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► **To cite this version:**

Marco Longinetti, Simone Naldi, Adriana Venturi. R-hulloid of the vertices of a tetrahedron. 2024.
hal-04597228

HAL Id: hal-04597228

<https://hal.science/hal-04597228>

Preprint submitted on 2 Jun 2024

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R-hulloid of the vertices of a tetrahedron

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June 2, 2024

Abstract

The R -hulloid, in the Euclidean space \mathbb{R}^3 , of the set of vertices V of a tetrahedron T is the minimal closed set containing V such that its complement is union of open balls of radius R . When R is greater than the circumradius of T , the boundary of the R -hulloid consists of V and possibly of four spherical subsets of well defined spheres of radius R through the vertices of T . The existence of a value R^* such that these subsets collapse into one point $O^* \notin V$ is investigated; in such case O^* is in the interior of T and belongs to four spheres of radius R^* , each one through three vertices of T and not containing the fourth one. As a consequence, the range of ρ such that V is a ρ -body is described completely. This work generalizes to three dimensions previous results, proved in the planar case and related to the three circles Johnson's Theorem.

Keywords. generalized convexity, tetrahedra, support cones.

Mathematics Subject Classification (MSC2020). 52A01, 52A30.

1 Introduction

Let R be a positive real number. The R -hulloid of a closed set E in the Euclidean space \mathbb{R}^d , denoted by $co_R(E)$ in [5], is the minimal closed set containing E such that its complement $(co_R(E))^c$ is union of open balls of radius R . If the complement of a body (non empty closed set) E , is union of open balls B of radius R , then E is called a R -body. The family of R -bodies is closed with respect to intersection, see [10], thus $co_R(E)$ always exists. If E is a R -body, $co_R(E) = E$.

The properties of the family of R -bodies were introduced by Perkal [10], then Cuevas, Fraiman and Pateiro-López in [2], more recently by Longinetti, Manselli and Venturi in [5].

In Proposition 2.8 the boundary of $co_\rho(E)$ is described by the ρ -supporting spheres of E : a ρ -supporting sphere S of a body E , at a point $a \in \partial E$, is the boundary $S = \partial B$ of an open ball B (of radius ρ) not intersecting E and with $a \in S$; B is called a ρ -supporting ball of E .

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In this paper we focus on discrete sets of points: $E = V$, the set of vertices of a simplex of \mathbb{R}^d . In the plane, V is the set of vertices of a triangle T and a complete description of $co_\rho(V)$, for all $\rho > 0$, was proved in [5] as elementary consequence of the three circles theorem by Johnson [4], here recalled in Proposition 2.7. This paper shows new results in three dimensions, as an extension of the planar case, proved in [5].

In three dimensions, for the vertices V of a tetrahedron T , the description of $co_\rho(V)$ looks similar: an explicit representation of $co_\rho(V)$, with $\rho > r(V)$, the circumradius of T , is obtained in Theorem 3.5 using special ρ -supporting spheres of V , see Definition 2.4. Since Johnson's Theorem cannot be applied, the proof relies on non-trivial geometric arguments in three dimensions. The supremum of ρ for which V is a ρ -body is considered; this value is the infimum of ρ such that $co_\rho(V)$ has not empty interior. We denote by $r_L(V)$ such critical value, see Definition 3.8. In dimension greater than two, $r_L(V)$ can be closely greater than $r(V)$. For a regular simplex $T = co(V)$, it is proved (see [5, Theorem 5.6]) in dimension $d \geq 2$ that $R^* := r_L(V) = \frac{d}{2}r(V)$; in such case, four R^* -supporting spheres of V must intersect in the center of T .

In §3.1 the geometry of the configurations of four spheres ρ -supporting the vertices of a generic tetrahedron T is investigated. In Theorem 3.11 it is proved that, if $r_L(V) > r(V)$, then there exists $R^* > r(V)$ such that $co_{R^*}(V)$ reduces to $V \cup \{O^*\}$, where O^* is the intersection of four special spheres R^* -supporting V . As consequence, in Corollary 3.12, a complete description of $co_\rho(V)$ is given for all ρ . The arguments here used for the definition of R^* and of O^* might be carried out in any dimension.

In §3.2 it is considered the configuration of four spheres S_j in \mathbb{R}^3 , of same radius, each S_j through three vertices $V \setminus \{v_j\}$; all S_j intersect at a single point O^L . If O^L does not lie on the circumsphere of T it is called a *four crossing point* of T ; O^L might not coincide with special central points of T , see Remark 3.15.

This configuration of spheres can be related to unit-sphere systems in \mathbb{R}^3 , see [9].

In §4, Theorem 4.3 proves an inclusive relationship between ρ -supporting spheres centered outside a dihedron. This inclusion is called a *ρ -supporting inclusion principle* for a dihedron.

In [7], using methods from symbolic computation, the existence of values for R^L and the related four-crossing points O^L are obtained for the family of triangular pyramids.

Sharp bounds for the value of R^* in the family of triangular pyramids and in the family of isosceles tetrahedra will be given in [8].

2 Definitions and preliminary on R-bodies

\mathbb{R}^d is the euclidean space of dimension $d \geq 2$. The elements of \mathbb{R}^d are called vectors. The origin of \mathbb{R}^d is denoted by $o = (0, \dots, 0)$. $B(x, r)$ is the open ball of center $x \in \mathbb{R}^d$ and radius $r > 0$, its boundary is the spherical surface $\partial B(x, r)$; the unit sphere $\partial B(o, 1) \subset \mathbb{R}^d$ is often denoted by S^{d-1} . In case R is a fixed positive real number, let us simply denote by $B(x)$ the open ball of radius R and center x , and by B any open ball of radius R . A body is a closed non empty set. The closure of a set A is $cl(A) = \overline{A}$, its interior is $int(A)$ and its complementary set A^c . The usual scalar product between vectors $u, v \in \mathbb{R}^d$ will be denoted by $\langle u, v \rangle$. A closed cone C , with

vertex o , is a subset of \mathbb{R}^d with the following property: when $x \in C$, then $\lambda x \in C \forall \lambda \geq 0$. C is a closed pointed cone if $C \cap (-C) = \{o\}$. The apex set of a closed cone C is $C \cap (-C)$ and it contains o .

Let A be a body and $q \in A$. The *tangent cone* of A at q is defined as:

$$\text{Tan}(A, q) = \{v \in \mathbb{R}^d : \forall \varepsilon > 0, \exists x \in A \cap B_\varepsilon(q), \exists r > 0 \text{ s.t. } |r(x - q) - v| < \varepsilon\}.$$

Let us recall that if $\text{Tan}(A, q) \neq \{o\}$ then

$$S^{d-1} \cap \text{Tan}(A, q) = \bigcap_{\varepsilon > 0} \text{cl} \left\{ \frac{x - q}{|x - q|} : x \in A \cap B(q, \varepsilon), x \neq q \right\}.$$

The *normal cone* at q to A is the non empty closed convex cone, given by:

$$\text{Nor}(A, q) = \{u \in \mathbb{R}^d : \langle u, v \rangle \leq 0 \quad \forall v \in \text{Tan}(A, q)\}.$$

The *dual cone* of a cone K is

$$K^* = \{y \in \mathbb{R}^d : \langle y, x \rangle \geq 0 \quad \forall x \in K\}.$$

Thus

$$\text{Nor}(A, q) = -\{\text{Tan}(A, q)\}^*.$$

Definition 2.1. A body $A \subset \mathbb{R}^d$ will be called a *R-body* if $\forall y \in A^c$, there exists an open ball $B \subset \mathbb{R}^d$ satisfying $y \in B \subset A^c$, that is:

$$A = \bigcap \{B^c : B \cap A = \emptyset\}.$$

Definition 2.2. Let $E \subset \mathbb{R}^d$ be a body and R be a positive real number. The set

$$co_R(E) := \bigcap \{B^c : B \cap E = \emptyset\}$$

will be called the *R-hulloid* of E .

If the family of all open balls not intersecting E is empty, let us assume $co_R(E) = \mathbb{R}^d$. The *R-hulloid* always exists and $co_R(E)$ is the minimal *R-body* containing E . Clearly every convex body E is a *R-body* (for all positive R), in such case $E = co(E) = co_R(E)$ for every R . A body A is a *R-body* iff $A = co_R(A)$. Moreover, see [10, formula (8)]:

$$(1) \quad R_1 \leq R_2 \Rightarrow co_{R_1}(E) \subseteq co_{R_2}(E).$$

Proposition 2.3. [5, Thm 3.10] The closed subsets of a sphere of radius r are *R-bodies* for all $R \leq r$.

Definition 2.4. ([6]) Let A be a closed set in \mathbb{R}^d and $a \in \partial A$. Let $v \in \partial B(o, 1) = S^{d-1}$. We say that the ball $B(a + Rv)$ (of radius R) is *R-supporting* A at a if

$$A \subset (B(a + Rv))^c;$$

let us notice that $a \in \partial B(a + rv)$, called a ρ -supporting sphere.

Definition 2.5. Let $\rho > 0$ and $S = \partial B(o, \rho)$. There is a one-to-one correspondence between closed cones $K \subset \mathbb{R}^d$ and closed subsets $\mathcal{K} \subset S \subset \mathbb{R}^d$: for any closed cone K of \mathbb{R}^d , let $\mathcal{K} = K \cap S$; conversely, for any closed set $\mathcal{K} \subseteq S$, let $K = \{\lambda v : v \in \mathcal{K}, \lambda \geq 0\}$ be the related cone in \mathbb{R}^d .

For any $\mathcal{K} \subset S$ let us define its **spherical convex hull** as

$$co_{sph}^S(\mathcal{K}) := co(K) \cap S.$$

Last definitions have similar extensions for a sphere S centered at a general point $p \in \mathbb{R}^d$, not necessarily at the origin o and for cones with vertex $p \neq o$.

2.1 Supporting balls and R -cones

Let us look at special sets $E = V$, where V is the set of the vertices of a simplex $T = \text{co}(V)$ in \mathbb{R}^d .

The following fact is well known: there exists a unique open ball $B(c(V), r(V))$ such that $V \subset \partial B$; it is called the circumball to $T = \text{co}(V)$, with radius $r(V)$ the circumradius of T and circumcenter $c(V)$, denoted below simply by c . Let us recall that its closure $D(V) := \overline{B(c, r(V))}$ does not coincide (in general) with the closed ball of minimum radius containing V .

The following proposition is a consequence of Proposition 2.3, with $r = r(V)$ and $R = \rho$:

Proposition 2.6. *Let $T = \text{co}(V)$ be a simplex in \mathbb{R}^d . Then,*

$$\text{co}_\rho(V) = V \quad \text{for all } \rho \leq r(V).$$

As consequence of Johnson's Theorem [4] (cf. fig. 1, left), for the set V of vertices of a simplex in the planar case, for $\rho > r(V)$, the set $\text{co}_\rho(V)$ is well understood (see fig. 1, right).

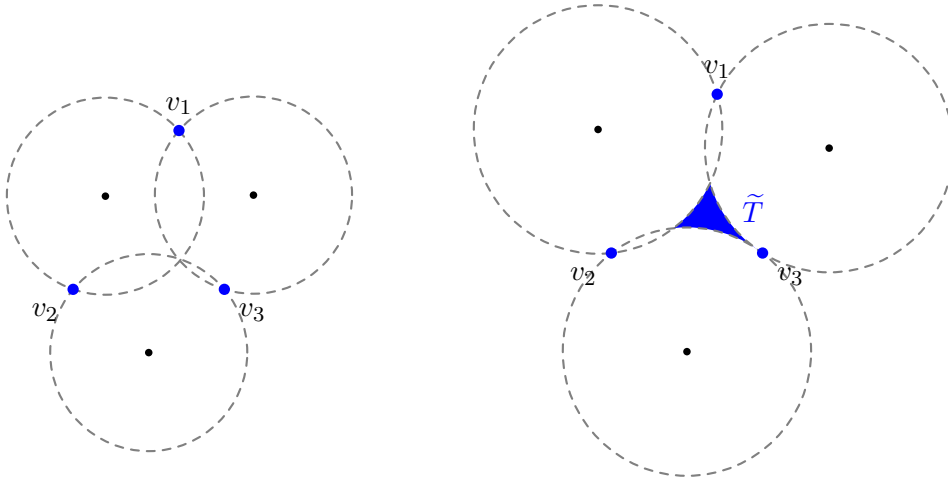


Figure 1: ρ -hulloid of three points in \mathbb{R}^2 , $\rho = r(V)$ (left) and $\rho > r(V)$ (right).

Proposition 2.7. [5, Theorem 4.2] *Let $V \subset \mathbb{R}^2$ be the set of vertices of a triangle T with circumradius $r(V)$. If $\rho > r(V)$, then*

$$\text{co}_\rho(V) = V \cup \tilde{T},$$

where $\tilde{T} \subset T$ is the curvilinear triangle bordered by three arcs of circles of radius ρ , each one through two vertices of T . If T is a right-angled or obtuse-angled then the vertex of the major angle of T is also a vertex of \tilde{T} .

The following properties of ρ -supporting balls $B(p, \rho)$ to the ρ -hulloid of a body $E \subset \mathbb{R}^3$ are a direct consequence of [6, Theorem 3.11 and Remark 3.12].

Proposition 2.8. *Let E be a body in \mathbb{R}^3 . Let $x \in \partial co_\rho(E) \setminus E$. Then there exists $p \in \mathbb{R}^3$ and $\rho > 0$ such that $B(p, \rho)$ is ρ -supporting $co_\rho(E)$ at x . Let us assume without loss of generality that $p = o$ and let $S = \partial B(o, \rho)$. Let H be the cone with vertex o related to $\mathcal{H} = \partial E \cap S$ and let $C = co(H)$. Then, there exist at least two points in \mathcal{H} and:*

- i) *if C is a pointed cone, then there exist x_1, \dots, x_s distinct points of \mathcal{H} , $2 \leq s \leq 3$, such that*

$$x \in co_{sph}^S(\{x_1, \dots, x_s\});$$

- ii) *if C is not a pointed cone, named $r = \dim(C \cap (-C))$, then it follows:*

- a) *if $r = 1$ there exist two opposite points $x_1, x_2 \in \mathcal{H}$, that is $|x_1 - x_2| = \rho$;*
b) *if $r = 2$ there exist three points $x_1, x_2, x_3 \in \mathcal{H}$, such that o is coplanar with them;*
c) *if $r = 3$ there exist four points $x_1, x_2, x_3, x_4 \in \mathcal{H}$ such that*

$$o \in \text{int}(co(\{x_1, x_2, x_3, x_4\})).$$

Let us recall a family of R -bodies, called R -cones, see [5, §5]. The R -cones are a generalization of convex cones, as the R -bodies are a generalization of convex sets.

Definition 2.9. *Let \mathcal{K} be a body in S^{d-1} . A R -cone with vertex o (more simply a R -cone) is the R -body:*

$$C_{\mathcal{K}} := \bigcap_{v \in \mathcal{K}} (B(Rv, R))^c.$$

Let $x \in \mathbb{R}^d$, a R -cone with vertex x is the R -body:

$$C_{\mathcal{K}}^x := x + C_{\mathcal{K}} = \bigcap_{v \in \mathcal{K}} (B(x + Rv, R))^c.$$

Proposition 2.10. [5, Theorem 5.6] *Let \mathcal{K} be a body in S^{d-1} and K its related cone in \mathbb{R}^d . Let $\mathcal{N} = \text{Nor}(C_{\mathcal{K}}) \cap S^{d-1}$. Then*

- a) $\text{Tan}(C_{\mathcal{K}}) = -K^*$;
b) $\mathcal{N} = co_{sph}(\mathcal{K})$;
c) $\text{Tan}(C_{\mathcal{K}}) \setminus \{o\} \subset \text{int}(C_{\mathcal{K}})$.

Lemma 2.11. *Let $B_j, j = 1, 2, 3 \subset \mathbb{R}^3$ be open balls of radius ρ , not necessarily distinct. Let $p^* \in \cap_{j=1}^3 \partial B_j$; then for every neighbourhood \mathcal{U} of p^* :*

$$\text{int}(\mathcal{U} \setminus \cup_{j=1}^3 B_j) \neq \emptyset.$$

Proof. Let u_j be the unit inner normal at p^* to B_j . Let $\mathcal{K} = \{u_1, u_2, u_3\} \subset S^2$. The closed set $\cap_{j=1}^3 (B_j)^c$ is called ρ -cone with vertex p^* in [6]. The ρ -cone $C_{\mathcal{K}} = \cap_{j=1}^3 (B_j)^c - p^*$ with vertex at $\{o\}$ is a generalization of the usual convex cone. Let $K = \{\lambda u_i, u_i \in \mathcal{K}, \lambda \geq 0\}$ be the related cone to \mathcal{K} . By a) and c) of Proposition 2.10

$$(2) \quad -K^* - \{o\} = \text{Tan}(C_{\mathcal{K}}) - \{o\} \subset \text{int}(C_{\mathcal{K}}).$$

Where $\text{Tan}(C_{\mathcal{K}})$ is the tangent cone to the ρ -cone $C_{\mathcal{K}}$ at o . Since \mathcal{K} is contained in a closed hemisphere, the dual cone K^* is not empty and the tangent cone is non empty too. Then, by inclusion (2), for every $v \neq o$ in the tangent cone: $v \in \text{int}(C_{\mathcal{K}})$. Let $y = p^* + v$, then $y \in \text{int}((\cup_{j=1}^3 B_j)^c) \neq \emptyset$ and

$$(3) \quad \text{dist}(y, \cup_{j=1}^3 B_j) > 0;$$

the thesis follows. \square

Lemma 2.12. *Let $B_j, j = 1, 2, 3 \subset \mathbb{R}^3$ be open balls of radius ρ . Let $p^* \in \cap_{j=1}^3 \partial B_j$; let $C_{\mathcal{K}}^{p^*} = \cap_{j=1}^3 (B_j)^c$ be the related ρ -cone with vertex p^* and let \mathcal{T} be a closed convex cone with vertex p^* . Then*

a) *if $\text{Tan}(\mathcal{T}, p^*) \cap \text{Tan}(C_{\mathcal{K}}^{p^*}, p^*) \setminus \{o\} \neq \emptyset$ then, for every neighbourhood \mathcal{U} of p^* , there exists a point $y \in \mathcal{T} \cap \mathcal{U}$ such that*

$$(4) \quad \text{dist}(y, (C_{\mathcal{K}}^{p^*})^c) > 0;$$

b) *if $\text{Tan}(\mathcal{T}, p^*) \cap \text{Tan}(C_{\mathcal{K}}^{p^*}, p^*) = \{o\}$, then p^* is an isolated point of $\mathcal{T} \cap C_{\mathcal{K}}^{p^*}$.*

Proof. It can be assumed that $p^* = o$. In case a), let r be a ray starting from o in the cone $\mathcal{T} \cap C_{\mathcal{K}}$. From (2) for every $y \in r, y \neq o$, (3) holds. Then for every neighbourhood \mathcal{U} of o and for every $y \in r \cap \mathcal{U}, y \neq o$ formula (4) holds.

In case b), by contradiction let $y_n \rightarrow o$, a sequence of points $y_n \in \mathcal{T} \cap C_{\mathcal{K}}, y_n \neq o$. Then up to a subsequence $v = \lim_n y_n/|y_n| \neq o$ belongs both to $\text{Tan}(\mathcal{T}) = \mathcal{T}$ and to $\text{Tan}(C_{\mathcal{K}})$; this is a contradiction. \square

3 R -huloids of vertices of a tetrahedron

Let $V = \{v_1, v_2, v_3, v_4\} \subset \mathbb{R}^3$ be the set of vertices of a simplex $T = \text{co} V \subset \mathbb{R}^3$. We look for a description of $\text{co}_{\rho}(V)$: formula (5) of Theorem 3.5. The formula (5) is a generalization of that one given in Proposition 2.6 for the planar case. Since Johnson's Theorem does not hold in \mathbb{R}^3 , the approach is different.

As in the planar case, by Proposition 2.6, $\text{co}_{\rho}(V) = V$ for $\rho \leq r(V)$.

From now on, let us assume $\rho > r(V)$ and let us denote by $c = c(V)$ the circumcenter of T , so that $B(c, r(V))$ is the circumball of T .

Let us denote by $V_i = V \setminus \{v_i\}$, for $1 \leq i \leq 4$, and by $T_i = \text{co}(V_i)$ the facets of T , by $H_i = \text{span}(T_i)$ and let c_i, r_i be the circumcenter and the circumradius respectively of T_i in H_i .

Definition 3.1. *The simplex $T = \text{co}(V)$ will be called **well-centered** if $c(V) \in \text{int}(T)$ (in dimension two, the well centered simplexes are the acute triangles).*

Let $D(V)$ be the closed circumball of T . The set $D(V) \cap H_i$ is a closed circle of radius r_i and center c_i . Let Ω_i be its relative interior.

As the vertices in V are affinely independent points, then v_i does not belong to $H_i, 1 \leq i \leq 4$. Let H_{i+} be the open half space bounded by H_i containing v_i . Let H_i^+ the closure of H_{i+} ; let l_i be the line orthogonal to H_i containing c_i .

Lemma 3.2. *For every $\rho > r(V)$ and for any face T_i of $T = \text{co}(V)$ and line l_i orthogonal to H_i at the circumcenter c_i of T_i , there exists a unique point $o_i(\rho)$ such that:*

$$V_i \subset \partial B(o_i(\rho), \rho) \cap H_i$$

and

$$v_i \notin B(o_i(\rho), \rho).$$

Moreover

$$\rho \mapsto \mathbf{H}_i^+ \cap (B(o_i(\rho), \rho))^c$$

is a continuous map of strictly nested ρ -bodies for $\rho > r(V)$.

Proof. In the sequel, let us denote, for simplicity, the balls

$$B_i(\rho) := B(o_i(\rho), \rho), \quad \text{for } \rho > r(V)$$

and

$$B(V) = \text{int}(D(V)) = B(c, r(V)), \quad B_i^+(r(V)) := \lim_{\rho \rightarrow r(V)^+} B_i(\rho).$$

Let us notice that two different cases can occur:

- a) T is well-centered. In this case $\forall i, c \in H_{i+}$; $o_i(\rho)$ lies on the half line l_i^* of l_i , with origin in \tilde{c}_i , the symmetric point of c with respect to H_i , not intersecting H_i^+ . The half line l_i^* is oriented as the outwards unit vectors n_i to the facet T_i of T . In this case $B(V) \neq B_i^+(r(V)) = B(\tilde{c}_i, r(V))$ and

$$\lim_{\rho \rightarrow r(V)^+} \mathbf{H}_i^+ \cap (B(o_i(\rho), \rho))^c = \mathbf{H}_i^+ \cap (B_i^+(r(V)))^c;$$

- b) T is not well-centered: then, there are indexes i such that the plane H_i separate c from v_i ; for these indexes, $o_i(\rho)$ lies on the half line l_i^* of l_i starting from c and not intersecting H_i^+ ; the half line l_i^* is again oriented as the outwards unit vectors n_i to the facet T_i of T .

For such indexes i it turns out that $B(V) = B_i^+(r(V))$, then

$$\lim_{\rho \rightarrow r(V)^+} \mathbf{H}_i^+ \cap (B(o_i(\rho), \rho))^c = \mathbf{H}_i^+ \cap (B(V))^c.$$

The proof of Lemma 3.2 follows from elementary geometric arguments. \square

Remark 3.3.

$$\bigcap_i \left(\mathbf{H}_i^+ \cap (B(c, r(V)))^c \right) = \left(\bigcap_i \mathbf{H}_i^+ \right) \cap (B(c, r(V)))^c = T \cap (B(c, r(V)))^c = V.$$

Lemma 3.4. *Let V be the set of the vertices of a tetrahedron $T = \text{co}(V)$. If $\rho > r(V)$, then every open ball $B(z, \rho)$, not intersecting V and intersecting $\text{co}(V)$, has the following properties:*

- b) $\text{dist}(z, c) > \rho - r(V)$;
c) the radical plane H through $\partial B(c, r(V)) \cap \partial B(z, \rho)$ separates z from T ;

d) $\text{dist}(z, T) > (\rho - r(V))/2$.

Proof. Since $\text{int}(T) \subset B(c, r(V))$, the spheres $\partial B(c, r(V))$ and $\partial B(z, \rho)$ have non empty intersection; thus they have a real radical plane H . Since every vertex v_i belongs to $\partial B(c, r(V)) \setminus B(z, \rho)$ then

$$\text{dist}(z, c) > \text{dist}(z, v_i) - \text{dist}(v_i, c) > \rho - r(V).$$

Moreover H separates $D(V) \setminus B(z, \rho)$ from z , then it separates V from z , this proves c). Since $\rho > r(V)$, then H has distance from c less than from z , therefore

$$\text{dist}(z, T) > \text{dist}(z, H) > \text{dist}(z, (c + z)/2) > (\rho - r(V))/2$$

and d) is proved, \square

Theorem 3.5. *Let $T = \text{co}(V)$ be a simplex in \mathbb{R}^3 and let $\rho > r(V)$. Then*

$$(5) \quad \text{co}_\rho(V) = \text{co}(V) \setminus \left(\bigcup_{i=1}^4 B_i(\rho) \right)$$

and $r(V) < \rho \mapsto \text{co}_\rho(V)$ is a not decreasing family of closed sets, with

$$(6) \quad \lim_{\rho \rightarrow +\infty} \text{co}_\rho(V) = V \cup \text{int}(T).$$

Proof. First let us notice that

$$G := \text{co}(V) \setminus \left(\bigcup_{i=1}^4 B_i(\rho) \right) = \bigcap_i H_i^+ \bigcap_i B_i^c(\rho).$$

Then G is a ρ -body as intersection of ρ -bodies and $G \supset V$. Then

$$F := \text{co}_\rho(V) \subset G.$$

Let us prove now that $G \subseteq F$. This is equivalent to prove that

$$(7) \quad F^c \subseteq G^c.$$

First $V \subset F$, by definition of ρ -hulloid. Let $y \notin F = \text{co}_\rho(V)$, then $y \notin V$. If $y \notin T$, then it is obvious that $y \notin G$; if $y \in \partial T \setminus V$, there exists a face T_i such that $y \in T_i \setminus V \subset B_i(\rho)$. Then, $y \in G^c$.

Let $y \in \text{int}(T)$ and $y \notin F = \text{co}_\rho(V)$; then, by definition of $\text{co}_\rho(V)$, there exists an open ball $\tilde{B}(z, \rho) \ni y, \tilde{B}(z, \rho) \cap V = \emptyset$. Let us consider the family

$$\Gamma_y = \{\tilde{B}(x, \rho) : \tilde{B}(x, \rho) \ni y, \tilde{B}(x, \rho) \cap V = \emptyset\}.$$

Since Γ_y is not empty let

$$r_y = \inf_z \{|z - y| : y \in B(z, \rho) \in \Gamma_y\}.$$

Obviously $r_y < \rho$. From d) of Lemma 3.4, $r_y > 0$; by continuity argument, r_y is a minimum and there exists $B(z^*, \rho) \in \Gamma_y$ such that

$$|z^* - y| = r_y.$$

Claim A: $\partial B(z^*, \rho) \cap V \neq \emptyset$.

By contradiction if $\partial B(z^*, \rho) \cap V = \emptyset$, moving z^* towards y , it is possible to get another ball $\tilde{B}(\tilde{x}, \rho) \in \Gamma_y$ such that

$$|\tilde{x} - y| < |z^* - y| = r_y.$$

Impossible, then Claim A is proved and there exists $v_3 \in \partial B(z^*, \rho) \cap V$. With a similar argument there exists at least another vertex $v_2 \neq v_3$, $v_2 \in \partial B(z^*, \rho)$, otherwise a rotation of $B(z^*, \rho)$ around v_3 towards y , decreases the value of r_y , impossible.

Claim B: $\{v_1, v_2, v_3\} \subset \partial B(z^*, \rho)$.

First let us notice that, since $y \in \text{int}(T)$, it does not belong to the line through v_2, v_3 ; moreover $|z^* - v_3| = |z^* - v_2|$. Let \mathcal{C} be the circle with center $(v_2 + v_3)/2$, through z^* , in the plane orthogonal to the axis v_2v_3 . Let Λ be the plane through y, v_2, v_3 and let z^+ be the closest point of $\mathcal{C} \cap \Lambda$ to y .

The function $z \rightarrow |z - y|$, such that $y \in B(z, \rho) \in \Gamma_y$, has minimum at z^* , then its restriction to \mathcal{C} has the same minimum value at z^* . Since the function $\mathcal{C} \ni z \rightarrow |z - y|$, with no restriction on z , attains its minimum on z^+ and decreases its value moving z on \mathcal{C} towards z^* ; then two cases need to be considered:

- a) $z^* = z^+$;
- b) there exists $v_1 \in \partial B(z^*, \rho) \cap V$.

Case a) cannot hold. Indeed, by item c) of Lemma 3.4, the radical plane of the two spheres $\partial B(z^+, \rho)$, $\partial B(c, r(V))$ (containing v_2, v_3) separates z^+ from y .

Then item b) holds: in such case $B(z^*, \rho)$ cannot rotate towards z^+ , to get another $B(z, \rho) \in \Gamma_y$, $B(z, \rho) \ni y$, to decrease further the value of r_y . Then y belongs to the open ball $B(z^*, \rho)$, which coincides with $B_4(\rho)$ of the family of balls defined in Lemma 3.2. Therefore $y \in G^c$. This proves (7).

The monotone property of $co_\rho(V)$ follows from (1). Since, see [10],

$$\bigcup_{\rho > r(V)} \text{int}(co_\rho(V)) = \text{int}(T),$$

then the limit property (6) holds. \square

Corollary 3.6. *If T is not well-centered, then*

$$(8) \quad \lim_{\rho \rightarrow r(V)^+} co_\rho(V) = V;$$

if T is well-centered, then

$$(9) \quad \lim_{\rho \rightarrow r(V)^+} co(V) \setminus \bigcup_{i=1}^4 B_i(\rho) = co(V) \setminus \bigcup_{i=1}^4 B_i^+(\rho(V)).$$

Proof. In case T is not well-centered, then there exists i such that H_i separates c from v_i , then

$$\lim_{\rho \rightarrow r(V)^+} B_i(\rho) = B(c, r(V)) \supset co(V) \setminus V$$

and (8) holds; the limit (9) follows from continuity and monotone properties of the map $\rho \mapsto co_\rho(V)$. \square

3.1 Critical configurations of supporting balls

Let V be the set of the vertices of a tetrahedron T . In this section we describe particular configurations of ρ -hulloid of V , that is when the ρ -supporting spheres intersect all in a single point which belongs to $\text{int}(T)$.

Definition 3.7. *Let $\rho > 0$. The ρ -hulloid of V will be called full if $\text{int}(\text{co}_\rho(V))$ is not empty.*

Definition 3.8. *Let us define*

$$r_L(V) = \inf\{\rho > r(V) : \text{co}_\rho(V) \text{ is full}\}.$$

Let us notice that, for $d = 2$ and $\rho > 0$, Proposition 2.6 and 2.7 imply that the set $\text{co}_\rho(V)$ is full if and only if $\rho > r(V)$.

Lemma 3.9. *For $\rho > r(V)$, the points $o_i(\rho)$, $1 \leq i \leq 4$, defined by Lemma 3.2, are the vertices of a simplex W .*

Proof. The points $o_i(\rho)$ are distinct. Otherwise if $o_i = o_j$ then $B_i(\rho) = B_j(\rho)$ would be an open ball of radius ρ containing all the vertices V on the boundary in contradiction with assumption $r(V) < \rho$. Moreover, see cases *a*) and *b*) in the proof of Lemma 3.2, by construction, each point o_i belongs to the corresponding half lines l_i^* contained in the lines l_i through the circumcenter c and orthogonal to the facet T_i . All distances of $o_i(\rho)$ from c are positive; the vectors from c to c_i are outwards normal vectors to the facet T_i of T . Then the simplex W , with vertex $o_i(\rho)$, contains, up to a translation and dilatation, the simplex T' whose vertices are the contact points of the inscribed sphere to T ; see also [1] for this simplicial vertex-normal duality. \square

Definition 3.10. *Let us denote $r(\rho)$ the circumradius of the simplex W , centered at $c(W)$, with vertices $o_i(\rho)$, defined in Lemma 3.9.*

Let us notice that, by definition of circumcenter $c(W)$ and $r(\rho)$, the spherical surfaces $\partial B(o_i(\rho), r(\rho))$ intersect all in $c(W)$; these spheres contain the vertices $V \setminus \{v_i\}$ if $\rho = r(\rho)$.

Theorem 3.11. *Let V be the set of the vertices of a tetrahedron T . If*

$$(10) \quad r_L(V) > r(V),$$

then it holds

$$(11) \quad \text{co}_{R^*}(V) = V \cup \{O^*\},$$

with $R^ := r_L(V)$ and O^* the circumcenter of the simplex W , with vertices the centers $o_i(R^*)$ of the spheres $B(o_i(R^*), R^*)$. Moreover $R^* = r(R^*)$ is the circumradius of W and $O^* \in \text{int}(T)$.*

Proof. For all $\rho > r(V)$ let us consider the centers $o_i(\rho)$ of the balls $B_i(\rho)$, $i = 1, \dots, 4$, defined by (5); let $r = r(\rho)$ be the circumradius of the simplex W with vertices $\{o_1(\rho), \dots, o_4(\rho)\}$ and $c(W)$ its circumcenter. Then for every i

$$r(\rho) = |c(W) - o_i(\rho)|$$

is a continuous function of ρ .

Then

$$r(R^*) = \lim_{\rho \rightarrow R^*} r(\rho).$$

Moreover

$$co_{R^*}(V) = \lim_{\rho \rightarrow (R^*)^+} co_{\rho}(V).$$

Let us prove now that

$$(12) \quad r(R^*) = R^*.$$

The set $T \setminus \cup_{i=1}^4 B_i(\rho)$ has non empty interior for $\rho > R^*$, and it is contained in a bounded set. Therefore there exists a sequence $\rho_n \rightarrow (R^*)^+$ and a point p^* such that

$$(13) \quad p^* \in \lim_n \text{int}(T \setminus \cup_{i=1}^4 B_i(\rho_n)).$$

Three cases could happen:

- i)* $p^* \in \text{int}(T)$;
- ii)* $p^* \in \partial T \setminus V$;
- iii)* $p^* \in V$;

First, let us consider case *i*). Then there exists a neighbourhood $\mathcal{U} \subset \text{int}(T)$ of p^* . In case $p^* \in \partial B_i(R^*)$ for every i , then p^* is the circumcenter O^* of W and (12) holds. Otherwise, by contradiction, if all four spheres do not cross at p^* , then there exists i and at most three spheres $\partial B_j(R^*), j \neq i$, intersecting at p^* . Then, by Lemma 2.11, $\tilde{\mathcal{U}} = \mathcal{U} \cap \cap_{j \neq i} (B_j(R^*))^c$ has non empty interior. Since $\text{dist}(p^*, B_i(R^*)) > 0$ then $B_i(R^*)^c \cap \tilde{\mathcal{U}}$ has non empty interior. Then $T \setminus \cup_{j=1}^4 B_j(R^*)$ has non empty interior too. By continuity argument, for $\varepsilon > 0$ small enough, $T \setminus \cup_{j=1}^4 B_j(R_\varepsilon^*)$ has non empty interior, with $R_\varepsilon^* = R^* - \varepsilon < R^*$. Then $co_{R_\varepsilon^*}(V)$ is full, in contradiction with the definition of $r_L(V) = R^*$.

Let us prove now that case *ii*) does not occur. Since the distance of $T \setminus \cup_{i=1}^4 B_i(\rho_n)$ from every $o_i(\rho_n)$ is ρ_n , then by (13)

$$\text{dist}(p^*, o_i(R^*)) \geq R^*, \quad \forall i.$$

Obviously $p^* \in T$. Since for every q on the face $T_i \setminus V$, $\text{dist}(q, o_i(R^*)) < R^*$, then p^* does not belong to $T_i \setminus V$, for every i .

Let us prove now that case *iii*) does not occur. By contradiction let $p^* = v_4 \in V$. If $p^* \in \partial B_4(R^*)$ then $\partial B_4(R^*)$ would be the circumsphere of T and $R^* = r(V)$, in contradiction with the assumption (10). Then $p^* \notin \partial B_4(R^*)$, therefore $\text{dist}(p^*, B_4(R^*)) > 0$.

Let us consider the R^* -hull of V :

$$co_{R^*}(V) = T \cap_{i=1}^4 (B_i(R^*))^c.$$

Since $\text{dist}(p^*, B_4(R^*)) > 0$ there exists a neighbourhood \mathcal{U} of p^* , such that

$$co_{R^*}(V) \cap \mathcal{U} = \mathcal{U} \cap (T \cap_{i=1}^3 B_i(R^*)^c) = \mathcal{U} \cap (T \cap (C_{\mathcal{K}}^{p^*})),$$

where $C_{\mathcal{K}}^{p^*}$ is the R^* -cone of Lemma 2.12, with $\rho = R^*$.

Let us consider Lemma 2.12 with $\mathcal{T} := \text{Tan}(T, p^*)$.

In case *a*) of Lemma 2.12, for every neighbourhood \mathcal{U} of p^* , small enough, there exists $y \in \mathcal{T} \cap \mathcal{U} \subset T$, $y \in \text{int}(C_{\mathcal{K}}^{p^*})$. By continuity argument there exists R_ε , $r(V) < R_\varepsilon < R^*$, such that

$$y \in \text{int} \cap_{i=1}^4 (B_i(R_\varepsilon))^c.$$

Since every neighbourhood of $y \in T$ has non empty intersection with $\text{int}(T)$, then

$$\text{int}(T) \cap \text{int}(\cap_{i=1}^4 (B_i(R_\varepsilon))^c) \neq \emptyset.$$

Then $\text{co}_{R_\varepsilon}(V) = T \cap_{i=1}^4 (B_i(R_\varepsilon))^c$ has non empty interior and it is full. This is impossible by definition of $r_L(V) = R^*$.

In case *b*) of Lemma 2.12 the vertex p^* is an isolated point of $T \setminus \cup_{i=1}^4 B_i(R^*) = \text{co}_{R^*}(V)$. This is in contradiction with (13).

Then, only case *i*) holds and $p^* = O^* \in \text{int}(T)$ is the circumcenter of W . Let us prove now (11).

Since

$$O^* \in \left(\bigcup_{i=1}^4 B_i(R^*) \right)^c \cap T,$$

by (5), it follows that $O^* \in \text{co}_{R^*}(V)$: then

$$(14) \quad V \cup \{O^*\} \subseteq \text{co}_{R^*}(V).$$

Let us consider the pyramid T_i^* with apex O^* and opposite face T_i . As $O^* \in \partial B(o_i(R^*), R^*)$, then

$$T_i^* \subseteq B(o_i(R^*), R^*) \cup V \cup \{O^*\};$$

thus

$$\text{co}_{R^*}(V) = T \setminus \bigcup_i B(o_i(R^*), R^*) \subseteq \left(T \setminus \bigcup_i T_i^* \right) \cup V \cup \{O^*\} = V \cup \{O^*\}.$$

This and (14) prove (11). \square

Corollary 3.12. *Let V be the set of the vertices of a simplex $T = T(V)$ in \mathbb{R}^3 , and let $R^* = r_L(V)$. Let us assume that $R^* > r(V)$, then*

$$\begin{aligned} \text{co}_\rho(V) &= V & \text{for } \rho < R^*; \\ \text{co}_\rho(V) &= V \cup \{O^*\} & \text{for } \rho = R^*; \\ \text{co}_\rho(V) &= V \cup \tilde{\Gamma} & \text{for } R^* < \rho, \end{aligned}$$

where

$$\tilde{\Gamma} = \text{int}(T) \setminus \bigcup_i B_i(\rho)$$

is a connected, not empty set, with $\partial \tilde{\Gamma}$ union of connected subsets of $\partial B_i(\rho)$.

3.2 The four-crossing point of a tetrahedron

The radius R^* in Theorem 3.11 is a suitable value of the radius of four spheres S_j (centered at $o_j(R^*)$) satisfying the following:

- (i) S_j contains the vertices v_i , for all $i \neq j$,
- (ii) S_j is the boundary of an open ball not containing v_j ,
- (iii) these four spheres intersect at a single point $\{O^*\}$.

Then O^* is the circumcenter of the simplex W of vertices $o_i(R^*)$.

Let us consider a more general question.

Definition 3.13. Given a tetrahedron T of vertices $V = \{v_i, 1 \leq i \leq 4\}$, let R^L be the radius of four distinct spheres S_j satisfying the following conditions, see fig. 2:

- a) S_j contains the vertices v_i , for all $i \neq j$,
- b) the intersection of the four spheres is one point $\{O^L\}$,
- c) O^L does not belong to the circumsphere of T .

R^L and O^L are called the **four-crossing radius** and the **four-crossing point** of T , respectively.

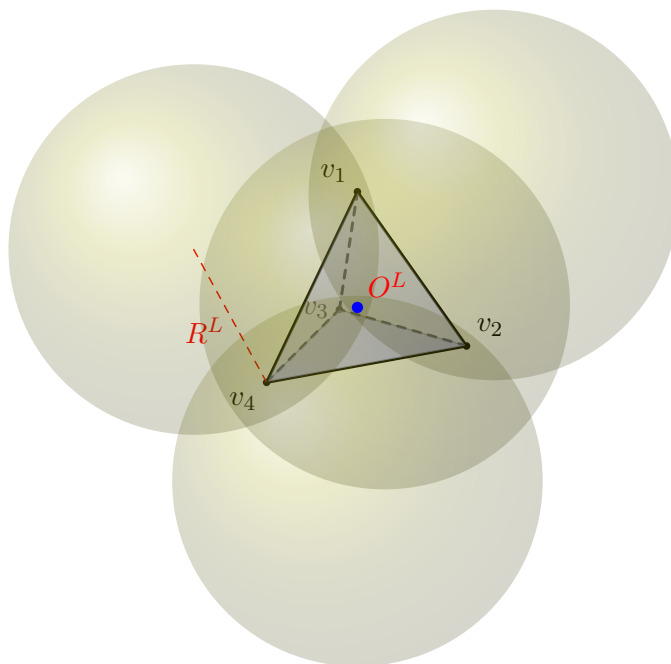


Figure 2: A four-crossing radius and a four-crossing point of the regular tetrahedron.

In [7] an algebraic approach using symbolic computation to determine of the values of R^L and O^L is described. Let us notice that the value $R^* = r_L(V)$, satisfying (10) and the point O^* , satisfying (11) in Theorem 3.11, are a four-crossing radius and a four-crossing point respectively of T , since $O^* \in \text{int}(T) \subset B(V)$. Moreover they are uniquely determined. More generally

Theorem 3.14. *Let (R^L, O^L) be a four-crossing radius and a four-crossing point for a simplex $T = \text{co}(V)$. If*

$$(15) \quad O^L \in \text{int}(T)$$

then R^L and O^L are uniquely determined.

Proof. Since $O^L \in \text{int}(T)$ every sphere $S_j = \partial B_j$ containing $V \setminus \{v_j\}$ and O^L is uniquely determined. Since $O^L \in \text{int}(T)$ then $v_j \notin \overline{B_j}$; by Lemma 3.2 the family

$$\rho \mapsto H_j^+ \cap (B(o_j(\rho), \rho))^c$$

is strictly nested. Then, B_j has radius R^L greater than $r(V)$. Therefore S_j is the boundary of an open ball B_j , which is R^L -supporting V at all vertices v_i with $i \neq j$. Then $R^L = r_L(V)$ of Definition 3.8 and satisfies (10) of Theorem 3.11. This implies that $R^L = R^*$ and $O^L = O^*$ defined by (11). \square

Remark 3.15. *Without condition (15), the four-crossing radius and point of a tetrahedron are not uniquely determined, as proved in [7] for the class of triangular pyramids. For a regular tetrahedron there are seven different four-crossing points, see [7, Example 1].*

A final question is considered.

In the plane the definition of three-crossing radius and three-crossing point for a triangle can be given in a similar way. From Johnson' Theorem $R^L = r(V)$, the circumradius of $T = \text{co}(V)$ and O^L is the orthocenter of T . In such case, condition c) of Definition 3.13 holds, except for the rectangular triangle for which O^L is a vertex. Then a uniqueness and existence result holds for three-crossing radius and point for all triangles, except for the triangular rectangles.

The following example shows a special triangular pyramid with a similar critical configuration.

Example 1. *Let $T = \text{co}(V)$ be the triangular pyramid with apex v_1 , equidistant 1 from the vertices $V \setminus \{v_1\}$ of an equilateral base $T_1 = \text{co}(V \setminus \{v_1\})$, with sides length $\sqrt{12/5}$. Then there exists a unique set of four distinct spheres S_j of radius $\rho = r(V) = \sqrt{5/4}$, the circumradius of T , where S_1 is the circumsphere of V and S_2, S_3, S_4 satisfy the following properties:*

1. S_j contains the vertices v_i , for all $i \neq j$;
2. $\bigcap_{j=2}^4 S_j = \{v_1\}$.

Proof. It has been shown in [7, Example 6] that T is the only triangular pyramid admitting a configuration of four spheres $S_j = \partial B(o_j(\rho), \rho)$, $j = 1, \dots, 4$, each supporting the vertices of a maximal face of T , of radius equal to the circumradius $r(V) = \sqrt{5/4}$ and that the intersection of the four spheres is the apex v_1 . In particular, S_1 is the circumsphere. In order to show item 2, assume without loss of generality that the vertices of T are

$$(16) \quad v_1 = \begin{bmatrix} 0 \\ 0 \\ \frac{\sqrt{5}}{5} \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 2\frac{\sqrt{5}}{5} \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -\frac{\sqrt{15}}{5} \\ -\frac{\sqrt{5}}{5} \\ 0 \end{bmatrix}, \quad \text{and} \quad v_4 = \begin{bmatrix} \frac{\sqrt{15}}{5} \\ -\frac{\sqrt{5}}{5} \\ 0 \end{bmatrix}.$$

Therefore the centers of the four spheres are respectively

$$(17) \quad o_1(\rho) = \begin{bmatrix} 0 \\ 0 \\ -\frac{3\sqrt{5}}{10} \end{bmatrix}, \quad o_2(\rho) = \begin{bmatrix} 0 \\ -\frac{\sqrt{5}}{2} \\ \frac{\sqrt{5}}{5} \end{bmatrix}, \quad o_3(\rho) = \begin{bmatrix} \frac{\sqrt{15}}{4} \\ \frac{\sqrt{5}}{4} \\ \frac{\sqrt{5}}{5} \end{bmatrix}, \quad \text{and} \quad o_4(\rho) = \begin{bmatrix} -\frac{\sqrt{15}}{4} \\ \frac{\sqrt{5}}{4} \\ \frac{\sqrt{5}}{5} \end{bmatrix}.$$

A direct computation shows that $\cap_{j=2}^4 S_j = \{v_1\}$. Let us notice that in this case property c) of Definition 3.13 is not satisfied and therefore, this configuration does not correspond to a four-crossing point. \square

Let us highlight the following fact. The mentioned question, whether there exists a pair (R^L, O^L) satisfying Definition 3.13 with $R^L = r(V)$, for some tetrahedron $T = co(V)$, has applications beyond the theory of R-bodies. Indeed, if such a sphere configuration should exist, it would lead to a counterexample of a conjecture by Maehara and Tokushige [9]: the four spheres S_j determining O^L and the circumsphere of T , would form a sphere system of five spheres of the same radius in \mathbb{R}^3 with no common intersection and four-by-four intersecting in a single point, in the sense of Conjecture 1 in [9]. It has been recently remarked in [7, Remark 1] that such a counterexample does not exist when restricting to the case of triangular pyramids.

4 ρ -supporting balls centered outside a dihedron

The following arguments generalize obvious properties concerning supporting planes used in convexity. We are interested in discrete sets of points. In classical convexity the definition of supporting hyperplane to a discrete sets of points V does not distinguish between V and $co(V)$. We are interested in R -supporting balls to a point of V (see Definition 2.4) that cut $co(V)$. The statement of Theorem 4.3, will be called a ρ -supporting inclusion principle for a dihedron $\mathfrak{D} \subset \mathbb{R}^3$.

Definition 4.1. *Let x, y be points on the boundary of a ball B of radius R , with $|x - y| < 2R$. The shorter arc on ∂B from x to y is denoted by $arc_{\partial B}(x, y)$, or simply $arc(x, y)$ when no ambiguity arises.*

Let H_i be non-parallel planes, $i = 1, 2$, in \mathbb{R}^3 and let $A, B \in H_1 \cap H_2$. Let $\mathfrak{D} = H_1^+ \cap H_2^+$ be the dihedron, with H_i^+ a fixed closed half space bounded by H_i , $i = 1, 2$. Let o_θ be a point on the axial plane Π of AB and outside \mathfrak{D} .

Definition 4.2. *A ball $B_\theta = B(o_\theta, \rho)$ is called ρ -supporting A, B centered outer $\mathfrak{D} = H_1^+ \cap H_2^+$, if:*

- i) $A, B \in \partial B(o_\theta, \rho)$, with $2\rho > |AB|$;*
- ii) $o_\theta \notin \mathfrak{D}$.*

Let $|AB| = 2h$. Let us choose the origin o of the coordinates at the middle point M of AB . From item *i*) in Definition 4.2, it follows that o_θ belongs to the axial plane Π of AB , therefore:

$$o_\theta = M + re(\theta) = re(\theta),$$

with $r = \sqrt{\rho^2 - h^2}$ and $e(\theta)$ a unit vector orthogonal to AB . Then by item *ii*) of Definition 4.2:

$$(18) \quad e(\theta) \in S^2 \cap \mathfrak{D}^c \cap \Pi.$$

Theorem 4.3. Let $\mathfrak{D} = \mathbb{H}_1^+ \cap \mathbb{H}_2^+$ be a dihedron in \mathbb{R}^3 and let $B_\theta, B_{\theta_1}, B_{\theta_2}$ be balls of radius ρ , centered outer \mathfrak{D} , which are ρ -supporting two points A, B . Let us assume that $o_\theta \in \text{arc}(o_{\theta_1}, o_{\theta_2}) \subset \Pi \cap \partial B(M, r)$; let H be the plane through A, B and orthogonal to the segment with end points M and $(o_{\theta_1} + o_{\theta_2})/2$.

If H separates $(o_{\theta_1} + o_{\theta_2})/2$ from \mathfrak{D} , then

$$(19) \quad B_\theta \cap \mathfrak{D} \subset (B_{\theta_1} \cup B_{\theta_2}) \cap \mathfrak{D}.$$

Proof. In a suitable reference system, let us assume that $o = M$, $A = (0, 0, h)$, $B = (0, 0, -h)$ and the axis x is the line on Π through $(o_{\theta_1} + o_{\theta_2})/2$ and M . Then

$$o_{\theta_1} = (r \cos \alpha, r \sin \alpha, 0), o_{\theta_2} = (r \cos \alpha, -r \sin \alpha, 0), o_\theta = (r \cos \theta, r \sin \theta, 0),$$

with $0 < \alpha < \pi$. Since o_θ lies on arc of radius r connecting o_{θ_1} with o_{θ_2} exterior to \mathfrak{D} , then $\alpha \leq \theta \leq 2\pi - \alpha$. Let $D_\theta = \overline{B_\theta}$, $o_\alpha = o_{\theta_1}$, $o_{-\alpha} = o_{\theta_2}$.

Let us assume by contradiction that (19) does not hold, then there exists $\eta, \alpha < \eta < 2\pi - \alpha$ and $p \in \partial B_\eta \cap \mathfrak{D}$, such that $\text{dist}(p, \{o_\alpha, o_{-\alpha}\}) > \rho$. Let $\bar{\rho} > \rho$ be the maximum of such distance, that is:

$$\rho < \bar{\rho} := \max \left\{ \text{dist}(q, \{o_\alpha, o_{-\alpha}\}) : q \in \bigcup_{\alpha \leq \theta \leq 2\pi - \alpha} \partial B_\theta \cap \mathfrak{D} \right\}.$$

Let \bar{p} such that $\text{dist}(\bar{p}, \{o_\alpha, o_{-\alpha}\}) = \bar{\rho}$.

Claim 1: $\bar{\rho} = |\bar{p} - o_\alpha| = |\bar{p} - o_{-\alpha}|$.

Otherwise if $\rho < \bar{\rho} = |\bar{p} - o_\alpha| < |\bar{p} - o_{-\alpha}|$, elementary variational arguments prove that

$$\partial B(o_\eta, \rho) \subset D(o_\alpha, \bar{\rho}).$$

As $\bar{p} \in \partial B(o_\eta, \rho) \cap \partial B(o_\alpha, \bar{\rho})$, the previous inclusion is satisfied if and only if o_η lies on the segment $[o_\alpha \bar{p}]$. Moreover all points $\bar{p}, o_\alpha, o_\eta$ lie on Π ; this is in contradiction with the fact that $\bar{p} \in \mathfrak{D}$ and *Claim 1* is proved. Then \bar{p} lies on the axial hyperplane of $o_\alpha o_{-\alpha}$, which has equation $y = 0$, therefore $\bar{p} = (x, 0, z)$.

Claim 2: $\Pi \cap \mathfrak{D} \subset \{(x, 0, 0) : x \geq 0\}$.

As o_π belongs to the arc of radius r , with center o , exterior to \mathfrak{D} , then the half line from o through o_π is orthogonal to H , which, by assumption, separates it from \mathfrak{D} ; then *Claim 2* is proved.

Let us notice that, as $\bar{p} \in \mathfrak{D}$, then

$$\rho^2 - z^2 \geq r^2.$$

Since $\bar{p} \in \partial B_\eta$ with center $o_\eta = (r \cos \eta, r \sin \eta, 0)$, then

$$\rho^2 = (x - r \cos \eta)^2 + (-r \sin \eta)^2 + z^2.$$

Since $p \in \partial B(o_\alpha, \bar{\rho})$, then

$$\bar{\rho}^2 = (x - r \cos \alpha)^2 + (-r \sin \alpha)^2 + z^2;$$

thus

$$(20) \quad \bar{\rho}^2 - \rho^2 = 2rx(\cos \eta - \cos \alpha).$$

Since $0 < \alpha < \eta < \pi$, then $(\cos \eta - \cos \alpha) < 0$; from *Claim 2* the value of x is non negative then from (20) it follows that $\bar{\rho}^2 \leq \rho^2$. This is in contradiction with the maximum property of $\bar{\rho}$. \square

Acknowledgments. This work has been partially supported by INDAM-GNAMPA (2023). The second author is supported by the ANR Project ANR-21-CE48-0006-01 “HYPERSPACE”.

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