Event-Triggered Finite-Dimensional Observer-Based Output Feedback Stabilization of Reaction-Diffusion PDEs

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***Abstract*—This paper addresses the topic of event-triggered output feedback stabilization of reaction-diffusion PDEs. The control applies at the boundary while the output is distributed. The control strategy consists of a finite-dimensional controller augmented with an adequate event-triggered mechanism. This event-triggered mechanism dictates the time instants at which the applied control input needs to be updated based on the occurrence of specific events monitored by the control law. The reported stability analysis relies on the use of two small gain arguments. A first small gain argument is used to establish that the employed finite-dimensional observer-based control strategy with continuous application of the control input (i.e., without triggering mechanism) satisfies an input-to-state stability (ISS) estimate with respect to an additive perturbation of the control input. Then, taking advantage of this latter ISS estimate along with the introduction of an adequate triggering mechanism, a second small-gain argument is developed to establish the exponential decay of the resulting closed-loop system trajectories. Finally, the feasibility of the reported event-triggered control approach is assessed through the establishment of a minimal dwell-time between two triggering instants.**

***Index Terms*—Reaction-diffusion PDE, event-triggered, finite-dimensional control, output feedback, boundary control.**

I. Introduction

The design of control strategies that combine a reduced complexity and an efficient usage of computational resources is a major concern. For finite-dimensional systems, event-triggered control strategies have emerged as efficient approaches to reduce computation and communication workloads [9], [23]. More recently, the possibility to develop event-triggered control methods for partial differential equations (PDEs) has attracted many attention. Efforts have been first devoted to the control of hyperbolic PDEs [3]–[5], [7], [19] with recent applications to mining cable elevators [26], deep sea construction [27], load-moving cable systems [24], and nonlinear manufacturing systems [2]. These efforts have been extended to parabolic PDE [10], [22], [25], [28]. A spectral reduction-based approach is proposed in [15] for output feedback by deriving sufficient LMI conditions. An emulation of backstepping control has been reported in [6] for the state-feedback event-triggered boundary control by taking advantage of an input-to-state stability (ISS) property. Emulation of backstepping control has been further studied for output feedback observer-based event-triggered boundary control [20].

All the above-mentioned event-triggered control approaches that embrace the problem of output feedback stabilization of a parabolic PDE rely on an infinite-dimensional observer that takes the form of a PDE. Therefore, the implementations of these control strategies require a late lumping approximation (i.e., an *a posteriori* finite-dimensional approximation) of the controller dynamics. To avoid the pitfall of late lumping approximations for the control of parabolic PDEs, efforts have been devoted to the development of procedures for the design of finite-dimensional observer-based control strategies. Such efforts root back to the pioneer work [21] and have then been extended in a number of directions [8], [13], [16], [17] using spectral reduction methods. This includes periodic sampled-data control [14] but at the expense of piecewise linear control inputs while digital implementations impose, in general, piecewise constant control inputs.

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This work addresses the topic of event-triggered finite-dimensional observer-based output feedback stabilization of reaction-diffusion PDEs by means of piecewise constant control inputs. We consider the case of Dirichlet/Neumann/Robin boundary control while the system output takes the form of a distributed measurement. The main contribution of this paper consists of the design of an event-triggered output-feeback strategy that relies, for the first time, on a finite-dimensional observer. Furthermore, the reported triggering mechanism presents a low complexity because its structure is reminiscent of those reported for finite-dimensional systems [9], [23]. Inspired by [6] in a state-feedback context, the proof of stability consists of the derivation of an ISS estimate with respect to a boundary perturbation coupled with small gain arguments. The first step of the proof is devoted to the study of the closed-loop system with a continuous application of the control strategy in the presence of a perturbation. By deriving an explicit small gain condition, we demonstrate that the order of the observer can always be selected such that the above closed-loop system is exponentially ISS with respect to an additive perturbation of the command. The developed approach extends spectral reduction-based ISS analyses methods for open-loop reaction-diffusion PDEs [11], [12] to a closed-loop PDE plant. Then, introducing an adequate event-triggered mechanism and interpreting the event-triggered-based control strategy as a perturbation of the continuous-time control strategy, a second explicit small-gain condition is derived to ensure the exponential decay of the closed-loop system trajectories. We show in the process that this small gain condition is always satisfied provided a suitable choice of one of the parameters of the event-triggered mechanism. We finally assess the existence of a minimal dwell-time, independent of the initial conditions of both the plant and controller, between two consecutive events. This ensures the avoidance of possible Zeno solutions that would prevent the practical feasibility of the reported control strategy [9], [23].

The paper is organized as follows. The problem description and the control strategy are described in Section II. An ISS estimate for the closed-loop system in the case of a continuous actuation is derived in Section III. The exponential decay of the closed-loop system trajectories with event-triggered control, along with the existence of a minimal dwell time, is reported in Section IV. A numerical illustration is presented in Section V. Finally, concluding remarks are formulated in Section VI.

Notation. Spaces $\mathbb{R}^n$ are endowed with the Euclidean norm $\|\cdot\|$. The corresponding induced norms of matrices are also denoted by $\|\cdot\|$. For any two vectors X and Y of arbitrary dimensions, $\mathrm{col}(X,Y)$ stands for the vector $[X^\top, Y^\top]^\top$. The space of square integrable functions on $(0,1)$ is denoted by $L^2(0,1)$ and is endowed with the usual inner product $\langle f,g\rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x$ and with associated norm denoted by $\|\cdot\|_{L^2}$. For an integer $m \geq 1$, $H^m(0,1)$ denotes the m-order Sobolev space and is equipped with its usual norm $\|\cdot\|_{H^m}$. For a given square matrix A, we denote by $\eta(A) = \max_{\lambda\in\mathrm{sp}(A)} \mathrm{Re}(\lambda)$ the spectral abscissa of the matrix A. We define the set $\mathcal{C}^1_{pw}(\mathbb{R}_+)$ as the the set of functions $d:\mathbb{R}_+\to\mathbb{R}$ such that

there exists a strictly increasing sequence $(t_j)_{j\geq 0}$ with $t_j = 0$ and $\lim_{j\to+\infty} t_j = +\infty$, so that $d|_{[t_j,t_{j+1})}$ can be continuously extended into a continuously differentiable function $\tilde{d}_j : [t_j, t_{j+1}] \to \mathbb{R}$ for all $j \geq 0$. For a given Hilbert basis $\{\phi_n\,,\, n \geq 1\}$ of $L^2(0,1)$, which will be specified later, we define $\mathcal{R}_N f = \sum_{n\geq N+1} \langle f, \phi_n\rangle \phi_n$ for any integer $N \geq 1$ and any $f \in L^2(0,1)$.

II. Problem description and control strategy

A. Problem setting

We consider in this paper the following reaction-diffusion equation:

$$z_t(t,x) = (pz_x)_x(t,x) - \tilde{q}(x)z(t,x), \tag{1a}$$
$$\cos(\theta_1)z(t,0) - \sin(\theta_1)z_x(t,0) = 0, \tag{1b}$$
$$\cos(\theta_2)z(t,1) + \sin(\theta_2)z_x(t,1) = u(t), \tag{1c}$$
$$z(0,x) = z_0(x), \tag{1d}$$
$$y(t) = \int_0^1 c(x)z(t,x)\,\mathrm{d}x, \tag{1e}$$

for $t > 0$ and $x \in (0,1)$. Here we have $p \in \mathcal{C}^2([0,1])$ with $p > 0$, $\tilde{q} \in \mathcal{C}^0([0,1])$, $\theta_1, \theta_2 \in [0, \pi/2]$, and $c \in L^2(0,1)$. The state at time t of the PDE is given by $z(t,\cdot)$, the command is $u(t)$, the output is $y(t)$, and the initial condition is z_0.

B. Spectral reduction

We introduce $q \in \mathcal{C}^0([0,1])$ and $q_c \in \mathbb{R}$ such that $\tilde{q}(x) = q(x) - q_c$ with $q(x) > 0$. This allows the introduction of the Sturm-Liouville operator $\mathcal{A} : D(\mathcal{A}) \subset L^2(0,1) \to L^2(0,1)$ associated with (1), defined by $\mathcal{A}f = -(pf')' + qf$ on the domain $D(\mathcal{A}) = \{f \in H^2(0,1) : \cos(\theta_1)f(0) - \sin(\theta_1)f'(0) = \cos(\theta_2)f(1) + \sin(\theta_2)f'(1) = 0\}$. Let λ_n and $\phi_n \in L^2(0,1)$, $n \geq 1$, be the eigenvalues and the associated unit eigenvectors of $\mathcal{A}$, respectively. Then the eigenvalues λ_n are simple, non negative, and form an increasing sequence with $\lambda_n \to +\infty$ as $n \to +\infty$. The eigenvectors $\phi_n \in L^2(0,1)$ form a Hilbert basis of $L^2(0,1)$. The domain of $\mathcal{A}$ is characterized by $D(\mathcal{A}) = \{f \in L^2(0,1) : \sum_{n\geq 1} |\lambda_n|^2 |\langle f, \phi_n\rangle|^2 < +\infty\}$. The growth of λ_n can be estimated as follows (see [18] for details): for any given $p_*, p^*, q^* \in \mathbb{R}$ so that $0 < p_* \leq p(x) \leq p^*$ and $0 \leq q(x) \leq q^*$ for all $x \in [0,1]$, we have that $0 \leq \pi^2(n-1)^2 p_* \leq \lambda_n \leq \pi^2 n^2 p^* + q^*$ for all $n \geq 1$. Moreover, owing to the assumption $p \in \mathcal{C}^2([0,1])$, we have (using the standard Bachmann–Landau notation) that $\phi_n(\xi) = O(1)$ and $\phi_n'(\xi) = O(\sqrt{\lambda_n})$ as $n \to +\infty$ for any given $\xi \in [0,1]$.

We introduce the coefficients of projection defined by $z_n(t) = \langle z(t,\cdot), \phi_n\rangle$ and $c_n = \langle c, \phi_n\rangle$. The projection of (1) into the Hilbert basis $\{\phi_n\,,\, n \geq 1\}$ gives (see, e.g., [11] or [17, Sec. 3.1] for details)

$$\dot{z}_n = (-\lambda_n + q_c)z_n + \beta_n u, \quad n \geq 1, \tag{2}$$

where $\beta_n = p(1)\{-\cos(\theta_2)\phi_n'(1) + \sin(\theta_2)\phi_n(1)\} = O(\sqrt{\lambda_n})$. Due to the Parseval identity, one has $\|z(t,\cdot)\|_{L^2}^2 = \sum_{n\geq 1} z_n(t)^2$. Finally, in view of (1e) we have $y(t) = \sum_{n\geq 1} c_n z_n(t)$.

C. Control strategy

Let $N_0 \geq 1$ be such that $-\lambda_{N_0+1} + q_c < 0$. Hence we have $-\lambda_n + q_c \leq -\lambda_{N_0+1} + q_c < 0$ for all $n \geq N_0 + 1$. Then, for an arbitrarily fixed integer $N \geq N_0 + 1$, which will be specified later, we consider the following observer dynamics:

$$\dot{\hat{z}}_n(t) = (-\lambda_n + q_c)\hat{z}_n(t) + \beta_n u(t) - l_n\left\{\sum_{k=1}^N c_k\hat{z}_k(t) - y(t)\right\},\ 1 \leq n \leq N_0, \tag{3a}$$

$$\dot{\hat{z}}_n(t) = (-\lambda_n + q_c)\hat{z}_n(t) + \beta_n u(t),\ N_0 + 1 \leq n \leq N, \tag{3b}$$

where $l_n \in \mathbb{R}$ are the observer gains. The initial condition of the observer (3) is denoted by $\hat{Z}_0 = \begin{bmatrix}\hat{z}_1(0) & \dots & \hat{z}_N(0)\end{bmatrix}^\top \in \mathbb{R}^N$. Note that the observer dynamics (3b) does not include a correction of the error of observation and is only introduced to improve the estimate $\sum_{k=1}^N c_k\hat{z}_k$ of the measurement y in (3a). The efficiency of such a finite-dimensional observer architecture was originally reported in [21] in the case of a bounded input operator.

In the case of a continuous application of the control, the input $u(t)$ of (1) is set as $u(t) = K\hat{Z}^{N_0}(t)$ where $\hat{Z}^{N_0} = \begin{bmatrix}\hat{z}_1 & \dots & \hat{z}_{N_0}\end{bmatrix}^\top$ and K is the feedback gain. In that case, introducing the matrices $A_0 = \mathrm{diag}(-\lambda_1 + q_c, \dots, -\lambda_{N_0} + q_c)$, $\mathcal{B}_0 = \begin{bmatrix}\beta_1 & \dots & \beta_{N_0}\end{bmatrix}^\top$, $C_0 = \begin{bmatrix}c_1 & \dots & c_{N_0}\end{bmatrix}$, and $L = \begin{bmatrix}l_1 & \dots & l_{N_0}\end{bmatrix}^\top$, the feedback gain K is selected so that the matrix $A_0 + \mathfrak{B}_0 K$ is Hurwitz. For the design of the observer, we make the following assumption.

Assumption 1: *$c_n \neq 0$ for all $1 \leq n \leq N_0$.*

Hence, we can select the observer gain L so that the matrix $A_0 - LC_0$ is Hurwitz. This approach ensures the output feedback exponential stabilization of the plant (1) provided the order of the observer $N \geq N_0 + 1$ is selected to be large enough, see [8], [17], [21].

Our objective is now to augment the above control design strategy with an event-triggered mechanism similar to those encountered for finite-dimensional systems [9], [23]. More precisely, we aim at defining events that produce a strictly increasing sequence of event-based instants $(t_j)_{j\geq 0}$, with $t_j = 0$ and $\lim_{j\to+\infty} t_j = +\infty$, so that the application of the sampled control

$$u(t) = u(t_j) = K\hat{Z}^{N_0}(t_j), \quad \forall t \in [t_j, t_{j+1}),\ \forall j \geq 0 \tag{4}$$

ensures the exponential decay of the trajectories of (1). To achieve this, we first introduce the exogenous dynamics described by

$$\dot{m}(t) = -\kappa m(t), \quad m(0) = m_0 > 0, \tag{5}$$

where the decay rate $\kappa > 0$ and the initial condition $m_0 > 0$ can be freely chosen. This signal is used in the proof of Theorem 2 to ensure the existence of a minimum dwell-time. Then, we introduce the actuation error induced by the event-triggered mechanism as

$$d(t) = K\hat{Z}^{N_0}(t_j) - K\hat{Z}^{N_0}(t), \quad \forall t \in [t_j, t_{j+1}),\ \forall j \geq 0. \tag{6}$$

Hence, the command (4) can be written as $u(t) = K\hat{Z}^{N_0}(t) + d(t)$ for all $t \geq 0$. We finally define $\hat{Z}^{N-N_0} = \begin{bmatrix}\hat{z}_{N_0+1} & \dots & \hat{z}_N\end{bmatrix}^\top$ and $\hat{Z} = \mathrm{col}(\hat{Z}^{N_0}, \hat{Z}^{N-N_0})$. Let $\beta > 0$ be a design parameter (of the event-triggered mechanism) that will be further specified later. We recursively define for any $j \geq 0$ the following set:

$$E(t_j) = \Big\{t \geq t_j\ :\ |d(t)| \geq \beta\|K\|\left(\|\hat{Z}(t)\| + \|\hat{Z}(t_j)\|\right) + \beta \sup_{s\in[t_j,t]} |y(s)| + m(t)\Big\}, \tag{7}$$

with system trajectories obtained from (1) and (3)-(5) for $t \geq t_j$ while starting from the initial conditions $z(t_j,\cdot)$ and $\hat{Z}(t_j)$ and where d is defined by (6).

Let $T_M > 0$ be arbitrarily given. We recursively define the sequence $(t_j)_{j\geq 0}$ of the time of the events as follows. We first define $t_0 = 0$. Then, we define iteratively the time of the events as[1]:

$$t_{j+1} = \min\{\inf E(t_j), t_j + T_M\}, \quad j \geq 0. \tag{8}$$

This allows the application of the event-triggered boundary control strategy (4). Note in particular that $t_{j+1} - t_j \leq T_M$ for all $j \geq 0$, which will be used in the proof of Theorem 3. A distinguished feature

[1]We use the convention $\inf\emptyset = +\infty$.

of the triggering mechanism (4)-(8) is its low complexity because reminiscent of those for finite-dimensional systems [9], [23].

Remark 1: Note that the control input $u(t)$ given by (4) is piecewise constant on $I = \cup_{j\geq 0}[t_j, t_{j+1})$. Consequently, the well-posedness of the closed-loop system formed by (1), (3)-(8) in terms of mild solutions with a continuous $z(t,\cdot) \in \mathcal{C}^0(I; L^2(0,1))$ is inferred from classical well-posedness results for boundary control systems [1, Sec. 3.3] and an induction argument. Note that $z(t,\cdot)$ is in general not of class C^1 due to discontinuity jumps of $z_t(t,\cdot)$ at times t_j. ∘

The stability analysis goes as follows. **Step 1:** consider the closed-loop system formed by the plant (1), the observer (3), and the disturbed control input:

$$u(t) = K\hat{Z}^{N_0}(t) + d(t), \tag{9}$$

for any arbitrary $d \in \mathcal{C}^1_{pw}(\mathbb{R}_+)$. Using a small gain-argument, we derive a sufficient condition on the dimension $N \geq N_0 + 1$ of the observer (3) so that the above closed-loop system satisfies an exponential ISS estimate w.r.t. d. **Step 2:** for an arbitrarily given $\beta > 0$, we assess for the closed-loop system composed of the plant (1), the observer (3), and the event-triggered control input (4) with events defined by (5)-(8) the existence of a minimal dwell-time $T_m > 0$ independent of the initial conditions $z_0, \hat{Z}_0, m_0$, i.e. such that $t_{j+1} - t_j \geq T_m$ for all $j \geq 0$. This will ensure the avoidance of possible Zeno solutions for the closed-loop system trajectories. **Step 3:** Using a second small gain argument, we finally derive a sufficient condition on the event-triggered parameter $\beta > 0$, see (7), so that we obtain the exponential stability of the latter closed-loop system with event-triggered boundary control.

Remark 2: The two above sufficient conditions, the first one concerning the choice of the dimension $N \geq N_0 + 1$ of the observer (3), and the second one regarding the choice of the event-triggered parameter $\beta > 0$, are both shown to be always feasible for an integer N selected to be large enough and $\beta > 0$ selected to be small enough, respectively. Hence, the reported control design procedure is systematic. ∘

In preparation of stability analysis, we fix $\sigma_M, \delta > 0$, independently of the integer $N \geq N_0 + 1$, such that

$$-\lambda_{N_0+1} + q_c < -(\sigma_M + 2\delta) < 0, \tag{10a}$$
$$\eta(A_0 + \mathfrak{B}_0 K) < -(\sigma_M + 2\delta) < 0, \tag{10b}$$
$$\eta(A_0 - LC_0) < -(\sigma_M + 2\delta) < 0. \tag{10c}$$

This implies that $-\lambda_n + q_c < -(\sigma_M + 2\delta) < 0$ for all $n \geq N_0 + 1$.

III. An input-to-state stability result

We use here the spectral representation of the PDE to establish an exponential ISS estimate w.r.t. d for the the closed-loop system formed by the plant (1), the observer (3), and the disturbed control input (9); see [11] for such an approach for the open-loop PDE.

We first introduce the errors of observation $e_n = z_n - \hat{z}_n$, the vectors of errors $E^{N_0} = \begin{bmatrix} e_1 & \dots & e_{N_0} \end{bmatrix}$ and $E^{N-N_0} = \begin{bmatrix} e_{N_0+1} & \dots & e_N \end{bmatrix}$, and the matrices $A_1 = \mathrm{diag}(-\lambda_{N_0+1}, \dots, -\lambda_N)$, $\mathfrak{B}_1 = \begin{bmatrix} \beta_{N_0+1} & \dots & \beta_N \end{bmatrix}^\top$, $C_1 = \begin{bmatrix} c_{N_0+1} & \dots & c_N \end{bmatrix}$, $C = \begin{bmatrix} C_0 & C_1 \end{bmatrix} = \begin{bmatrix} c_1 & \dots & c_N \end{bmatrix}$, $F = F_1 + F_2$, $F_1 = \mathrm{diag}(A_0 + \mathfrak{B}_0 K, A_0 - LC_0, A_1)$,

$$F_2 = \begin{bmatrix} 0 & LC_0 & LC_1 \\ 0 & 0 & -LC_1 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathcal{L} = \begin{bmatrix} L \\ -L \\ 0 \end{bmatrix}, \ \mathcal{L}_d = \begin{bmatrix} \mathfrak{B}_0 \\ 0 \\ 0 \end{bmatrix}.$$

Then, introducing the vector $X = \mathrm{col}(\hat{Z}^{N_0}, E^{N_0}, E^{N-N_0})$, we infer from (2) and (3) along with (9) that

$$\dot{X} = FX + \mathcal{L}\zeta + \mathcal{L}_d d, \tag{11}$$

where $\zeta = \sum_{n\geq N+1} c_n z_n$ represents the residue of measurement. Moreover we have that

$$\dot{\hat{Z}}^{N-N_0} = A_1 \hat{Z}^{N-N_0} + \mathfrak{B}_1 K \hat{Z}^{N_0} + \mathfrak{B}_1 d. \tag{12}$$

Noting that the matrix F is Hurwitz with spectral abscissa $\eta(F) < -(\sigma_M + 2\delta)$, we have for any integer $N \geq N_0 + 1$ the existence of a constant $\mathfrak{C}_{1,N} > 0$ so that

$$\|e^{Ft}\| \leq \mathfrak{C}_{1,N} e^{-(\sigma_M+\delta)t}, \quad \forall t \geq 0, \ \forall N \geq N_0 + 1. \tag{13}$$

Note that an explicit, but conservative, estimate of $\mathfrak{C}_{1,N}$ is derived in Appendix. In practice, tighter estimates can be derived numerically (e.g., based on Lyapunov's equation). We can now state the following theorem whose proof is placed in Appendix.

Theorem 1: *Let $N_0 \geq 1$ be such that $-\lambda_{N_0+1} + q_c < 0$. Under Assumption 1, let $K \in \mathbb{R}^{1\times N_0}$ and $L \in \mathbb{R}^{N_0}$ be such that the matrices $A_0 + \mathfrak{B}_0 K$ and $A_0 - LC_0$ are Hurwitz. We fix $\sigma_M, \delta > 0$ such that (10) holds. Let $N \geq N_0 + 1$ be fixed such that*

$$\Delta_N \triangleq \sqrt{2}\delta^{-1}\mathfrak{C}_{1,N}\|L\|\|K\|\|\mathcal{R}_N c\|_{L^2}\Sigma_N < 1, \tag{14}$$

where $\Sigma_N = \left(\sum_{n\geq N+1} \left|\frac{\beta_n}{\lambda_n - q_c - \sigma_M}\right|^2\right)^{1/2} < \infty$. Then there exist $\mathfrak{C}_2, \mathfrak{C}_3 > 0$ such that, for any initial conditions $z_0 \in L^2(0,1)$ and $\hat{z}_n(0) \in \mathbb{R}$, and any disturbance $d \in \mathcal{C}^1_{pw}(\mathbb{R}_+)$, the trajectories of the plant (1), the observer dynamics (3), and the disturbed control input (9) satisfy, for all $t \geq 0$,

$$\|z(t,\cdot)\|_{L^2} + \|\hat{Z}(t)\| \leq \mathfrak{C}_2 e^{-\sigma_M t}\left(\|z_0\|_{L^2} + \|\hat{Z}_0\|\right) + \mathfrak{C}_3 \sup_{s\in[0,t]} e^{-\sigma_M(t-s)}|d(s)|. \tag{15}$$

An explicit realization of the constant $\mathfrak{C}_3$ appearing in the latter theorem is provided by (31) in Appendix and will be used in the statement of the next results. We now assess that the small gain condition (14) can always be satisfied for N selected to be large enough. We first introduce the following result.

Lemma 1: *There exists $\mathfrak{C}_1 > 0$, independent of $N \geq N_0 + 1$, such that $\|e^{Ft}\| \leq \mathfrak{C}_1 e^{-(\sigma_M+\delta)t}$ for all $t \geq 0$ and all $N \geq N_0 + 1$.*

The proof of Lemma 1 is placed in Appendix. Therefore, without loss of generality, we can select $\mathfrak{C}_{1,N}$ appearing in the small gain condition (14) such that $\mathfrak{C}_{1,N} \leq \mathfrak{C}_1$ for all $N \geq N_0 + 1$. We can now infer from (14) that $\Delta_N \to 0$ as $N \to +\infty$. This is a direct consequence of: (i) δ, L and K are independent of N; (ii) $\mathfrak{C}_{1,N} = O(1)$ as $N \to +\infty$ due to Lemma 1; and (iii) $\|\mathcal{R}_N c\|_{L^2} \to 0$ and $\Sigma_N \to 0$ when $N \to +\infty$. Hence the small gain condition (14) is always achievable for $N \geq N_0 + 1$ selected to be large enough.

IV. Exponential decay under event-triggered actuation

We now focus on the analysis of the closed-loop system composed of the plant (1), the observer (3), and the event-triggered control input (4) with events defined by (5)-(8).

A. Minimal dwell-time

Our objective in this section is to ensure the avoidance of possible Zeno phenomena, i.e., the possible occurrence of an accumulation point for the sequence of triggering events $(t_j)_{j\geq 0}$, as this would present any possible practical implementation of the control strategy. To do so, we assess the existence of a minimal dwell-time for the closed-loop system by using a second small gain argument [6].

Theorem 2: *Let $\kappa > 0$ and $N_0 \geq 1$ be such that $-\lambda_{N_0+1} + q_c < 0$. Under Assumption 1, let $K \in \mathbb{R}^{1\times N_0}$ and $L \in \mathbb{R}^{N_0}$ be such that the matrices $A_0 + \mathfrak{B}_0 K$ and $A_0 - LC_0$ are Hurwitz. For arbitrarily given $N \geq N_0 + 1$, $\beta > 0$, and $T_M > 0$, consider the closed-loop system*

composed of the plant (1), the observer (3), and the event-triggered control input (4) with events defined by (5)-(8). Then, there exists a constant $T_m > 0$, independent of the initial conditions $z_0 \in L^2(0,1)$, $\hat{z}_n(0) \in \mathbb{R}$, and $m_0 > 0$, such that $t_{j+1} - t_j \geq T_m$ for all $j \geq 0$.

Proof. We fix again $\sigma_M, \delta > 0$ such that (10) holds.

The case $K = 0$ is trivial as it implies from (4) that $u(t) = 0$ for all $t \geq 0$. This corresponds to the case where the open-loop plant (1) is exponentially stable. In this configuration, since $m(t) > 0$, (8) implies that $t_{j+1} - t_j = T_M > 0$. We focus on the case $K \neq 0$.

Let $j \geq 0$ be arbitrarily fixed. From the definition (6) of d we have that $d(t_j) = 0$. Hence, owing to the definition (8) of the triggering event t_{j+1} and because $m(t) > 0$ for all $t \geq 0$, we have by a continuity argument that $t_{j+1} > t_j$. Furthermore, we have either $t_{j+1} - t_j = T_M$ or

$$\lim_{t \to (t_{j+1})^-} |d(t)| \geq \Phi_j \triangleq \beta \|K\| \left(\|\hat{Z}(t_{j+1})\| + \|\hat{Z}(t_j)\| \right) + \beta \sup_{s \in [t_j, t_{j+1}]} |y(s)| + m(t_{j+1}). \tag{16}$$

Let us focus on the second scenario, i.e. when (16) holds.

We note from (3) that

$$\dot{\hat{Z}}^{N_0}(t) = (A_0 - LC_0)\hat{Z}^{N_0}(t) + \mathfrak{B}_0 K \hat{Z}^{N_0}(t_j) - LC_1 \hat{Z}^{N-N_0}(t) + Ly(t), \tag{17a}$$

$$\dot{\hat{Z}}^{N-N_0}(t) = A_1 \hat{Z}^{N-N_0}(t) + \mathfrak{B}_1 K \hat{Z}^{N_0}(t_j), \tag{17b}$$

for all $t \in [t_j, t_{j+1})$. Recalling that $\|e^{A_1 t}\| \leq e^{-(\sigma_M + 2\delta)t} \leq 1$ for all $t \geq 0$, we infer from (17b) that

$$\|\hat{Z}^{N-N_0}(t)\| \leq M_2 \|\hat{Z}(t_j)\|, \tag{18}$$

for all $t \in [t_j, t_{j+1})$ with $M_2 = 1 + \frac{\|\mathfrak{B}_1 K\|}{\sigma_M + 2\delta} > 0$. Owing to (10), we have the existence of a constant $M_3 > 0$ such that $\|e^{(A_0 - LC_0)t}\| \leq M_3 e^{-(\sigma_M + 2\delta)t}$ for all $t \geq 0$. Hence we infer from (17a) while using (18) that

$$\|\hat{Z}^{N_0}(t)\| \leq M_4 \|\hat{Z}(t_j)\| + M_5 \sup_{s \in [t_j, t]} |y(s)|, \tag{19}$$

for all $t \in [t_j, t_{j+1})$. Here the constants are defined by $M_4 = M_3 \left\{ 1 + \frac{\|\mathfrak{B}_0 K\|}{\sigma_M + 2\delta} + \frac{\|LC_1\|}{\sigma_M + 2\delta} M_2 \right\} > 0$ and $M_5 = \frac{\|L\|}{\sigma_M + 2\delta} M_3 \geq 0$.

From the definition (6) of d, we have $d(t) = -K \int_{t_j}^t \dot{\hat{Z}}^{N_0}(s)\, \mathrm{d}s$, for all $t \in [t_j, t_{j+1})$. In view of (17a), this implies that $|d(t)| \leq \|K(A_0 - LC_0)\| \sup_{s \in [t_j, t]} \|\hat{Z}^{N_0}(s)\| |t - t_j| + \|K\mathfrak{B}_0 K\| \|\hat{Z}^{N_0}(t_j)\| |t - t_j| + \|KLC_1\| \sup_{s \in [t_j, t]} \|\hat{Z}^{N-N_0}(s)\| |t - t_j| + \|KL\| \sup_{s \in [t_j, t]} |y(s)| |t - t_j|$. Combining this latter estimate with (18)-(19), we deduce that

$$|d(t)| \leq \left\{ M_6 \|\hat{Z}(t_j)\| + M_7 \sup_{s \in [t_j, t]} |y(s)| \right\} |t - t_j|, \tag{20}$$

for all $t \in [t_j, t_{j+1})$ with $M_6 = M_4 \|K(A_0 - LC_0)\| + \|K\mathfrak{B}_0 K\| + M_2 \|KLC_1\| > 0$ and $M_7 = M_5 \|K(A_0 - LC_0)\| + \|KL\| \geq 0$. Letting $t \to (t_{j+1})^-$ in (20) and comparing with (16) we infer that

$$\Phi_j \leq \left\{ M_6 \|\hat{Z}(t_j)\| + M_7 \sup_{s \in [t_j, t_{j+1}]} |y(s)| \right\} |t_{j+1} - t_j| \leq M_8 \Phi_j |t_{j+1} - t_j|,$$

with $M_8 = \max\left(\frac{M_6}{\beta \|K\|}, \frac{M_7}{\beta} \right) > 0$ where we have used that $m(t_{j+1}) > 0$. Using the fact that $m(t_{j+1}) > 0$, we infer that $\Phi_j > 0$. Then, the combination of the latter estimates gives $t_{j+1} - t_j \geq 1/M_8 > 0$. We have shown that $t_{j+1} - t_j \geq T_m \triangleq \min(T_M, 1/M_8)$ for all $j \geq 0$ where $T_m > 0$ is a constant independent of the initial conditions. This completes the proof.

B. Exponential decay assessment

We are now in position to state our main result.

Theorem 3: *Let $\kappa > 0$ and $N_0 \geq 1$ be such that $-\lambda_{N_0+1} + q_c < 0$. Under Assumption 1, let $K \in \mathbb{R}^{1 \times N_0}$ and $L \in \mathbb{R}^{N_0}$ be such that the matrices $A_0 + \mathfrak{B}_0 K$ and $A_0 - LC_0$ are Hurwitz. We fix $\sigma_M, \delta > 0$ such that (10) holds. Let $N \geq N_0 + 1$ be selected such that the small gain condition (14) holds. Let $\beta > 0$ be selected such that*

$$\mathfrak{C}_3 \beta \{2\|K\| + \|c\|_{L^2}\} < 1, \tag{21}$$

where the constant $\mathfrak{C}_3 > 0$ is provided by Theorem 1 with explicit estimate given by (31) in Appendix. Then, for any given $T_M > 0$, there exist constants $\sigma, \mathfrak{C}_4 > 0$ such that, for any initial conditions $z_0 \in L^2(0,1)$, $\hat{z}_n(0) \in \mathbb{R}$ and $m_0 > 0$, the trajectories of the closed-loop system composed of the plant (1), the observer (3), and the event-triggered control input (4) with events defined by (5)-(8) satisfy, for all $t \geq 0$,

$$\|z(t,\cdot)\|_{L^2} + \|\hat{Z}(t)\| + |m(t)| \leq \mathfrak{C}_4 e^{-\sigma t} \left(\|z_0\|_{L^2} + \|\hat{Z}_0\| + |m_0| \right) \tag{22}$$

Remark 3: Introducing

$$\beta_{\max} = 1/(\mathfrak{C}_3 \{2\|K\| + \|c\|_{L^2}\}), \tag{23}$$

the small gain condition (21), hence the conclusion of Theorem 3, holds for any $\beta \in (0, \beta_{\max})$. ∘

Proof. Let $\beta > 0$ be such that (21) holds. For a fixed $T_M > 0$, a continuity argument at $\sigma = 0$ shows the existence of $\sigma \in (0, \min(\sigma_M, \kappa))$ so that

$$\Gamma_\beta = \mathfrak{C}_3 \beta \left\{ (1 + e^{\sigma T_M}) \|K\| + e^{\sigma T_M} \|c\|_{L^2} \right\} < 1. \tag{24}$$

By definition of the event-triggered control input (4) with events defined by (5)-(8), we have

$$|d(t)| \leq \beta \|K\| \left(\|\hat{Z}(t)\| + \|\hat{Z}(t_j)\| \right) + \beta \sup_{s \in [t_j, t]} |y(s)| + m(t),$$

for all $t \in [t_j, t_{j+1})$ and for all $j \geq 0$. We introduce the notation $\tau(t) = t_j$ for all $t \in [t_j, t_{j+1})$ and for all $j \geq 0$. Hence, since Theorem 2 ensures that $\lim_{j \to +\infty} t_j = +\infty$, we have

$$e^{\sigma t}|d(t)| \leq \beta \|K\| e^{\sigma t} \|\hat{Z}(t)\| + \beta \|K\| e^{\sigma(t - \tau(t))} e^{\sigma \tau(t)} \|\hat{Z}(\tau(t))\| + \beta e^{\sigma t} \sup_{s \in [\tau(t), t]} |y(s)| + e^{\sigma t} m(t),$$

for all $t \geq 0$. For any given $t \geq 0$, we have $t \in [t_j, t_{j+1})$ for some integer $j \geq 0$. Thus, by definition of $\tau(t)$, we have $0 \leq t - \tau(t) < t_{j+1} - t_j \leq T_M$ hence $e^{\sigma(t - \tau(t))} \leq e^{\sigma T_M}$. Moreover, for any $s \in [\tau(t), t]$ we have $e^{\sigma(s - \tau(t))} \geq 1$, hence $e^{\sigma t} \sup_{s \in [\tau(t), t]} |y(s)| \leq e^{\sigma(t - \tau(t))} \sup_{s \in [\tau(t), t]} e^{\sigma s} |y(s)| \leq e^{\sigma T_M} \sup_{s \in [\tau(t), t]} e^{\sigma s} |y(s)|$. Consequently we have

$$e^{\sigma t}|d(t)| \leq \beta \|K\| e^{\sigma t} \|\hat{Z}(t)\| + \beta \|K\| e^{\sigma T_M} e^{\sigma \tau(t)} \|\hat{Z}(\tau(t))\| + \beta e^{\sigma T_M} \sup_{s \in [\tau(t), t]} e^{\sigma s} |y(s)| + e^{\sigma t} m(t),$$

for all $t \geq 0$. This implies that

$$\sup_{s \in [0,t]} e^{\sigma s}|d(s)| \leq \beta \|K\| (1 + e^{\sigma T_M}) \sup_{s \in [0,t]} e^{\sigma s} \|\hat{Z}(s)\| + \beta e^{\sigma T_M} \sup_{s \in [0,t]} e^{\sigma s} |y(s)| + \sup_{s \in [0,t]} e^{\sigma s} m(s), \tag{25}$$

for all $t \geq 0$. In view of (1e) we have $|y(t)| \leq \|c\|_{L^2} \|z(t,\cdot)\|_{L^2}$. Using this latter estimate into (25) and combining with (15), while recalling that $0 < \sigma < \sigma_M$, we deduce that

$$\sup_{s \in [0,t]} e^{\sigma s} \left(\|z(s,\cdot)\|_{L^2} + \|\hat{Z}(s)\| \right) \leq \mathfrak{C}_2 \left(\|z_0\|_{L^2} + \|\hat{Z}_0\| \right)$$

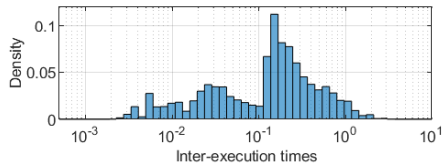
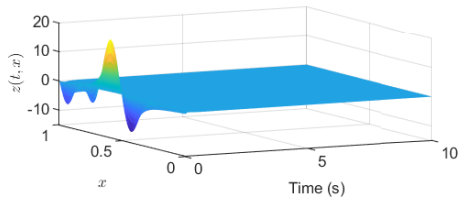
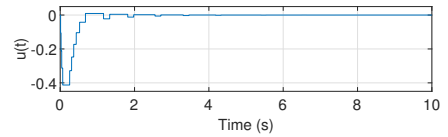


Fig. 1. Density of the inter-execution times for 1000 randomly chosen initial conditions on a simulation time frame of 10 seconds



(a) State $z(t,x)$



(b) Command $u(t)$

Fig. 2. Closed-loop system under event-triggered control

$$+\Gamma_\beta \sup_{s\in[0,t]} e^{\sigma s}\left(\|z(s,\cdot)\|_{L^2}+\|\hat{Z}(s)\|\right)+\mathfrak{C}_3 \sup_{s\in[0,t]} e^{\sigma s} m(s).$$

Recall from (24) that $\Gamma_\beta < 1$. Hence, based on (5) with $0 < \sigma < \kappa$, we infer that (22) holds with $\mathfrak{C}_4 = \max(\mathfrak{C}_2, \mathfrak{C}_3)/(1-\Gamma_\beta)$.

V. Numerical illustration

We consider $p = 1$, $\tilde{q} = -9$, $\theta_1 = 0.1$, $\theta_2 = 0$, and $c(x) = \frac{1}{5}x(x-1)$ for the measurement (1e). The open-loop plant (1) is exponentially unstable. For feedback control we set $N_0 = 1$, $K = -2.025$, $L = 217.1$, $\kappa = 1$, and $T_M = 5$. In this case (10) holds for $\sigma_M = 0.01$ and $\delta = 3.5$. Then, the small gain condition (14) holds for a dimension of the observer $N = 2$. The computation of $\beta_{\max}$ defined by (23) gives $\beta_{\max} \approx 0.0083$; this value is inferred based on $\mathfrak{C}_3 = 29.59$ as provided by Theorem 1 through (31) reported in Appendix. This allows us to set $\beta = 0.008$. Based on (24), we deduce numerically that (22) holds for $\sigma = 0.01$. Finally, following the proof of Theorem 2, it is found that the guaranteed minimal dwell-time is $T_m = 9.09 \cdot 10^{-6}$ s. Such an *a priori* dwell-time is known to be, in general, very conservative.

We consider for numerical simulations initial conditions of the form $z_0(x) = 10x\sum_{n=1}^{4}(\alpha_{0,n} + \alpha_{1,n}x + \alpha_{2,n}x^2)\sin(2n\pi x)$ with $\alpha_{k,n} \in [-1,1]$ randomly chosen. We fix $\hat{z}_n(0) = 0$ and $m_0 = 0.1$. The density of inter-execution times for 1000 randomly chosen initial conditions are depicted in Fig. 1. The exponential stabilization of the closed-loop system for one specific initial condition is depicted in Fig. 2. As shown by Fig. 2(b), this is achieved with a command presenting 29 aperiodic updates on a time interval of 10 seconds.

VI. Conclusion

This paper discussed the event-triggered output feedback stabilization of reaction-diffusion PDEs. The main feature of the reported control procedure is that the event-triggered control strategy relies for the first time on a finite-dimensional observer. Since the reported procedure applies for bounded measurement operators (in particular due to Theorem 1), future research directions may be concerned with extensions to unbounded measurement operators. The relaxation of the conservatism of the small gain conditions is also a reseach direction of interest.

Appendix
Proof of Theorem 1

Denoting by X_0 the initial condition of (11), we have $X(t) = e^{Ft}X_0 + \int_0^t e^{F(t-s)}\{\mathcal{L}\zeta(s) + \mathcal{L}_d d(s)\}\,\mathrm{d}s$. In view of (13), we infer that

$$\|X(t)\| \le \mathfrak{C}_{1,N}e^{-\sigma_M t}\|X_0\| + \delta^{-1}\mathfrak{C}_{1,N}\|\mathcal{L}\| \sup_{s\in[0,t]} e^{-\sigma_M(t-s)}|\zeta(s)| + \delta^{-1}\mathfrak{C}_{1,N}\|\mathcal{L}_d\| \sup_{s\in[0,t]} e^{-\sigma_M(t-s)}|d(s)|. \quad (26)$$

Now, the integration of (2) for $n \ge N+1$ with input u given by (9) gives $z_n(t) = e^{(-\lambda_n+q_c)t}z_n(0) + \int_0^t e^{(-\lambda_n+q_c)(t-s)}\beta_n u(s)\,\mathrm{d}s$. Recalling that $-\lambda_n + q_c \le -\lambda_{N_0+1} + q_c < -(\sigma_M + 2\delta)$ for all $n \ge N_0+1$, we have for all $n \ge N+1$ that $\int_0^t e^{(-\lambda_n+q_c)(t-s)}|u(s)|\,\mathrm{d}s = e^{(-\lambda_n+q_c)t}\int_0^t e^{(\lambda_n-q_c-\sigma_M)s} \times e^{\sigma_M s}|u(s)|\,\mathrm{d}s \le \frac{1}{\lambda_n-q_c-\sigma_M}\sup_{s\in[0,t]} e^{-\sigma_M(t-s)}|u(s)|$ where we have used that $\lambda_n - q_c - \sigma_M \ge 2\delta > 0$. This implies that $|z_n(t)| \le e^{-\sigma_M t}|z_n(0)| + \frac{|\beta_n|}{\lambda_n-q_c-\sigma_M}\sup_{s\in[0,t]} e^{-\sigma_M(t-s)}|u(s)|$. Introducing $S_N(t) = \left(\sum_{n\ge N+1}|z_n(t)|^2\right)^{1/2}$, the use of the triangular inequality gives $S_N(t) \le e^{-\sigma_M t}S_N(0) + \Sigma_N \sup_{s\in[0,t]} e^{-\sigma_M(t-s)}|u(s)|$ where $\Sigma_N = \left(\sum_{n\ge N+1}\left|\frac{\beta_n}{\lambda_n-q_c-\sigma_M}\right|^2\right)^{1/2} < \infty$. In view of (9), we deduce that

$$\sup_{s\in[0,t]} e^{\sigma_M s}S_N(s) \le S_N(0) + \Sigma_N\|K\| \sup_{s\in[0,t]} e^{\sigma_M s}\|\hat{Z}^{N_0}(s)\| + \Sigma_N \sup_{s\in[0,t]} e^{\sigma_M s}|d(s)|, \quad (27)$$

for all $t \ge 0$.

Since $\zeta(t) = \sum_{n\ge N+1} c_n z_n(t)$, Cauchy-Schwarz inequality gives $\zeta(t) \le \|\mathcal{R}_N c\|_{L^2} S_N(t)$. Substituting (27) into the latter estimate and using it in the right-hand side of (26), we obtain that

$$\|X(t)\| \le \mathfrak{C}_{1,N}e^{-\sigma_M t}\|X_0\| + C_1 e^{-\sigma_M t}S_N(0) + \Delta_N \sup_{s\in[0,t]} e^{-\sigma_M(t-s)}\|\hat{Z}^{N_0}(s)\| + C_2 \sup_{s\in[0,t]} e^{-\sigma_M(t-s)}|d(s)|,$$

with the constants $C_1 = \delta^{-1}\mathfrak{C}_{1,N}\|\mathcal{L}\|\|\mathcal{R}_N c\|_{L^2}$ and $C_2 = \delta^{-1}\mathfrak{C}_{1,N}\{\|\mathcal{L}\|\|\mathcal{R}_N c\|_{L^2}\Sigma_N + \|\mathcal{L}_d\|\}$ while Δ_N is defined by (14). Using the fact that $\|\hat{Z}^{N_0}(t)\| \le \|X(t)\|$, we infer that $\sup_{s\in[0,t]} e^{\sigma_M s}\|X(s)\| \le \mathfrak{C}_{1,N}\|X_0\| + C_1 S_N(0) + \Delta_N \sup_{s\in[0,t]} e^{\sigma_M s}\|X(s)\| + C_2 \sup_{s\in[0,t]} e^{\sigma_M s}|d(s)|$. Using the small gain condition (14), we deduce that $\sup_{s\in[0,t]} e^{\sigma_M s}\|X(s)\| \le C_3\|X_0\| + C_4 S_N(0) + C_5 \sup_{s\in[0,t]} e^{\sigma_M s}|d(s)|$ with $C_3 = \mathfrak{C}_{1,N}/(1-\Delta_N)$, $C_4 = C_1/(1-\Delta_N)$, and $C_5 = C_2/(1-\Delta_N)$. This gives

$$\|X(t)\| \le C_3 e^{-\sigma_M t}\|X_0\| + C_4 e^{-\sigma_M t}S_N(0) + C_5 \sup_{s\in[0,t]} e^{-\sigma_M(t-s)}|d(s)|, \quad (28)$$

for all $t \ge 0$. Using (28) into (27) we also have

$$S_N(t) \le C_6 e^{-\sigma_M t}\|X_0\| + C_7 e^{-\sigma_M t}S_N(0) + C_8 \sup_{s\in[0,t]} e^{-\sigma_M(t-s)}|d(s)|, \quad (29)$$

with $C_6 = \Sigma_N\|K\|C_3$, $C_7 = 1 + \Sigma_N\|K\|C_4$, and $C_8 = \Sigma_N(1 + \|K\|C_5)$. Finally, from (12) and using $\|e^{A_1 t}\| \le e^{-(\sigma_M+2\delta)t}$ for all $t \ge 0$, we have $\|\hat{Z}^{N-N_0}(t)\| \le$

$e^{-\sigma_M t}\|\hat{Z}^{N-N_0}(0)\| + \frac{\|\mathfrak{B}_1 K\|}{2\delta}\sup_{s\in[0,t]} e^{-\sigma_M(t-s)}\|\hat{Z}^{N_0}(s)\| + \frac{\|\mathfrak{B}_1\|}{2\delta}\sup_{s\in[0,t]} e^{-\sigma_M(t-s)}|d(s)|$. In view of (28) this implies that

$$\|\hat{Z}^{N-N_0}(t)\| \leq e^{-\sigma_M t}\|\hat{Z}^{N-N_0}(0)\| + C_9 e^{-\sigma_M t}\|X_0\| + C_{10}e^{-\sigma_M t}S_N(0) + C_{11}\sup_{s\in[0,t]} e^{-\sigma_M(t-s)}|d(s)|, \quad (30)$$

with $C_9 = \frac{\|\mathfrak{B}_1 K\|}{2\delta}C_3$, $C_{10} = \frac{\|\mathfrak{B}_1 K\|}{2\delta}C_4$, and $C_{11} = \frac{\|\mathfrak{B}_1 K\|}{2\delta}C_5 + \frac{\|\mathfrak{B}_1\|}{2\delta}$.

We can now complete the proof. We first note that $\|z(t,\cdot)\|_{L^2} = \sqrt{\sum_{n\geq 1} z_n(t)^2} \leq \sqrt{\sum_{n=1}^{N} z_n(t)^2 + S_N(t)}$. Moreover we have $\sqrt{\sum_{n=1}^{N} z_n^2} = \sqrt{\sum_{n=1}^{N}(e_n+\hat{z}_n)^2} \leq \sqrt{2\sum_{n=1}^{N_0}(e_n^2+\hat{z}_n^2) + \sum_{n=N_0+1}^{N} e_n^2} + \sqrt{\sum_{n=N_0+1}^{N}\hat{z}_n^2} \leq \sqrt{2}\|X\| + \|\hat{Z}^{N-N_0}\|$. Hence we infer that $\|z(t,\cdot)\|_{L^2} + \|\hat{Z}(t)\| \leq (1+\sqrt{2})\|X(t)\| + 2\|\hat{Z}^{N-N_0}(t)\| + S_N(t)$. The claimed stability estimate (15) is now obtained by substituting (28)-(30) in the right-hand side of the latter estimate, giving in particular :

$$\mathfrak{C}_3 = (1+\sqrt{2})C_5 + 2C_{11} + C_8. \quad (31)$$

APPENDIX
PROOF OF LEMMA 1

The proof follows arguments reported in [13]. Since $F_2^3 = 0$, we have $F_1^{n_1}F_2F_1^{n_2}F_2F_1^{n_3}F_2 = 0$ for all integers $n_1, n_2, n_3 \geq 0$. Using series expansions, this implies that $e^{F_1 t_1}F_2e^{F_1 t_2}F_2e^{F_1 t_3}F_2 = 0$ for all $t_1, t_2, t_3 \geq 0$. Consider now the function $x(t) = e^{Ft}x_0$ for some arbitrarily given vector $x_0 \in \mathbb{R}^{N_0+N}$. Then, we have $\dot{x}(t) = Fx(t) = F_1x(t) + u(t)$ with $u(t) = F_2x(t) = F_2e^{Ft}x_0$. A direct integration gives $x(t) = e^{F_1 t}x(0) + \int_0^t e^{F_1(t-s)}u(s)\,\mathrm{d}s$. Because x_0 is arbitrary, we infer that $e^{Ft} = e^{F_1 t} + \int_0^t e^{F_1(t-s)}F_2e^{Fs}\,\mathrm{d}s$. Applying three times in a row this identity, we deduce that $e^{Ft} = e^{F_1 t} + \int_0^t e^{F_1(t-s)}F_2e^{F_1 s}\,\mathrm{d}s + \int_0^t\int_0^{s_1} e^{F_1(t-s_1)}F_2e^{F_1(s_1-s_2)}F_2e^{F_1 s_2}\,\mathrm{d}s_2\,\mathrm{d}s_1$. Since F_1 is block diagonal, we have $e^{F_1 t} = \mathrm{diag}(e^{(A_0+\mathfrak{B}_0K)t}, e^{(A_0-LC_0)t}, e^{A_1 t})$. As A_1 is diagonal with simple real eigenvalues not larger than $-\lambda_{N_0+1} + q_c < -(\sigma_M + 2\delta)$, we have $\|e^{A_1 t}\| \leq e^{-(\sigma_M+2\delta)t}$ for all $t \geq 0$ and all $N \geq N_0 + 1$. Hence, in view of (10), there exists a constant $M_1 > 0$, independent of the integer N, so that $\|e^{F_1 t}\| \leq M_1 e^{-(\sigma_M+2\delta)t}$ for all $t \geq 0$ and all $N \geq N_0 + 1$. Therefore we obtain that $\|e^{Ft}\| \leq M_1\left(1 + M_1\|F_2\|t + \frac{M_1^2\|F_2\|^2}{2}t^2\right)e^{-(\sigma_M+2\delta)t}$. The claimed conclusion is obtained by noting that (i) $t^n e^{-\delta t} \leq (n/\delta)^n e^{-n}$ for all $t \geq 0$ and all integer $n \geq 1$; and (ii) from the definition of the matrix F_2, $\|F_2\| \leq \|L\|\sqrt{\|C\|^2 + \|C_1\|^2} \leq \sqrt{2}\|L\|\|c\|_{L^2}$ where L and c are independent of N. In particular one can take $\mathfrak{C}_{1,N} = M_1\sup_{t\geq 0}\left\{\left(1 + M_1\|F_2\|t + \frac{M_1^2\|F_2\|^2}{2}t^2\right)e^{-\delta t}\right\} \leq M_1\{1 + \delta^{-1}e^{-1}M_1\|F_2\| + 2\delta^{-2}e^{-2}M_1^2\|F_2\|^2\}$ and $\mathfrak{C}_1 = M_1\{1+\sqrt{2}\delta^{-1}e^{-1}M_1\|L\|\|c\|_{L^2} + 4\delta^{-2}e^{-2}M_1^2\|L\|^2\|c\|_{L^2}^2\}$.

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