in[0,1]}\{\gamma_i(x) : i = 1,2\} > 0$, and v is the control.

Such boundary conditions of Kirchhoff type is used widely in physics, electrical engineering, and biology to describe systems involving network. For instance, in [15] and [16], authors have studied the mathematical aspects of Kirchhoff's rule for quantum wires, while in [8], the author has studied the FitzHugh-Nagumo-Rall model of a neuronal network with a Kirchhoff type boundary condition. It

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should be noted that the usual Kirchhoff condition involves only first order derivatives (i.e, $\alpha = 0$), however, in the control system (1.1) a general Kirchhoff condition with $\alpha \geqslant 0$ has been chosen.

The system (1.1) without control (i.e., $v = 0$) can be written in an abstract form as

$$\begin{pmatrix} \partial_t u_1 \\ \partial_t u_2 \end{pmatrix} = \mathcal{A} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

where the differential operator $\mathcal{A}$ is given by

$$\mathcal{A} := \begin{pmatrix} \partial_x(\gamma_1 \partial_x u_1) & 0 \\ 0 & \partial_x(\gamma_2 \partial_x u_2) \end{pmatrix},$$

with its domain

$$D(\mathcal{A}) := \left\{ (v_1, v_2) \in (H^2(0,1))^2 : v_1(0) = v_2(0) = 0, v_1(1) = v_2(1), \sum_{i=1}^{2} \gamma_i(1) \partial_x v_i(1) + \alpha v_1(1) = 0 \right\}.$$

Define the space $\mathcal{H} := D(\mathcal{A}^{1/2})$ as the completion of $D(\mathcal{A})$ with respect to the norm

$$\|v\|_{\mathcal{H}} := \langle \mathcal{A}\, v, v \rangle_{L^2 \times L^2}^{1/2} = \left(\sum_{i=1}^{2} \int_0^1 \gamma_i(1) |v_i'(x)|^2 \, dx + \alpha |v_1(1)|^2 \right)^{1/2}, \text{ for } v \in D(\mathcal{A}).$$

Note that this norm is equivalent to $H^1 \times H^1$ norm. Moreover, the Hilbert space $\mathcal{H}$ with the inner product

$$\langle u, v \rangle_{\mathcal{H}} := \left(\sum_{i=1}^{2} \int_0^1 \gamma_i(1)\, u_i'(x)\, v_i'(x)\, dx + \alpha\, u_1(1)\, v_1(1) \right)^{1/2}, \quad \text{for } u,\, v \in \mathcal{H},$$

can be explicitly represented as

$$\mathcal{H} = \Big\{ (v_1, v_2) \in (H^1(0,1))^2 : v_1(0) = v_2(0), v_1(1) = v_2(1) \Big\}.$$

Finally, we denote $\mathcal{H}'$ as the dual space of $\mathcal{H}$ with respect to the pivot space $E := \big[L^2(0,1)\big]^2$.

1.1. **Controllability of continuous system.** Before introducing the discrete control system and the associated control problem, let us first state the existing null controllability result for the continuous control system (1.1), which we shall recover in this article using the null controllability of the corresponding discrete control system.

Let us first introduce the notion of solution for the control system (1.1).

Definition 1.1 (Transposition solution). *For $u_0 = (u_{1,0}, u_{2,0}) \in \mathcal{H}'$, a function $u \in L^2(0,T;E)$ is said to be a solution of the above control system (1.1) if it satisfies*

$$\int_0^T \langle u(t,\cdot), g(t,\cdot) \rangle_E \, dt - \int_0^T \gamma_2(0)\, v(s)\, \partial_x \varphi_2(s,0)\, ds - \langle y_0, \varphi(0,\cdot) \rangle_{\mathcal{H}',\mathcal{H}} = 0, \quad \forall\, g \in L^2(0,T;E), \tag{1.2}$$

where (φ_1, φ_2) solves

$$\begin{cases} -\partial_t \varphi_1 - \partial_x \left(\gamma_1\, \partial_x \varphi_1 \right) = g_1, & \text{on } (0,T) \times (0,1), \\ -\partial_t \varphi_2 - \partial_x \left(\gamma_2\, \partial_x \varphi_2 \right) = g_2, & \text{on } (0,T) \times (0,1), \\ \varphi_1(t,0) = 0, \varphi_2(t,0) = 0, & t \in (0,T), \\ \varphi_1(t,1) = \varphi_2(t,1), & t \in (0,T), \\ \gamma_1(1) \partial_x \varphi_1(t,1) + \gamma_2(1) \partial_x \varphi_2(t,1) + \alpha \varphi_1(t,1) = 0, & t \in (0,T), \\ \varphi_1(T,x) = 0, \varphi_2(T,x) = 0, & x \in (0,1). \end{cases} \tag{1.3}$$

We have the following well-posedness result for the system (1.1).

Proposition 1.2 (Well-posedness). *For any $u_0 \in \mathcal{H}'$, and $v \in L^2(0,T;\mathbb{R})$, the control system (1.1) admits a unique solution $u \in L^2(0,T;E)$ in the sense of transposition defined above. Moreover, the solution $u \in C^0([0,T];\mathcal{H}')$.*

Now, we define the notion of null controllability for the system (1.1).

Definition 1.3 (Null controllability). *The control system (1.1) is said to be null controllable in time $T > 0$ if for any initial data $u_0 \in \mathcal{H}'$, there exists a control $v \in L^2(0,T)$ such that the solution of (1.1) satisfies*

$$u(T,\cdot) = 0 \text{ on } (0,1).$$

The null controllability of system (1.1) has been addressed in [9, Chapter 8], [2, Remark 3.6] and [1] with the usual Kirchhoff boundary condition, i.e., $\alpha = 0$. This was further generalized for $\alpha \geqslant 0$ in the work [3] via duality approach, which will be followed in this article as well for proving a similar notion of null controllability for the discrete control system. As per controllability-observability duality property, proving null controllability of the control system (1.1) is equivalent to proving the following observability inequality

$$\|\varphi(0)\|^2_{\mathcal{H}} \leqslant C(T) \int_0^T \left|\partial_x \varphi_2(0)\right|^2 dt, \quad \forall\, \varphi_T = (\varphi_{1,T}, \varphi_{2,T}) \in \mathcal{H}, \tag{1.4}$$

where φ solve the following adjoint system

$$\begin{cases} \partial_t \varphi_1 - \partial_x(\gamma_1 \partial_x \varphi_1) = 0, & (t,x) \in (0,T) \times (0,1), \\ \partial_t \varphi_2 - \partial_x(\gamma_2 \partial_x \varphi_2) = 0, & (t,x) \in (0,T) \times (0,1), \\ \varphi_1(t,0) = 0, \varphi_2(t,0) = 0, & t \in (0,T), \\ \varphi_1(t,1) = \varphi_2(t,1), & t \in (0,T), \\ \gamma_1(1)\partial_x \varphi_1(t,1) + \gamma_2(1)\partial_x \varphi_2(t,1) + \alpha \varphi_1(t,1) = 0, & t \in (0,T), \\ \varphi_1(T,x) = \varphi_{1,T}(x), \varphi_2(T,x) = \varphi_{2,T}(x), & x \in (0,1). \end{cases} \tag{1.5}$$

The authors in [3] derived a Carleman inequality to deduce the observability inequality (1.4), which subsequently gives the following controllability result.

Proposition 1.4 (Null controllability, see [3]). *The control system (1.1) is null controllable in any time $T > 0$ with the control $v \in L^2(0,T)$ satisfying the following estimate*

$$\|v\|_{L^2(0,T)} \leqslant C e^{C/T} \|u_0\|_{\mathcal{H}'},$$

where the constant $C > 0$ depends on the system parameters $\gamma_1, \gamma_2, \alpha$ but is independent of controllability time T and initial data u_0.

The present article addresses similar question of null controllability for a time discretized version of the control system (1.1), and uses it to produce a sequence of controls which approximates the control obtained in [3].

1.2. **Notations and symbols.** To set the framework for posing and studying the problem of this article, let us introduce some relevant notations and symbols. For any $M \in \mathbb{N}$, we define the discretization parameter $\Delta t = \frac{T}{M}$. For $a, b \in \mathbb{R}$, we define the set $[\![a, b]\!]$ as

$$[\![a, b]\!] := [a, b] \cap \mathbb{N}.$$

We define $t_n = n\Delta t$ and $t_{n+\frac{1}{2}} = \left(n + \frac{1}{2}\right) \Delta t$, for $n \in [\![0, M-1]\!]$. Let us now denote the set of primal and dual discrete points, respectively as

$$\mathcal{P} := \{t_n : n \in [\![1, M]\!]\}, \quad \mathcal{D} := \{t_{n+\frac{1}{2}} : n \in [\![0, M-1]\!]\}. \tag{1.6}$$

Further, we denote

$$\overline{\mathcal{P}} := \mathcal{P} \cup \{t_0\}, \quad \overline{\mathcal{D}} := \mathcal{D} \cup \{t_{M+\frac{1}{2}}\}, \tag{1.7}$$

where $t_{M+1/2}$ is an extra point outside the interval $[0, T]$.

We denote the sets of all real-valued discrete functions defined on $\mathcal{P}$ and $\mathcal{D}$ by $\mathbb{R}^{\mathcal{P}}$ and $\mathbb{R}^{\mathcal{D}}$, respectively. For $u \in \mathbb{R}^{\mathcal{P}}$ $\left(\text{resp. } u \in \mathbb{R}^{\mathcal{D}}\right)$, we denote by u^n $\left(\text{resp. } u^{n+\frac{1}{2}}\right)$, its value at the point t_n $\left(\text{resp. } t_{n+\frac{1}{2}}\right)$. We define the time-discrete integrals as

$$\int_0^T u := \sum_{n=1}^{M} \Delta t\, u^n, \quad \text{for } u \in \mathbb{R}^{\mathcal{P}}, \tag{1.8}$$

and

$$-\!\!\!\!\!\!\int_0^T u := \sum_{n=0}^{M-1} \Delta t\, u^{n+\frac{1}{2}}, \quad \text{for } u \in \mathbb{R}^{\mathcal{D}}. \tag{1.9}$$

Let $(X, \|\cdot\|_X)$ be a real Banach space. We denote by $X^{\mathcal{P}}$ and $X^{\mathcal{D}}$ the sets of vector-valued functions defined on $\mathcal{P}$ and $\mathcal{D}$ respectively. We introduce the spaces $L^p_{\mathcal{P}}(0,T;X)$ and $L^p_{\mathcal{D}}(0,T;X)$, $1 \leqslant p < +\infty$, to denote the spaces $X^{\mathcal{P}}$ and $X^{\mathcal{D}}$, endowed with the norms

$$\|u\|_{L^p_{\mathcal{P}}(0,T;X)} := \left(\int_0^T \|u\|_X^p\right)^{1/p} \quad \text{and} \quad \|u\|_{L^p_{\mathcal{D}}(0,T;X)} := \left(-\!\!\!\!\!\!\int_0^T \|u\|_X^p\right)^{1/p}, \text{ respectively,} \tag{1.10}$$

where the discrete integrals are defined by (1.8) and (1.9). More precisely,

$$\|u\|_{L^p_{\mathcal{P}}(0,T;X)} = \left(\sum_1^M \Delta t \, \|u^n\|_X^p\right)^{1/p} \quad \text{and} \quad \|u\|_{L^p_{\mathcal{D}}(0,T;X)} = \left(\sum_0^{M-1} \Delta t \, \left\|u^{n+\frac{1}{2}}\right\|_X^p\right)^{1/p}.$$

For $p = \infty$, we define the spaces $L^\infty_{\mathcal{P}}(0,T;X)$ and $L^\infty_{\mathcal{D}}(0,T;X)$ by means of the norms

$$\|u\|_{L^\infty_{\mathcal{P}}(0,T;X)} := \sup_{t_n \in \mathcal{P}} \|u(t_n)\|_X \text{ and } \|u\|_{L^\infty_{\mathcal{D}}(0,T;X)} := \sup_{t_{n+\frac{1}{2}} \in \mathcal{D}} \|u(t_{n+\frac{1}{2}})\|_X. \tag{1.11}$$

For $p = 2$ and $X = L^2(0,1)$, the spaces $L^p_{\mathcal{P}}(0,T;X)$ and $L^p_{\mathcal{D}}(0,T;X)$ are Hilbert spaces with the inner products

$$\left\langle u, v \right\rangle_{L^2_{\mathcal{P}}(0,T;L^2(0,1))} = \iint_Q u\, v = \sum_1^M \Delta t \, \langle u^n, v^n \rangle_{L^2(0,1)}, \tag{1.12}$$

$$\text{and } \left\langle u, v \right\rangle_{L^2_{\mathcal{D}}(0,T;L^2(0,1))} = -\!\!\!\!\!\!\iint_Q u\, v = \sum_0^{M-1} \Delta t \, \left\langle u^{n+\frac{1}{2}}, v^{n+\frac{1}{2}} \right\rangle_{L^2(0,1)}, \tag{1.13}$$

respectively.

Let us now introduce the translation operators $\mathtt{t}^+$ and $\mathtt{t}^-$ as

$$\mathtt{t}^+ : X^{\mathcal{P}} \to X^{\mathcal{D}} : (\mathtt{t}^+ u)^{n+\frac{1}{2}} = u^{n+1}, \text{for } n \in [\![0, M-1]\!],$$
$$\mathtt{t}^- : X^{\overline{\mathcal{P}}} \to X^{\mathcal{D}} : (\mathtt{t}^- u)^{n+\frac{1}{2}} = u^n, \quad \text{for } n \in [\![0, M-1]\!].$$

Next we define the difference operator D_t as a map from $X^{\overline{\mathcal{P}}}$ into $X^{\mathcal{D}}$ given by

$$\left(D_t u\right)^{n+\frac{1}{2}} := \frac{u^{n+1} - u^n}{\Delta t} = \left(\frac{1}{\Delta t}(\mathtt{t}^+ - \mathtt{t}^-)u\right)^{n+\frac{1}{2}}, \quad n \in [\![0, M-1]\!]. \tag{1.14}$$

Analogously, we define the translation operators $\bar{\mathtt{t}}^+ : X^{\overline{\mathcal{D}}} \to X^{\mathcal{P}}$ and $\bar{\mathtt{t}}^- : X^{\mathcal{D}} \to X^{\mathcal{P}}$ by

$$\left(\bar{\mathtt{t}}^+ u\right)^n := u^{n+\frac{1}{2}} \quad \text{and} \quad \left(\bar{\mathtt{t}}^- u\right)^n := u^{n-\frac{1}{2}}, \quad n \in [\![1, M]\!], \tag{1.15}$$

respectively, as well as an difference operator $\overline{D}_t$ from $X^{\overline{\mathcal{D}}}$ to $X^{\mathcal{P}}$ given by

$$\left(\overline{D}_t u\right)^n := \frac{u^{n+\frac{1}{2}} - u^{n-\frac{1}{2}}}{\Delta t} = \left(\frac{1}{\Delta t}(\bar{\mathtt{t}}^+ - \bar{\mathtt{t}}^-)u\right)^n, \quad n \in [\![1, M]\!]. \tag{1.16}$$

1.3. **Time-discrete control system.** Let us now consider the following time-discrete control system

$$\begin{cases} \dfrac{u_1^{n+1} - u_1^n}{\Delta t} - \partial_x(\gamma_1 \partial_x u_1^{n+1}) = 0, & n \in [\![0, M-1]\!], \\ \dfrac{u_2^{n+1} - u_2^n}{\Delta t} - \partial_x(\gamma_2 \partial_x u_2^{n+1}) = 0, & n \in [\![0, M-1]\!], \\ u_1^{n+1}(0) = 0,\ u_2^{n+1}(0) = v^{n+1}, & n \in [\![0, M-1]\!], \\ u_1^{n+1}(1) = u_2^{n+1}(1), & n \in [\![0, M-1]\!], \\ \gamma_1(1)\partial_x u_1^{n+1}(1) + \gamma_2(1)\partial_x u_2^{n+1}(1) + \alpha u_1^{n+1}(1) = 0, & n \in [\![0, M-1]\!]. \\ u_1^0 = u_{1,0},\ u_2^0 = u_{2,0}. \end{cases} \tag{1.17}$$

This discrete system can be obtained from (1.1) using standard implicit Euler scheme for the time derivative, where u_i^n (resp. v^n) denote the approximate value of u_i for $i = 1, 2$ (resp. v) at time t_n.

The discretized differential equation can be written in shorthand form as

$$\begin{cases} (D_t u_1)^{n+\frac{1}{2}} - \partial_x(\gamma_1(\mathtt{t}^+ \partial_x u_1))^{n+\frac{1}{2}} = 0, \\ (D_t u_2)^{n+\frac{1}{2}} - \partial_x(\gamma_2(\mathtt{t}^+ \partial_x u_2))^{n+\frac{1}{2}} = 0, \end{cases} \quad n \in [\![0, M-1]\!]. \tag{1.18}$$

Using the Lax-Milgram theorem, one can easily prove the following existence-uniqueness result.

Proposition 1.5 (Well-posedness). *For any given $u^0 \in \mathcal{H}'$, the above discrete system (1.17) can be solved inductively to get a unique sequence of weak solution $u^{n+1} \in \left[H^1(0,L)\right]^2$ for $n \in [\![0, M-1]\!]$, satisfying the boundary conditions*

$$\begin{cases} u_1^{n+1}(0) = 0, u_2^{n+1}(0) = v^{n+1}, \\ u_1^{n+1}(1) = u_2^{n+1}(1), \end{cases}$$

and the identity

$$\sum_{i=1}^{2} \int_0^1 \gamma_i \, \partial_x u_i^{n+1} \, \partial_x \xi_i + \frac{1}{\Delta t} \sum_{i=1}^{2} \int_0^1 u_i^{n+1} \xi_i + \alpha \, u_1^{n+1}(1) \, \xi(1) = \frac{1}{\Delta t} \langle u^n, \xi \rangle_{\mathcal{H}',\mathcal{H}}, \quad \forall \xi \in \mathcal{H}. \tag{1.19}$$

Motivated by Definition 1.3, we now define the null controllability for the above discrete control system.

Definition 1.6 (Discrete null controllability). *The time discrete control system (1.17) is said to be null controllable at $n = M$ if for any $u_0 \in \mathcal{H}'$, there exist controls $\{v^{n+1}\}_{[\![0,M-1]\!]}$ such that the solution $\{u^{n+1}\}_{[\![0,M-1]\!]}$ satisfies $u^M = 0$ in $L^2(0,1)$.*

As pointed out in the work [22] for single heat equation with interior control, one cannot expect the null controllability of the discrete control system (1.17) to hold in the above sense. In fact, we have proved so in Section 2. This motivates us to study a weaker notion, known as $\phi(\Delta t)$- controllability, which roughly means that there exists a discrete control $v \in L^2_{\mathcal{P}}(0,T)$ such that the norm of u^M is small, and tends to 0 as $\Delta t \to 0$ at a rate of $\sqrt{\phi(\Delta t)}$, for any function $\phi : (0,\infty) \to (0,\infty)$ satisfying

$$\liminf_{\Delta t \to 0} \frac{\phi(\Delta t)}{e^{-C/(\Delta t)^{1/4}}} > 0, \text{for some constant } C. \tag{1.20}$$

This notion of $\phi(\Delta t)$-controllability is an appropriate analogue of null controllability for discrete control systems (see Remark 1.8), and such notion has been studied in literature for time-discrete parabolic systems, for instance, see the works [22], [4] for heat equation with interior control, and the recent work [13] for stabilized Kuramoto-Sivashinsky system.

1.4. **Main Results.** We prove the $\phi(\Delta t)$-controllability of the discrete control system (1.17) by following the duality approach, analogue to the duality approach used in [3] to study of null controllability of the main control system (1.1). First, we derive a time-discrete Carleman estimate, and then use it to deduce a relaxed observability inequality for the following adjoint system

$$\begin{cases} -(D_t\varphi_1)^{n+\frac{1}{2}} - \partial_x(\gamma_1(t^-\partial_x\varphi_1))^{n+\frac{1}{2}} = 0, \\ -(D_t\varphi_2)^{n+\frac{1}{2}} - \partial_x(\gamma_2(t^-\partial_x\varphi_2))^{n+\frac{1}{2}} = 0, \\ \varphi_1^n(0) = \varphi_2^n(0) = 0, \ \varphi_1^n(1) = \varphi_2^n(1), \qquad n \in [\![0, M-1]\!]. \\ \gamma_1(1)\partial_x\varphi_1^n(1) + \gamma_2(1)\partial_x\varphi_2^n(1) + \alpha\varphi_1^n(1) = 0, \\ \varphi_1^M(x) = \varphi_{1,M}(x), \varphi_2^M(x) = \varphi_{2,M}(x), \end{cases} \tag{1.21}$$

More precisely, we obtain the following weak observability inequality, analogue to the estimate (1.4).

Theorem 1.7 (Relaxed observability inequality). *There exist constants $K_0, K_1, K_2 > 0$ such that for all $T \in (0,1)$ and $\Delta t = \widetilde{\Delta t} \leqslant K_0(T^2+T)^{-4}T^6$, any solution to (1.21) with $\varphi^M = (\varphi_{1,M}, \varphi_{2,M}) \in \mathcal{H}$ satisfies*

$$\|\varphi^0\|^2_{\mathcal{H}} \leqslant C_{obs}\left(\int_0^T \left|\partial_x(t^-\varphi_2)(0)\right|^2 + e^{-\frac{K_2}{(\Delta t)^{1/4}}} \|\varphi^M\|^2_{\mathcal{H}} \right), \tag{1.22}$$

where $C_{obs} = e^{K_1(1+1/T)}$.

Remark 1.8. *The second term appearing on the right side of above inequality is specific to the discrete system, and does not appear in the observability inequality (1.4). Due to the presence of this extra term, the inequality (1.22) is referred as relaxed observability inequality in the literature. But note that this extra term appearing in (1.22) goes to zero as the discrete parameter $\Delta t \to 0$. Moreover, one can show that*

$$\int_0^T \left|\partial_x(t^-\varphi_2)(0)\right|^2 \to \int_0^T \left|\partial_x\varphi_2(0)\right|^2, \text{ as } \Delta t \to 0,$$

where φ_2 appearing on right is a solution component of the continuous adjoint system (1.5), see Lemma 6.7. This shows that the inequality (1.22) approximates the observability inequality (1.4). This points out that $\phi(\Delta t)$-controllability is a suitable notion of controllability for discrete control systems.

Remark 1.9. *The assumption $T \in (0,1)$ in the above theorem is made as we are mainly interested in small time controllability result. Although, one can follow the same procedure for $T \geqslant 1$, with a small change in the definition of $\widetilde{\Delta t}$.*

Next, we show that this weak observability inequality is enough to guarantee the $\phi(\Delta t)$-controllability of the time-discrete system (1.17) using the standard variational approach. More precisely, we are able to obtain the following controllability result.

Theorem 1.10 ($\phi(\Delta t)$-controllability in $\mathcal{H}'$)**.** *Let $T > 0$ and choose the discretization parameter Δt sufficiently small. Then, for any initial data $u_0 \in \mathcal{H}'$ and any function ϕ satisfying*

$$\liminf_{\Delta t \to 0} \frac{\phi(\Delta t)}{e^{-K_2/(\Delta t)^{1/3}}} > 0, \tag{1.23}$$

there exists a time-discrete control $v \in L^2_{\mathcal{P}}(0,T)$ satisfying

$$\|v\|_{L^2_{\mathcal{P}}(0,T)} \leqslant C \|u_0\|_{\mathcal{H}'},$$

such that the associated solution $u \in L^2_{\mathcal{P}}(0,T;\mathcal{H}')$ of (1.17) satisfies

$$\left\|u^M\right\|_{\mathcal{H}'} \leqslant C\sqrt{\phi(\Delta t)}\, \|u_0\|_{\mathcal{H}'}, \tag{1.24}$$

where $C > 0$ is a constant, depending on ϕ and T.

Using the sequence of controls $\{v^{n+1}\}_{n\in[\![0,M-1]\!]}$ responsible for $\phi(\Delta t)$-controllability of the discrete control system (1.17) and the associated solution $\{u^n\}_{n\in[\![0,M]\!]}$, we define approximations $V_M \in L^2(0,T)$, and $U_M \in L^2(0,T;E)$ as

$$V_M(t) = \sum_{n=0}^{M-1} 1_{(t_n,t_{n+1}]}(t)\, v^{n+1}, \quad t \in (0,T), \tag{1.25}$$

$$U_M(t,x) = 1_{[t_0,t_1]}(t)\frac{u^1(x)}{2} + \sum_{n=1}^{M-1} 1_{(t_n,t_{n+1}]}(t)\left(\frac{u^{n+1}+u^n}{2}\right)(x), \quad (t,x) \in (0,T)\times(0,1), \tag{1.26}$$

where $U_M = \Big(\big(U_M\big)_1, \big(U_M\big)_2\Big)$, and $u^k = \big(u_1^k, u_2^k\big)$. Then, we show that these sequence of functions (V_M, U_M) has a weak limit $(v,u) \in L^2(0,T)\times L^2(0,T;E)$ as $M \to \infty$, which satisfies the identity (1.2), i.e., the limit function u solves system (1.1) with the limit function v as non-homogeneous boundary data. Moreover, the limit function v of V_M turns out to be a control responsible for the null controllability of main continuous control system (1.1). We obtain the following convergence result.

Theorem 1.11 (Discrete to continuous)**.** *There exist functions $v \in L^2(0,T)$ and $u \in L^2(0,T;E)$ such that*

$$V_M \rightharpoonup v \text{ weakly in } L^2(0,T),$$
$$\text{and } U_M \rightharpoonup u \text{ weakly in } L^2(0,T;E).$$

Furthermore, the pair of functions (u,v) solves the control system (1.1) such that the state u satisfies

$$u(T,x) = (u_1(T,x), u_2(T,x)) = (0,0), \quad \text{for } x \in (0,1) \text{ a.e..}$$

1.5. **Short literature survey concerning parabolic systems.** The paper [22] is first one to address the question of uniform controllability for time discrete heat equation. The proof is based on the Lebeau-Robbiano's strategy for the continuous heat equation (see [17], [18]), which relies on an observability estimate for the eigenfunctions of the Dirichlet Laplacian. Later, the work [10] proved similar controllability result for an abstract controllable parabolic equation (discrete or continuous in space). In particular, using the controllability result of space-discrete heat equation proved in [18], they conclude about the controllability for fully discrete heat equation. In the article [7], the authors studied the controllability problem for space-discrete and fully discrete heat equation, and computed the controls using the variational approach. Moreover, they proved the convergence of control for the fully discrete system to that of semi-discrete one. For proving the controllability result for fully discrete system, they assumed spectral inequality for space-discrete system and then used the Lebeau-Robbiano's strategy to yield the

controllability for the fully discrete system. Such spectral inequality for space-discrete system has been proved by them in their earlier works [5], [6]. All the works mentioned above deals with an appropriate filtering of high-frequencies. More precisely, for the lower frequency, one proves the null controllability and for high frequencies, dissipation property of parabolic equation is utilized. This is where study of similar control problems for hyperbolic systems becomes different from the parabolic ones (see [23] for details).

Recently, a Carleman-type inequality for a time-discrete heat equation has been derived in [4] to deduce $\phi(\Delta t)$-controllability result for the time-discrete heat equation with localized interior control. Unlike the works mentioned before, this approach does not uses the spectrum of the underlying operator. Since then, there has been few more works utilizing this method, for instance, see [12] for fully discrete heat equation, and [13] for a time-discrete coupled system of fourth and second order parabolic equations.

Concerning the wave equation, the authors in [21] established some uniform boundary controllability of trapezoidal time discrete wave equation in higher dimension using the discrete version of multiplier method introduced in [14]. In [11], the authors proved controllability for general time-discrete conservative linear systems, which includes wave equation, in particular. Recently, the authors in [19] has studied the boundary controllability of time-discrete approximation schemes for a class of integro-differential equations in one dimension, which has been further extended in two dimensions in [20].

1.6. **Outline.** The remaining of the article is organized as follows. We first prove the lack of null controllability of the discrete control system (1.17) in the sense of Definition 1.6 in Section 2. In the next Section 3, we prove a suitable time-discrete Carleman estimate. Such estimate plays a very crucial role in proving the relaxed observability inequality as stated in Theorem 1.7, which has been proved in Section 4. In Section 5, we prove the $\phi(\Delta t)$-controllability result as stated in Theorem 1.10. Lastly, we dedicate Section 6 for the convergence analysis mentioned in Theorem 1.11.

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2. Lack of classical null controllability

Motivated by the work presented in [22], we use a contradictory argument to prove the lack of null controllability of the discrete control system (1.17) in the sense of Definition 1.6.

If possible, suppose that the control system (1.17) is null controllable, i.e., for any $u_0 \in \mathcal{H}'$, there exists a control $v \in \mathbb{R}^{\mathcal{P}}$ such that the solution $\left\{\left(u_1^{n+1}, u_2^{n+1}\right) : n \in [\![0, M-1]\!]\right\}$ of (1.17) satisfies

$$(u_1^M, u_2^M) = 0 \text{ in } L^2(0,1). \tag{2.1}$$

Using discrete control system (1.17) and integration by parts w.r.t. x, one can easily obtain the following identity:

$$\left\langle (D_t u)^{n+\frac{1}{2}}, \varphi^n \right\rangle_{\mathcal{H}',\mathcal{H}} = \sum_{i=1}^{2} \int_0^1 u_i^{n+1} \partial_x(\gamma_i \partial_x \varphi_i^n) + \gamma_2(0)\, \partial_x \varphi_2^n(0)\, v^{n+1}, \quad \text{for } n \in [\![0, M-1]\!].$$

Multiplying the above identity by Δt, and summing over $n \in [\![0, M-1]\!]$, we get

$$\int_0^T \left\langle D_t u, \mathtt{t}^- \varphi \right\rangle_{\mathcal{H}',\mathcal{H}} - \sum_{i=1}^{2} \int_0^T \left\langle \mathtt{t}^+ u_i, \partial_x(\gamma_i \mathtt{t}^- \partial_x \varphi) \right\rangle_{L^2} = \gamma_2(0) \int_0^T \left(\mathtt{t}^- \partial_x \varphi_2(0)\right) (\mathtt{t}^+ v).$$

Now using discrete integration by parts (A.6), and the discrete adjoint system (1.21), the above identity gives

$$-\left\langle u^0, \varphi^0 \right\rangle_{\mathcal{H}',\mathcal{H}} + \left\langle u^M, \varphi^M \right\rangle_{\mathcal{H}',\mathcal{H}} = \gamma_2(0) \int_0^T \left(\mathtt{t}^- \partial_x \varphi_2(0)\right) (\mathtt{t}^+ v),$$

which due to assumption (2.1) becomes

$$\big\langle u^0, \varphi^0 \big\rangle_{\mathcal{H}',\mathcal{H}} = -\gamma_2(0) \sum_{n=0}^{M-1} \Delta t\, \partial_x \varphi_2^n(0) v^{n+1}. \tag{2.2}$$

Claim: The identity (2.2) cannot be true.

Proof. Let $\Phi^0 \in C_c^\infty(0,1) \times C_c^\infty(0,1)$ with $\|\Phi^0\|_E > 0$. For $n \in [\![0, M-1]\!]$, define

$$\Phi_i^{n+1} := \Phi_i^n - \Delta t\, \partial_x(\gamma_i \partial_x \Phi_i^n), \quad \text{for } i \in \{1,2\}.$$

Note that $\Phi^n \in \big(C_c^\infty(0,1)\big)^2$ for all $n \in [\![1, M]\!]$. So, multiplying the above equation by Φ_i^n and using integration by parts, we get

$$\sum_{i=1}^2 \left[\Delta t \int_0^1 \gamma_i |\partial_x \Phi_i^n|^2 + \int_0^1 |\Phi_i^n|^2 \right] = \sum_{i=1}^2 \int_0^1 \Phi_i^n \Phi_i^{n+1} \leqslant \sum_{i=1}^2 \left[\frac{1}{2} \int_0^1 |\Phi_i^n|^2 + \frac{1}{2} \int_0^1 |\Phi_i^{n+1}|^2 \right],$$

which gives

$$\sum_{i=1}^2 \int_0^1 |\Phi_i^n|^2 \leqslant \sum_{i=1}^2 \int_0^1 |\Phi_i^{n+1}|^2, \quad \forall n \in [\![0, M-1]\!].$$

This shows that $\Phi^M \in C_c^\infty(0,1) \times C_c^\infty(0,1)$ satisfy $\|\Phi^M\|_E > 0$.

Now let us choose the terminal data $\varphi^M = \Phi^M \in \mathcal{H}$, in particular. Then by the uniqueness of solution of the adjoint system (1.21) we get

$$\varphi^n = \Phi^n, \quad \forall n \in [\![0, M]\!].$$

Due to the fact that $\varphi^n \in C_c^\infty(0,1) \times C_c^\infty(0,1)$, the identity (2.2) becomes

$$\big\langle u_0, \varphi^0 \big\rangle_{\mathcal{H}',\mathcal{H}} = 0,$$

and thus by choosing the initial data $u_0 = \varphi^0$, we get $\|\varphi^0\|_E = \|\Phi^0\|_E = 0$, i.e., $\Phi^0 = 0$, which is a contradiction. □

3. Time-discrete Carleman estimate

This section is devoted to derive a time-discrete version of the Carleman estimate obtained in [3] for the associated adjoint system to (1.1). The proof would be done by mimicking the steps from the continuous setup, as done in [4] and [13].

To state the time-discrete Carleman estimate, let us first introduce the relevant weight functions which is similar to the one used in [3] for the continuous system, and plays a very important role in the estimate. Assume a constant $\nu_0 \in (0,1)$, sufficiently close to 1 so that

$$\left(\frac{216\nu_0}{(1-\nu_0)^3} \gamma_2^2(1) - 7\gamma_1^2(1) \right) \geqslant 1. \tag{3.1}$$

For $i \in \{1,2\}$, consider the functions β_i on $[0,1]$ given by

$$\begin{cases} \beta_i(x) = 2 + c_i(x-1), \\ \text{with} \quad c_1 = 1,\ c_2 = c_2(\gamma_1,\gamma_2) := -\dfrac{6}{(1-\nu_0)} < 0. \end{cases} \tag{3.2}$$

Note that

$$\begin{cases} \beta_2(x) \geqslant \beta_1(x) > 0,\ x \in [0,1], \\ \beta_2(1) = \beta_1(1),\ \text{and } |\beta_2'| > |\beta_1'|. \end{cases}$$

Let $K = 2\max\big\{\|\beta_1\|_\infty, \|\beta_2\|_\infty\big\}$, and assume $\lambda > 1$. Then for $i \in \{1,2\}$, we define the functions η_i and μ_i over the interval $[0,1]$ as

$$\eta_i(x) = e^{\lambda K} - e^{\lambda \beta_i(x)}, \quad \mu_i(x) = e^{\lambda \beta_i(x)}. \tag{3.3}$$

Note that $\mu_1(1) = \mu_2(1)$ and $\eta_1(1) = \eta_2(1)$, since $\beta_1(1) = \beta_2(1)$. Next, given that

$$\eta_i(x) = \Big(e^{\lambda(K - \beta_i(x))} - 1 \Big) \mu_i(x),$$

we can establish the following relations:

$$\eta_i \leqslant e^{\lambda K}\mu_i \text{ on } [0,1], \tag{3.4}$$

$$\mu_i \leqslant C\eta_i \text{ on } [0,1]. \tag{3.5}$$

Observe that the constant appearing in (3.5) is independent of λ as $\lambda \geqslant 1$.

Let $\delta \in (0,1/2)$. Then, we define the function θ as

$$\theta(t) = \frac{1}{(t+\delta T)(T+\delta T - t)}, \quad t \in [0,T]. \tag{3.6}$$

One can easily observe that θ satisfies the following estimates

$$\frac{1}{T^2} \leqslant \max_{[0,T]} \theta(t) \leqslant \frac{1}{\delta T^2}. \tag{3.7}$$

Remark 3.1. *Note that the δT term is absent in the weight function associated to the continuous Carleman estimate. This is added here to avoid the singularities of θ at $t = 0$ and T, and thus this number δ is specific to the discrete system.*

For $\tau > 0$, let $s(t) = \tau\theta(t)$. Finally for $i \in \{1,2\}$, we define the functions r_i and ρ_i as

$$r_i(t,x) = e^{-\tau\theta(t)\eta_i(x)}, \quad \rho_i(t,x) = (r_i(t,x))^{-1}, \quad (t,x) \in (0,T)\times(0,1). \tag{3.8}$$

Then the spatial derivatives of μ_i, η_i and r_i can be given by

$$\partial_x\mu_i = \lambda c_i \mu_i, \quad \partial_x\eta_i = -\lambda c_i\mu_i, \quad \partial_x r_i = \tau\lambda c_i\theta\mu_i r_i. \tag{3.9}$$

Let $Q = (0,T)\times(0,1)$. We introduce the space $\mathcal{F}$, defined as

$$\mathcal{F} := \Big\{(\varphi_1,\varphi_2) \in \big(C^2(\overline{Q})\big)^2 : \varphi_1(t,0) = \varphi_2(t,0) = 0, \varphi_1(t,1) = \varphi_2(t,1), \sum_{i=1}^{2}\gamma_i(1)\partial_x\varphi_i(t,1) + \alpha\varphi_1(t,1) = 0\Big\}. \tag{3.10}$$

Let us denote the discrete adjoint differential operator as

$$(L_{\mathcal{P}}\varphi_i)^{n+\frac{1}{2}} := -(D_t\varphi_i)^{n+\frac{1}{2}} - \partial_x(\gamma_1(t^-\partial_x\varphi_i))^{n+\frac{1}{2}}, \text{ for } i \in \{1,2\}.$$

Theorem 3.2 (Time-discrete Carleman estimate)**.** *Let η_i for $i = 1,2$ and θ be the weight function as defined in (3.3) and (3.6) resp. Then for $\lambda \geqslant 1$ large enough, there exists $\tau_0 \geqslant 1$, $\epsilon_0 > 0$ and a constant $C > 0$, all depending on $\gamma_1, \gamma_2, \alpha, T$ and λ, such that the following estimate holds true*

$$\begin{aligned}
\tau^3\sum_{i=1}^{2}\iint_Q (t^-\theta)^3(t^-r_i)^2\,|t^-\varphi_i|^2 + \tau\sum_{i=1}^{2}\iint_Q (t^-\theta)(t^-r_i)^2\,|\partial_x(t^-\varphi_i)|^2 + \tau^2\int_0^T (t^-\theta)(t^-r_1(1))^2\big|t^-\varphi_1(1)\big|^2 \\
\leqslant C\sum_{i=1}^{2}\big\|(t^-r_i)\,(L_{\mathcal{P}}\varphi_i)\big\|^2_{L^2_{\mathcal{D}}(Q)} + C\tau\int_0^T (t^-\theta)\,\big|t^-\,(r_2(0))\big|^2\,\big|\partial_x(t^-\varphi_2)(0)\big|^2 \\
+ C\,(\Delta t)^{-1}\sum_{i=1}^{2}\Big(\int_0^1\big|(r_i\,\varphi_i)^0\big|^2 + \int_0^1\big|(r_i\,\varphi_i)^M\big|^2 + \int_0^1\big|(r_i\,\partial_x\varphi_i)^M\big|^2\Big),
\end{aligned} \tag{3.11}$$

for every $\tau \geqslant \tau_0(T^2+T)$, $\Delta t \in (0,1)$, and for any $\delta \in (0,1/2]$ satisfying the condition

$$\frac{\tau^3\Delta t}{\delta^4(\min\{T^3,T^6\})} \leqslant \epsilon_0,$$

and for all time discrete function $(\varphi_1,\varphi_2) \in \mathcal{F}^{\overline{\mathcal{P}}}$.

Proof. The proof of this theorem is quite long and so it has been broken in steps for better readability.

Step-1: Change of variable:

Let us first make the following change of variables. For $i \in \{1,2\}$, we denote

$$q_i^n(x) := r_i^n(x)\,\varphi_i^n(x),\ n \in [\![0,M]\!],$$

where $r_i^n(x) = r_i(t_n,x)$. Then, $q_i \in \mathcal{C}^2([0,1])^{\overline{\mathcal{P}}}$, and so for $n \in [\![0,M-1]\!]$ we have

$$\gamma_i\,\partial_x q_i^n = \tau\lambda c_i\,\theta^n\mu_i\gamma_i\,q_i^n + r_i^n\gamma_i\,\partial_x\varphi_i^n.$$

Recall that $(\varphi_1, \varphi_2) \in \mathcal{F}^{\overline{\mathcal{P}}}$, and so (q_1, q_2) satisfies the following boundary conditions:

$$\begin{cases} q_1^n(0) = q_2^n(0) = 0, \\ q_1^n(1) = r_1^n(1)\varphi_1^n(1) = r_2^n(1)\varphi_2^n(1) = q_2^n(1), \\ (\gamma_1\, \partial_x q_1^n)(1) + (\gamma_2\, \partial_x q_2^n)(1) + \alpha\, q_1^n(1) = \tau\lambda \left(\sum_{i=1}^{2} c_i \gamma_i(1) \right) \mu_1(1) \theta^n q_1^n(1), \end{cases} \tag{3.12}$$

for $n \in [\![0, M-1]\!]$.

Now, we compute

$$\begin{aligned} \partial_x(\gamma_i \partial_x(\mathtt{t}^- q_i)) = {} & \tau\lambda^2 c_i^2(\mathtt{t}^-\theta)\mu_i\gamma_i(\mathtt{t}^- q_i) + \tau\lambda c_i(\mathtt{t}^-\theta)\mu_i\gamma'(\mathtt{t}^- q_i) + \tau\lambda c_i(\mathtt{t}^-\theta)\mu_i\gamma_i\partial_x(\mathtt{t}^- q_i) \\ & - \tau^2\lambda^2 c_i^2(\mathtt{t}^-\theta)^2\mu_i^2\gamma_i(\mathtt{t}^- q_i) - \tau(\mathtt{t}^-\theta)\eta_i'\gamma_i\partial_x(\mathtt{t}^- q_i) + (\mathtt{t}^- r)\partial_x(\gamma_i\partial_x(\mathtt{t}^-\varphi)). \end{aligned}$$

Next, using Lemma A.1 and Lemma B.1, we get

$$\begin{aligned} D_t(q_i) &= (\mathtt{t}^- r_i) D_t\varphi_i + D_t r_i (\mathtt{t}^+\varphi_i) \\ &= (\mathtt{t}^- r_i) D_t\varphi_i + (\mathtt{t}^+\rho)(D_t r_i)(\mathtt{t}^+ q_i) \\ &= (\mathtt{t}^- r_i) D_t\varphi_i + \left[-\tau(\mathtt{t}^+\theta')\eta_i + \Delta t \left(\frac{\tau}{\delta^3 T^4} - \frac{\tau^2}{\delta^4 T^6} \right) \mathcal{O}_\lambda(1) \right] (\mathtt{t}^+ q_i). \end{aligned}$$

Combining the above computations, we get the following identity:

$$A q_i + B q_i = F_i, \quad i \in \{1, 2\}, \tag{3.13}$$

where

$$A q_i = \partial_x\big(\gamma_i \partial_x(\mathtt{t}^- q_i)\big) + \tau^2\lambda^2 c_i^2(\mathtt{t}^-\theta)^2\mu_i^2\, \gamma_i(\mathtt{t}^- q_i) + \tau\, \eta_i\big(\mathtt{t}^+\theta'\big)\big(\mathtt{t}^- q_i\big), \tag{3.14}$$

$$B q_i = D_t q_i - 2\tau\lambda\, c_i(\mathtt{t}^-\theta)\mu_i\gamma_i\, \partial_x\big(\mathtt{t}^- q_i\big) - 2\tau\lambda^2 c_i^2(\mathtt{t}^-\theta)\mu_i\gamma_i(\mathtt{t}^- q_i), \tag{3.15}$$

$$\begin{aligned} F_i = {} & (\mathtt{t}^- r_i)\Big(D_t\varphi_i + \partial_x\big(\gamma_i\, \mathtt{t}^-(\partial_x\varphi_i)\big)\Big) + \tau\lambda c_i(\mathtt{t}^-\theta)\mu_i\gamma_i'\,(\mathtt{t}^- q_i) - \tau\lambda^2 c_i^2(\mathtt{t}^-\theta)\mu_i\gamma_i(\mathtt{t}^- q_i) \\ & + \Delta t \left(\frac{\tau}{\delta^3 T^4} - \frac{\tau^2}{\delta^4 T^6} \right) \mathcal{O}_\lambda(1)(\mathtt{t}^+ q_i) - \Delta t\, \tau\, \big(\mathtt{t}^+\theta'\big)\, \eta_i (D_t q_i). \end{aligned} \tag{3.16}$$

Taking L^2-norm in (3.13), we have:

$$\|A q_i\|^2_{L^2_{\mathcal{D}}(Q)} + \|B q_i\|^2_{L^2_{\mathcal{D}}(Q)} + 2\langle A q_i, B q_i\rangle_{L^2_{\mathcal{D}}(Q)} = \|F_i\|^2_{L^2_{\mathcal{D}}(Q)}, \quad i \in \{1, 2\}. \tag{3.17}$$

We denote the inner product $2\langle A q_i, B q_i\rangle_{L^2_{\mathcal{D}}(Q)}$ as the sum $\sum_{j,k=1}^{3} I^i_{jk}$ with $I^i_{jk} = \langle (A q_i)_j, (B q_i)_k\rangle_{L^2_{\mathcal{D}}(Q)}$, for $i \in \{1, 2\}$.

Step-2: Expanding and estimating the cross product terms:

Estimating the inner product $\langle A q_i, B q_i\rangle_{L^2_{\mathcal{D}}(Q)}$ and $\|F_i\|_{L^2_{\mathcal{D}}(Q)}$ properly, one can obtain the estimate mentioned in the lemma below.

Lemma 3.3. *Let $\epsilon_0 > 0$ be any real number. Then for a fixed $\lambda = \lambda_0$, there exists $\tau_0(\gamma_1, \gamma_2, \alpha) \geqslant 1$ and a constant $C > 0$ such that*

$$\begin{aligned} \sum_{i=1}^{2} \Bigg(\|B q_i\|^2_{L^2_{\mathcal{D}}(Q)} + \tau^3 \iint_Q (\mathtt{t}^-\theta)^3 |\mathtt{t}^- q_i|^2 + \tau \iint_Q (\mathtt{t}^-\theta) |\partial_x(\mathtt{t}^- q_i)|^2 + \int_0^1 |q_i^M|^2 - \widehat{Z}_i \Bigg) \\ \leqslant C \sum_{i=1}^{2} \left(\Big\| (\mathtt{t}^- r_i)\Big(D_t\varphi_i + \partial_x\big(\gamma_i\, \partial_x(\mathtt{t}^-\varphi)\big)\Big) \Big\|^2_{L^2_{\mathcal{D}}(Q)} + \widehat{W}_i \right) \\ + \epsilon_0\, C \sum_{i=1}^{2} \left(\iint_Q (\mathtt{t}^- s)^3\, |\mathtt{t}^- q_i|^2 + \iint_Q (\mathtt{t}^- s)^{-1}\, |D_t q_i|^2 \right), \end{aligned} \tag{3.18}$$

for all $\tau \geqslant \tau_0(T^2+T)$, *and for all* $\delta \in (0,1/2]$ *and* Δt *satisfying the condition* $\frac{\tau^3\,\Delta t}{\delta^4 \min\{T^3,T^6\}} \leqslant \epsilon_0$, *where*

$$\widehat{W}_i = \int_0^1 (\partial_x q_i^M)^2 + \tau \int_0^1 \mu_i^2(\theta^0)^2(q_i^0)^2 + (\Delta t)^2\left(\frac{\tau^2}{\delta^6 T^8} + \frac{\tau^4}{\delta^8 T^{12}}\right)\int_0^1 (q_i^M)^2, \tag{3.19}$$

$$\begin{aligned}\widehat{Z}_i = &-\int_0^T \gamma_i(1)\,\mathtt{t}^-(\partial_x q_i)(1) D_t q_i(1) + \tau c_i \int_0^T (\mathtt{t}^-\theta)\mu_i(1)\,(\gamma_i(1))^2\left(\mathtt{t}^-(\partial_x q_i(1))\right)^2\\ &-\tau c_i\int_0^T (\mathtt{t}^-\theta)\mu_i(0)\,(\gamma_i(0))^2\left(\mathtt{t}^-(\partial_x q_i(0))\right)^2 + 2c_i^2\,(\gamma_i(1))^2\,\tau\mu_i(1)\int_0^T (\mathtt{t}^-\theta)(\mathtt{t}^- q_i)(1)\,\mathtt{t}^-(\partial_x q_i)(1)\\ &+\tau^3 c_i^3(\mu_i(1))^3\int_0^T (\mathtt{t}^-\theta)^3\,(\gamma_i(1))^2\,(\mathtt{t}^- q_i)^2(1) + \tau^2|c_i|\gamma_i(1)\mu_i(1)\eta_i(1)T\int_0^T (\mathtt{t}^-\theta)^3(\mathtt{t}^- q_i(1))^2\\ &+\tau^2|c_i|\gamma_i(1)\mu_i(1)\eta_i(1)\frac{\Delta t}{\delta^3 T^4}\int_0^T (\mathtt{t}^-\theta)(\mathtt{t}^- q_i(1))^2.\end{aligned} \tag{3.20}$$

Further, we have following auxiliary estimate for the boundary terms $\widehat{Z}_i$.

Lemma 3.4 (Estimates on the boundary terms of $\widehat{Z}_i$)**.** *Let* $\tau_0 > 0$ *be a sufficiently large number. Then for* $\tau \geqslant \tau_0(T+T^4)$, *there exists a constant* $C>0$ *such that*

$$\begin{aligned}-\sum_{i=1}^2 \widehat{Z}_i \geqslant &-C\overline{W}_1 - C\bar{E}_1 + C\tau\int_0^T (\mathtt{t}^-\theta)\left|\partial_x(\mathtt{t}^- q_2(1))\right|^2 + C\tau^3\int_0^T (\mathtt{t}^-\theta)^3\left|\mathtt{t}^- q_1(1)\right|^2\\ &-C\tau\int_0^T (\mathtt{t}^-\theta)\left|\mathtt{t}^-(r_2(0)\right|^2\left|\partial_x(\mathtt{t}^-\varphi_2)(0)\right|^2 - C\tau^2\frac{\Delta t}{\delta^3 T^4}\sum_{i=1}^2\int_0^T (\mathtt{t}^-\theta)(\mathtt{t}^- q_i(1))^2,\end{aligned} \tag{3.21}$$

where

$$\overline{W}_1 = \frac{\alpha}{2}\left(\left(q_1^M(1)\right)^2 - \left(q_1^0(1)\right)^2\right) - \frac{\tau\lambda_0}{2}\left(\sum_{i=1}^2 c_i\gamma_i(1)\right)\left\{\theta^M\left(q_1^M(1)\right)^2 - \theta^0\left(q_1^0(1)\right)^2\right\}, \tag{3.22}$$

$$\begin{aligned}\bar{E}_1 = \Delta t\Bigg\{&-\frac{\alpha}{2}\int_0^T (D_t q_1(1))^2\,dt + \frac{\tau\lambda_0}{2}\left(\sum_{i=1}^2 c_i\gamma_i(1)\right)\int_0^T \mu_1(1)(\mathtt{t}^-\theta)\left(D_t q_1(1)\right)^2 dt\\ &+C\mu_1(1)\frac{\tau\lambda_0}{\delta^3 T^4}\int_0^T (\mathtt{t}^- q_1(1))^2 + C\tau\Delta t\lambda_0\mu_1(1)\int_0^T\left(T(\mathtt{t}^-\theta)^2 + \frac{\Delta t}{\delta^3 T^4}\right)\left(D_t q_1(1)\right)^2\Bigg\}.\end{aligned} \tag{3.23}$$

The proofs of Lemma 3.3 and Lemma 3.4 have been postponed to Appendix C and Appendix D, respectively for better readability.

Now, using the above estimate (3.21) in (3.18), we get the following:

$$\begin{aligned}&\sum_{i=1}^2\left(\|Bq_i\|^2_{L^2_{\mathcal{D}}(Q)} + \tau^3\iint_Q (\mathtt{t}^-\theta)^3|\mathtt{t}^- q_i|^2 + \tau\iint_Q (\mathtt{t}^-\theta)|\partial_x(\mathtt{t}^- q_i)|^2 + \int_0^1 |q_i^M|^2\right) + \tau^3\int_0^T (\mathtt{t}^-\theta)^3|\mathtt{t}^- q_1(1)|^2\\ &\leqslant C\sum_{i=1}^2\left\|(\mathtt{t}^- r_i)\left(D_t\varphi_i + \partial_x\left(\gamma_i\,\partial_x(\mathtt{t}^-\varphi)\right)\right)\right\|^2_{L^2_{\mathcal{D}}(Q)} + C\tau\int_0^T (\mathtt{t}^-\theta)\left|\mathtt{t}^-(r_2(0)\right|^2\left|\partial_x(\mathtt{t}^-\varphi_2)(0)\right|^2\\ &\quad+\epsilon_0\,C\sum_{i=1}^2\left(\iint_Q (\mathtt{t}^- s)^3\,|\mathtt{t}^- q_i|^2 + \iint_Q (\mathtt{t}^- s)^{-1}\,|D_t q_i|^2\right) + C\sum_{i=1}^2\widehat{W}_i + \overline{W}_1 + \bar{E}_1\\ &\quad+C\tau^2\frac{\Delta t}{\delta^3 T^4}\sum_{i=1}^2\int_0^T (\mathtt{t}^-\theta)(\mathtt{t}^- q_i(1))^2,\end{aligned} \tag{3.24}$$

for all $\tau \geqslant \tau_0(T^2+T)$ with some constant $C := C(\gamma_1,\gamma_2,\alpha) > 0$, where τ_0 is large enough to absorb the unwanted terms from right by the l.h.s. terms to get the above estimate.

Step-3: Absorption of $(D_t q_i)$ term from the r.h.s.:

Let us recall the expression of $D_t q_i$ from equation (3.15),

$$D_t q_i = Bq_i + 2\tau\lambda_0 c_i(\mathtt{t}^-\theta)\mu_i\gamma_i\,\partial_x(\mathtt{t}^- q_i) + 2\tau\lambda_0^2 c_i^2(\mathtt{t}^-\theta)\mu_i\gamma_i(\mathtt{t}^- q_i).$$

As $\tau \geqslant \tau_0 T^2$, we note that

$$\tau^{-1}(\mathtt{t}^-\theta)^{-1}\mu_i^{-1} \leqslant \tau^{-1}T^2 e^{-\lambda_0\beta_1} \leqslant C.$$

Using this, we have

$$\tau^{-1}\iint_Q (\mathtt{t}^-\theta)^{-1}\mu_i^{-1}|D_t q_i|^2 \leqslant C\left(\|Bq_i\|^2_{L^2_{\mathcal{D}}(Q)} + \tau\lambda_0^2\iint_Q (\mathtt{t}^-\theta)\mu_i\left(\mathtt{t}^-(\partial_x q_i)\right)^2 + \tau\lambda_0^4\iint_Q (\mathtt{t}^-\theta)\mu_i(\mathtt{t}^- q_i)^2\right).$$

By utilizing the above estimate in (3.24) and choosing ϵ_0 small enough, we get

$$\begin{aligned}\sum_{i=1}^2\left(\tau^3\iint_Q (\mathtt{t}^-\theta)^3|\mathtt{t}^- q_i|^2 + \tau\iint_Q (\mathtt{t}^-\theta)|\partial_x(\mathtt{t}^- q_i)|^2 + \int_0^1 |q_i^M|^2\right) + \tau\int_0^T (\mathtt{t}^-\theta)|\mathtt{t}^- q_1(1)|^2 \\ \leqslant C\sum_{i=1}^2\left\|(\mathtt{t}^- r_i)\Big(D_t\varphi_i + \partial_x\big(\gamma_i\,\partial_x(\mathtt{t}^-\varphi)\big)\Big)\right\|^2_{L^2_{\mathcal{D}}(Q)} + C\tau\int_0^T (\mathtt{t}^-\theta)\left|\mathtt{t}^-(r_2(0)\right|^2\left|\partial_x(\mathtt{t}^-\varphi_2)(0)\right|^2 \\ + C\sum_{i=1}^2\widehat{W}_i + \overline{W}_1 + \bar{E}_1 + C\tau^2\frac{\Delta t}{\delta^3 T^4}\sum_{i=1}^2\int_0^T (\mathtt{t}^-\theta)(\mathtt{t}^- q_i(1))^2, \end{aligned} \tag{3.25}$$

for all $\tau \geqslant \tau_0(T^2+T)$.

Step-4: Estimating the remaining discrete terms of r.h.s.:

We begin this step by estimating the last two terms of the r.h.s. of inequality (3.25). From the expression of $\bar{E}_1$ given by (3.23), we see

$$\bar{E}_1 \leqslant C\Delta t\left\{-\frac{\alpha}{2} + \frac{\tau}{\delta T^2}\left(\sum_{i=1}^2 c_i\gamma_i(1)\right) + \tau\Delta t\left(\frac{T}{\delta^2 T^4} + \frac{\Delta t}{\delta^3 T^4}\right)\right\}\int_0^T (D_t q_1(1))^2\,dt + C\frac{\tau\,\Delta t}{\delta^3 T^4}\int_0^T (\mathtt{t}^- q_1(1))^2\,dt.$$

Now using the condition $\dfrac{\tau^3\,\Delta t}{\delta^4\min\{T^3,T^6\}} \leqslant \epsilon_0$, $\tau \geqslant \tau_0 T^2$ and the fact $\theta^{-1} \leqslant T^2$, we get

$$\begin{aligned}\bar{E}_1 + \tau^2\frac{\Delta t}{\delta^3 T^4}\sum_{i=1}^2\int_0^T (\mathtt{t}^-\theta)(\mathtt{t}^- q_i(1))^2 \\ \leqslant C\Delta t\left\{-\frac{\alpha}{2} + \frac{\tau}{\delta T^2}\left(\sum_{i=1}^2 c_i\gamma_i(1)\right) + \epsilon_0\right\}\int_0^T (D_t q_1(1))^2\,dt \\ + C\Delta t\left(\frac{\tau}{\delta^3 T^2} + \frac{\tau^2}{\delta^3 T^4}\right)\int_0^T (\mathtt{t}^-\theta)(\mathtt{t}^- q_1(1))^2\,dt \\ \leqslant C\Delta t\left\{-\frac{\alpha}{2} + \frac{\tau}{\delta T^2}\left(\sum_{i=1}^2 c_i\gamma_i(1)\right) + \epsilon_0\right\}\int_0^T (D_t q_1(1))^2\,dt + C\epsilon_0\int_0^T (\mathtt{t}^-\theta)(\mathtt{t}^- q_1(1))^2\,dt.\end{aligned}$$

Note that the first term is negative as $\sum_{i=1}^2 c_i\gamma_i(1)$ can be made so by choosing ν_0 close to 1 (see (3.2)), and so

$$\bar{E}_1 + \tau^2\frac{\Delta t}{\delta^3 T^4}\sum_{i=1}^2\int_0^T (\mathtt{t}^-\theta)(\mathtt{t}^- q_i(1))^2 \leqslant \epsilon_0\int_0^T (\mathtt{t}^-\theta)(\mathtt{t}^- q_1(1))^2\,dt. \tag{3.26}$$

Using the facts $\tau \geqslant \tau_0 T^2$, $0<\delta<1/2$, and $\theta \leqslant 1/(\delta T^2)$, we now estimate the terms $\widehat{W}_i, \overline{W}_1$ appearing in the r.h.s. of (3.25) as follows

$$\begin{aligned}\sum_{i=1}^2\widehat{W}_i + \overline{W}_1 \leqslant C\sum_{i=1}^2\left\{\int_0^1\left(\partial_x q_i^M\right)^2 + \frac{(\Delta t)^2\tau^4}{\delta^8(\min\{T^8,T^{12}\})}\int_0^1 (q_i^M)^2 + \frac{\tau}{\delta^2 T^4}\int_0^1 (q_i^0)^2\right\} \\ + \frac{\alpha}{2}\left(q_1^M(1)\right)^2 - \frac{\tau\lambda_0}{2}\left(\sum_{i=1}^2 c_i\gamma_i(1)\right)\left(\theta^M\left(q_1^M(1)\right)^2 - \theta^0\left(q_1^0(1)\right)^2\right),\end{aligned}$$

where $\widehat{W}_i, \overline{W}_1$ are given by (3.19) and (3.22), respectively. Further we use the fact that $\sum_{i=1}^2 c_i\gamma_i(1) < 0$, to remove the term $\theta^0 \left(q_1^0(1)\right)^2$ from the right side, and thus we obtain:

$$\sum_{i=1}^2 \widehat{W}_i + \overline{W}_1 \leqslant C\sum_{i=1}^2 \int_0^1 \left(\partial_x q_i^M\right)^2 + C\epsilon_0^2 \sum_{i=1}^2 \int_0^1 (q_i^M)^2 + C\frac{\tau}{\delta^2 T^4}\sum_{i=1}^2 \int_0^1 (q_i^0)^2 + C\Big(1+\frac{\tau}{\delta T^2}\Big)\left(q_1^M(1)\right)^2.$$

Now as $q_1^M(0) = 0$, so we have:

$$q_1^M(1) = \int_0^1 \partial_x\left(q_1^M(x)\right) dx,$$

and so using Hölder's inequality, we get

$$\left(q_1^M(1)\right)^2 \leqslant \int_0^1 \left|\partial_x q_1^M\right|^2.$$

Using this estimate and the bound $\frac{\tau}{\delta T^2} \geqslant \frac{1}{\tau_0}$, we get

$$\sum_{i=1}^2 \widehat{W}_i + \overline{W}_1 \leqslant C\epsilon_0^2 \sum_{i=1}^2 \int_0^1 (q_i^M)^2 + C\frac{\tau}{\delta^2 T^4}\sum_{i=1}^2 \int_0^1 (q_i^0)^2 + C\frac{\tau}{\delta T^2}\sum_{i=1}^2 \int_0^1 \left|\partial_x q_i^M\right|^2. \tag{3.27}$$

Finally, substituting the estimates (3.26) and (3.27) in (3.25), we obtain

$$\begin{aligned}
&\sum_{i=1}^2 \left(\tau^3 \iint_Q (\mathtt{t}^-\theta)^3 |\mathtt{t}^- q_i|^2 + \tau \iint_Q (\mathtt{t}^-\theta)|\partial_x(\mathtt{t}^- q_i)|^2\right) + \tau\int_0^T (\mathtt{t}^-\theta)|\mathtt{t}^- q_1(1)|^2 \\
&\qquad \leqslant C\Bigg(\sum_{i=1}^2 \Big\|(\mathtt{t}^- r_i)\Big(D_t\varphi_i + \partial_x\big(\gamma_i\,\partial_x(\mathtt{t}^-\varphi)\big)\Big)\Big\|^2_{L^2_{\mathcal{D}}(Q)} + \tau\int_0^T (\mathtt{t}^-\theta)\left|\mathtt{t}^-\left(r_2(0)\right)\right|^2 \left|\partial_x(\mathtt{t}^-\varphi_2)(0)\right|^2 \\
&\qquad\qquad + \frac{\tau}{\delta^2 T^4}\sum_{i=1}^2 \int_0^1 (q_i^0)^2 + \frac{\tau}{\delta T^2}\sum_{i=1}^2 \int_0^1 \left|\partial_x q_i^M\right|^2\Bigg),
\end{aligned} \tag{3.28}$$

for $\tau \geqslant \tau_0(T^2+T)$ and $\frac{\tau^3\Delta t}{\delta^4 \min\{T^3, T^6\}} \leqslant \epsilon_0$.

This completes the estimation of the cross product terms.

Step-5: Returning back to the original variables:

This is the final step, where we return to the original variable φ to get the desired Carleman estimate (3.11). For that, let us recall the expressions

$$(\mathtt{t}^- q_i) = (\mathtt{t}^- r_i)\,(\mathtt{t}^-\varphi_i). \tag{3.29}$$

$$(\mathtt{t}^- r_i)\,\mathtt{t}^-(\partial_x\varphi_i) = \mathtt{t}^-(\partial_x q_i) - \tau\lambda_0 c_i \mathtt{t}^-(\theta)\mu_i(x)\mathtt{t}^-(r_i)\mathtt{t}^-(\varphi_i). \tag{3.30}$$

$$\left|\partial_x(\mathtt{t}^- q_i)\right| \leqslant (\mathtt{t}^- r_i)\Big(|\partial_x(\mathtt{t}^-\varphi_i)| + \tau\lambda_0|c_i|\mu_i(\mathtt{t}^-\theta)|\varphi_i|\Big) \leqslant (\mathtt{t}^- r_i)\Big(|\partial_x(\mathtt{t}^-\varphi_i)| + C\frac{\tau}{\delta T^2}|\varphi_i|)\Big). \tag{3.31}$$

From the identity (3.30), we have

$$(\mathtt{t}^- r_i)^2\,|\mathtt{t}^-(\partial_x\varphi_i)|^2 \leqslant 2|\mathtt{t}^-(\partial_x q_i)|^2 + 2\tau^2\lambda^2 c_i^2(\mu_i)^2(\mathtt{t}^-(\theta))^2\mathtt{t}^-(q_i)^2,$$

and hence

$$\tau\lambda^2\sum_{i=1}^2 \iint_Q (\mathtt{t}^-\theta)\,|\mathtt{t}^- r_i|^2\,|\mathtt{t}^-(\partial_x\varphi_i)|^2 \leqslant C\tau\lambda^2\sum_{i=1}^2 \iint_Q (\mathtt{t}^-\theta)\,|\mathtt{t}^-(\partial_x q_i)|^2 + C\tau^3\lambda^4\sum_{i=1}^2 \iint_Q (\mathtt{t}^-\theta)^3\,|\mathtt{t}^- q_i|^2.$$

Thus, using this estimate and (3.29), the inequality (3.28) can be rewritten as

$$\begin{aligned}
&\sum_{i=1}^2 \left(\tau^3 \iint_Q (\mathtt{t}^-\theta)^3(\mathtt{t}^- r_i)^2\,|\mathtt{t}^-\varphi_i|^2 + \tau\iint_Q (\mathtt{t}^-\theta)(\mathtt{t}^- r_i)^2\,|\partial_x(\mathtt{t}^-\varphi_i)|^2\right) + \tau\int_0^T (\mathtt{t}^-\theta)(\mathtt{t}^- r_1(1))^2|\mathtt{t}^-\varphi_1(1)|^2 \\
&\qquad \leqslant C\Bigg(\sum_{i=1}^2 \Big\|(\mathtt{t}^- r_i)\Big(D_t\varphi_i + \partial_x\big(\gamma_i\,\partial_x(\mathtt{t}^-\varphi)\big)\Big)\Big\|^2_{L^2_{\mathcal{D}}(Q)} + \tau\int_0^T (\mathtt{t}^-\theta)\left|\mathtt{t}^-\left(r_2(0)\right)\right|^2\left|\partial_x(\mathtt{t}^-\varphi_2)(0)\right|^2 \\
&\qquad\qquad + \frac{\tau}{\delta^2 T^4}\sum_{i=1}^2 \int_0^1 \left|(r_i\,\varphi_i)^0\right|^2 + \frac{\tau}{\delta T^2}\sum_{i=1}^2 \int_0^1 \left|(r_i\,\partial_x\varphi_i)^M\right|^2 + \frac{\tau^3}{\delta^3 T^6}\sum_{i=1}^2 \int_0^1 \left|(r_i\,\varphi_i)^M\right|^2\Bigg).
\end{aligned}$$

Finally, using the fact that $\tau \geqslant \tau_0\, T$ and $\frac{\tau^3 \Delta t}{\delta^4 \min\{T^3, T^6\}} \leqslant \epsilon_0$, we get the desired estimate

$$\sum_{i=1}^{2} \left(\tau^3 \iint_Q (\mathtt{t}^- \theta)^3 (\mathtt{t}^- r_i)^2 \, |\mathtt{t}^- \varphi_i|^2 + \tau \iint_Q (\mathtt{t}^- \theta)(\mathtt{t}^- r_i)^2 \, |\partial_x (\mathtt{t}^- \varphi_i)|^2 \right) + \tau \int_0^T (\mathtt{t}^- \theta)(\mathtt{t}^- r_1(1))^2 |\mathtt{t}^- \varphi_1(1)|^2$$
$$\leqslant C \sum_{i=1}^{2} \Big\| (\mathtt{t}^- r_i) \Big(D_t \varphi_i + \partial_x \big(\gamma_i \, \partial_x (\mathtt{t}^- \varphi) \big) \Big) \Big\|^2_{L^2_{\mathcal{D}}(Q)} + C\tau \int_0^T (\mathtt{t}^- \theta) \, |\mathtt{t}^- (r_2(0))|^2 \, |\partial_x (\mathtt{t}^- \varphi_2)(0)|^2$$
$$+ C \, (\Delta t)^{-1} \sum_{i=1}^{2} \left(\int_0^1 \left| (r_i \, \varphi_i)^0 \right|^2 + \int_0^1 \left| (r_i \, \varphi_i)^M \right|^2 + \int_0^1 \left| (r_i \, \partial_x \varphi_i)^M \right|^2 \right), \tag{3.32}$$

for $\tau \geqslant \tau_0 (T^2 + T)$, and for all $\Delta t \in (0, 1)$ and $\delta \in (0, 1/2]$ satisfying the condition

$$\frac{\tau^3 \, \Delta t}{\delta^4 \min(\{T^3, T^6\})} \leqslant \epsilon_0.$$

□

4. Relaxed Observability Inequality

In this section, we prove Theorem 1.7. The proof uses the discrete Carleman inequality (3.11), energy type estimates for the adjoint system (1.21), and proper bounds for the weight functions. Before going to the proof, let us first see the relevant energy estimates.

Lemma 4.1 (Energy type estimates). *For $\varphi^M \in \mathcal{H}$, the solution of the adjoint system (1.21) satisfies*

$$\|\varphi^n\|^2_{\mathcal{H}} \leqslant \|\varphi^{n+1}\|^2_{\mathcal{H}}, \quad \text{for } n \in [\![0, M-1]\!], \tag{4.1}$$
$$\|\varphi^n\|_E \leqslant \|\varphi^{n+1}\|_E, \quad \text{for } n \in [\![0, M-1]\!]. \tag{4.2}$$

Proof. We multiply the first and second equation of (1.21) by $(D_t \varphi_1)$ and $(D_t \varphi_2)$ respectively, and add them together to get

$$(D_t \varphi_1)^2 + (D_t \varphi_2)^2 + \partial_x \left(\gamma_1 \left(\mathtt{t}^- \partial_x \varphi_1 \right) \right) (D_t \varphi_1) + \partial_x \left(\gamma_2 \left(\mathtt{t}^- \partial_x \varphi_2 \right) \right) (D_t \varphi_2) = 0.$$

Now integrating it w.r.t. x over $(0, 1)$, we get

$$\int_0^1 (D_t \varphi_1)^2 + \int_0^1 (D_t \varphi_2)^2 - \int_0^1 \gamma_1 \, \mathtt{t}^- (\partial_x \varphi_1) D_t (\partial_x \varphi_1) - \int_0^1 \gamma_2 \, \mathtt{t}^- (\partial_x \varphi_2) D_t (\partial_x \varphi_2)$$
$$+ \gamma_1 \, \mathtt{t}^- (\partial_x \varphi_1) D_t (\varphi_1) + \gamma_2 \, \mathtt{t}^- (\partial_x \varphi_2) D_t (\varphi_2) \Big|_{x=0}^{1} = 0.$$

Further, performing discrete integration w.r.t. t over (t_n, t_{n+1}), and using the boundary conditions of adjoint system (1.21), we have

$$-\!\!\!\!\!\!\int_{t_n}^{t_{n+1}} \int_0^1 (D_t \varphi_1)^2 + -\!\!\!\!\!\!\int_{t_n}^{t_{n+1}} \int_0^1 (D_t \varphi_2)^2 - -\!\!\!\!\!\!\int_{t_n}^{t_{n+1}} \int_0^1 \gamma_1 \, \mathtt{t}^- (\partial_x \varphi_1) D_t (\partial_x \varphi_1)$$
$$- -\!\!\!\!\!\!\int_{t_n}^{t_{n+1}} \int_0^1 \gamma_2 \, \mathtt{t}^- (\partial_x \varphi_2) D_t (\partial_x \varphi_2) - \alpha -\!\!\!\!\!\!\int_0^T \mathtt{t}^- \left(\varphi_1(1) \right) D_t \left(\varphi_1(1) \right) = 0.$$

Lastly, we use the discrete product rule (A.4) to get

$$-\!\!\!\!\!\!\int_{t_n}^{t_{n+1}} \int_0^1 \gamma_1 \left\{ \frac{1}{2} D_t \left(\partial_x \varphi_1 \right)^2 - \frac{\Delta t}{2} \left(D_t (\partial_x \varphi_1) \right)^2 \right\} + -\!\!\!\!\!\!\int_{t_n}^{t_{n+1}} \int_0^1 \gamma_2 \left\{ \frac{1}{2} D_t \left(\partial_x \varphi_2 \right)^2 - \frac{\Delta t}{2} \left(D_t (\partial_x \varphi_2) \right)^2 \right\}$$
$$+ \alpha -\!\!\!\!\!\!\int_{t_n}^{t_{n+1}} \left\{ \frac{1}{2} D_t \left(\varphi_1(1) \right)^2 - \frac{\Delta t}{2} \left(D_t (\varphi_1(1)) \right)^2 \right\} = -\!\!\!\!\!\!\int_{t_n}^{t_{n+1}} \int_0^1 (D_t \varphi_1)^2 + -\!\!\!\!\!\!\int_{t_n}^{t_{n+1}} \int_0^1 (D_t \varphi_2)^2,$$

which proves the estimate (4.1).

Next we multiply the first and second equations of (1.21) by φ_1^n and φ_2^n respectively, and add them to obtain

$$\sum_{i=1}^{2} \left\{ \frac{\varphi_i^n - \varphi_i^{n+1}}{\Delta t} \varphi_i^n - \partial_x \left(\gamma_i \partial_x \varphi_i^n \right) \varphi_i^n \right\} = 0, \quad n \in [\![0, M-1]\!].$$

Upon rearranging the terms and applying integration by parts in the above identity, we obtain

$$\sum_{i=1}^{2}\left\{\int_0^1 (\varphi_i^n)^2 - \int_0^1 \varphi_i^{n+1}\varphi_i^n + \Delta t\int_0^1 \gamma_i\,(\partial_x\varphi_i^n)^2 - \Delta t\,\gamma_i(1)\,(\partial_x\varphi_i^n(1))\,\varphi_i^n(1)\right\} = 0.$$

Now, using Hölder's and Young's inequalities in the second term, we get

$$\sum_{i=1}^{2}\left\{\frac{1}{2}\int_0^1 (\varphi_i^n)^2 - \frac{1}{2}\int_0^1 \left(\varphi_i^{n+1}\right)^2 + \Delta t\int_0^1 \gamma_i\,(\partial_x\varphi_i^n)^2 - \Delta t\gamma_i(1)\,(\partial_x\varphi_i^n(1))\,\varphi_i^n(1)\right\} \leqslant 0.$$

Using non-negativity of the third term, and the boundary conditions of (1.21), the above inequality gives

$$\begin{aligned}\sum_{i=1}^{2}\int_0^1 (\varphi_i^n)^2 &\leqslant \sum_{i=1}^{2}\int_0^1 \left(\varphi_i^{n+1}\right)^2 + 2\Delta t\,(-\alpha\varphi_1^n(1))\,\varphi_1^n(1)\\ &\leqslant \sum_{i=1}^{2}\int_0^1 \left(\varphi_i^{n+1}\right)^2,\end{aligned}$$

which proves (4.2). □

Let us now proceed to the proof of weak observability inequality (1.22).

Proof of Theorem 1.7. Applying the Carleman estimate (3.11) to the solution of the adjoint system (1.21) with $\varphi^M \in \mathcal{H}$ and ignoring the integral term associated to $(\mathtt{t}^-\varphi_i)$, we get

$$\begin{aligned}\tau\sum_{i=1}^{2}\iint_Q (\mathtt{t}^-\theta)(\mathtt{t}^-r_i)^2\,|\partial_x(\mathtt{t}^-\varphi_i)|^2 + \tau\int_0^T (\mathtt{t}^-\theta)(\mathtt{t}^-r_1(1))^2\left|\mathtt{t}^-\varphi_1(1)\right|^2\\ \leqslant C\tau\int_0^T (\mathtt{t}^-\theta)\left|\mathtt{t}^-\left(r_2(0)\right)\right|^2\left|\partial_x(\mathtt{t}^-\varphi_2)(0)\right|^2\\ + C\,(\Delta t)^{-1}\sum_{i=1}^{2}\left(\int_0^1\left|(r_i\,\varphi_i)^0\right|^2 + \int_0^1\left|(r_i\,\varphi_i)^M\right|^2 + \int_0^1\left|(r_i\,\partial_x\varphi_i)^M\right|^2\right),\end{aligned}\tag{4.3}$$

for $\tau \geqslant \tau_0(T^2+T)$, and for all $\Delta t \in (0,1)$ and $\delta \in (0,1/2]$ satisfying the condition $\frac{\tau^3\,\Delta t}{\delta^4 T^6} \leqslant \epsilon_0$.

Now we find lower bounds for the l.h.s. terms. Using the fact $\tau \geqslant 1$, and the estimate

$$\frac{1}{T^2} \leqslant \theta\left(\frac{T}{2}\right) \leqslant \theta(t) \leqslant \theta\left(\frac{T}{4}\right) \leqslant \frac{16}{3}\frac{1}{T^2}, \quad \text{for } t \in \left[\frac{T}{4}, \frac{3T}{4}\right],$$

we get

$$\begin{aligned}\tau\sum_{i=1}^{2}\iint_Q (\mathtt{t}^-\theta)(\mathtt{t}^-r_i)^2\,|\partial_x(\mathtt{t}^-\varphi_i)|^2 + \tau\int_0^T (\mathtt{t}^-\theta)(\mathtt{t}^-r_1(1))^2\left|\mathtt{t}^-\varphi_1(1)\right|^2\\ \geqslant \tau\frac{1}{T^2}e^{-C_1\tau/T^2}\int_{T/4}^{3T/4}\left(\sum_{i=1}^{2}\int_0^1 |\mathtt{t}^-(\partial_x\varphi_i)|^2 + |\mathtt{t}^-\varphi_1(1)|^2\right),\end{aligned}$$

where $C_1 = \frac{32\,\|\eta_1\|_\infty}{3}$. Next using the estimate (4.1), we have:

$$\int_{T/4}^{3T/4}\left(\sum_{i=1}^{2}\int_0^1 |\mathtt{t}^-(\partial_x\varphi_i)|^2 + |\mathtt{t}^-\varphi_1(1)|^2\right) = \int_{T/4}^{3T/4}\|\mathtt{t}^-\varphi\|_{\mathcal{H}}^2 \geqslant \frac{T}{2}\,\|\varphi^0\|_{\mathcal{H}}^2.$$

Hence, combining the last two estimates and using the fact that $\tau \geqslant \tau_0\,T$, we have

$$\tau\sum_{i=1}^{2}\iint_Q (\mathtt{t}^-\theta)(\mathtt{t}^-r_i)^2\,|\partial_x(\mathtt{t}^-\varphi_i)|^2 + \tau\int_0^T (\mathtt{t}^-\theta)(\mathtt{t}^-r_1(1))^2\left|\mathtt{t}^-\varphi_1(1)\right|^2 \geqslant \frac{\tau_0}{2}e^{-C_1\tau/T^2}\,\|\varphi^0\|_{\mathcal{H}}^2.\tag{4.4}$$

Note that $\theta(t)e^{-2\tau\theta(t)\eta_2(0)}$ attains its maximum value at $t = T/2$ and so we have:

$$\tau(\mathtt{t}^-\theta)(\mathtt{t}^-r_2(0))^2 \leqslant \frac{4\tau}{T^2}e^{\frac{-2\tau\eta_2(0)}{T^2}} \leqslant C.\tag{4.5}$$

Using the bounds (4.4) and (4.5) in (4.3), we get

$$\|\varphi^0\|^2_{\mathcal{H}} \leqslant C\, e^{C_1\tau/T^2} \int_0^T \left|\partial_x(\mathtt{t}^-\varphi_2)(0)\right|^2 + C\, e^{C_1\tau/T^2}\,(\Delta t)^{-1} \sum_{i=1}^{2} \left(\int_0^1 \left|(r_i\,\varphi_i)^0\right|^2 + \int_0^1 \left|(r_i\,\varphi_i)^M\right|^2 + \int_0^1 \left|(r_i\,\partial_x\varphi_i)^M\right|^2 \right), \tag{4.6}$$

for $\tau \geqslant \tau_0(T^2+T)$.

Thus, in order to get the observability inequality (1.22) we only need to estimate the last three terms. To do that, we use $\theta(0) = \theta(T) \geqslant \frac{2}{3}\frac{1}{\delta T^2}$ to obtain

$$\left|(r_i\,\varphi_i)^0\right|^2 + \left|(r_i\,\varphi_i)^M\right|^2 + \left|(r_i\,\partial_x\varphi_i)^M\right|^2 \leqslant e^{-\frac{4\eta_0}{3}\frac{\tau}{\delta T^2}} \sum_{i=1}^{2} \left(|\varphi_i^0|^2 + |\varphi_i^M|^2 + |\partial_x\varphi_i^M| \right),$$

where, $\eta_0 = \min_{[0,1]}\{\eta_i : i \in \{1,2\}\}$. Further we use this estimate in (4.6) to get:

$$\|\varphi^0\|^2_{\mathcal{H}} \leqslant C\, e^{C_1\tau/T^2} \int_0^T \left|\partial_x(\mathtt{t}^-\varphi_2)(0)\right|^2 + C\, e^{-C_2\tau/\delta T^2}\,(\Delta t)^{-1} \sum_{i=1}^{2} \int_0^1 \left(|\varphi_i^0|^2 + |\varphi_i^M|^2 + |\partial_x\varphi_i^M|^2 \right),$$

where $C_2 = \frac{4}{3}\eta_0 - C_1\delta > 0$, for $\delta \in (0, \delta_0]$, for sufficiently small δ_0. Lastly, we use the Poincaré inequality and the estimate (4.2) in the last inequality to obtain

$$\|\varphi^0\|^2_{\mathcal{H}} \leqslant C\, e^{C_1\tau/T^2} \int_0^T \left|\partial_x(\mathtt{t}^-\varphi_2)(0)\right|^2 + C\, e^{-C_2\tau/\delta T^2}\,(\Delta t)^{-1} \sum_{i=1}^{2} \int_0^1 |\partial_x\varphi_i^M|^2, \tag{4.7}$$

Let us now fix $\tau = \tau_0(T^2+T)$, and define

$$\widetilde{\Delta t} := \frac{\epsilon_0}{\tau_0^4}\delta_0^4\,(T^2+T)^{-4}T^6,$$

which implies

$$\frac{\tau^4\widetilde{\Delta t}}{\delta_0^4 T^6} = \epsilon_0.$$

Next for any $\Delta t \leqslant \widetilde{\Delta t}$, we define $\delta := \frac{(\Delta t)^{1/4}\delta_0}{(\widetilde{\Delta t})^{1/4}} \leqslant \delta_0$ so that we get

$$\frac{\tau^4\Delta t}{\delta^4 T^6} = \epsilon_0,$$

which further implies

$$\frac{\tau}{\delta T^2} = \left(\frac{\tau^4}{\delta^4 T^8}\right)^{1/4} \geqslant \left(\frac{\tau^4}{\delta^4 T^6}\right)^{1/4} = \epsilon_0^{1/4}(\Delta t)^{-1/4},$$

as $T < 1$. Now using this inequality and then fixing $\tau = \tau_0(T^2+T)$ in (4.7), we have

$$\|\varphi^0\|^2_{\mathcal{H}} \leqslant e^{C_1\tau_0(T^2+1+1/T)} \left(\int_0^T \left|\partial_x(\mathtt{t}^-\varphi_2)(0)\right|^2 + e^{-\frac{\tilde{C}_2}{(\Delta t)^{1/4}}} \sum_{i=1}^{2} \int_0^1 |\partial_x\varphi_i^M|^2 \right), \tag{4.8}$$

for some constant $\tilde{C}_2 > 0$. This completes the proof. □

5. $\phi(\Delta t)$-Null Controllability Results

This section is devoted to prove the $\phi(\Delta t)$-controllability result mentioned in Theorem 1.10, which mainly relies on the relaxed observability inequality (1.22).

Proof of Theorem 1.10. To prove the $\phi(\Delta t)$-controllability result, let us first recall the condition (1.23) on ϕ, given by

$$\liminf_{\Delta t\to 0} \frac{\phi(\Delta t)}{e^{-K_2/(\Delta t)^{1/4}}} > 0.$$

This means there exists a constant $C > 0$ such that for sufficiently small Δt (depending on ϕ), we have

$$\phi(\Delta t) \geqslant C e^{-K_2/(\Delta t)^{1/4}}.$$

Thus, from the relaxed observability inequality (1.22) we have

$$\left\|\varphi^0\right\|_{\mathcal{H}}^2 \leqslant C_{obs}\left(\int_0^T |\mathtt{t}^-(\partial_x\varphi_2)(0)|^2 + \phi(\Delta t)\left\|\varphi^M\right\|_{\mathcal{H}}^2\right), \text{ for sufficiently small } \Delta t. \tag{5.1}$$

Let us define the functional

$$J_{\Delta t}(\varphi^M) = \frac{1}{2}\int_0^T |\mathtt{t}^-(\partial_x\varphi_2(0))|^2 + \frac{\phi(\Delta t)}{2}\left\|\partial_x\varphi^M\right\|_E^2 + \left\langle u_0, \varphi^0\right\rangle_{\mathcal{H}',\mathcal{H}}, \text{ for } \varphi^M \in \mathcal{H}, \tag{5.2}$$

where $\varphi = (\varphi_1, \varphi_2)$ is solution of the discrete adjoint system (1.21). This functional $J_{\Delta t}$ satisfies the following properties:

(a) $J_{\Delta t}$ is continuous and strictly convex. The continuity follows by sequential approach using the well-posedness result of the adjoint system, and the strict convexity by the properties of norm induced in the Hilbert space.

(b) Using Cauchy-Schwarz and Young's inequalities in the expression of $J_{\Delta t}$, we get

$$J_{\Delta t}(\varphi^M) \geqslant \frac{1}{2}\int_0^T \left|\mathtt{t}^-(\partial_x\varphi_2(0))\right|^2 + \frac{\phi(\Delta t)}{2}\left\|\partial_x\varphi^M\right\|_E^2 - \frac{1}{4\,C_{obs}}\left\|\varphi^0\right\|_{\mathcal{H}}^2 - C_{obs}\|u_0\|_{\mathcal{H}'}^2.$$

Now using inequality (5.1) and the equivalence of the norms $\|\varphi^M\|_{\mathcal{H}}$ and $\|\partial_x\varphi^M\|_E$ in the above estimate proves coercivity of the functional $J_{\Delta t}$.

These properties guarantee the existence of a unique minimizer of the functional $J_{\Delta t}$. Let us denote the minimizer by $\widehat{\varphi}^M$ and the associated solution of (1.21) by $\widehat{\varphi}$. Then, the Euler-Lagrange equation associated with $J_{\Delta t}$ is given by

$$\int_0^T \mathtt{t}^-\left(\partial_x\widehat{\varphi}_2(0)\right)\mathtt{t}^-\left(\partial_x\varphi_2(0)\right) + \phi(\Delta t)\left\langle\partial_x\widehat{\varphi}^M, \partial_x\varphi^M\right\rangle_E + \left\langle u_0, \varphi^0\right\rangle_{\mathcal{H}',\mathcal{H}} = 0, \quad \forall\varphi^M \in \mathcal{H}. \tag{5.3}$$

Let us now define a linear map $\mathcal{L}_T^{\Delta t} : \mathcal{H}' \to L^2_{\mathcal{P}}(0,T)$ given by

$$\left(\mathcal{L}_T^{\Delta t}(u_0)\right)^n := \left(\partial_x\widehat{\varphi}_2(0)\right)^n, \text{ for } n \in [\![0, M-1]\!], \tag{5.4}$$

and set the control v^{n+1} in the control system (1.17) as

$$v^{n+1} = \left(\mathcal{L}_T^{\Delta t}(y_0)\right)^n = (\partial_x\widehat{\varphi}_2(0))^n, \text{ for } n \in [\![0, M-1]\!]. \tag{5.5}$$

Now taking duality product in (1.17) with the solution φ^n of the adjoint system with $\varphi_M \in \mathcal{H}$, for each $n \in [\![0, M-1]\!]$, and then summing over $n \in [\![0, M-1]\!]$, we obtain

$$\int_0^T (\mathtt{t}^+ v)\,\mathtt{t}^-\left(\partial_x\varphi_2(0)\right) - \left\langle u^M, \varphi^M\right\rangle_{\mathcal{H}',\mathcal{H}} + \left\langle u_0, \varphi^0\right\rangle_{\mathcal{H}',\mathcal{H}} = 0, \tag{5.6}$$

where the control v is given by (5.5). On comparing this identity (5.6) with (5.3), we get

$$\left\langle u^M, \varphi^M\right\rangle_{\mathcal{H}',\mathcal{H}} = -\phi(\Delta t)\left\langle\partial_x\widehat{\varphi}^M, \partial_x\varphi^M\right\rangle_E, \quad \forall\varphi^M \in \mathcal{H},$$

which implies

$$u^M = \phi(\Delta t)\,\partial_x^2(\widehat{\varphi}^M) \text{ in } \mathcal{H}'. \tag{5.7}$$

Now, we take $\varphi^M = \widehat{\varphi}^M$ in (5.3) and use the Cauchy-Schwarz inequality to get

$$\int_0^T \left|\mathtt{t}^-\left(\partial_x\widehat{\varphi}_2\right)(0)\right|^2 + \phi(\Delta t)\left\|\widehat{\varphi}^M\right\|_{\mathcal{H}}^2 \leqslant \|u_0\|_{\mathcal{H}'}\left\|\widehat{\varphi}^0\right\|_{\mathcal{H}}.$$

Next substituting (5.1) for $\varphi^M = \widehat{\varphi}^M$ in the last estimate, we obtain

$$\int_0^T \left|\mathtt{t}^-\left(\partial_x\widehat{\varphi}_2\right)(0)\right|^2 + \phi(\Delta t)\left\|\widehat{\varphi}^M\right\|_{\mathcal{H}}^2 \leqslant \sqrt{C_{obs}}\,\|u_0\|_{\mathcal{H}'}\left(\int_0^T |\mathtt{t}^-(\partial_x\widehat{\varphi}_2)(0)|^2 + \phi(\Delta t)\left\|\widehat{\varphi}^M\right\|_{\mathcal{H}}^2\right)^{\frac{1}{2}},$$

which yields

$$\|v\|_{L^2_{\mathcal{P}}} = \left\|\mathtt{t}^-(\partial_x\widehat{\varphi}_2)(0)\right\|_{L^2_{\mathcal{D}}} \leqslant \sqrt{C_{obs}}\,\|y_0\|_{\mathcal{H}'}, \tag{5.8}$$

$$\sqrt{\phi(\Delta t)}\left\|\widehat{\varphi}^M\right\|_{\mathcal{H}} \leqslant \sqrt{C_{obs}}\,\|y_0\|_{\mathcal{H}'}. \tag{5.9}$$

This proves the well-definedness and continuity of the control map $\mathcal{L}_T^{\Delta t}$ given by (5.4). Finally, we use the expression (5.7) and the estimate (5.9) to get

$$\left\|u^M\right\|_{\mathcal{H}'} \leqslant \sqrt{C_{obs}}\sqrt{\phi(\Delta t)}\left\|y_0\right\|_{\mathcal{H}'}.$$

This completes the proof. □

6. Convergence analysis

The main goal of this section is to prove the convergence result as stated in Theorem 1.11. Note that the transposition definition for the control system (1.1) mentioned in Definition 1.1 involves the adjoint system as well. Due to this reason, we study the convergence of discrete adjoint system to the continuous system in Section 6.1. Further, we carry out the study of convergence of the discrete control to the control for the continuous system in Section 6.2.

6.1. Convergence of the adjoint system.

For $n \in [\![0, M-1]\!]$, let us consider the following time-discrete adjoint system

$$\begin{cases} -\frac{\varphi_1^{n+1}-\varphi_1^n}{\Delta t} - \partial_x\left(\gamma_1\,\partial_x\varphi_1^n\right) = g_1^{n+1}, \\ -\frac{\varphi_2^{n+1}-\varphi_2^n}{\Delta t} - \partial_x\left(\gamma_2\,\partial_x\varphi_2^n\right) = g_2^{n+1}, \\ \varphi_1^n(0) = \varphi_2^n(0) = 0, \\ \varphi_1^n(1) = \varphi_2^n(1), \\ \gamma_1(1)\,\partial_x\varphi_1^n + \gamma_2(1)\,\partial_x\varphi_2^n(1) + \alpha\,\varphi_1^n(1) = 0, \\ \varphi^M = \varphi_T, \end{cases} \tag{6.1}$$

where the functions g^{n+1} and φ_T are given.

The following lemma gives stability of the above discrete system (1.3).

Lemma 6.1. *For $\{g^{n+1}\}_{n\in[\![0,M-1]\!]} \subset E$ and $\varphi_T \in \mathcal{H}$, system (6.1) admits a unique set of solutions $\{\varphi^n\}_{n\in[\![0,M-1]\!]}$ satisfying the following estimates*

(a) $\|\varphi^n\|_{\mathcal{H}} \leqslant C\left(\|g\|_{L^2_{\mathcal{P}}(0,T;E)} + \|\varphi_T\|_{\mathcal{H}}\right), \quad \forall\, n \in \{0,1,\dots,M-1\}.$

(b) $\displaystyle\sum_{n=0}^{M-1} \Delta t\,\|\partial_x(\gamma_2\,\partial_x\varphi_2^n)\|_{L^2}^2 \leqslant C\left(\|g\|^2_{L^2_{\mathcal{P}}(0,T;E)} + \|\varphi_T\|^2_{\mathcal{H}}\right), \quad$ *for* $n_0 \in \{0,1,\dots,M-1\}.$

(c) $\displaystyle\sum_{n=0}^{M-1} \Delta t\left|\gamma_2(0)\,\partial_x\varphi_2^n(0)\right|^2 \leqslant C\left(\|g\|^2_{L^2_{\mathcal{P}}(0,T;E)} + \|\varphi_T\|^2_{\mathcal{H}}\right).$

Proof. The existence and uniqueness of solution for system (6.1) is guaranteed by the well-known Lax-Milgram theorem. To obtain the above estimates, let us multiply the differential equations in (6.1) by $\partial_x(\gamma_i\,\partial_x\varphi_i^n)$ for $i \in \{1,2\}$. Then, performing integration by parts and summing over i, we get

$$\begin{aligned} &\sum_{i=1}^{2}\Delta t\int_0^1 |\partial_x(\gamma_i\,\partial_x\varphi_i^n)|^2 + \sum_{i=1}^{2}\int_0^1 \gamma_i\,|\partial_x\varphi_i^n|^2 + \alpha\,|\varphi_1^n(1)|^2 \\ &\qquad\leqslant \frac{\Delta t}{2}\sum_{i=1}^{2}\int_0^1 |g_i^{n+1}|^2 + \frac{\Delta t}{2}\sum_{i=1}^{2}\int_0^1 |\partial_x(\gamma_i\,\partial_x\varphi_i^n)|^2 + \frac{1}{2}\sum_{i=1}^{2}\int_0^1 \gamma_i\,|\partial_x\varphi_i^{n+1}|^2 \\ &\qquad\qquad + \frac{1}{2}\sum_{i=1}^{2}\int_0^1 \gamma_i\,|\partial_x\varphi_i^n|^2 + \frac{\alpha}{2}|\varphi_1^{n+1}|^2 + \frac{\alpha}{2}|\varphi_1^n|^2, \end{aligned}$$

which gives

$$\|\varphi^n\|^2_{\mathcal{H}} - \|\varphi^{n+1}\|^2_{\mathcal{H}} \leqslant \Delta t\,\|g^{n+1}\|^2_E, \quad \forall\, n \in \{0,1,\dots,M-1\}, \tag{6.2}$$

$$\text{and}\quad \sum_{i=1}^{2}\Delta t\,\|\partial_x(\gamma_i\,\partial_x\varphi_i^n)\|^2_{L^2} \leqslant \Delta t\,\|g^{n+1}\|^2_E + \|\varphi^{n+1}\|^2_{\mathcal{H}} - \|\varphi^n\|^2_{\mathcal{H}}. \tag{6.3}$$

Summing the above estimates over n, we get the estimates (a) and (b). Finally, using the trace inequality and the estimates (a), (b), one can easily get (c). □

Using the solution $\{\varphi^n = (\varphi_1^n, \varphi_2^n)\}_{n\in[\![0,M-1]\!]}$ of the discrete system (6.1), we define the functions

$$(\varphi_M)_i(t,x) := \sum_{n=0}^{M-1} 1_{[t_n,t_{n+1}]}(t)\left(\frac{(t-t_n)}{\Delta t}\varphi_i^{n+1}(x) + \frac{(t_{n+1}-t)}{\Delta t}\varphi_i^n(x)\right), \quad \text{for } i\in\{1,2\}. \tag{6.4}$$

Let us now recall the non-homogeneous adjoint system (1.3)

$$\begin{cases} -\partial_t\varphi_1 - \partial_x(\gamma_1\partial_x\varphi_1) = g_1, & (t,x)\in(0,T)\times(0,1),\\ -\partial_t\varphi_2 - \partial_x(\gamma_2\partial_x\varphi_2) = g_2, & (t,x)\in(0,T)\times(0,1),\\ \varphi_1(t,0) = 0, \varphi_2(t,0) = 0, & t\in(0,T),\\ \varphi_1(t,1) = \varphi_2(t,1), & t\in(0,T),\\ \gamma_1(1)\,\partial_x\varphi_1(t,1) + \gamma_2(1)\,\partial_x\varphi_2(t,1) + \alpha\varphi_1(t,1) = 0, & t\in(0,T),\\ \varphi_1(T,x) = 0, \varphi_2(T,x) = 0, & x\in(0,1). \end{cases} \tag{6.5}$$

Define the space:

$$\mathcal{C}_\alpha := \Big\{(\varphi_1,\varphi_2)\in\left(C^\infty(\overline{Q})\right)^2 : \varphi_1(t,0) = \varphi_2(t,0) = 0, \varphi_1(t,1) = \varphi_2(t,1),$$
$$\sum_{i=1}^{2}\gamma_i(1)\partial_x\varphi_i(t,1) + \alpha\varphi_1(t,1) = 0, \text{ for } t\in(0,T)\Big\}.$$

Definition 6.2. *$\varphi\in L^2(0,T;\mathcal{H})\cap H^1(0,T;E)$ is a solution of (6.5) if it satisfies*

(i)

$$-\sum_{i=1}^{2}\iint_Q \partial_t\varphi_i\,\psi_i + \sum_{i=1}^{2}\iint_Q \gamma_i(x)\,\partial_x\varphi_i\,\partial_x\psi_i + \alpha\int_0^T \varphi_1(t,1)\,\psi_1(t,1) = \sum_{i=1}^{2}\iint_Q g_i\,\psi_i, \quad \forall\,\psi\in\mathcal{C}_\alpha,$$

(ii) $\varphi(T,x) = 0$.

Theorem 6.3. *Let $g = (g_1,g_2)\in L^2(0,T;E)$. For $n\in[\![0,M-1]\!]$, define the functions*

$$g_i^{n+1}(x) = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} g_i(t,x)\,dt, \quad \text{for } i\in\{1,2\}.$$

Then the function φ_M, defined by (6.4) corresponding to above g^{n+1} and $\varphi_T = 0$, converges weakly to the solution φ of (6.5) in $H^1(0,T;E)\cap L^2(0,T;\mathcal{H}\cap H^2\times H^2)$ as $M\to\infty$.

Proof. The proof of this theorem is done in two steps. First step is to show the uniform boundedness of $\{\varphi_M\}_{M\in\mathbb{N}}$, which gives a convergent subsequence with some weak limit φ. In the next step, we show that this weak limit φ solves the adjoint system (6.5).

Step-1: Uniform boundedness.

Using the estimates mentioned in Lemma 6.1, one can easily obtain the following bounds for φ_M defined by (6.4):

(a) $\|\varphi_M\|^2_{L^2(\mathcal{H})} \leqslant C\,\|g\|^2_{L^2(0,T;E)}$.

(b) $\|\partial_t\varphi_M\|^2_{L^2(E)} \leqslant C\,\|g\|^2_{L^2(0,T;E)}$.

Due to these uniform bounds, we get a function $\varphi\in L^2(0,T;\mathcal{H})$ with $\partial_t\varphi\in L^2(0,T;E)$ such that

$$\begin{cases} \varphi_M \rightharpoonup \varphi \text{ in } L^2(0,T;\mathcal{H}),\\ \partial_t\varphi_M \rightharpoonup \partial_t\varphi \text{ in } L^2(0,T;E),\\ \varphi_M(t,1) \rightharpoonup \varphi(t,1) \text{ in } L^2(0,T). \end{cases} \tag{6.6}$$

up to a subsequence. For the last convergence, we have used the continuity of the trace map

$$\mathcal{T} : L^2(0,T;\mathcal{H}) \to L^2(0,T) : \mathcal{T}(f) = f(t,1),$$

with the convergence result $\varphi_M \rightharpoonup \varphi$ in $L^2(0,T;\mathcal{H})$.

Step-2: The limit φ is the solution.

Let $\psi \in \mathcal{C}_\alpha$. Then using the differential equation (6.1) and performing integration by parts w.r.t. x, we obtain

$$\int_0^T \langle \partial_t \varphi_M, \psi \rangle_E = \sum_{i=1}^{2} \sum_{n=0}^{M-1} \int_0^1 \gamma_i(x)\, \partial_x \varphi_i^n(x) \left(\int_{t_n}^{t_{n+1}} \partial_x \psi_i(t,x)\, dt \right) dx + \alpha \sum_{n=0}^{M-1} \varphi_1^n(1) \left(\int_{t_n}^{t_{n+1}} \psi_1(t,1)\, dt \right) - \sum_{i=1}^{2} \sum_{n=0}^{M-1} \int_0^1 g_i^{n+1}(x) \left(\int_{t_n}^{t_{n+1}} \psi_i(t,x)\, dt \right) dx.$$

Thus using the above identity, we have

$$\begin{aligned}
&- \int_0^T \langle \partial_t \varphi_M, \psi \rangle_E + \sum_{i=1}^{2} \iint_Q \gamma_i(x)\, \partial_x (\varphi_M)_i\, \partial_x \psi_i + \alpha \int_0^T (\varphi_M)_1(t,1)\, \psi_1(t,1) - \sum_{i=1}^{2} \iint_Q g_i\, \psi_i \\
&= \left[\sum_{i=1}^{2} \sum_{n=0}^{M-1} \int_0^1 \gamma_i(x) \int_{t_n}^{t_{n+1}} \left(\frac{(t-t_n)}{\Delta t} \partial_x \varphi_i^{n+1} + \frac{(t_{n+1}-t)}{\Delta t} \partial_x \varphi_i^n \right) \partial_x \psi_i \right. \\
&\qquad \left. - \sum_{i=1}^{2} \sum_{n=0}^{M-1} \int_0^1 \gamma_i(x)\, \partial_x \varphi_i^n(x) \left(\int_{t_n}^{t_{n+1}} \partial_x \psi_i(t,x)\, dt \right) dx \right] \\
&\quad + \left[\alpha \sum_{n=0}^{M-1} \int_{t_n}^{t_{n+1}} \left(\frac{(t-t_n)}{\Delta t} \varphi_1^{n+1}(1) + \frac{(t_{n+1}-t)}{\Delta t} \varphi_1^n(1) \right) \psi_1(t,1)\, dt \right. \\
&\qquad \left. - \alpha \sum_{n=0}^{M-1} \varphi_1^n(1) \left(\int_{t_n}^{t_{n+1}} \psi_1(t,1)\, dt \right) \right] \\
&\quad + \left[\sum_{i=1}^{2} \sum_{n=0}^{M-1} \int_0^1 g_i^{n+1}(x) \left(\int_{t_n}^{t_{n+1}} \psi_i(t,x) \right) - \sum_{i=1}^{2} \sum_{n=0}^{M-1} \int_0^1 \int_{t_n}^{t_{n+1}} g_i(t,x) \psi_i(t,x) \right] \\
&=: I + II + III.
\end{aligned}$$

These three terms can be estimated as listed below:

- Using Lemma E.1, we can easily conclude that

$$|III| \leqslant C \sqrt{\Delta t}\, \|\psi\|_{L^2(E)}.$$

- Next observe that using $\varphi^M = 0$, the first term can be written as

$$\begin{aligned}
I &= \sum_{i=1}^{2} \sum_{n=0}^{M-1} \int_0^1 \gamma_i\, \partial_x \varphi^{n+1} \int_{t_n}^{t_{n+1}} \frac{(t-t_n)}{\Delta t} \partial_x \psi_i - \sum_{i=1}^{2} \sum_{n=0}^{M-1} \int_0^1 \gamma_i\, \partial_x \varphi_i^n \int_{t_n}^{t_{n+1}} \frac{(t-t_n)}{\Delta t} \partial_x \psi_i \\
&= \sum_{i=1}^{2} \sum_{n=0}^{M-1} \int_0^1 \gamma_i\, \partial_x \varphi_i^{n+1} \left(\int_{t_n}^{t_{n+1}} \frac{(t-t_n)}{\Delta t} \partial_x \psi_i - \int_{t_{n+1}}^{t_{n+2}} \frac{(t-t_{n+1})}{\Delta t} \partial_x \psi_i \right) - \sum_{i=1}^{2} \int_0^1 \gamma_i\, \partial_x \varphi_i^0 \int_0^{t_1} \frac{t}{\Delta t} \partial_x \psi_i.
\end{aligned}$$

Now we use change of variable and triangle inequality to get

$$|I| \leqslant \sum_{i=1}^{2} \sum_{n=0}^{M-1} \int_0^1 \gamma_i \left| \partial_x \varphi_i^{n+1} \right| \int_{t_n}^{t_{n+1}} \frac{(t-t_n)}{\Delta t} \left| \partial_x \psi_i(t,\cdot) - \psi_i(t+\Delta t,\cdot) \right| dt + \sum_{i=1}^{2} \int_0^1 \gamma_i \left| \partial_x \varphi_i^0 \right| \Delta t\, \|\partial_x \psi_i\|_\infty.$$

Using the mean value theorem and Hölders's inequality in the first term, we get

$$|I| \leqslant C \left((\Delta t)^2 \sum_{n=0}^{M-1} \left\| \varphi^{n+1} \right\|_{\mathcal{H}} + \Delta t \sum_{i=1}^{2} \int_0^1 \gamma_i(x) \left| \partial_x \varphi_i^0 \right| \right).$$

- Similarly, we have:

$$|II| \leqslant C \left((\Delta t)^2 \sum_{n=0}^{M-1} \left(\int_0^1 \alpha \left| \varphi_1^{n+1}(1) \right|^2 \right)^{1/2} + \Delta t \left(\alpha \left| \varphi_1^0(1) \right|^2 \right)^{1/2} \right).$$

Now we combine the last two estimates and use Lemma 6.1 to obtain

$$\begin{aligned}|I| + |II| &\leqslant C\left((\Delta t)^2 \sum_{n=0}^{M-1} \left\|\varphi^{n+1}\right\|_{\mathcal{H}} + \Delta t \left\|\varphi^0\right\|_{\mathcal{H}}\right) \\ &\leqslant C\,\Delta t\,\|g\|_{L^2(0,T;E)}.\end{aligned}$$

Using the estimates of I, II and III, we have

$$\lim_{M\to\infty}\left\{-\int_0^T \langle \partial_t \varphi_M, \psi\rangle_E + \sum_{i=1}^{2} \iint_Q \gamma_i\, \partial_x(\varphi_M)_i\, \partial_x\psi_i + \alpha \int_0^T (\varphi_M)_1(t,1)\,\psi_1(t,1)\,dt - \sum_{i=1}^{2}\iint_Q g_i\psi_i\right\} = 0. \tag{6.7}$$

We now use the convergence result (6.6) to get

$$-\int_0^T \langle \partial_t \varphi, \psi\rangle_E + \sum_{i=1}^{2} \iint_Q \gamma_i(x)\, \partial_x\varphi_i\, \partial_x\psi_i + \alpha \int_0^T \varphi_1(t,1)\,\psi_1(t,1)\,dt - \sum_{i=1}^{2}\iint_Q g_i\,\psi_i = 0. \tag{6.8}$$

This proves the first point of the Definition 6.2.

Next, using integration by parts w.r.t. t in (6.7) and (6.8) for the $\psi \in \mathcal{C}_\alpha$ such that $\psi(0,x) = 0$, with the fact $\varphi^M = 0$ and the convergence result (6.6), we get

$$\sum_{i=1}^{2}\int_0^1 \varphi_i(T,x)\psi_i(T,x)\,dx = 0, \quad \forall\,\psi \in \mathcal{C}_\alpha \text{ with } \psi(0,x) = 0, \tag{6.9}$$

which gives

$$\varphi(T,\cdot) = 0 \text{ on } (0,1).$$

This proves the second point of the Definition 6.2, which subsequently shows that $\varphi \in L^2(0,T;\mathcal{H}) \cap H^1(0,T;E)$ solves the system (6.5) for $g \in L^2(0,T)$. □

The sequence φ_M has few more convergence properties as mentioned in the lemma below.

Lemma 6.4. *Consider the sequence $\{\varphi_M\}_{M\in\mathbb{N}}$ as mentioned in Theorem 6.3, and let $\varphi \in L^2(0,T;\mathcal{H}) \cap H^1(0,T;E)$ be the associated limit which solves the system (6.5). Then, the following convergence results hold for a subsequence:*

(a) $\varphi_M(0,x) \rightharpoonup \varphi(0,x)$ *in* $\mathcal{H}$.
(b) $\partial_x(\varphi_M)_2(t,0) \to \partial_x\varphi_2(t,0)$ *in* $L^2(0,T)$.
(c) $\varphi_M\left(\frac{T}{M},\cdot\right) - \varphi_M(0,\cdot) \rightharpoonup 0$ *in* $\mathcal{H}$.

Proof of Lemma 6.4. (a) Using Lemma 6.1, we have:

$$\|\varphi_M(0,x)\|_{\mathcal{H}}^2 \leqslant C\,\|g\|_{L^2(E)}^2, \quad \forall\, n \in \{0,1,\dots M-1\}.$$

So, there exists a function $\varphi^\# \in \mathcal{H}$ such that $\varphi_M(0,x) \rightharpoonup \varphi^\#$ in $\mathcal{H}$ and hence in E, up to a subsequence. Further, using the arguments similar to the one used in the proof of Theorem 6.3 to get (6.9), we obtain

$$\lim_{M\to\infty}\sum_{i=1}^{2}\int_0^1 (\varphi_M)_i\,(0,x)\psi_i(0,x)\,dx = \sum_{i=1}^{2}\int_0^1 \varphi_i(0,x)\psi_i(0,x)\,dx, \quad \forall\,\psi \in \mathcal{C}_\alpha \text{ with } \psi(T,x) = 0,$$

which proves that $\varphi_M(0,\cdot) \rightharpoonup \varphi(0,\cdot)$ in E. Thus, by uniqueness of weak limit, we have

$$\varphi(0,\cdot) = \varphi^\#(\cdot) \in \mathcal{H},$$

which proves (a).

(b) Using the estimate

$$|f(0)| \leqslant \|f\|_{L^2(0,1)} + \|f\|_{H^1(0,1)}^{1/2}\|f\|_{L^2(0,1)}^{1/2}, \text{ for } f \in H^1(0,1),$$

we have

$$\begin{aligned}\|\partial_x(\varphi_M)_2(t,0)-\partial_x\varphi_2(t,0)\|^2_{L^2(0,T)} &= \int_0^T |\partial_x(\varphi_M)_2(t,0)-\partial_x\varphi_2(t,0)|^2\,dt\\ &\leqslant \int_0^T \|(\varphi_M)_2(t,\cdot)-\varphi_2(t,\cdot)\|_{H^1(0,1)}\,\|(\varphi_M)_2(t,\cdot)-\varphi_2(t,\cdot)\|_{H^2(0,1)} + \int_0^T \|(\varphi_M)_2(t,\cdot)-\varphi_2(t,\cdot)\|^2_{H^1}\\ &\leqslant \|(\varphi_M)_2-\varphi_2\|_{L^2(0,T;H^1(0,1))}\,\|(\varphi_M)_2-\varphi_2\|_{L^2(0,T;H^2(0,1))} + \|(\varphi_M)_2-\varphi_2\|^2_{L^2(0,T;H^1(0,1))}. \qquad (6.10)\end{aligned}$$

Due to Lemma 6.1, we have

$$\begin{aligned}\|(\varphi_M)_2\|_{L^2(0,T;H^2)} &\leqslant C\,\|g\|_{L^2(0,T;E)},\\ \text{and } \|\partial_t(\varphi_M)_2\|_{L^2(0,T;L^2)} &\leqslant C\,\|g\|_{L^2(0,T;E)}.\end{aligned}$$

Note that the embedding $H^2\subset H^1$ and of $H^1\subset L^2$ are compact and continuous, respectively. So by Aubin-Lions lemma, $(\varphi_M)_2\to\varphi_2\in L^2(H^1)$ up to a subsequence. Using this result in the estimate (6.10), we conclude

$$\partial_x(\varphi_M)_2(t,0)\to\partial_x\varphi_2(t,0) \text{ in } L^2(0,T) \text{ up to a subsequence.}$$

This proves (b).

(c) Using fundamental theorem of calculus and Hölder's inequality, we have

$$|(\varphi_M)_i(\Delta t,x)-(\varphi_M)_i(0,x)|^2 \leqslant \Delta t\int_0^{\Delta t}|\partial_t(\varphi_M)_i(s,x)|^2\,ds.$$

Further, integrating w.r.t. x over $(0,1)$ and summing over $i\in\{1,2\}$, we get

$$\|\varphi_M(\Delta t,\cdot)-\varphi_M(0,\cdot)\|^2_E \leqslant \Delta t\,\|\partial_t\varphi_M\|^2_{L^2(0,\Delta t;E)} \leqslant C\,\Delta t\,\|g\|_{L^2(0,T;E)}.$$

This shows

$$\left\|\varphi_M\left(\frac{T}{M},\cdot\right)-\varphi_M(0,\cdot)\right\|_E \to 0 \text{ as } M\to\infty. \qquad (6.11)$$

Also due to Lemma 6.1, we have

$$\left\|\varphi_M\left(\frac{T}{M},\cdot\right)-\varphi_M(0,\cdot)\right\|_{\mathcal{H}} = \|\varphi^1-\varphi^0\|_{\mathcal{H}} \leqslant C\|g\|_E,$$

which says that there exists a function $\varphi^\circ\in\mathcal{H}$ such that

$$\varphi_M\left(\frac{T}{M},\cdot\right)-\varphi_M(0,\cdot)\rightharpoonup\varphi^\circ \text{ in } \mathcal{H} \text{ as } M\to\infty.$$

Due to (6.11), we must have $\varphi^\circ=0$, and hence proves (c) . □

Let us finally step into the proof of main theorem of this section, i.e., Theorem 1.11 in the following subsection.

6.2. **Convergence of the control system.** The proof has been split into two theorems. The first theorem concerns about the convergence of (V_M,U_M) to the solution pair (v,u) of the system (1.1), while the second one deals with proving the function v as a control for the null controllability of (1.1).

Theorem 6.5 (Uniform boundedness for V_M and U_M)**.** *Let $M\in\mathbb{N}$ and $u_0\in\mathcal{H}'$. Then the functions V_M and U_M given by (1.25) and (1.26) respectively satisfies the following uniform bound*

(a) $\|V_M\|_{L^2(0,T)} \leqslant \sqrt{C_{obs}}\|u_0\|_{\mathcal{H}'}$.
(b) $\|U_M\|_{L^\infty(0,T;\mathcal{H}')} \leqslant C\,\|u_0\|_{\mathcal{H}'}$.
(c) $\|U_M\|_{L^2(0,T,E)} \leqslant C\,\|u_0\|_{\mathcal{H}'}$.

Furthermore, there exist $v\in L^2(0,T)$ and $u\in L^2(0,T;E)$ such that the pair (v,u) solves the continuous control system (1.1).

Proof. (a) Due to Theorem 1.10 the approximate control V_M, given by (1.25) satisfies

$$\|V_M\|_{L^2(0,T)} = \|v\|_{L^2_{\widehat{p}}(0,T)} \leqslant C\|u_0\|_{\mathcal{H}'}.$$

(b) For $k \in \{0, 1, \dots, M-1\}$, and $\varphi_\tau \in \mathcal{H}$, consider the system

$$\begin{cases} \varphi_1^n - \Delta t\, \partial_x(\gamma_1\, \partial_x \varphi_1^n) = \varphi_1^{n+1}, \\ \varphi_2^n - \Delta t\, \partial_x(\gamma_2\, \partial_x \varphi_2^n) = \varphi_2^{n+1}, \\ \varphi_1^n(0) = \varphi_2^n(0) = 0, \\ \varphi_1^n(1) = \varphi_2^n(1), \\ \gamma_1(1)\, \partial_x \varphi_1^n(1) + \gamma_2(1)\, \partial_x \varphi_2^n(1) + \alpha\, \varphi_1^n(1) = 0, \\ \varphi^{k+1} = \varphi_\tau, \end{cases} \qquad n \in [\![0, k]\!].$$

Then to estimate $\|u^{k+1}\|_{\mathcal{H}'}$, let us consider the duality product

$$\big\langle u^{k+1}, \varphi_\tau \big\rangle_{\mathcal{H}',\mathcal{H}} = \Big\langle \Big(u_1^{k+1}, u_2^{k+1}\Big), \Big(\varphi_1^k - \Delta t\, \partial_x(\gamma_1\, \partial_x \varphi_1^k), \varphi_2^k - \Delta t\, \partial_x(\gamma_2\, \partial_x \varphi_2^k)\Big) \Big\rangle_{\mathcal{H}',\mathcal{H}}.$$

Note that since $u^{k+1} \in \mathcal{H} \subset E$, the duality product on r.h.s. is basically the $\langle \cdot, \cdot \rangle_E$ inner product, and so we have

$$\big\langle u^{k+1}, \varphi_\tau \big\rangle_{\mathcal{H}',\mathcal{H}} = \Big\langle \Big(u_1^{k+1}, u_2^{k+1}\Big), \Big(\varphi_1^k - \Delta t\, \partial_x\big(\gamma_1\, \partial_x \varphi_1^k\big), \varphi_2^k - \Delta t\, \partial_x\big(\gamma_2\, \partial_x \varphi_2^k\big)\Big) \Big\rangle_{E}.$$

Now using integration by parts, we have

$$\big\langle u^{k+1}, \varphi_\tau \big\rangle_{\mathcal{H}',\mathcal{H}} = \sum_{i=1}^{2} \left(\int_0^L u_i^{k+1} \varphi_i^k + \Delta t \int_0^L \gamma_i\, \partial_x u_i^{k+1}\, \partial_x \varphi_i^k \right) + \alpha\, \Delta t\, \varphi_1^k(1)\, u_1^{k+1}(1) + \Delta t\, \gamma_2(0)\, \partial_x \varphi_2^k(0)\, v^{k+1}.$$

Finally, we use the identity (1.19) in the above expression to obtain

$$\big\langle u^{k+1}, \varphi_\tau \big\rangle_{\mathcal{H}',\mathcal{H}} = \big\langle u^k, \varphi^k \big\rangle_{\mathcal{H}',\mathcal{H}} + \Delta t\, \gamma_2(0)\, \partial_x \varphi_2^k(0)\, v^{k+1},$$

which iteratively gives

$$\big\langle u^{k+1}, \varphi_\tau \big\rangle_{\mathcal{H}',\mathcal{H}} = \big\langle u^0, \varphi^0 \big\rangle_{\mathcal{H}',\mathcal{H}} + \sum_{n=0}^{k} \Delta t\, \gamma_2(0)\, \partial_x \varphi_2^n(0)\, v^{n+1}. \tag{6.12}$$

Using Cauchy-Schwarz and triangle inequalities in the above identity, we have

$$\Big| \big\langle u^{k+1}, \varphi_\tau \big\rangle_{\mathcal{H}',\mathcal{H}} \Big| \leqslant \|u_0\|_{\mathcal{H}'} \|\varphi^0\|_{\mathcal{H}} + \sum_{n=0}^{k} \Delta t\, \gamma_2(0)\, \big|\partial_x \varphi_2^n(0)\big|\, \big|v^{n+1}\big|.$$

Next, we use the trace regularity and the Hölder's inequality in the second term of r.h.s. to obtain

$$\Big| \big\langle u^{k+1}, \varphi_\tau \big\rangle_{\mathcal{H}',\mathcal{H}} \Big| \leqslant \|u_0\|_{\mathcal{H}'} \|\varphi_\tau\|_{\mathcal{H}} + \left(\sum_{n=0}^{k} \Delta t \Big\| \partial_x\big(\gamma_2\, \partial_x \varphi_2^n\big) \Big\|_{L^2}^2 \right)^{1/2} \left(\sum_{n=0}^{k} \Delta t\, |v^{n+1}|^2 \right)^{1/2}.$$

Using Lemma 6.1 and (a) in the second term of r.h.s. of above inequality

$$\Big| \big\langle u^{k+1}, \varphi_\tau \big\rangle_{\mathcal{H}',\mathcal{H}} \Big| \leqslant C\, \|u_0\|_{\mathcal{H}'} \|\varphi_\tau\|_{\mathcal{H}},$$

which gives

$$\|u^{k+1}\|_{\mathcal{H}'} \leqslant C\, \|u_0\|_{\mathcal{H}'}, \quad \forall\, k \in \{0, 1, \dots, M-1\}. \tag{6.13}$$

Now, due to the uniform estimate (6.13) we get

$$\|U_M(\tau)\|_{\mathcal{H}'} \leqslant C\, \|u_0\|_{\mathcal{H}'}, \quad \forall\, \tau \in [0, T],$$

which establish the estimate (b).

(c) Let $g \in L^2(0, T, E)$ be arbitrary, and let $\varphi^n := (\varphi_1^n, \varphi_2^n)$ be the solution of system (6.1) with $\varphi_T = 0$ and $g^{n+1} := \big(g_1^{n+1}, g_2^{n+1}\big)$ given by

$$g_i^{n+1}(x) = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} g_i(t, x)\, dt, \quad n \in \{0, 1, \dots, M-1\}, \text{ for } i \in \{1, 2\}.$$

To get $L^2(0,T;E)$ bound for U_M, let us consider

$$\begin{aligned}\int_0^T \langle U_M(t), g(t)\rangle_E \, dt &= \sum_{i=1}^{2} \left\{ \frac{\Delta t}{2} \int_0^1 u_i^1(x)\, g_i^1(x)\, dx + \sum_{n=1}^{M-1} \Delta t \int_0^1 \frac{(u_i^{n+1}+u_i^n)}{2}(x)\, g_i^{n+1}(x)\, dx \right\} \\ &= \sum_{i=1}^{2} \left[\frac{\Delta t}{2} \int_0^1 u_i^1(x)\, g_i^1(x)\, dx + \frac{1}{2} \sum_{n=1}^{M-1} \Delta t \int_0^1 u_i^{n+1}(x)\, g_i^{n+1}(x)\, dx \right. \\ &\qquad \left. + \frac{1}{2} \sum_{n=1}^{M-1} \Delta t \int_0^1 u_i^n(x)\, g_i^{n+1}(x)\, dx \right] \\ &=: A + B + C.\end{aligned}$$

Using the differential equation of (6.1), and performing integration by parts, we get:

$$\begin{aligned}A &= \frac{1}{2} \sum_{i=1}^{2} \left\{ -\int_0^1 u_i^1(\varphi_i^1 - \varphi_i^0) - \Delta t \int_0^1 u_i^1\, \partial_x(\gamma_i\, \partial_x \varphi_i^0) \right\} \\ &= \frac{1}{2} \sum_{i=1}^{2} \left\{ \int_0^1 u_i^1 \varphi_i^0 + \Delta t \int_0^1 \gamma_i\, \partial_x u_i^1\, \partial_x \varphi_i^0 \right\} + \frac{\Delta t}{2} \alpha\, u_1^1(1) \varphi_1^0(1) + \frac{\Delta t}{2} \gamma_2(0)\, v^1\, \partial_x \varphi_2^0(0) \\ &\qquad - \frac{1}{2} \sum_{i=1}^{2} \int_0^1 u_i^1 \varphi_i^1.\end{aligned}$$

Next we use (1.19) in the above identity to get

$$A = \frac{1}{2} \left\langle u^0, \varphi^0 \right\rangle_{\mathcal{H}',\mathcal{H}} + \frac{\Delta t}{2} \gamma_2(0)\, v^1\, \partial_x \varphi_2^0(0) - \frac{1}{2} \sum_{i=1}^{2} \int_0^1 u_i^1 \varphi_i^1.$$

Following similar steps as in the case of A, and noting the fact that $u^{n+1} \in \mathcal{H}$, B can be simplified to get

$$B = \frac{1}{2} \sum_{i=1}^{2} \int_0^1 u_i^1 \varphi_i^1 + \frac{1}{2} \gamma_2(0) \sum_{n=1}^{M-1} \Delta t\, v^{n+1}\, \partial_x \varphi_2^n(0).$$

Again we follow the same steps as in A, keeping the regularity of $\{u^n\}_{n \in [\![0, M-1]\!]}$ in mind, to get

$$C = \frac{1}{2} \left\langle u^0, \varphi^1 \right\rangle_{\mathcal{H}',\mathcal{H}} + \frac{1}{2} \gamma_2(0) \sum_{n=1}^{M-1} \Delta t\, v^n\, \partial_x \varphi_2^n(0).$$

Summing these expressions together, we get

$$\begin{aligned}\int_0^T \langle U_M(t), g(t)\rangle_E \, dt &= \gamma_2(0) \sum_{n=1}^{M-1} \Delta t \left(\frac{v^{n+1}+v^n}{2} \right) \partial_x \varphi_2^n(0) + \frac{\Delta t}{2} v^1\, \gamma_2(0)\, \partial_x \varphi_2^0(0) \\ &\qquad + \frac{1}{2} \left\langle u^0, \varphi^0 + \varphi^1 \right\rangle_{\mathcal{H}',\mathcal{H}}. \qquad (6.14)\end{aligned}$$

Using summation by parts and the fact $\varphi^M = \varphi_T = 0$, the first term of r.h.s. can be rewritten as

$$\begin{aligned}\gamma_2(0) \sum_{n=1}^{M-1} \Delta t \left(\frac{v^{n+1}+v^n}{2} \right) \partial_x \varphi_2^n(0) &= \gamma_2(0) \sum_{n=0}^{M-1} \Delta t\, v^{n+1} \left(\frac{\partial_x \varphi_2^{n+1}(0) + \partial_x \varphi_2^n(0)}{2} \right) \\ &\qquad - \frac{\Delta t}{2} \gamma_2(0)\, v^1\, \partial_x \varphi_2^0(0).\end{aligned}$$

Now substitute this identity in the expression (6.14) and recall the definition (1.25) of V_M to get

$$\begin{aligned}\int_0^T \langle U_M(t), g(t)\rangle_E \, dt &= \gamma_2(0) \int_0^T V_M(t)\, \partial_x (\varphi_M)_2(t,0)\, dt + \frac{1}{2} \left\langle u_0, \varphi^0 + \varphi^1 \right\rangle_{\mathcal{H}',\mathcal{H}} \\ &= \left\langle u_0, \varphi^0 \right\rangle_{\mathcal{H}',\mathcal{H}} + \gamma_2(0) \int_0^T V_M(t)\, \partial_x (\varphi_M)_2(t,0)\, dt + \frac{1}{2} \left\langle u_0, \varphi^1 - \varphi^0 \right\rangle_{\mathcal{H}',\mathcal{H}}. \qquad (6.15)\end{aligned}$$

Using Cauchy-Schwarz, Hölder's inequality, and the estimates from Lemma 6.1, we get

$$\left| \int_0^T \langle U_M(t), g(t) \rangle_E \, dt \right| \leqslant C \left(\|u_0\|_{\mathcal{H}'} + \|V_M\|_{L^2(0,T)} \right) \|g\|^2_{L^2(E)},$$

which gives

$$\|U_M\|_{L^2(0,T;E)} \leqslant C \left(\|u_0\|_{\mathcal{H}'} + \|V_M\|_{L^2(0,T)} \right).$$

Finally using the estimate (a), we get

$$\|U_M\|_{L^2(0,T,E)} \leqslant C \, \|u_0\|_{\mathcal{H}'}, \text{ for some } C > 0.$$

Now, due to the uniform bounds of V_M and U_M obtained above, there exist $v \in L^2(0,T)$ and $u \in L^2(0,T;E)$ such that $V_M \rightharpoonup v$ in $L^2(0,T)$ and $U_M \rightharpoonup u$ in $L^2(0,T;E)$, up to a subsequence. From (6.15), we have

$$\begin{aligned} \int_0^T \langle U_M(t), g(t) \rangle_E \, dt - \langle u_0, \varphi_M(0,\cdot) \rangle_{\mathcal{H}',\mathcal{H}} - \gamma_2(0) \int_0^T V_M(t) \, \partial_x (\varphi_M)_2(t,0) \, dt \\ = -\frac{1}{2} \langle u_0, \varphi_M(\Delta t, \cdot) - \varphi_M(0,\cdot) \rangle_{\mathcal{H}',\mathcal{H}}, \quad \forall g \in L^2(0,T;E). \end{aligned} \tag{6.16}$$

Thus, taking limit as $M \to \infty$ in the above identity and using Lemma 6.4, we get

$$\int_0^T \langle u(t), g(t) \rangle_E \, dt - \langle u_0, \varphi(0,\cdot) \rangle_{\mathcal{H}',\mathcal{H}} - \gamma_2(0) \int_0^T v(t) \, \partial_x \varphi_2(t,0) \, dt = 0,$$

where φ is solution of the adjoint system (6.5). This proves the pair (u,v) solves the system (1.1). □

The next and final theorem is to show that the limit function v serves as a control for the null controllability of the continuous system (1.1).

Theorem 6.6. *The limit function v of sequence V_M is a control for the continuous control system (1.1).*

Proof. Let u and φ be the solution of the control system (1.1) and its adjoint system (1.5), respectively. Then, using the differential equations of u and φ, and performing integration by parts w.r.t. x, we have

$$\frac{d}{dt} \langle u(t,\cdot), \varphi(t,\cdot) \rangle_{\mathcal{H}',\mathcal{H}} = v(t) \, \partial_x \varphi_2(t,0).$$

Further integrating w.r.t. t over $[0,T]$ we get

$$\langle u(T), \varphi_T \rangle_{\mathcal{H}',\mathcal{H}} - \langle u_0, \varphi(0,\cdot) \rangle_{\mathcal{H}',\mathcal{H}} = \int_0^T v(t) \, \partial_x \varphi_2(t,0) \, dt, \quad \forall \, \varphi_T \in \mathcal{H}.$$

Now, note that the null controllability of the system (1.1) is equivalent to the identity

$$\int_0^T v(t) \, \partial_x \varphi_2(t,0) \, dt + \langle u_0, \varphi(0,x) \rangle_{\mathcal{H}',\mathcal{H}} = 0, \quad \forall \, \varphi_T \in \mathcal{H}. \tag{6.17}$$

Thus to prove the theorem, we shall prove that v satisfies the identity (6.17).

Let $\varphi_T \in \mathcal{H}$ be arbitrarily chosen. For $n \in [\![-1, M]\!]$, let $\varphi^n = (\varphi_1^n, \varphi_2^n)$ be the solution of system (6.1) with $g^{n+1} = 0$ and terminal data φ_T. Then, from the Euler-Lagrange identity (5.3), we have

$$\sum_{n=0}^{M-1} \Delta t \, v^{n+1} \, \partial_x \varphi_2^n(0) + \phi(\Delta t) \left\langle \partial_x \widehat{\varphi}^M, \partial_x \varphi_T \right\rangle_E + \left\langle y_0, \varphi^0 \right\rangle_{\mathcal{H}',\mathcal{H}} = 0, \quad \forall \, \varphi_T \in \mathcal{H}, \tag{6.18}$$

where $\widehat{\varphi}^M$ is the minimizer of the functional $J_{\Delta t}$ given by (5.2).

Note that $\{\varphi^{n-1}\}_{n \in [\![0,M]\!]}$ solves system (6.1) with $g^{n+1} = 0$ and terminal data φ^{M-1}. Thus, using the identity (6.18) for $\varphi_T = \varphi^{M-1}$, we have

$$\sum_{n=0}^{M-1} \Delta t \, v^{n+1} \, \partial_x \varphi_2^{n-1}(0) + \phi(\Delta t) \left\langle \partial_x \widehat{\varphi}^M, \partial_x \varphi^{M-1} \right\rangle_E + \left\langle y_0, \varphi^{-1} \right\rangle_{\mathcal{H}',\mathcal{H}} = 0. \tag{6.19}$$

Adding the identities (6.18) and (6.19), we have

$$\sum_{n=0}^{M-1} \Delta t \, v^{n+1} \Big(\partial_x \varphi_2^n(0) + \partial_x \varphi_2^{n-1}(0) \Big) + \phi(\Delta t) \left\langle \partial_x \widehat{\varphi}^M, \partial_x \varphi^M + \partial_x \varphi^{M-1} \right\rangle_E + \left\langle y_0, \varphi^0 + \varphi^{-1} \right\rangle_{\mathcal{H}',\mathcal{H}} = 0. \tag{6.20}$$

Recall that

$$V_M(t) = \sum_{n=0}^{M-1} 1_{[t_n,t_{n+1})} v^{n+1},$$

and define the approximation φ_M as

$$\varphi_M(t,x) = \sum_{n=0}^{M-1} 1_{[t_n,t_{n+1}]} \left(\frac{(t-t_n)}{\Delta t}\varphi^n + \frac{(t_{n+1}-t)}{\Delta t}\varphi^{n-1} \right). \tag{6.21}$$

Using these approximations, we have

$$\int_0^T V_M(t)\,\partial_x\big(\varphi_M\big)_2(t,0)\,dt = -\phi(\Delta t)\left\langle \partial_x\widehat{\varphi}^M, \frac{\partial_x\varphi^M + \partial_x\varphi^{M-1}}{2} \right\rangle_E - \left\langle y_0, \frac{\varphi^0+\varphi^{-1}}{2} \right\rangle_{\mathcal{H}',\mathcal{H}}. \tag{6.22}$$

Finally, using the convergence results mentioned in the lemma below, we get the desired identity (6.17). □

Lemma 6.7. *Let φ solves (1.5). Then the following convergence results holds up to a subsequence.*

(a) $\phi(\Delta t)\langle \partial_x\widehat{\varphi}^M, \partial_x\varphi^M + \partial_x\varphi^{M-1}\rangle_E \to 0$ *as* $M\to\infty$.
(b) $\varphi^{M-1}(\cdot) \rightharpoonup \varphi_T(\cdot)$ *in* $\mathcal{H}$ *as* $M\to\infty$.
(c) $\partial_x\big(\varphi_M\big)_2(\cdot,0) \to \partial_x\varphi_2(\cdot,0)$ *in* $L^2(0,T)$ *as* $M\to\infty$.
(d) $\frac{\varphi^{-1}(\cdot)+\varphi^0(\cdot)}{2} \rightharpoonup \varphi(0,\cdot)$ *in* $\mathcal{H}$ *as* $M\to\infty$.

Proof. The proof of this lemma follows easily using the property of function ϕ and analysis similar to the one used in the proof of Lemma 6.4.

(a) Using Cauchy-Schwarz inequality and the uniform estimate (5.9), we have

$$\begin{aligned}\left|\phi(\Delta t)\langle \partial_x\widehat{\varphi}^M, \partial_x\varphi^M + \partial_x\varphi^{M-1}\rangle_E\right| &\leqslant \varphi(\Delta t)\|\widehat{\varphi}^M\|_{\mathcal{H}}\|\varphi^M+\varphi^{M-1}\|_{\mathcal{H}} \\ &\leqslant 2\sqrt{C_{obs}}\sqrt{\phi(\Delta t)}\,\|y_0\|_{\mathcal{H}'}\,\|\varphi_T\|_{\mathcal{H}},\end{aligned}$$

Note that $\phi(\Delta t)\to 0$ as $M\to\infty$, and hence we get the result.

(b) Using the equation, and the estimate (6.3) we have

$$\|\varphi^{M-1}-\varphi_T\|_E = \Delta t\left\|\Big(\partial_x\big(\gamma_1\,\partial_x\varphi_i^{M-1}\big), \partial_x\big(\gamma_1\,\partial_x\varphi_i^{M-1}\big)\Big)\right\|_E \leqslant \sqrt{\Delta t}\,\|\varphi_T\|_{\mathcal{H}}.$$

This shows that $\varphi^{M-1}\to\varphi_T$ in E. Moreover, due to the uniform bound $\|\varphi^{M-1}\|_{\mathcal{H}}\leqslant C\|\varphi_T\|_{\mathcal{H}}$, we can conclude $\varphi^{M-1}\rightharpoonup\varphi_T$ in $\mathcal{H}$.

(c) The proof of this is similar to the proof of (b) of Lemma 6.4.

(d) Using the argument similar to the one used in proving (a) and (c) of Lemma 6.4, one can easily get

$$\begin{cases}\varphi_M(0,\cdot) \rightharpoonup \varphi(0,\cdot) & \text{in } \mathcal{H},\\ \varphi_M(\Delta t,\cdot)-\varphi_M(0,\cdot)\rightharpoonup 0 & \text{in } \mathcal{H}.\end{cases}$$

Thus, using the definition of φ_M given by (6.21), we get

$$\varphi^{-1}(\cdot)+\varphi^0(\cdot) = \varphi_M(0,\cdot)+\varphi_M(\Delta t,\cdot) \rightharpoonup 2\varphi(0,\cdot) \text{ in } \mathcal{H}.$$

This completes the proof of the claim.

□

Appendix A. Discrete calculus

For any continuous function h defined over $\mathbb{R}$, we introduce the following continuous operators, with similar notations as in the discrete case,

$$\mathtt{t}^+h(t) := h(t+\Delta t),\ \ \mathtt{t}^-h(t) := h(t-\Delta t),\ \text{ and } D_th := \frac{1}{\Delta t}(\mathtt{t}^+-\mathtt{t}^-)h.$$

Lemma A.1 (Discrete Product rule)**.** *Let f and g be continuously defined function over $\mathbb{R}$. Then, we have*

$$D_t(fg) = \mathtt{t}^+f\,D_tg + D_tf\,\mathtt{t}^-g. \tag{A.1}$$
$$D_t(fg) = \mathtt{t}^-f\,D_tg + D_tf\,\mathtt{t}^+g. \tag{A.2}$$

Using the above formulas with $f = g$, we obtain the following identities

$$\mathtt{t}^+ f\, D_t f = \frac{1}{2} D_t(f^2) + \frac{\Delta t}{2}(D_t f)^2. \tag{A.3}$$

$$\mathtt{t}^- f\, D_t f = \frac{1}{2} D_t(f^2) - \frac{\Delta t}{2}(D_t f)^2. \tag{A.4}$$

Proof. We compute

$$\begin{aligned} D_t(fg)(\tau) &= \frac{1}{\Delta t}\left((fg)(\tau+\Delta t) - (fg)(\tau-\Delta t)\right) \\ &= \frac{1}{\Delta t}\left((fg)(\tau+\Delta t) + f(\tau+\Delta t)g(\tau-\Delta t) - f(\tau+\Delta t)g(\tau-\Delta t) - (fg)(\tau-\Delta t)\right) \\ &= f(\tau+\Delta t)\frac{1}{\Delta t}\left(g(\tau+\Delta t) - g(\tau-\Delta t)\right) + \frac{1}{\Delta t}\left(f(\tau+\Delta t) - f(\tau-\Delta t)\right) g(\tau-\Delta t) \\ &= (\mathtt{t}^+ f)(\tau)\,(D_t g)(\tau) + (D_t f)(\tau)\,(\mathtt{t}^- g)(\tau), \end{aligned}$$

which proves (A.1). Similarly, replacing the term $f(\tau+\Delta t)g(\tau-\Delta t) - f(\tau+\Delta t)g(\tau-\Delta t)$ by $f(\tau-\Delta t)g(\tau+\Delta t) - f(\tau-\Delta t)g(\tau+\Delta t)$ in the second line, we get (A.2). □

Lemma A.2 (Discrete integration by parts)**.** *Let H be any Hilbert space.*

$$-\!\!\!\!\!\!\int_0^T \langle D_t f, g\rangle = -\int_0^T \langle \overline{D}_t g, f\rangle + \left\langle f^M, g^{M+\frac{1}{2}}\right\rangle - \left\langle f^0, g^{\frac{1}{2}}\right\rangle, \quad \forall f \in H^{\overline{\mathcal{P}}},\ g \in H^{\overline{\mathcal{D}}}, \tag{A.5}$$

$$-\!\!\!\!\!\!\int_0^T \left\langle D_t f, \mathtt{t}^+ g\right\rangle = -\!\!\!\!\!\!\int_0^T \left\langle \mathtt{t}^- f, D_t g\right\rangle + \left\langle f^M, g^M\right\rangle - \left\langle f^0, g^0\right\rangle, \quad \forall f, g \in H^{\overline{\mathcal{P}}}, \tag{A.6}$$

where we use the common notation $\langle\cdot,\cdot\rangle$ to denote inner product and duality product (depending on the situation).

Proof. For $f \in H^{\overline{\mathcal{P}}}$, $g \in H^{\overline{\mathcal{D}}}$, we have

$$\begin{aligned} -\!\!\!\!\!\!\int_0^T \langle D_t f, g\rangle &= \sum_{n=0}^{M-1} \Delta t \left\langle (D_t f)^{n+\frac{1}{2}}, g^{n+\frac{1}{2}}\right\rangle \\ &= \sum_{n=0}^{M-1} \Delta t \left\langle \frac{1}{\Delta t}(f^{n+1} - f^n), g^{n+\frac{1}{2}}\right\rangle \\ &= \sum_{n=1}^{M} \Delta t \left\langle f^n, \frac{1}{\Delta t}(-g^{n+\frac{1}{2}} + g^{n-\frac{1}{2}})\right\rangle - \left\langle f^0, g^{\frac{1}{2}}\right\rangle + \left\langle f^M, g^{M+\frac{1}{2}}\right\rangle \\ &= -\left\langle f^0, g^{\frac{1}{2}}\right\rangle + \left\langle f^M, g^{M+\frac{1}{2}}\right\rangle - \int_0^T \langle \overline{D}_t g, f\rangle. \end{aligned}$$

This proves (A.5). Similarly, one can prove (A.6) as well. □

Appendix B. Estimates on weight functions

Lemma B.1 (see Lemma B.4 of [4])**.** *Provided $\frac{\Delta t \tau}{T^3\delta^2} \leqslant 1$, we have*

$$(\mathtt{t}^+\rho_i)\,(D_t r_i) = -\tau(\mathtt{t}^+\theta')\eta_i + \Delta t\left(\frac{\tau}{\delta^3 T^4} - \frac{\tau^2}{\delta^4 T^6}\right)\mathcal{O}_\lambda(1), \quad \textit{for } i \in \{1,2\}.$$

Proof. For any sufficiently smooth function f, we have the following n^{th} order Taylor's formula

$$f(x+y) = \sum_{k=0}^{n-1} f^{(k)}(x)\,\frac{y^k}{k!} + y^n \int_0^1 \frac{(1-\alpha)^{n-1}}{(n-1)!} f^{(n)}(x+\alpha y)\, d\alpha.$$

Using this formula for $f_i(t,x) = e^{-s(t)\eta_i(x)}$ w.r.t. t and $y = \Delta t$ with n=2, we get

$$\begin{aligned}
&\frac{e^{-s(t+\Delta t)\eta_i} - e^{-s(t)\eta_i}}{\Delta t} \\
&= -s'(t)\,\eta_i\, e^{-s(t)\eta_i} + \Delta t \int_0^1 (1-\alpha) e^{-s(t+\alpha\Delta t)\eta_i} \left[(s'(t+\alpha\Delta t)\eta_i)^2 - s''(t+\alpha\Delta t)\eta_i\right] d\alpha \\
&= e^{-\tau\theta(t)\eta_i}\Big[-\tau\theta'(t)\,\eta_i - \Delta t \int_0^1 (1-\alpha) e^{-\tau\theta(t+\Delta t)\eta_i + \tau\theta(t)\eta_i} \tau\theta''(t+\alpha\,\Delta t)\,\eta_i\, d\alpha \\
&\qquad\qquad + \Delta t \int_0^1 (1-\alpha) e^{-\tau\theta(t+\Delta t)\eta_i + \tau\theta(t)\eta_i} \tau^2 \left(\theta'(t+\alpha\,\Delta t)\right)^2 \eta_i^2\, d\alpha \Big].
\end{aligned}$$

Using the facts $\|\eta_i\|_{\mathcal{C}^0([0,1])} = \mathcal{O}(1)$ and $\max\limits_{t\in[0,T]} \theta^{(j)} \leqslant \dfrac{1}{\delta^{j+1}\, T^{j+2}}$ for $j = 0, 1, \ldots$ in the above expression, and choosing the parameters satisfying $\Delta t\, \tau (T^3\delta^2)^{-1} \leqslant 1$ gives the required result. □

Lemma B.2. *There exists a constant $C > 0$ independent of the discrete parameters δ, Δt, and controllability time T such that*

$$|D_t(\theta^{(k)})| \leqslant kT(\mathtt{t}^-\theta^{k+1}) + C\frac{\Delta t}{\delta^{k+2}T^{2k+2}}, \quad k = 1, 2, \ldots . \tag{B.1}$$

$$|\,\mathtt{t}^+(\theta')| \leqslant T(\mathtt{t}^-\theta)^2 + C\frac{\Delta t}{\delta^3 T^4}. \tag{B.2}$$

$$0 \leqslant D_t(\theta') \leqslant CT^2\,\mathtt{t}^-(\theta^3) + C\frac{\Delta t}{\delta^4 T^5}. \tag{B.3}$$

The proof of this lemma is elementary and can be found in [4, Lemma B.5].

Appendix C. Proof of Lemma 3.3

Proof. The proof has been divided in few steps for the ease of reading. We do the estimation for $i \in \{1,2\}$, and at last sum the estimates over $i \in \{1,2\}$ to get the desired estimate (C.7). First recall that

$$I^i_{jk} = \langle (Aq_i)_j, (Bq_i)_k \rangle_{L^2_{\mathcal{D}}(Q)}, \quad i \in \{1,2\}.$$

Step-1: Estimating the terms with no discrete operation:

• Using the integration by parts, we get:

$$\begin{aligned}
I^i_{12} &= -2\tau\lambda c_i \iint_Q \mu_i\Big(\mathtt{t}^-\theta\Big)\Big(\gamma_i\,\partial_x\big(\mathtt{t}^-q_i\big)\Big)\,\partial_x\Big(\gamma_i\,\partial_x\big(\mathtt{t}^-q_i\big)\Big) \\
&= \tau\lambda^2 c_i^2 \iint_Q (\mathtt{t}^-\theta)\,\mu_i\,\big|\gamma_i\partial_x(\mathtt{t}^-q_i)\big|^2 - \tau\lambda c_i \int_0^T (\mathtt{t}^-\theta)\,\mu_i\,\big|\gamma_i\partial_x(\mathtt{t}^-q_i)\big|^2\Big|_{x=0}^1.
\end{aligned}$$

• Next using integration by parts and the fact that $q_i^n(0) = 0$, for $n \in [\![0, M-1]\!]$, we have:

$$\begin{aligned}
I^i_{13} &= -2\tau\lambda^2 c_i^2 \iint_Q (\mathtt{t}^-\theta)\,\mu_i\,(\gamma_i(\mathtt{t}^-q_i))\,\partial_x(\gamma_i(\mathtt{t}^-(\partial_x q_i))) \\
&= J^i_{13} + 2\tau\lambda^2 c_i^2 \iint_Q (\mathtt{t}^-\theta)\,\mu_i\,|\gamma_i\,\partial_x(\mathtt{t}^-q_i)|^2 - 2\tau\lambda^2 c_i^2 \int_0^T (\mathtt{t}^-\theta)\,(\mathtt{t}^-q_i)(1)\,\partial_x\big(\mathtt{t}^-q_i\big)(1),
\end{aligned}$$

where

$$J^i_{13} = 2\tau\lambda^2 c_i^2 \iint_Q (\mathtt{t}^-\theta)\Big(\mu_i'\,\gamma_i\,(\mathtt{t}^-q_i) + \mu_i\,\gamma_i'\,(\mathtt{t}^-q_i)\Big)\gamma_i\,\partial_x(\mathtt{t}^-q_i).$$

Using the facts $\gamma_i \in C^1([0,1])$, and $\mu_i' = \lambda c_i \mu_i$ we have

$$|J^i_{13}| \leqslant C\tau\lambda^3 \iint_Q (\mathtt{t}^-\theta)\mu_i(\mathtt{t}^-q_i)\Big(\gamma_i\partial_x(\mathtt{t}^-q_i)\Big).$$

Next we first use the Young's Inequality and then the estimates $(\mathtt{t}^-\theta) \leqslant T^4(\mathtt{t}^-\theta)^3$, $\mu_i \leqslant \mu_i^3$ to get

$$|J^i_{13}| \leqslant C_\epsilon\tau T^4\lambda^4 \iint_Q \mu_i^3(\mathtt{t}^-\theta)^3(\mathtt{t}^-q_i)^2 + \epsilon\tau\lambda^2 \iint_Q \mu_i\,(\mathtt{t}^-\theta)\,|\gamma_i\,\partial_x(\mathtt{t}^-q_i)|^2, \text{ for some } \epsilon > 0.$$

Choosing the ϵ sufficiently small, we get

$$I^i_{13} \geqslant C\tau\lambda^2 c_i^2 \iint_Q (\mathtt{t}^-\theta)\,\mu_i\,|\gamma_i\,\partial_x(\mathtt{t}^- q_i)|^2 - C\,\tau\lambda^4 T^4 \iint_Q \mu_i^3(\mathtt{t}^-\theta)^3\,|\mathtt{t}^- q_i|^2 \\ - 2\tau\lambda^2 c_i^2\,(\gamma_i(1))^2\mu_i(1)\int_0^T (\mathtt{t}^-\theta)(\mathtt{t}^- q_i)(1)\,\mathtt{t}^-(\partial_x q_i)(1).$$

• Again using integration by parts, boundary conditions on q and Young's inequality, we get

$$I^i_{22} = -2\tau^3\lambda^3 c_i^3 \iint_Q (\mathtt{t}^-\theta)^3\,\mu_i^3\,\gamma_i^2\,(\mathtt{t}^- q_i)\,\partial_x(\mathtt{t}^- q_i) \\ = \tau^3\lambda^3 c_i^3 \iint_Q (\mathtt{t}^-\theta)^3\Big(3\mu_i^2\,\mu_i'\,\gamma_i^2 + 2\mu_i^3\,\gamma_i\,\gamma_i'\Big)|\mathtt{t}^- q_i|^2 - \tau^3\lambda^3 c_i^3 \int_0^T (\mathtt{t}^-\theta)^3(\mu_i(1))^3(\gamma_i(1))^2|\mathtt{t}^- q_i(1)|^2.$$

Further, using (3.9) and the fact that $\gamma_i \in C^1([0,1])$ such that $\gamma_i \geqslant \gamma_{min} > 0$, we get

$$I^i_{22} \geqslant 3\tau^3\lambda^4 c_i^4 \iint_Q (\mathtt{t}^-\theta)^3\mu_i^3\,\gamma_i^2\,|\mathtt{t}^- q_i|^2 - C\tau^3\lambda^3 \iint_Q (\mathtt{t}^-\theta)^3\mu_i^3\,(\mathtt{t}^- q_i)^2 \\ - \tau^3\lambda^3 c_i^3 \int_0^T (\mathtt{t}^-\theta)^3(\mu_i(1))^3(\gamma_i(1))^2|\mathtt{t}^- q_i(t,1)|^2.$$

• Next, we have

$$I^i_{23} = -2\tau^3\,\lambda^4\,c_i^4 \iint_Q (\mathtt{t}^-\theta)^3\mu_i^3\,\gamma_i^2\,|\mathtt{t}^- q_i|^2.$$

Step-2: Estimating the terms which involve discrete operations:
• Using integration by parts w.r.t.. x, and (A.4) we get

$$I^i_{11} = \iint_Q \partial_x\big(\gamma_i\,\partial_x(\mathtt{t}^- q_i)\big)D_t(q_i) \\ = -\iint_Q \gamma_i\,\mathtt{t}^-(\partial_x q_i)\,D_t(\partial_x q_i) + \int_0^T \gamma_i\,\mathtt{t}^-(\partial_x q_i)D_t q_i\,\big|_{x=0}^1 \\ = -\frac{1}{2}\iint_Q \gamma_i\,D_t(|\partial_x q_i|^2) + \frac{\Delta t}{2}\iint \gamma_i\,|D_t(\partial_x q_i)|^2 + \sum_{n=0}^{M-1}\gamma_i(\partial_x q_i)^n\,\big(q_i^{n+1} - q_i^n\big)\,\big|_{x=0}^1 \\ \geqslant -\frac{1}{2}\iint_Q \gamma_i\,D_t(|\partial_x q_i|^2) + \int_0^T \gamma_i\,\mathtt{t}^-(\partial_x q_i)\,D_t q_i\,\big|_{x=1}.$$

In the last step we have used the non-negativity of second terms and the boundary conditions $q_i^n(0) = 0$ for $n \in [\![0, M-1]\!]$, along with the fact that $q_i^M(0) = q_{i,M}(0) = 0$ as $\varphi = \rho\,q \in \mathcal{F}^{\mathcal{P}}$. Further, evaluating the time integral of the first term, we have:

$$I^i_{11} \geqslant -C\int_0^1 \big(\partial_x q_i^M\big)^2 + \int_0^T \gamma_i\,\mathtt{t}^-(\partial_x q_i)\,D_t q_i\,\big|_{x=1}.$$

• Using (A.4), we have

$$I^i_{21} = \tau^2\lambda^2 c_i^2 \iint_Q (\mathtt{t}^-\theta)^2\mu_i^2\gamma_i(\mathtt{t}^- q_i)(D_t q_i) \\ = \frac{1}{2}\tau^2\lambda^2 c_i^2 \iint_Q (\mathtt{t}^-\theta)^2\mu_i^2\gamma_i D_t(q_i^2) - \frac{\Delta t}{2}\tau^2\lambda^2 c_i^2 \iint_Q (D_t q_i)^2 =: J^i_{21} + K^i_{21}.$$

Now, using the discrete integration by parts (A.6), (3.7), and the fact $\gamma_i \geqslant \gamma_{min} > 0$, we have

$$J^i_{21} = -\frac{1}{2}\tau^2\lambda^2 c_i^2 \iint_Q \gamma_i\,\mu_i^2 D_t(\theta^2)(\mathtt{t}^+ q_i)^2 + \frac{1}{2}\tau^2\lambda^2 c_i^2 \int_0^1 \gamma_i\mu_i^2\Big(\big(\theta^M\big)^2\big(q_i^M\big)^2 - \big(\theta^0\big)^2\big(q_i^0\big)^2\Big) \\ \geqslant -\frac{1}{2}\tau^2\lambda^2 c_i^2 \iint_Q \gamma_i\,\mu_i^2 D_t(\theta^2)(\mathtt{t}^+ q_i)^2 + C\frac{\tau^2\lambda^2}{T^4}\int_0^1 \mu_i^2\big(q_i^M\big)^2 - \frac{1}{2}\tau^2\lambda^2 c_i^2 \int_0^1 \gamma_i\,\mu_i^2\big(\theta^0\big)^2\big(q_i^0\big)^2.$$

Further, using the estimate (B.1) and $\mathtt{t}^+q = \mathtt{t}^-q + \Delta t\, D_t q$, we can estimate the first term of the above estimate to get

$$\begin{aligned}\Big|-\frac{1}{2}\tau^2\lambda^2 c_i^2 \iint_Q \gamma_i\,\mu_i^2 D_t(\theta^2)(\mathtt{t}^+q_i)^2\Big| \\ \leqslant C\tau^2\lambda^2 \iint_Q \mu_i^2\left[T(\mathtt{t}^-\theta)^3 + \frac{\Delta t}{T^6\delta^4}\right](\mathtt{t}^-q_i)^2 + C(\Delta t)^2\tau^2\lambda^2 \iint_Q \mu_i^2\left[T(\mathtt{t}^-\theta)^3 + \frac{\Delta t}{T^6\delta^4}\right](D_t q_i)^2.\end{aligned}$$

Hence, we get the following estimate on I_{21}^i

$$\begin{aligned}I_{21}^i \geqslant -C\tau^2\lambda^2 T\iint_Q \mu_i^2(\mathtt{t}^-\theta)^3(\mathtt{t}^-q_i)^2 - C\Delta t\frac{\tau^2\lambda^2}{T^6\delta^4}\iint_Q \mu_i^2(\mathtt{t}^-q_i)^2 \\ - C(\Delta t)^2 T\tau^2\lambda^2\iint_Q \mu_i^2(\mathtt{t}^-\theta)^3(D_t q_i)^2 - C(\Delta t)^3\frac{\tau^2\lambda^2}{T^6\delta^4}\iint_Q \mu_i^2(D_t q_i)^2 - C\Delta t\tau^2\lambda^2\iint_Q (D_t q_i)^2 \\ + C\frac{\tau^2\lambda^2}{T^4}\int_0^1 \mu_i^2\big(q_i^M\big)^2 - \frac{1}{2}\tau\lambda^2 c_i^2\int_0^1 \gamma_i\mu_i^2\big(\theta^0\big)^2\big(q_i^0\big)^2.\end{aligned}$$

Finally combining the third and fourth terms, and using (3.7) and the fact that $\Delta t/T \leqslant 1$ in the combined term, the above estimate can be rewritten as

$$\begin{aligned}I_{21}^i \geqslant -C\tau^2\lambda^2 T\iint_Q \mu_i^2(\mathtt{t}^-\theta)^3\,|\mathtt{t}^-q_i|^2 - C\Delta t\frac{\tau^2\lambda^2}{T^6\delta^4}\iint_Q \mu_i^2\,|\mathtt{t}^-q_i|^2 - C(\Delta t)^2\frac{\tau^2\lambda^2}{\delta^4T^5}\iint \mu_i^2\,|D_t q_i|^2 \\ - C\Delta t\,\tau^2\lambda^2\iint |D_t q_i|^2 + C\frac{\tau^2\lambda^2}{T^4}\int_0^1 \mu_i^2\,|q_i^M|^2 - \frac{1}{2}\tau\lambda^2 c_i^2\int_0^1 \gamma_i\,\mu_i^2\,\big(\theta^0\big)^2|q_i^0|^2.\end{aligned}$$

• Using the discrete product rule (A.4), we get

$$I_{31}^i = \iint_Q \tau(\mathtt{t}^+\theta')\eta_i(\mathtt{t}^-q_i)(D_t q_i) = \frac{\tau}{2}\iint_Q \eta_i(\mathtt{t}^+\theta')D_t(q_i^2) - \frac{\tau\Delta t}{2}\iint_Q \eta_i(\mathtt{t}^+\theta')(D_t q_i)^2.$$

Further using discrete integration by parts (A.6), and the estimates (B.1),(B.2) of θ, we have:

$$\begin{aligned}I_{31}^i &= -\frac{\tau}{2}\iint_Q \eta_i(D_t\theta')(\mathtt{t}^-q_i)^2 - \frac{\tau}{2}\int_0^1 \eta_i(\theta')^0(q_i^2)^0 + \frac{\tau}{2}\int_0^1 \eta_i(\theta')^M(q_i^2)^M - \frac{\tau\Delta t}{2}\iint_Q \eta_i(\mathtt{t}^+\theta')(D_t q_i)^2 \\ &\geqslant -C\frac{\tau}{2}\iint_Q \eta_i\left(T(\mathtt{t}^-\theta)^3 + \frac{\Delta t}{\delta^4T^5}\right)(\mathtt{t}^-q_i)^2 - \frac{\tau\Delta t}{2}\iint_Q \eta_i\left(T(\mathtt{t}^-\theta)^2 + C\frac{\Delta t}{\delta^3T^4}\right)(D_t q_i)^2.\end{aligned}$$

Lastly, using the relation (3.4) we get:

$$\begin{aligned}I_{31}^i \geqslant -Ce^{\lambda K}T\tau\iint_Q (\mu_i)^3(\mathtt{t}^-\theta)^3(\mathtt{t}^-q_i)^2 - C\frac{\tau\Delta t}{\delta^4T^5}\iint_Q \eta_i(\mathtt{t}^-q_i)^2 - C\tau\Delta t\,T\iint_Q \eta_i(\mathtt{t}^-\theta)^2(D_t q_i)^2 \\ - C\frac{(\Delta t)^2\tau}{\delta^3T^4}\iint_Q \eta_i(D_t q_i)^2.\end{aligned}$$

• Using integration by parts w.r.t. x, boundary conditions given by (3.12), and the expression for η' and μ', we get

$$\begin{aligned}I_{32}^i &= -2\tau^2\lambda c_i\iint_Q (\mathtt{t}^+\theta')(\mathtt{t}^-\theta)\mu_i\,\eta_i\,\gamma_i(\mathtt{t}^-q_i)\,\partial_x(\mathtt{t}^-q_i) \\ &= \tau^2\lambda^2c_i^2\iint_Q (\mathtt{t}^+\theta')(\mathtt{t}^-\theta)\mu_i\,\eta_i\,\gamma_i(\mathtt{t}^-q_i)^2 - \tau^2\lambda^2c_i^2\iint_Q (\mathtt{t}^+\theta')(\mathtt{t}^-\theta)\mu_i^2\,\gamma_i(\mathtt{t}^-q_i)^2 \\ &\quad + \tau^2\lambda c_i\iint_Q (\mathtt{t}^+\theta')(\mathtt{t}^-\theta)\mu_i\,\eta_i\,\gamma_i'(\mathtt{t}^-q_i)^2 - \tau^2\lambda c_i\int_0^T (\mathtt{t}^+\theta')(\mathtt{t}^-\theta)\mu_i\,\eta_i\,\gamma_i(\mathtt{t}^-q_i)^2\,\big|_{x=1}.\end{aligned}$$

Next using the estimate (B.2), relation (3.4), and the facts that $\mu_i \geqslant 1$ on $[0,1]$ and $\lambda \geqslant 1$, we get:

$$\begin{aligned}|I_{32}^i| \leqslant C\tau^2\lambda^2c_i^2\iint_Q \left(T(\mathtt{t}^-\theta)^2 + \frac{\Delta t}{\delta^3T^4}\right)(\mathtt{t}^-\theta)\big(|\eta_i\mu_i| + |\mu_i|^2\big)|\gamma_i|\,|\mathtt{t}^-q_i|^2 \\ + C\tau^2\lambda|c_i|\int_0^T \left(T(\mathtt{t}^-\theta)^2 + \frac{\Delta t}{\delta^3T^4}\right)|\mathtt{t}^-\theta||\mu_i(1)\,\eta_i(1)\,\gamma_i(1)|\,|\mathtt{t}^-q_i(1)|^2.\end{aligned}$$

Thus, we have

$$I^i_{32} \geqslant -C\tau^2\lambda^2 T e^{\lambda K}\iint_Q (\mathtt{t}^-\theta)^3\mu_i^3|\mathtt{t}^-q_i|^2 - C\frac{\tau^2\lambda^2\Delta t}{\delta^3 T^4}\iint_Q (\mathtt{t}^-\theta)\big(|\eta_i\mu_i| + |\mu_i|^2\big)|\mathtt{t}^-q_i|^2$$
$$- C\tau^2\lambda\int_0^T \left(T(\mathtt{t}^-\theta)^2 + \frac{\Delta t}{\delta^3 T^4}\right)|\mathtt{t}^-\theta||\mu_i(1)\,\eta_i(1)\,\gamma_i(1)|\,|\mathtt{t}^-q_i(1)|^2.$$

- Using (B.2), the last term can be estimated as

$$|I^i_{33}| = -2\tau^2\lambda^2 c_i^2\iint_Q (\mathtt{t}^+\theta')(\mathtt{t}^-\theta)\eta_i\,\mu_i\,\gamma_i(\mathtt{t}^-q_i)^2$$
$$\leqslant C\tau^2\lambda^2 T\iint_Q (\mathtt{t}^-\theta)^3|\eta_i\,\mu_i|\,|\mathtt{t}^-q_i|^2 + C\frac{\tau^2\lambda^2\Delta t}{\delta^3 T^4}\iint_Q (\mathtt{t}^-\theta)|\eta_i\,\mu_i|\,|\mathtt{t}^-q_i|^2.$$

Thus, using (3.4) we get

$$I^i_{33} \geqslant -C\tau^2\lambda^2 T e^{\lambda K}\iint_Q (\mathtt{t}^-\theta)^3\mu_i^3\,|\mathtt{t}^-q_i|^2 - C\frac{\tau^2\lambda^2\Delta t}{\delta^3 T^4}e^{\lambda K}\iint_Q (\mathtt{t}^-\theta)\mu_i^3|\mathtt{t}^-q_i|^2$$

Step-3: Combining the above estimates:

(a) Terms corresponding to $\partial_x(\mathtt{t}^-q_i)$:
Collecting the first terms from the estimates of I^i_{12} and I^i_{13}, we have

$$\Big(\tau\lambda^2(c_i)^2 + C\tau\lambda^2(c_i)^2\Big)\iint_Q (\mathtt{t}^-\theta)\mu_i|\gamma_i\,\partial_x(\mathtt{t}^-q_i)\big|^2 \geqslant C\tau\lambda^2\iint_Q (\mathtt{t}^-\theta)\mu_i|\gamma_i\,\partial_x(\mathtt{t}^-q_i)\big|^2.$$

(a) Terms corresponding to $(\mathtt{t}^-q_i)$:
Collecting the terms from the estimates of $I^i_{13}, I^i_{22}, I^i_{23}, I^i_{21}, I^i_{31}, I^i_{32}$, and I^i_{33}, and using $\tau, \lambda, e^{\lambda K} \geqslant 1$ we have

$$\Big(-\tau\lambda^4T^4 + \tau^3\lambda^4 - \tau^3\lambda^3 - \tau^2\lambda^2T - \tau e^{\lambda K}T - 2\tau^2\lambda^2e^{\lambda K}T\Big)\iint_Q \mu_i^3\,(\mathtt{t}^-\theta)^3|\mathtt{t}^-q_i|^2$$
$$\geqslant \Big(\tau^3\lambda^4 - \tau^3\lambda^3 - \tau\lambda^4T^4 - \tau^2\lambda^2e^{\lambda K}T\Big)\iint_Q \mu_i^3\,(\mathtt{t}^-\theta)^3|\mathtt{t}^-q_i|^2.$$

Next, for $\lambda \geqslant \lambda_{0,i} >> 1$ we have

$$\Big(\tau^3\lambda^4 - \tau^3\lambda^3 - \tau\lambda^4T^4 - \tau^2\lambda^2e^{\lambda K}T\Big) \geqslant C\Big(\tau^3\lambda^4 - \tau\lambda^4T^4 - \tau^2\lambda^2e^{\lambda K}T\Big),$$

and for $\tau \geqslant \tau_{0,i}\,T^4$ with $\tau_{0,i} >> 1$, we have

$$\tau^3\lambda^4 - \tau\lambda^4T^4 \geqslant \tau^3\lambda^4 - \frac{1}{\tau_0}\tau^2\lambda^4 \geqslant C\tau^3\lambda^4.$$

Hence, we obtain

$$\Big(-\tau\lambda^4T^4 + \tau^3\lambda^4 - \tau^3\lambda^3 - \tau^2\lambda^2T - \tau e^{\lambda K}T - 2\tau^2\lambda^2e^{\lambda K}T\Big)\iint_Q \mu_i^3\,(\mathtt{t}^-\theta)^3|\mathtt{t}^-q_i|^2$$
$$\geqslant C\Big(\tau^3\lambda^4 - \tau^2\lambda^2e^{\lambda K}T\Big)\iint_Q \mu_i^3\,(\mathtt{t}^-\theta)^3|\mathtt{t}^-q_i|^2.$$

Now, we sum up all the above estimates to get a lower bound for $\langle Aq_i, Bq_i\rangle$ and use it in (3.17) to get

$$\frac{1}{2}\|F_i\|^2_{L^2_{\mathcal{D}}(Q)} \geqslant \frac{1}{2}\|Aq_i\|^2_{L^2_{\mathcal{D}}(Q)} + \frac{1}{2}\|Bq_i\|^2_{L^2_{\mathcal{D}}(Q)} + C\tau\lambda^2\iint_Q (\mathtt{t}^-\theta)\mu_i|\gamma_i\partial_x(\mathtt{t}^-q_i)|^2$$
$$+ C\tau^3\lambda^4\iint_Q \mu_i^3(\mathtt{t}^-\theta)^3(\mathtt{t}^-q_i)^2 - C\tau^2\lambda^2Te^{\lambda K}\iint_Q (\mathtt{t}^-\theta)^3\mu_i^3\,|\mathtt{t}^-q_i|^2$$
$$+ C\frac{\tau^2\lambda^2}{T^4}\int_0^1 \mu_i^2(q_i^2)^M - C\Big(W_i + X_i + Y_i\Big) - Z_i, \tag{C.1}$$

where

$$W_i = \int_0^1 (\partial_x q_i^M)^2 + \tau\lambda^2 \int_0^1 \mu_i^2(\theta^0)^2(q_i^0)^2, \tag{C.2}$$

$$X_i = \frac{\Delta t\,\tau^2\lambda^2}{T^6\delta^4} \iint_Q \mu_i^2(\mathtt{t}^- q_i)^2 + \frac{\Delta t\tau}{\delta^4 T^5} \iint_Q \eta_i\,(\mathtt{t}^- q_i)^2 + \frac{\tau^2\,\lambda^2\,\Delta t}{\delta^3 T^4} \iint_Q (\mathtt{t}^-\theta)|\mu_i\,\eta_i|(\mathtt{t}^- q_i)^2, \tag{C.3}$$

$$\begin{aligned} Y_i = (\Delta t)^2 \frac{\tau^2\lambda^2}{\delta^4 T^6} \iint \mu_i^2 (D_t q_i)^2 + \Delta t\,\tau^2\lambda^2 \iint (D_t q_i)^2 + \tau\Delta t T \iint \eta_i(\mathtt{t}^-\theta)^2 (D_t q_i)^2 \\ + \frac{(\Delta t)^2\tau}{\delta^3 T^4} \iint_Q \eta_i (D_t q_i)^2, \end{aligned} \tag{C.4}$$

$$\begin{aligned} Z_i &= \tau\lambda c_i \int_0^T (\mathtt{t}^-\theta)\mu_i(1)\,(\gamma_i(1))^2\,\big(\mathtt{t}^-(\partial_x q_i(1))\big)^2 - \tau\lambda c_i \int_0^T (\mathtt{t}^-\theta)\mu_i(0)\,(\gamma_i(0))^2\,\big(\mathtt{t}^-(\partial_x q_i(0))\big)^2 \\ &\quad + 2c_i^2\,(\gamma_i(1))^2\,\tau\lambda^2\mu_i(1) \int_0^T (\mathtt{t}^-\theta)(\mathtt{t}^- q_i)(1)\mathtt{t}^-(\partial_x q_i)(1) - \int_0^T \gamma_i(1)\,\mathtt{t}^-(\partial_x q_i)(1) D_t q_i(1) \\ &\quad + \tau^2\lambda|c_i|\gamma_i(1)\mu_i(1)\eta_i(1) \int_0^T \left(T(\mathtt{t}^-\theta)^2 + \frac{\Delta t}{\delta^3 T^4}\right)(\mathtt{t}^-\theta)(\mathtt{t}^- q_i(1))^2 \\ &\quad + \tau^3\lambda^3 c_i^3 (\mu_i(1))^3 \int_0^T (\mathtt{t}^-\theta)^3\,(\gamma_i(1))^2\,(\mathtt{t}^- q_i)^2(1). \end{aligned} \tag{C.5}$$

Step-4: Estimating the $\|F_i\|^2_{L^2_{\mathcal{D}}(Q)}$:
Recall the expression of F_i from (3.16), and then use triangle inequality to obtain

$$\begin{aligned} \|F_i\|^2_{L^2_{\mathcal{D}}(Q)} &\leqslant C \Big\|(\mathtt{t}^- r_i)\Big(D_t\varphi_i + \partial_x\big(\gamma_i\,\partial_x(\mathtt{t}^-\varphi)\big)\Big)\Big\|^2_{L^2_{\mathcal{D}}(Q)} + C\tau^2\lambda^2 c_i^2 \iint_Q (\mathtt{t}^-\theta)^2\mu_i^2(\gamma_i')^2(\mathtt{t}^- q_i)^2 \\ &\quad + C\tau^2\lambda^4 c_i^4 \iint_Q (\mathtt{t}^-\theta)^2\mu_i^2(\gamma_i)^2(\mathtt{t}^- q_i)^2 + C(\Delta t)^2\left(\frac{\tau}{\delta^3 T^4} - \frac{\tau^2}{\delta^4 T^6}\right)^2 \mathcal{O}_\lambda(1) \iint_Q (\mathtt{t}^+ q_i)^2 \\ &\quad + C(\Delta t)^2\tau^2 \iint_Q (\mathtt{t}^+\theta')^2\eta_i^2(D_t q_i)^2 \\ &\leqslant C \Big\|(\mathtt{t}^- r_i)\Big(D_t\varphi_i + \partial_x\big(\gamma_i\,\partial_x(\mathtt{t}^-\varphi)\big)\Big)\Big\|^2_{L^2_{\mathcal{D}}(Q)} + C\tau^2\lambda^4 T^2 \iint_Q (\mathtt{t}^-\theta)^3\mu_i^3(\mathtt{t}^- q_i)^2 \\ &\quad + C(\Delta t)^2\left(\frac{\tau^2}{\delta^6 T^8} + \frac{\tau^4}{\delta^8 T^{12}}\right)\mathcal{O}_\lambda(1)\left[\iint_Q (\mathtt{t}^- q_i)^2 + \int_0^1 (q_i^M)^2 - \int_0^1 (q_i^0)^2\right] \\ &\quad + C\tau^2(\Delta t)^2 \iint_Q (\mathtt{t}^-\theta)^4 (D_t q_i)^2 + C\frac{(\Delta t)^4\tau^2}{\delta^6 T^8} \iint_Q (D_t q_i)^2. \end{aligned} \tag{C.6}$$

In the last inequality, we have merged the second and third term together and then used the facts $(\mathtt{t}^-\theta)^2 \leqslant T^2(\mathtt{t}^-\theta)^3$, $\mu_i \geqslant 1$.

Step-5: Combining the estimate of $\|F_i\|_{L^2_{\mathcal{D}}(Q)}$ with (C.1):
Using (C.6) in (C.1), and absorbing the second term of r.h.s. of the above inequality in left, we get:

$$\begin{aligned} &\frac{1}{2}\|Aq_i\|^2_{L^2_{\mathcal{D}}(Q)} + \frac{1}{2}\|Bq_i\|^2_{L^2_{\mathcal{D}}(Q)} + C\tau^3\lambda^4 \iint_Q \mu_i^3(\mathtt{t}^-\theta)^3(\mathtt{t}^- q_i)^2 - C\tau^2\lambda^2 T e^{\lambda K} \iint_Q (\mathtt{t}^-\theta)^3\mu_i^3\,|\mathtt{t}^- q_i|^2 \\ &\qquad + C\tau\lambda^2 \iint_Q (\mathtt{t}^-\theta)\mu_i|\gamma_i\partial_x(\mathtt{t}^- q_i)|^2 + C\frac{\tau^2\lambda^2}{T^4}\int_0^1 \mu_i^2(q_i^2)^M - Z_i \\ &\leqslant C\Big\|(\mathtt{t}^- r_i)\Big(D_t\varphi_i + \partial_x\big(\gamma_i\,\partial_x(\mathtt{t}^-\varphi)\big)\Big)\Big\|^2_{L^2_{\mathcal{D}}(Q)} + C(\Delta t)^2\left(\frac{\tau^2}{\delta^6 T^8} + \frac{\tau^4}{\delta^8 T^{12}}\right)\left(\iint_Q (\mathtt{t}^- q_i)^2 + \int_0^1 (q_i^M)^2\right) \\ &\qquad + C\tau^2(\Delta t)^2 \iint_Q (\mathtt{t}^-\theta)^4 (D_t q_i)^2 + C\frac{(\Delta t)^4\tau^2}{\delta^6 T^8} \iint_Q (D_t q_i)^2 + C\Big(W_i + X_i + Y_i\Big), \end{aligned}$$

for $\lambda \geqslant \lambda_{0,i}(\gamma_i) > 0, \tau \geqslant \tau_{0,i}(T^2 + T)$.

Step-6: Summing the above estimate over $i \in \{1, 2\}$:

Define $\lambda_0 := \max\{\lambda_{0,i} : i = 1, 2\}$ and $\tau_0 := \max\{\tau(\gamma_i)_{0,i} : i = 1, 2\}$. Now, summing the above estimate over $i \in \{1, 2\}$, we get

$$\sum_{i=1}^{2} \Bigg(\|Aq_i\|^2_{L^2_{\mathcal{D}}(Q)} + \|Bq_i\|^2_{L^2_{\mathcal{D}}(Q)} + \tau^3\lambda^4 \iint_Q \mu_i^3 (\mathtt{t}^- \theta)^3 (\mathtt{t}^- q_i)^2 - \tau^2\lambda^2 T e^{\lambda K} \iint_Q (\mathtt{t}^- \theta)^3 \mu_i^3 \, |\mathtt{t}^- q_i|^2 + \tau\lambda^2 \iint_Q (\mathtt{t}^- \theta)\mu_i |\gamma_i \partial_x (\mathtt{t}^- q_i)|^2 + \frac{\tau^2\lambda^2}{T^4} \int_0^1 \mu_i^2 (q_i^2)^M - Z_i \Bigg)$$
$$\leqslant C \sum_{i=1}^{2} \Bigg(\Big\| (\mathtt{t}^- r_i) \Big(D_t\varphi_i + \partial_x \big(\gamma_i \, \partial_x (\mathtt{t}^- \varphi) \big) \Big) \Big\|^2_{L^2_{\mathcal{D}}(Q)} + \widetilde{W}_i + \widetilde{X}_i + \widetilde{Y}_i \Bigg), \tag{C.7}$$

for all $\tau \geqslant \tau_0(T^2 + T)$, $\lambda \geqslant \lambda_0$ and for all $\delta \in (0, 1/2]$ and Δt satisfying the condition $\frac{\tau \, \Delta t}{T^3 \delta^2} \leqslant 1$. The expression $\widetilde{W}_i, \widetilde{X}_i$ and $\widetilde{Y}_i$ are given by

$$\widetilde{W}_i = W_i + (\Delta t)^2 \left(\frac{\tau^2}{\delta^6 T^8} + \frac{\tau^4}{\delta^8 T^{12}} \right) \int_0^1 (q_i^M)^2,$$
$$\widetilde{X}_i = X_i + (\Delta t)^2 \left(\frac{\tau^2}{\delta^6 T^8} + \frac{\tau^4}{\delta^8 T^{12}} \right) \iint_Q (\mathtt{t}^- q_i)^2,$$
$$\widetilde{Y}_i = Y_i + \tau^2 (\Delta t)^2 \iint_Q (\mathtt{t}^- \theta)^4 (D_t q_i)^2 + \frac{(\Delta t)^4 \tau^2}{\delta^6 T^8} \iint_Q (D_t q_i)^2.$$

Let us fix $\lambda = \lambda_0$. Then by ignoring few positive terms from l.h.s., using $\gamma_i \geqslant \gamma_{min} > 0, \mu_i \geqslant 1$ and the fact $\tau \geqslant \tau_0 \, T^2$, the above estimate (C.7) can be rewritten as

$$\sum_{i=1}^{2} \Bigg(\|Bq_i\|^2_{L^2_{\mathcal{D}}(Q)} + \tau^3 \iint_Q (\mathtt{t}^- \theta)^3 (\mathtt{t}^- q_i)^2 + \tau \iint_Q (\mathtt{t}^- \theta) |\partial_x (\mathtt{t}^- q_i)|^2 + \int_0^1 \big| q_i^M \big|^2 - \widehat{Z}_i \Bigg)$$
$$\leqslant C_{\lambda_0} \sum_{i=1}^{2} \Bigg(\Big\| (\mathtt{t}^- r_i) \Big(D_t\varphi_i + \partial_x \big(\gamma_i \, \partial_x (\mathtt{t}^- \varphi) \big) \Big) \Big\|^2_{L^2_{\mathcal{D}}(Q)} + \widehat{W}_i + \widehat{X}_i + \widehat{Y}_i \Bigg), \tag{C.8}$$

where $\widehat{Z}_i, \widehat{W}_i, \widehat{X}_i$ and $\widehat{Y}_i$ are nothing but $Z_i, \widetilde{W}_i, \widetilde{X}_i$ and $\widetilde{Y}_i$, respectively with $\lambda = \lambda_0$ in their expression.
Claim: Let $\epsilon_0 > 0$ be any real number. Then for any $\Delta t > 0$, and $\tau \geqslant \tau_0(T^2 + T)$ satisfying the relation

$$\frac{\tau^3 \, \Delta t}{\delta^4 \min\{T^4, T^6\}} < \epsilon_0,$$

we have the following:

$$\widehat{X}_i \leqslant \epsilon_0 \iint_Q (\mathtt{t}^- s)^3 \, |\mathtt{t}^- q_i|^2, \tag{C.9}$$
$$\text{and } \widehat{Y}_i \leqslant \epsilon_0 \iint_Q (\mathtt{t}^- s)^{-1} \, |D_t q_i|^2. \tag{C.10}$$

Proof of the claim. Let us first write $\widehat{X}_i$ and $\widehat{Y}_i$ as

$$\widehat{X}_i = \sum_{j=1}^{5} \widehat{X}_{ij}, \quad \widehat{Y}_i = \sum_{j=1}^{6} \widehat{Y}_{ij}.$$

Now, using the fact $1 \leqslant \tau\theta(t) = s(t)$, we can estimate the $\tilde{X}_{ij}$ as follows:

- $\widehat{X}_{i1} \leqslant \dfrac{\Delta t\tau^2}{\delta^4 T^6} \iint_Q (\mathtt{t}^- q_i)^2 \leqslant \dfrac{\Delta t\tau^2}{\delta^4 T^6} \iint_Q (\mathtt{t}^- s)^3 (\mathtt{t}^- q_i)^2 \leqslant \epsilon_0 \iint_Q (\mathtt{t}^- s)^3 (\mathtt{t}^- q_i)^2.$
- $\widehat{X}_{i2} \leqslant \dfrac{\Delta t\tau}{\delta^4 T^5} \iint_Q (\mathtt{t}^- s)^3 (\mathtt{t}^- q_i)^2 \leqslant \epsilon_0 \iint_Q (\mathtt{t}^- s)^3 (\mathtt{t}^- q_i)^2.$
- $\widehat{X}_{i3} \leqslant \dfrac{\Delta t\tau}{\delta^3 T^4} \iint_Q (\mathtt{t}^- s)^3 (\mathtt{t}^- q_i)^2 \leqslant \epsilon_0 \iint_Q (\mathtt{t}^- s)^3 (\mathtt{t}^- q_i)^2.$
- $\widehat{X}_{i4} \leqslant \dfrac{(\Delta t)^2\tau^2}{\delta^6 T^8} \iint_Q (\mathtt{t}^- s)^3 (\mathtt{t}^- q_i)^2 \leqslant \epsilon_0^2 \iint_Q (\mathtt{t}^- s)^3 (\mathtt{t}^- q_i)^2.$
- $\widehat{X}_{i5} \leqslant \dfrac{(\Delta t)^2\tau^4}{\delta^8 T^{12}} \iint_Q (\mathtt{t}^- s)^3 (\mathtt{t}^- q_i)^2 \leqslant \epsilon_0^2 \iint_Q (\mathtt{t}^- s)^3 (\mathtt{t}^- q_i)^2.$

Thus, combining the estimates, we have

$$\sum_{j=1}^{5} \widehat{X}_{ij} \leqslant \epsilon_0 \iint_Q (\mathtt{t}^- s)^3 (\mathtt{t}^- q_i)^2. \tag{C.11}$$

Next we estimate the $\widetilde{Y}_{ij}$ terms, and in order to do so we use the fact that $|\eta_i| \leqslant C\mu_i^2$ and $(\mathtt{t}^-\theta) \leqslant \frac{1}{\delta T^2}$.

- $\widehat{Y}_{i1} \leqslant \dfrac{(\Delta t)^2\tau^2}{\delta^4 T^6} \iint_Q \mu_i^2 (D_t q_i)^2 \leqslant \dfrac{(\Delta t)^2\tau^3}{\delta^5 T^8} \iint_Q \mu_i^2 (\mathtt{t}^- s)^{-1} (D_t q_i)^2 \leqslant \epsilon_0^2 \iint_Q \mu_i^2 (\mathtt{t}^- s)^{-1} (D_t q_i)^2.$
- $\widehat{Y}_{i2} \leqslant \Delta t\, \tau^2 \iint_Q \mu_i^2 (D_t q_i)^2 \leqslant \dfrac{(\Delta t)\tau^3}{\delta T^2} \iint_Q \mu_i^2 (\mathtt{t}^- s)^{-1} (D_t q_i)^2 \leqslant \epsilon_0 \iint_Q \mu_i^2 (\mathtt{t}^- s)^{-1} (D_t q_i)^2.$
- $\widehat{Y}_{i3} \leqslant \tau\Delta t \iint_Q \mu_i^2 (\mathtt{t}^-\theta)^2 (D_t q_i)^2 = \tau\Delta t \iint_Q \mu_i^2 (\mathtt{t}^-\theta)^3 \tau (\mathtt{t}^- s)^{-1} (D_t q_i)^2$
  $\leqslant \dfrac{\tau^2\Delta t}{\delta^3 T^6} \iint_Q \mu_i^2 (\mathtt{t}^- s)^{-1} (D_t q_i)^2 \leqslant \epsilon_0 \iint_Q \mu_i^2 (\mathtt{t}^- s)^{-1} (D_t q_i)^2.$
- $\widehat{Y}_{i4} = \dfrac{(\Delta t)^2\tau}{\delta^3 T^4} \iint_Q (\mu_i)^2 (D_t q_i)^2 = \dfrac{(\Delta t)^2\tau}{\delta^3 T^4} \iint_Q (\mu_i)^2 (\mathtt{t}^-\theta) \tau (\mathtt{t}^- s)^{-1} (D_t q_i)^2$
  $\leqslant \dfrac{(\Delta t)^2\tau^2}{\delta^4 T^6} \iint_Q (\mathtt{t}^- s)^{-1} (D_t q_i)^2 \leqslant \epsilon_0^2 \iint_Q (\mathtt{t}^- s)^{-1} (D_t q_i)^2.$
- $\widehat{Y}_{i5} = \tau^2 (\Delta t)^2 \iint_Q (\mathtt{t}^-\theta)^4 (D_t q_i)^2 = \tau^2 (\Delta t)^2 \iint_Q (\mathtt{t}^-\theta)^5 \tau (\mathtt{t}^- s)^{-1} (D_t q_i)^2$
  $\leqslant \dfrac{(\Delta t)^2\tau^3}{\delta^5 T^{10}} \iint_Q (\mathtt{t}^- s)^{-1} (D_t q_i)^2 \leqslant \epsilon_0^2 \iint_Q (\mathtt{t}^- s)^{-1} (D_t q_i)^2.$
- $\widehat{Y}_{i6} = \dfrac{(\Delta t)^4\tau^2}{\delta^6 T^8} \iint_Q (D_t q_i)^2 \leqslant \dfrac{(\Delta t)^4\tau^3}{\delta^7 T^{10}} \iint_Q (\mathtt{t}^- s)^{-1} (D_t q_i)^2 \leqslant \epsilon_0^4 \iint_Q (\mathtt{t}^- s)^{-1} (D_t q_i)^2.$

On combining these estimates, we get:

$$\sum_{j=1}^{6} \widehat{Y}_{ij} \leqslant \epsilon_0 \iint_Q (\mathtt{t}^- s)^{-1} (D_t q_i)^2. \tag{C.12}$$

Thus, the estimates (C.11) and (C.12) proves the claim. □

Finally, the claim reduces the estimate (C.8) to

$$\sum_{i=1}^{2} \left(\|Bq_i\|^2_{L^2_{\mathcal{D}}(Q)} + \tau^3 \iint_Q (\mathtt{t}^-\theta)^3 |\mathtt{t}^- q_i|^2 + \tau \iint_Q (\mathtt{t}^-\theta) |\partial_x (\mathtt{t}^- q_i)|^2 + \int_0^1 |q_i^M|^2 - \widehat{Z}_i \right)$$
$$\leqslant C_{\lambda_0} \sum_{i=1}^{2} \left(\left\| (\mathtt{t}^- r_i) \Big(D_t\varphi_i + \partial_x \big(\gamma_i\, \partial_x (\mathtt{t}^-\varphi) \big) \Big) \right\|^2_{L^2_{\mathcal{D}}(Q)} + \widehat{W}_i \right)$$
$$+ \epsilon_0\, C_{\lambda_0} \sum_{i=1}^{2} \left(\iint_Q (\mathtt{t}^- s)^3 \, |\mathtt{t}^- q_i|^2 + \iint_Q (\mathtt{t}^- s)^{-1} \, |D_t q_i|^2 \right), \tag{C.13}$$

for all $\tau \geqslant \tau_0(T^2+T)$, and $\frac{\tau^3\Delta t}{\delta^4 \min\{T^3,T^6\}} \leqslant \epsilon_0$. This completes the proof of Lemma 3.3. □

Appendix D. Proof of Lemma 3.4

Proof. To begin the proof, let us first club the boundary terms together as

$$-\sum_{i=1}^{2} \widehat{Z}_i := \sum_{k=1}^{7} J_k.$$

Now, we aim at simplifying these J_k's for $1 \leqslant k \leqslant 7$.

• Using the boundary condition (3.12), J_1 can be rewritten as

$$\begin{aligned} J_1 &= \sum_{i=1}^{2} \gamma_i(1) \fint_0^T \partial_x(\mathtt{t}^- q_i)(1) D_t q_i(1) \\ &= \fint_0^T \Big(-\alpha\,(\mathtt{t}^- q_1)(1) + \tau\lambda_0 \Big(\sum_{i=1}^{2} c_i\gamma_i(1) \Big) \mu_1(1)(\mathtt{t}^-\theta)\,(\mathtt{t}^- q_1)(1) \Big) (D_t q_1)(1). \end{aligned}$$

Next, using the product rule (A.2) we obtain

$$\begin{aligned} J_1 &= -\alpha \fint_0^T \Big[\frac{1}{2} D_t(q_1^2(1)) - \frac{\Delta t}{2} (D_t q_1(1))^2 \Big] \\ &\qquad + \tau\lambda_0 \Big(\sum_{i=1}^{2} c_i\gamma_i(1) \Big) \fint_0^T \mu_1(1)(\mathtt{t}^-\theta) \Big[\frac{1}{2} D_t(q_1^2)(1) - \frac{\Delta t}{2} (D_t q_1(1))^2 \Big] \\ &= -\frac{\alpha}{2} \fint_0^T D_t(q_1^2(1)) + \frac{\tau\lambda_0}{2} \Big(\sum_{i=1}^{2} c_i\gamma_i(1) \Big) \fint_0^T \mu_1(1)(\mathtt{t}^-\theta) D_t(q_1^2(1)) + E_1, \end{aligned}$$

where

$$E_1 = \Delta t \left\{ \frac{\alpha}{2} \fint_0^T (D_t q_1(1))^2\, dt - \frac{\tau\lambda_0}{2} \Big(\sum_{i=1}^{2} c_i\gamma_i(1) \Big) \fint_0^T \mu_1(1)(\mathtt{t}^-\theta)(D_t q_1(1))^2\, dt \right\}. \tag{D.1}$$

Further, expanding the integral in first term and using discrete integration by parts (A.6) in the second one, we get

$$\begin{aligned} J_1 &= -\frac{\alpha}{2} \Big(\big(q_1^M(1)\big)^2 - \big(q_1^0(1)\big)^2 \Big) \\ &\qquad + \frac{\tau\lambda_0}{2} \Big(\sum_{i=1}^{2} c_i\gamma_i(1) \Big) \mu_1(1) \left[-\fint_0^T (D_t\theta)(\mathtt{t}^+(q_1(1))^2 + \theta^M \big(q_1^M(1)\big)^2 - \theta^0 \big(q_1^0(1)\big)^2 \right] + E_1 \\ &= -\frac{\tau\lambda_0}{2} \Big(\sum_{i=1}^{2} c_i\gamma_i(1) \Big) \mu_1(1) \fint_0^T (D_t\theta)(\mathtt{t}^+ q_1(1))^2 - \overline{W}_1 + E_1, \end{aligned}$$

where

$$\overline{W}_1 = \frac{\alpha}{2} \Big(\big(q_1^M(1)\big)^2 - \big(q_1^0(1)\big)^2 \Big) - \frac{\tau\lambda_0}{2} \Big(\sum_{i=1}^{2} c_i\gamma_i(1) \Big) \Big\{ \theta^M \big(q_1^M(1)\big)^2 - \theta^0 \big(q_1^0(1)\big)^2 \Big\}.$$

Now using (B.1), and the relation $\mathtt{t}^+ q_1 = \mathtt{t}^- q_1 + \Delta t D_t q_1$, the first term can be further estimated as

$$\begin{aligned} &\left| \frac{\tau\lambda_0}{2} \Big(\sum_{i=1}^{2} c_i\gamma_i(1) \Big) \fint_0^T \mu_1(1)(D_t\theta)(\mathtt{t}^+ q_1(1))^2 \right| \\ &\qquad \leqslant C\tau \fint_0^T \mu_1(1) \left(T(\mathtt{t}^-\theta)^2 + C\frac{\Delta t}{\delta^3 T^4} \right) \Big[\big(\mathtt{t}^- q_1(1)\big)^2 + (\Delta t)^2 \big(D_t q_1(1)\big)^2 \Big]. \end{aligned}$$

Finally using the above estimate along with the fact $\theta^2 \leqslant 2T^2\theta^3$, we get

$$J_1 \geqslant -C\tau T^3 \left(\mu_1(1)\right)^3 \fint_0^T (\mathtt{t}^-\theta)^3 (\mathtt{t}^- q_1(1))^2 - \overline{W}_1 - \bar{E}_1,$$

where

$$\bar{E}_1 = \Delta t \left\{ -\frac{\alpha}{2} \int_0^T (D_t q_1(1))^2 \, dt + \frac{\tau\lambda_0}{2} \left(\sum_{i=1}^2 c_i \gamma_i(1) \right) \int_0^T \mu_1(1)(\mathtt{t}^- \theta) \big(D_t q_1(1)\big)^2 \, dt \right.$$
$$\left. + C\mu_1(1) \frac{\tau\lambda_0}{\delta^3 T^4} \int_0^T (\mathtt{t}^- q_1(1))^2 + C\tau\Delta t \lambda_0 \mu_1(1) \int_0^T \left(T(\mathtt{t}^-\theta)^2 + \frac{\Delta t}{\delta^3 T^4} \right) \big(D_t q_1(1)\big)^2 \right\}.$$

• Recall the expression of J_2

$$J_2 = -\tau\lambda_0 \sum_{i=1}^2 c_i \int_0^T (\mathtt{t}^-\theta)\mu_i(1)(\gamma_i(1)\partial_x(\mathtt{t}^- q_i)(1))^2 := J_{21} + J_{22}.$$

Now, using the boundary condition (3.12) J_{21} can be rewritten as

$$J_{21} = -\tau\lambda_0 \int_0^T (\mathtt{t}^-\theta)\mu_1(1) \left\{ -\alpha \, (\mathtt{t}^- q_1)(1) - \Big(\gamma_2(\mathtt{t}^- \partial_x q_2)\Big)(1) + \tau\lambda_0 \left(\sum_{i=1}^2 c_i \gamma_i(1) \right) \mu_1(1) \, (\mathtt{t}^-\theta) \, (\mathtt{t}^- q_1)(1) \right\}^2.$$

Using $(a+b+c)^2 \leqslant 3\big(a^2+b^2+c^2\big), (a+b)^2 \leqslant 2(a^2+b^2)$ and $\theta \leqslant T^4\theta^3$, we have

$$|J_{21}| \leqslant 3\tau\lambda_0 \, \alpha^2 T^4 \int_0^T (\mathtt{t}^-\theta)^3 \, (\mu_1(1))^3 \, (\mathtt{t}^- q_1(1))^2 + 3\tau\lambda_0 \int_0^T (\mathtt{t}^-\theta)\mu_1(1) |\gamma_2(1)\mathtt{t}^-(\partial_x q_2(1))|^2$$
$$+ 6\tau^3\lambda_0^3 \left(\sum_{i=1}^2 c_i^2 \gamma_i^2(1) \right) \int_0^T (\mu_1(1))^3 (\mathtt{t}^-\theta)^3 (\mathtt{t}^- q_1(1))^2$$
$$:= J_{21}^{(1)} + J_{21}^{(2)} + J_{21}^{(3)}.$$

Next using the fact $\mu_2(1) = \mu_1(1)$, we have

$$J_{22} = -\tau\lambda_0 c_2 \int_0^T (\mathtt{t}^-\theta)\mu_1(1) \Big(\gamma_2(1) \, (\partial_x \mathtt{t}^- q_2)(1)\Big)^2.$$

• The term J_3 can be written as

$$J_3 = \tau\lambda_0 \sum_{i=1}^2 c_i \int_0^T (\mathtt{t}^-\theta)\mu_i(0) \, \big(\gamma_i(0)\partial_x(\mathtt{t}^- q_i)(0)\big)^2 := J_{31} + J_{32}.$$

Then using the fact that $c_1 = 1$, we have

$$J_{31} = \tau\lambda_0 \, \mu_1(0) \int_0^T (\mathtt{t}^-\theta) \big(\gamma_1(0) \, \partial_x(\mathtt{t}^- q_1)(0)\big)^2 \geqslant 0.$$

The second term J_{22} can be further estimated as

$$|J_{32}| \leqslant C\tau\mu_2(0) \int_0^T (\mathtt{t}^-\theta) \left|\mathtt{t}^-(r_2(0)\right|^2 \left|\partial_x(\mathtt{t}^-\varphi_2)(0)\right|^2.$$

• Now, recall

$$J_4 = -2\tau\lambda_0^2 \sum_{i=1}^2 c_i^2 \, (\gamma_i(1))^2 \int_0^T \mu_i(1) \, (\mathtt{t}^-\theta) \, (\mathtt{t}^- q_i)(1)\mathtt{t}^-(\partial_x q_i)(1) := J_{41} + J_{42}.$$

Then, using the fact $\theta \leqslant T^2\theta^2$ we have

$$|J_{41}| \leqslant 2\tau\lambda_0^2 \, c_1^2 \, (\gamma_1(1))^2 \, T^2 \int_0^T (\mu_1(1))^2 \, (\mathtt{t}^-\theta)^2 |(\mathtt{t}^- q_1)(1)| \left|\mathtt{t}^-(\partial_x q_1)(1)\right|.$$

Let $\epsilon > 0$. Using Young's inequality and (3.12), we have

$$|J_{41}| \leqslant \epsilon\tau \int_0^T \mu_1(1)(\mathtt{t}^-\theta)|\gamma_1(1) \, \mathtt{t}^-(\partial_x q_1)(1)|^2 + C_\epsilon \tau T^4 \int_0^T (\mu_1(1))^3 (\mathtt{t}^-\theta)^3 |\mathtt{t}^- q_1(1)|^2$$
$$\leqslant C\epsilon\tau \int_0^T \mu_1(1)(\mathtt{t}^-\theta)|\gamma_2(1) \, \mathtt{t}^-(\partial_x q_2)(1)|^2 + C\epsilon\tau^3 \big(\mu_1(1)\big)^3 \int_0^T (\mathtt{t}^-\theta)^3 |\mathtt{t}^- q_1(1)|^2$$
$$+ C\tau T^4 (\mu_1(1))^3 \int_0^T (\mathtt{t}^-\theta)^3 |\mathtt{t}^- q_1(1)|^2$$
$$:= J_{41}^{(1)} + J_{41}^{(2)} + J_{41}^{(3)}.$$

Similarly, for $\epsilon > 0$ noting the fact $q_1^n(1) = q_2^n(1)$, the second term of term J_4 can be estimated as

$$|J_{42}| \leqslant \epsilon\tau \int_0^T \mu_2(1)(\mathtt{t}^-\theta)\left|\gamma_2(1)\mathtt{t}^-(\partial_x q_2)(1)\right|^2 + C_\epsilon \tau T^4 \int_0^T (\mu_2(1))^3(\mathtt{t}^-\theta)^3|\mathtt{t}^- q_1(1)|^2$$
$$:= J_{42}^{(1)} + J_{42}^{(2)}.$$

• Using the facts $\mu_1(1) = \mu_2(1)$, and $\mathtt{t}^- q_1(1) = \mathtt{t}^- q_2(1)$ the term J_5 follows to

$$J_5 = \tau^3\lambda_0^3\left(-c_1^3\left(\gamma_1(1)\right)^2 - c_2^3\left(\gamma_2(1)\right)^2\right)\int_0^T (\mathtt{t}^-\theta)^3(\mu_1(1))^3\left(\mathtt{t}^- q_1(1)\right)^2.$$

• Again using $\mu_1(1) = \mu_2(1), \eta_1(1) = \eta_2(1)$ and $\mathtt{t}^- q_1(1) = \mathtt{t}^- q_2(1)$, the sixth boundary term can be estimated as

$$|J_6| \leqslant C\tau^2\, T\, \mu_1(1)\, \eta_1(1) \int_0^T (\mathtt{t}^-\theta)^3\left(\mathtt{t}^- q_1(1)\right)^2.$$

• To estimate the last term J_7, we again use the boundary condition $\big(\mathtt{t}^- q_1\big)(1) = \big(\mathtt{t}^- q_2\big)(1)$ to get

$$|J_7| \leqslant C\tau^2\, \frac{\Delta t}{\delta^3 T^4} \int_0^T (\mathtt{t}^-\theta)\big(\mathtt{t}^- q_1(1)\big)^2.$$

We now combine the similar terms from above estimates together.

(a) Terms corresponding to the derivative $\partial_x\big(\mathtt{t}^- q_1(1)\big)$:

Looking at the leading terms, we have:

$$J_{22} - J_{21}^{(2)} = \Big(-c_2\tau\lambda_0 - 3\tau\lambda_0\Big)\mu_1(1)\int_0^T (\mathtt{t}^-\theta)\left(\gamma_2(1)\right)^2\left|\partial_x(\mathtt{t}^- q_2(1))\right|^2.$$

Using $-c_2 = \frac{6}{(1-\nu_0)} \geqslant 6$, we have

$$J_{22} - J_{21}^{(2)} \geqslant 3\,\tau\lambda_0\,\mu_1(1)\left(\gamma_2(1)\right)^2\int_0^T (\mathtt{t}^-\theta)\left|\partial_x(\mathtt{t}^- q_2(1))\right|^2.$$

Thus for sufficiently small $\epsilon > 0$, all the boundary terms corresponding to $\partial_x\big(\mathtt{t}^- q_1(1)\big)$ can be combined together to get

$$J_{22} - J_{21}^{(2)} - J_{41}^{(1)} - J_{42}^{(1)} \geqslant C\tau \int_0^T (\mathtt{t}^-\theta)\left|\partial_x(\mathtt{t}^- q_2(1))\right|^2. \tag{D.2}$$

(b) Terms corresponding to the derivative $\big(\mathtt{t}^- q_1(1)\big)$:

Again looking at the leading terms, we have

$$J_5 - J_{21}^{(3)} = M\tau^3\lambda_0^3\big(\mu_1(1)\big)^3\int_0^T (\mathtt{t}^-\theta)^3\left|\mathtt{t}^- q_1(1)\right|^2,$$

where

$$M = \Big(\left(-c_2^3 - 6c_2^2\right)\left(\gamma_2(1)\right)^2 - 7\big(\gamma_1(1)\big)^2\Big) = \left(\frac{216\,\nu_0}{(1-\nu_0)^3}\gamma_2^2(1) - 7\gamma_1^2(1)\right).$$

So, using (3.1) we have

$$J_5 - J_{21}^{(3)} = \tau^3\lambda_0^3\left(\mu_1(1)\right)^3\int_0^T (\mathtt{t}^-\theta)^3\left|\mathtt{t}^- q_1(1)\right|^2.$$

Now, lower order terms can be combined together as

$$J_1 - J_{21}^{(1)} - J_{41}^{(2)} - J_{41}^{(3)} - J_{42}^{(2)} + J_6 \geqslant -C\epsilon\tau^3\int_0^T (\mathtt{t}^-\theta)^3\left|\mathtt{t}^- q_1(1)\right|^2$$
$$- C\tau\max\{T^3, T^4\}\int_0^T (\mathtt{t}^-\theta)^3\left|\mathtt{t}^- q_1(1)\right|^2$$
$$- C\tau^2\int_0^T (\mathtt{t}^-\theta)^3\left|\mathtt{t}^- q_1(1)\right|^2.$$

Thus for sufficiently small and $\epsilon > 0$ and $\tau \geqslant C\max\{T^3, T^4\}$, we have

$$J_1 - J_{21}^{(1)} - J_{21}^{(3)} - J_{41}^{(2)} - J_{41}^{(3)} - J_{42}^{(2)} + J_5 + J_6 \geqslant C\tau^3\int_0^T (\mathtt{t}^-\theta)^3\left|\mathtt{t}^- q_1(1)\right|^2. \tag{D.3}$$

Finally, we can conclude the following

$$-\sum_{i=1}^{2}\widehat{Z}_i \geqslant -\overline{W}_1-\bar{E}_1+C\tau\int_0^T(\mathtt{t}^-\theta)\left|\partial_x(\mathtt{t}^-q_2(1))\right|^2+C\tau^3\int_0^T(\mathtt{t}^-\theta)^3\left|\mathtt{t}^-q_1(1)\right|^2$$
$$-C\tau\int_0^T(\mathtt{t}^-\theta)\left|\mathtt{t}^-(r_2(0)\right|^2\left|\partial_x(\mathtt{t}^-\varphi_2)(0)\right|^2-C\tau^2\,\frac{\Delta t}{\delta^3T^4}\int_0^T(\mathtt{t}^-\theta)\big(\mathtt{t}^-q_1(1)\big)^2.$$

This completes the proof. □

Appendix E. A technical lemma

Lemma E.1. *Let $f\in L^2(0,T;E)$. Define for $M\in\mathbb{N}$,*

$$f_M(t,x)=\sum_{n=0}^{M-1}1_{[t_n,t_{n+1})}f^{n+1}(x),$$

where $f^{n+1}(x)=\dfrac{1}{\Delta t}\displaystyle\int_{t_n}^{t_{n+1}}f(t,x)\,dt$, $n\in\{0,1,\dots,M-1\}$. Then,

(a) $\displaystyle\int_0^T|f_M|^2=\sum_{n=0}^{M-1}\Delta t|f^{n+1}|^2\leqslant\int_0^T|f|^2.$

(b) $\left|\displaystyle\int_0^T\langle(f_M-f)(t,\cdot),\psi(t,\cdot)\rangle_E\,dt\right|\leqslant C\sqrt{\Delta t}\,\|\psi\|_{L^2(0,T;L^2(0,1))},\quad\forall\,\psi\in L^2(0,T;E),$

which implies

$$f_M\rightharpoonup f\ \text{ in } L^2(0,T;E).$$

Proof. Using the expression of f^{n+1} and Hölder's inequality, we have

$$\left|f^{n+1}\right|^2\leqslant\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}|f|^2,$$

which on summing over $n\in[\![0,M-1]\!]$ gives (a).

Let us now prove second part of the lemma. Note that for $\psi=1_{(a,b)}(t)\,\psi^x(x)$, where $\psi^x\in E$, $a\in(t_p,t_{p+1})$ and $b\in(t_q,t_{q+1})$ for some $p,q\in[\![0,M-1]\!]$ with $p\neq q$, we have

$$\begin{aligned}
&\int_0^T\langle(f_M-f)(t,\cdot),\psi(t,\cdot)\rangle_E\,dt\\
&\quad=\sum_{i=1}^{2}\int_a^b\int_0^1(f_M-f)_i(t,x)\,\psi_i^x(x)\,dx\,dt\\
&\quad=\sum_{i=1}^{2}\int_a^{t_{p+1}}\int_0^1(f_M-f)_i(t,x)\,\psi_i^x(x)\,dx\,dt+\sum_{i=1}^{2}\int_{t_{p+1}}^{t_q}\int_0^1(f_M-f)_i(t,x)\,\psi_i^x(x)\,dx\,dt\\
&\qquad\qquad+\sum_{i=1}^{2}\int_{t_q}^b\int_0^1(f_M-f)_i(t,x)\,\psi_i^x(x)\,dx\,dt.
\end{aligned}$$

Note that the second integral term is 0. Further, using the definition of f_M, we have

$$\begin{aligned}
\int_0^T\langle(f_M-f)(t,\cdot),\psi(t,\cdot)\rangle_E\,dt&=\sum_{i=1}^{2}\int_0^1\int_a^{t_{p+1}}\left(\frac{1}{\Delta t}\int_{t_p}^{t_{p+1}}f_i(s,x)\,ds-f_i(t,x)\right)\psi_i^x(x)\,dx\,dt\\
&\quad+\sum_{i=1}^{2}\int_0^1\int_{t_q}^b\left(\frac{1}{\Delta t}\int_{t_q}^{t_{q+1}}f_i(s,x)\,ds-f_i(t,x)\right)\psi_i^x(x)\,dx\,dt.
\end{aligned}$$

Next, we use Hölder's inequality in each of the four terms to obtain

$$\begin{aligned}
\int_0^T\langle(f_M-f)(t,\cdot),\psi(t,\cdot)\rangle_E\,dt&\leqslant\sum_{i=1}^{2}\int_0^1\sqrt{\Delta t}\Big(\|f_i(\cdot,x)\|_{L^2(t_p,t_{p+1})}+\|f_i(\cdot,x)\|_{L^2(t_q,t_{q+1})}\Big)\psi^x(x)\,dx\\
&\leqslant C\sqrt{\Delta t}\,\|\psi\|_{L^2(0,T;L^2(0,1))}.
\end{aligned}$$

The above relation is true even for simple functions of the form $\psi = \sum_{1}^{n_0} 1_{(a_i,b_i)} c_i \psi^x(x)$. Finally, using the fact that simple functions are dense in L^2 space, we can conclude

$$\left| \int_0^T \langle (f_M - f)(t,\cdot), \psi(t,\cdot) \rangle_E \, dt \right| \leqslant C \sqrt{\Delta t} \, \|\psi\|_{L^2(0,T;L^2(0,1))}, \quad \forall \, \psi \in L^2(0,T;E).$$

Hence, $f_M \rightharpoonup f$ in $L^2(0,T;E)$, which completes the proof. □

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