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# FREDHOLM BACKSTEPPING AND RAPID STABILIZATION OF GENERAL LINEAR SYSTEMS

AMAURY HAYAT AND EPIPHANE LOKO

ABSTRACT. We study the rapid stabilization of general linear systems, when the differential operator  $\mathcal{A}$  has a Riesz basis of eigenvectors. We find simple sufficient conditions for the rapid stabilization and the construction of a relatively explicit feedback operator. We use an  $F$ -equivalence (or Fredholm backstepping) approach to show a stronger result: under these sufficient conditions the system is equivalent to a simple exponentially stable system, with arbitrarily large decay rate. In the particular, our conditions improve the existing conditions of rapid stabilization for non-parabolic operators such as skew-adjoint systems.

## 1. INTRODUCTION

Stabilization is one of the three main problems in control theory, together with controllability and optimal control. The goal is to drive a system of differential equations to a given long-term behavior by acting on it with a control, with the specificity that the control has to be a function of the state. This creates a feedback loop (the control influence the system which in turns influence the control), which can make the achieved stability relatively robust. However, this feedback loop makes the mathematical problem of finding a suitable control difficult and even the well-posedness can be a challenge. From a practical point of view, stabilization has many applications [4], in engineering [1], transportation [18, 23, 21], chemistry [44], biology [36], population dynamics [10], fluid mechanics [4], economics [25] etc.

From an abstract point of view, the stabilization problem in a linear framework is the following: given a system

$$(1) \quad \partial_t u = \mathcal{A}u + Bw(t),$$

where  $\mathcal{A}$  is a differential operator and  $B$  is a given operator, we would like to find a control feedback law  $w(t) = K(u(t, \cdot))$  such that the system (1) is exponentially stable. That is, we want all solutions of the system converge exponentially quickly to 0. A more ambitious problem is to require that for any  $\lambda > 0$  there exists a control feedback law  $w(t) = K_\lambda(u(t, \cdot))$  such that all solutions of the system converge exponentially quickly to 0 with decay rate at least  $\lambda > 0$ . This is the so-called *rapid stabilization* (or *complete stabilization*). Solving this problem is all the more challenging when  $v$  is finite dimensional (in particular this implies that  $v$  is not a function of the space variable  $x$ ) and  $B$  is unbounded. This is the case, for instance, when the control is located at a boundary, or when the control is a distributed force where only the amplitude can be controlled (see for instance [5, 16, 15] for particular examples). This linear problem is studied since at least the work of Slemrod in 1972 [38] and powerful results concerning the exponential stabilization of generic linear systems were obtained using tools from optimal control and LQ

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theory by, among others, Lions, Barbu, Lasiecka and Triggiani [31, 3, 29, 30], and several of these results can be extended to rapid stabilization and an unbounded operator  $B$  [42, 43, 40]. Nevertheless, the control feedback laws obtained rely often on the knowledge of the semigroup  $e^{A^*t}$  or solving a minimization problem and a nonlinear Riccati equation, and are not always explicit [27]. For this reason, many works on the stabilization problem have focused on particular systems, especially when  $B$  is unbounded, see for instance (such as [4, 27]).

In the recent years, a new approach has been introduced to tackle this problem more generally: the *Fredholm backstepping* or *F-equivalence*. Instead of trying directly to find a feedback  $K$ , this method consists in solving a different mathematical problem: finding an isomorphism-feedback pair  $(T, K)$  such that  $T$  maps the original system of interest to an exponentially stable target system. If such a pair exists, then the original system with feedback  $K$  is exponentially stable, thanks to the isomorphism property of  $T$ . In the particular case where  $T$  is a Volterra transform of the second kind, this method corresponds to the well investigated *backstepping method* introduced in [2, 7, 28] (and inspired from the final dimensional method [9, 26, 41] and the adaptation [14]) which has known a large success in the last 20 years. For this reason the *F-equivalence* can be seen as a generalized backstepping.

Introduced in [16] for the special case of the Korteweg-de Vries equation and the Kuramoto-Sivashinsky equation [?], the *F-equivalence* has been successfully extended to many different frameworks: the (linear) Schroedinger equation in [11], the 1-d heat equation in [19], the transport equation in [47], the linearized Saint-Venant equations in [15]. The hope that this approach could be used in a general framework, rather than in specific cases, materialized in [19] where the authors managed to generalize the approach to any (linear) skew-adjoint system when the differential operator has eigenvalues  $\lambda_n$  scaling as  $n^\alpha$  with  $\alpha > 1$ . This impressive result resulted from overcoming the structural limitations of the original method in [16] and as a side result allowed to deal with systems like the capillary gravity water-wave equations that was an open question presented in [13]. Nevertheless, these results are restricted to skew-adjoint systems which enjoy very nice properties (such as a basis of orthonormal eigenvectors, pure imaginary eigenvalues, a  $C^0$  group, etc.)

In this paper we generalize the *F-equivalence* to a much more generic class of systems: essentially any system where  $\mathcal{A}$  generates a  $C^0$  semigroup, has a Riesz basis of eigenvectors with eigenvalues with finite multiplicities and satisfying a growth assumption (see (6)–(7)). The main point of this approach is to give explicit feedback controls and conditions under which the PDE system under consideration is equivalent to a simpler PDE system, rather than trying to improve the sufficient conditions of stabilizability coming from the successful existing abstract approaches (such as [29, 30, 40, 43, 33, 32]). Nevertheless, in some cases, our method results in very weak generic sufficient conditions for rapid stabilization which, to the best of our knowledge, were yet unknown. In particular, for skew-adjoint systems it shows that the system can be rapidly stabilized even if the control operator  $B$  is not admissible **and** the system is not exactly null controllable. This is discussed in Section 3.1.

## 2. NOTATIONS AND ASSUMPTIONS

Consider the control system described by (1). Let  $(H, \langle \cdot, \cdot \rangle)$  be a Hilbert space. The operator  $\mathcal{A} : \mathcal{D}(\mathcal{A}) \rightarrow H$  is assumed to be the generator of a  $C^0$  semigroup  $e^{At}$  with finite growth bound in  $\mathcal{D}(\mathcal{A})$  where  $\mathcal{D}(\mathcal{A})$  is defined as

$$\mathcal{D}(\mathcal{A}) := \{f \in H, \mathcal{A}f \in H\}.$$

We assume the following

(H)  $\mathcal{A}$  is diagonal, i.e. there exists a Riesz basis of  $H$  consisting of eigenvectors of  $\mathcal{A}$ , and the associated eigenvalues have bounded multiplicities.

Let us denote by  $m \in \mathbb{N}^*$  the highest multiplicity, one can decompose the space  $H$  into

$$(2) \quad H = H_1 + \dots + H_m,$$

such that for any  $t \geq 0$ ,  $e^{t\mathcal{A}}H_i \subset H_i$ , and on which  $\mathcal{A}$  has a spectral decomposition with simple eigenvalues (see [19, Appendix G] for an explicit construction of such a decomposition<sup>1</sup>). In other words, on each of the  $H_i$ , there exists a Riesz basis (of  $H_i$ ) of eigenvectors  $(\varphi_n^i)_{n \in \mathbb{N}^*}$  with associated eigenvalues  $(\lambda_n^i)_{n \in \mathbb{N}^*}$  such that  $\lambda_n^i \neq \lambda_p^i$  for any  $n \neq p$ , and  $(\varphi_n^i)_{(i,n) \in \{1, \dots, m\} \times \mathbb{N}^*}$  is a Riesz basis of  $H$ . In what follow, we assume without loss of generality that any  $H_i$  is generated by infinitely number of  $\varphi_n^i$ . Otherwise the spaces  $H_i$  generated by finite number of  $\varphi_n^i$ , can be easily treated as finite dimension spaces [12] and the following works with a finite set  $\mathcal{I}$  instead of  $\mathbb{N}^*$ . We define for any  $s \in \mathbb{R}$ , the space

$$\mathcal{H}^s := \mathcal{H}_1^s + \dots + \mathcal{H}_m^s,$$

where

$$\mathcal{H}_i^s = \left\{ f = \sum_{n \in \mathbb{N}^*} f_n^i \varphi_n^i \in H_i, \sum_{n \in \mathbb{N}^*} n^{2s} |f_n^i|^2 < \infty \right\}.$$

These spaces can be endowed with a natural Hilbert space structure:

**LEMMA 2.1.** *For any  $s \in \mathbb{R}$ , and  $i \in \{1, \dots, m\}$ , the function*

$$(3) \quad \|\cdot\|_{\mathcal{H}_i^s} : f \rightarrow \|f\|_{\mathcal{H}_i^s} := \left( \sum_{n \in \mathbb{N}^*} n^{2s} |f_n^i|^2 \right)^{1/2},$$

*is a norm of  $\mathcal{H}_i^s$  and  $(\mathcal{H}_i^s, \|\cdot\|_{\mathcal{H}_i^s})$  is a Hilbert space, with inner product*

$$(4) \quad \langle f, g \rangle_{\mathcal{H}_i^s} := \sum_{n \in \mathbb{N}^*} n^{2s} f_n^i \overline{g_n^i}, \quad \forall f = \sum_{n \in \mathbb{N}^*} f_n^i \varphi_n^i, \quad g = \sum_{n \in \mathbb{N}^*} g_n^i \varphi_n^i \in \mathcal{H}_i^s.$$

This is shown in Appendix A.1. As a consequence, for any  $s = (s_1, \dots, s_m) \in \mathbb{R}^m$  we can define the Hilbert space

$$\mathcal{H}^{\vec{s}} := \mathcal{H}_1^{s_1} + \dots + \mathcal{H}_m^{s_m}.$$

These spaces are intimately related to the operator  $\mathcal{A}$ . Indeed, if the assumption 6 (see below) holds, then  $\mathcal{H}^\alpha = D(\mathcal{A})$ , and for any  $s \in \mathbb{Z}$ ,  $\mathcal{H}^{s\alpha} = D(\mathcal{A}^s)$  (this can even be extended to  $s \in \mathbb{R}$  if  $\mathcal{A}$  is sectorial, see [35, Section 2.6] for more details). These spaces also coincide with the usual Sobolev spaces  $H^s$  whenever  $(\varphi_n)_{n \in \mathbb{N}}$  is an orthonormal basis of  $L^2$ . Similarly to the usual Sobolev spaces, these spaces satisfy the following embedding property

**LEMMA 2.2.** *For any  $s \in \mathbb{R}$  and  $\varepsilon > 0$ ,*

$$\mathcal{H}^{s+\varepsilon} \text{ is compactly embedded in } \mathcal{H}^s.$$

This actually holds for the  $\mathcal{H}_i^s$  taken separately and the proof is identical to [19, Section 1.8.1]. Recall that if  $s = 0$ ,  $\mathcal{H}^s = H$ . Since  $(\varphi_n)_{n \in \mathbb{N}}$  is not necessarily an orthonormal basis of  $H$  with the canonical norm  $\|\cdot\|_H$ , then  $\|\cdot\|_H$  is different from  $\|\cdot\|_{\mathcal{H}^0}$ . However, we have the following

**LEMMA 2.3.** *If  $s = 0$ , then  $\|\cdot\|_{\mathcal{H}^s}$  and  $\|\cdot\|_H$  are equivalent norms.*

<sup>1</sup>the only difference being that (2) is not a direct sum when the basis of eigenvectors of  $\mathcal{A}$  is not orthogonal and the  $(\varphi_n^{i+1})$  are chosen outside  $Span((\varphi_n^i)_{n \in \mathbb{N}^*})$  but not necessarily orthogonal

This is a direct consequence of the fact that  $\varphi_n$  is a Riesz basis. Indeed, by definition there exists positive constants  $c$  and  $C$  such that for any  $(f_n)_{n \in \mathbb{N}^*} \in l^2$ ,

$$(5) \quad c \left( \sum_{n \in \mathbb{N}} |f_n|^2 \right)^{1/2} \leq \left\| \sum_{n \in \mathbb{N}} f_n \varphi_n \right\|_H \leq C \left( \sum_{n \in \mathbb{N}} |f_n|^2 \right)^{1/2}.$$

Since  $\left\| \sum_{n \in \mathbb{N}} f_n \varphi_n \right\|_{\mathcal{H}^0} = \left( \sum_{n \in \mathbb{N}} |f_n|^2 \right)^{1/2}$ , Lemma 2.3 follows.

From [5], for each  $i \in \{1, \dots, m\}$ , the family  $(\varphi_n^i)_{n \in \mathbb{N}^*}$  admits a unique bi-orthonormal family  $(\widetilde{\varphi}_n^i)_{n \in \mathbb{N}^*}$  in  $H_i$  which forms a Riesz basis of  $H_i$  and corresponds to the family of eigenvectors of the adjoint  $\mathcal{A}^*$  of  $\mathcal{A}$  (with respect to the scalar product  $\langle \cdot, \cdot \rangle$  of  $H$ ). Thus any  $f \in H_i$  admits a unique decomposition in  $H_i$  as follows

$$f = \sum_{n \in \mathbb{N}^*} f_n \varphi_n = \sum_{n \in \mathbb{N}^*} \langle f, \widetilde{\varphi}_n^i \rangle \varphi_n^i,$$

and, as a consequence  $\langle f, \widetilde{\varphi}_n^i \rangle = \langle f, \varphi_n^i \rangle_{\mathcal{H}_i^0}$  and this extends to any function  $f \in \mathcal{H}_i^s$  for  $s \in \mathbb{R}$ .

**Assumptions** Thorough the paper, we assume the following assumptions: there exists  $(\alpha_1, \dots, \alpha_m) \in (1, +\infty)^m$ , such that

- there exist  $c_1, c_2 > 0$

$$(6) \quad c_1 n^{\alpha_i} \leq |\lambda_n^i| + 1 \leq c_2 n^{\alpha_i}, \quad \forall n \in \mathbb{N}^*, i \in \{1, \dots, m\}$$

- there exists  $C > 0$  such that for any  $n, p \in \mathbb{N}^*, i \in \{1, \dots, m\}$ ,

$$(7) \quad |\lambda_n^i - \lambda_p^i| \geq C n^{\alpha_i - 1} |n - p|.$$

### 3. MAIN RESULTS

Our main result generates the exponential stabilization of system (1) for any operator  $\mathcal{A}$  satisfying (H), (6)–(7).

**THEOREM 3.1.** *Consider the control system (1) with operator  $\mathcal{A}$  satisfying the assumptions (6) and (7). If  $B = (B_1, \dots, B_m) \in \mathcal{H}^{-\alpha_1/2} \times \dots \times \mathcal{H}^{-\alpha_m/2}$  is such that*

$$(8) \quad c_1 \leq |\langle B_i, \widetilde{\varphi}_n^i \rangle| \leq c_2 n^{\gamma_i}, \quad \forall n \in \mathbb{N}^*, i \in \{1, \dots, m\}$$

for some constant  $c_1, c_2 > 0$ , and some  $\gamma_i \in [0, (\alpha_i - 1)/2]$  for  $i \in \{1, \dots, m\}$  and set  $\alpha := \min \alpha_i$  and  $\gamma := \min \gamma_i$ . Then, for any  $\lambda_0 > 0$ , there exist  $\lambda > \lambda_0$  and a bounded linear feedback  $K \in \mathcal{L}(\mathcal{H}^{\alpha_1/2} \times \dots \times \mathcal{H}^{\alpha_m/2}, \mathbb{C}^m)$  and a mapping  $T$  that is an isomorphism from  $\mathcal{H}^r$  to itself for any  $r \in (1/2 - \alpha + \gamma, \alpha - 1/2 - \gamma)$  such that  $T$  maps the system

$$(9) \quad \partial_t u = \mathcal{A}u + BK(u)$$

to the system

$$(10) \quad \partial_t v = \mathcal{A}v - \lambda v.$$

In particular, for any  $\mu > 0$ ,  $\lambda_0$  can be chosen sufficiently large such that the closed loop system (9) is exponentially stable with decay rate  $\mu > 0$  in  $\mathcal{H}^r$  for any  $r \in (1/2 - \alpha, \alpha - 1/2)$ .

**REMARK 3.2** (Regularity of the feedback operator). *While the linear feedback  $K$  belongs a priori to  $\mathcal{L}(\mathcal{H}^{\alpha_1/2} \times \dots \times \mathcal{H}^{\alpha_m/2}, \mathbb{C}^m)$ , the feedback we construct is actually more regular and can actually be extended on  $\mathcal{L}((\mathcal{H}^{1/2+\varepsilon})^n, \mathbb{C}^m)$  for any  $\varepsilon > 0$ . (see (64) and Lemma 5.8)*

In fact, Theorem 3.1 can be generalized as follows:

**THEOREM 3.3.** *Consider the control system (1) with operator  $\mathcal{A}$  satisfying the assumptions (6) and (7). Let  $(\beta_1, \dots, \beta_m) \in \mathbb{R}^m$ ,  $(\gamma_1, \dots, \gamma_m) \in [0, (\alpha_1 - 1)/2] \times \dots \times [0, (\alpha_m - 1)/2]$ , and  $B = (B_1, \dots, B_m) \in \mathcal{H}^{\beta_1 - \frac{\alpha_1}{2}} \times \dots \times \mathcal{H}^{\beta_m - \frac{\alpha_m}{2}}$  such that there exist  $c_1, c_2 > 0$  satisfying*

$$(11) \quad c_1 n^{-\beta_i} \leq |\langle B_i, \widetilde{\varphi}_n^i \rangle| \leq c_2 n^{-\beta_i + \gamma_i}, \quad \forall n \in \mathbb{N}^*, i \in \{1, \dots, m\}.$$

*Then, for any  $\lambda > 0$  there exist  $\lambda > \lambda_0$  and a bounded linear feedback  $K \in \mathcal{L}(\mathcal{H}^{\beta_1 + \frac{\alpha_1}{2}} \times \dots \times \mathcal{H}^{\beta_m + \frac{\alpha_m}{2}}, \mathbb{C}^m)$  and a mapping  $T$  that is an isomorphism from  $\mathcal{H}^{\vec{r}}$  to itself with*

$$(12) \quad \mathcal{H}^{\vec{r}} = \mathcal{H}^{\beta_1 + r_1} + \dots + \mathcal{H}^{\beta_m + r_m}$$

*for any*

$$(13) \quad r_i \in (1/2 - \alpha_i + \gamma_i, \alpha_i - 1/2 - \gamma_i).$$

*Moreover  $T$  maps the system*

$$\partial_t u = \mathcal{A}u + BK(u)$$

*to the system*

$$\partial_t v = \mathcal{A}v - \lambda v.$$

*In particular, for any  $\mu > 0$ ,  $\lambda_0$  can be chosen sufficiently large such that the closed loop system (9) is exponentially stable with decay rate  $\mu$  in  $\mathcal{H}^{\vec{r}}$ , for any  $\vec{r}$  satisfying (13).*

We give some illustrations from this Theorem in Sections 3.2–3.3: an application to the Schroedinger equation considered and the water wave equations, and an extension of the linear Laplacian operator studied in [20].

In the particular case where  $\mathcal{A}$  is skew-adjoint operator, we recover the result of [19] but our conditions are strictly less restrictive. One can also recover the result on the linearized Schrodinger equation of [11], here again with less restrictive conditions of stabilization since [11, Hypothesis 1.1] is not necessarily needed depending on the regularity of the control operator).

From Theorem 3.3 we deduce the following conditions for the rapid stabilization of system 1.

**COROLLARY 3.4.** *Consider the control system (1) with operator  $\mathcal{A}$  satisfying the assumptions (6) and (7). Let  $(\gamma_1, \dots, \gamma_m) \in [0, (\alpha_1 - 1)/2] \times \dots \times [0, (\alpha_m - 1)/2]$ ,  $r_i \in (1/2 - \alpha_i + \gamma_i, \alpha_i - 1/2 - \gamma_i)$ , and  $B = (B_1, \dots, B_m) \in \mathcal{H}^{-\alpha_1} \times \dots \times \mathcal{H}^{-\alpha_m}$  such that there exist  $c_1, c_2 > 0$  satisfying*

$$(14) \quad c_1 n^{r_i} \leq |\langle B_i, \widetilde{\varphi}_n^i \rangle| \leq c_2 n^{r_i + \gamma_i},$$

*then for any  $\mu > 0$  there exists a (constructive) bounded linear feedback  $K_\gamma \in \mathcal{L}(\mathcal{H}^{\alpha_1/2 - r_1} \times \dots \times \mathcal{H}^{\alpha_m/2 - r_m}, \mathbb{C}^m)$  such that  $K_\gamma \in \mathcal{L}(\mathcal{H}^{1/2 - r_1 + \varepsilon} \times \dots \times \mathcal{H}^{1/2 - r_m + \varepsilon}, \mathbb{C}^m)$  for any  $\varepsilon > 0$  and the system (9) is exponentially stable in  $H$  with decay rate  $\mu$ .*

**3.1. Link with controllability and existing conditions for stabilizability.** Usually, a typical “good” condition for a generic rapid stabilization result (in  $H$ ) would be exact null controllability (in  $H$ ) and admissibility of the control operator  $B$ . These are the conditions typically required when using Riccati methods for stabilization [42, 29, 30], or other methods such as the one introduced by Komornik in [27]. The result of [40, Proposition 1] using an observability approach to investigate rapid stabilizability in a general framework with also uses this as assumption (see also [39] in the case where  $A$  generates a  $C^0$  group and not only a semigroup). In our case, because we require  $BK$  to shift all the spectrum<sup>2</sup> of  $\mathcal{A}$  our statement is stronger than the sole

<sup>2</sup>since the system should be mapped to  $\partial_t v = (\mathcal{A} - \lambda Id)v$

rapid stabilization and we could expect to require a stronger condition. Indeed, because we are shifting the entire spectrum, our problem has similarities with the pole placement theorem, for which a typical condition would be the exact controllability [?] (which is stronger than exact null controllability in general when dealing with PDE and coincide when  $\mathcal{A}$  is skew-adjoint, see [40, 39, 46]). Surprisingly, our condition (14) is in fact less restrictive: in our framework the system is not necessarily exactly controllable in  $H$  and  $B$  is not necessarily admissible.

This is particularly striking when the system is skew-adjoint: in this case our sufficient conditions (14) are strictly less restrictive than those of all the existing results so far. In particular, the system (1) can be stabilized rapidly even it is not exactly (null) controllable and is  $B$  is not admissible.

To go more in detail, let us recall the following results from [45] (see also [37])

**LEMMA 3.5** ([45]). *If  $B$  is admissible then there exists  $C > 0$  such that*

$$(15) \quad |\langle B_i, \varphi_n^i \rangle| \leq C(1 + |\operatorname{Re}(\lambda_n^i)|)^{1/2}, \quad \forall (i, n) \in \{1, \dots, m\} \times \mathbb{N}^*,$$

*and if the system is in addition exactly controllable in  $H$  then there exists  $c > 0$  such that*

$$(16) \quad |\langle B_i, \varphi_n^i \rangle| \geq c(1 + |\operatorname{Re}(\lambda_n^i)|)^{1/2}, \quad \forall (i, n) \in \{1, \dots, m\} \times \mathbb{N}^*.$$

These conditions are necessary (and not necessarily sufficient [22] in general) to admissibility and exact controllability. Comparing with our condition (14) we see the following:

- If the  $\lambda_n^i$  have bounded real part (for instance if  $\mathcal{A}$  is a skew-adjoint operator, but not only).
  - For  $r_i = 0, \gamma_i = 0$  we recover the usual condition of admissibility and exact controllability which coincide with the regularity and observability conditions of [27, 42, 43] (and [11, 34] for the bilinear Schroedinger equation).
  - For  $r_i + \gamma_i \in (1/2, \alpha_i - 1/2)$ ,  $B$  is not necessarily admissible, in particular the regularity condition required in [43, 42, 27] is not satisfied.
  - For  $r_i - \gamma_i \in (1/2 - \alpha, 0)$ , then the exact controllability of the system is not satisfied. This is could seem surprising given the usual result according which rapid stabilizability implies exact (null) controllability for skew-adjoint systems given in [46, Theorem 16.5] (see also [40]). In fact, since in these references, the definition of rapid stabilizability additionally requires that  $K \in \mathcal{L}(X, \mathbb{C}^m)$ , which is helpful in the analysis, in particular to ensure that the system is well-posed, but is not assumed here (compared to the previous case  $r_i \in (1/2, \alpha - 1/2)$ ): we only require the weaker condition that  $A + BK$  generates a  $C^0$  semigroup on  $H$ . This less restrictive and relaxed definition can also be found in [32, 33] where the authors provide very nice necessary conditions in terms of observability when  $\mathcal{A}$  is a skew-adjoint operators (but under the assumption that  $B$  is admissible, which is not the case here).

Note that both  $r_i - \gamma_i \in (1/2 - \alpha, 0)$  and  $r_i - \gamma_i \in (1/2 - \alpha, 0)$  can be satisfied at the same time. As a consequence, our condition sufficient condition for rapid stabilization (14) can hold even in cases where the system is **not** exactly controllable **and**  $B$  is not admissible.

- If  $\operatorname{Re}(\lambda_n^i) \geq c|\lambda_n^i|$  for some constant  $c > 0$  (for instance if  $\mathcal{A}$  is a self-adjoint operator, but not only)
  - Interestingly if  $r_i \in (1/2, \alpha_i/2)$  then  $K \in \mathcal{L}(X, \mathbb{C}^m)$  but the system is not exactly controllable. This is not a surprise from a stabilization point of view, but it is still interesting since it is strictly weaker than the conditions for pole placement [?] although we shift here the entire spectrum.

- If  $r_i \in (\alpha/2, \alpha - 1/2)$  then the system is exactly controllable but  $B$  does not necessarily need to be admissible.

Of course, if  $B$  is admissible and the system exactly controllable, then our condition on  $B$  is always satisfied, as one could expect.

**3.2. Application to Schroedinger equation around the ground state.** We consider the bilinear Schroedinger equation linearized around the ground state studied in [11, 34] (see also [5, 6]):

$$(17) \quad \begin{aligned} i\partial_t \Psi &= -\Delta \Psi - \lambda_1 \Psi + u(t)\mu(x)\varphi_1(x), \\ \Psi(0) &= \Psi(1) = 0, \end{aligned}$$

where  $\lambda_1 = \pi^2$  and  $\varphi_1 = \sqrt{2}\sin(\pi x)$  are respectively the first eigenvalue and eigenvector of the Laplace operator with Dirichlet boundary conditions. The total family of eigenvalues and eigenvectors are the solution of

$$(18) \quad \begin{aligned} \Delta \varphi_n &= \lambda_n \varphi_n \\ \varphi_n(0) &= \varphi_n(1) = 0, \end{aligned}$$

and is given by

$$(19) \quad \lambda_n = \pi^2 n^2, \quad \varphi_n(x) = \sqrt{2}\sin(n\pi x), \quad \forall n \in \mathbb{N}^*.$$

Here, the space under consideration is, as in [11, 34]

$$(20) \quad H = \{f \in H^3((0, 1); \mathbb{R}^2), f(0) = f(1) = f'(0) = f'(1) = 0, \operatorname{Re}\langle f, \varphi_1 \rangle_{L^2(I)} = 0\}.$$

One can easily check that the associated operator  $\mathcal{A} = i(\Delta - \lambda_1 Id)$  is diagonal with eigenvectors  $\tilde{\varphi}_n = (n+1)^{-3}\varphi_{n+1}$  associated to eigenvalues  $\tilde{\lambda}_n = i(\lambda_{n+1} - \lambda_1)$ , for  $n \in \mathbb{N}^*$  on  $H$  (note that the eigenvector corresponding to  $\lambda_1 - \lambda_1 = 0$  is excluded by the choice of the space  $H$ , which is denoted  $\mathcal{H}_0$  in [11], see also [34, (1.9)-(1.10)]) and satisfy (6)–(7) with  $\alpha = 2$ .

In [11, 34] it was showed that the system is rapidly stabilizable –i.e. for any  $\lambda > 0$  there exists a feedback operator  $K$  such that the system (17) with  $u(t) = K(\Psi(t))$  is exponentially stable with decay rate  $\lambda$ – provided that the following sufficient condition is satisfied

$$(21) \quad \begin{aligned} \mu &\text{ belongs to } H^3(0, 1) \\ |\langle \mu\varphi_1, \varphi_n \rangle_{L^2(0,1)}| &\geq cn^{-3}, \quad \forall n \in \mathbb{N}^* \end{aligned}$$

for some  $c > 0$ . This implies in particular (see [11, Remark 1.2]) that

$$(22) \quad cn^{-3} \leq |\langle \mu\varphi_1, \varphi_n \rangle_{L^2(0,1)}| \leq Cn^{-3},$$

for some  $c, C > 0$ . The left equality is equivalent to the exact (null) controlability of the system (17) [11]. Since  $(\tilde{\varphi}_n)_{n \in \mathbb{N}^*}$  is a Riesz (and in fact orthonormal) basis of  $H$  this is also

$$(23) \quad c \leq |\langle \mu\varphi_1, \tilde{\varphi}_n \rangle_H| \leq C,$$

which corresponds to our condition (14) with  $\gamma = \beta = 0$ . From Corollary 3.4 we can relax the condition (21) of [11, 19] and show that the system (17) with  $u(t) = K(\Psi(t))$  is rapidly stabilizable as soon as there exists  $\gamma \in [0, 1/2)$  and  $r \in (-3/2 + \gamma, 3/2 - \gamma)$  such that

$$(24) \quad cn^r \leq |\langle \mu\varphi_1, \tilde{\varphi}_n \rangle_H| \leq Cn^{r+\gamma}.$$



In particular, a sufficient condition for rapid stabilization is

$$(25) \quad \begin{aligned} &\mu \text{ belongs to } H^3(0,1) \\ &|\langle \mu \varphi_1, \varphi_n \rangle_{L^2(0,1)}| \geq cn^{-5/2+\varepsilon}, \quad \forall n \in \mathbb{N}^*, \text{ for some } \varepsilon > 0. \end{aligned}$$

**REMARK 3.6.** *Note that in [34] the author uses a very different method and shows an impressive quantitative estimate of  $K$  with respect to  $\lambda$ , which allows a finite-time stabilization of this system. Since the  $F$ -equivalence construct relatively explicitly the control  $K$  and the isomorphism  $T$ , it would be interesting to see if it possible to obtain in our case a quantitative estimate on  $K$  and  $T$  and, in turn, a finite time stabilization.*

**3.3. Application to the linear diffusion equation.** Let us first consider the heat equation on a torus  $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$  with two scalar controls

$$(26) \quad \partial_t u = \Delta u + \phi w(t),$$

where  $\phi = (\phi_1, \phi_2)^T$  such that  $\phi$  is even and  $\phi_2$  is odd and  $w(t) \in \mathbb{R}^2$ . The space of consideration is  $H^m(\mathbb{T})$  and the eigenvectors and associated eigenvalues of the operator  $\mathcal{A} = \Delta$  are

$$(27) \quad \lambda_n = -n^2, \quad \varphi_n^1(x) = n^{-m} c_{1,n} \sin(nx), \quad \varphi_{n+1}^2(x) = n^{-m} c_{2,n} \cos(nx), \quad \forall n \in \mathbb{N}^*,$$

and the constant function  $\varphi_0^2(x) = (2\pi)^{-1}$  associated to the eigenvalue 0. Note that all eigenvalues have multiplicity 2 except 0. In [20] it was shown that for arbitrarily large  $\lambda > 0$  there exists a feedback operator  $K \in \mathcal{L}(H^{m+1/2+\varepsilon}; \mathbb{R}^2)$  for any  $\varepsilon > 0$  such that this system with  $w(t) = K(u(t))$  can be invertibly mapped to the exponentially stable system

$$(28) \quad \partial_t u = \Delta u - \lambda u,$$

in  $\mathcal{H}^{m+r}(\mathbb{T})$  for any  $r \in (-1/2, 1/2)$  provided that the following sufficient condition holds:

$$(29) \quad c \leq |\langle \phi_i, \varphi_n^i \rangle_{H^m(\mathbb{T})}| \leq C, \quad \forall (i, n) \in \{1, 2\} \times \mathbb{N}^*$$

We recover the same with our Theorem 3.3, but with a weaker condition:

**COROLLARY 3.7.** *Let  $\gamma \in [0, 1/2)$ , if*

$$(30) \quad c \leq |\langle \phi_i, \varphi_n^i \rangle_{H^m(\mathbb{T})}| \leq Cn^\gamma, \quad \forall (i, n) \in \{1, 2\} \times \mathbb{N}^*,$$

*then for arbitrarily large  $\lambda > 0$  there exists a feedback operator  $K \in \mathcal{L}(H^{m+1/2+\varepsilon}; \mathbb{R}^2)$  such that the system with  $w(t) = K(u(t))$  can be invertibly mapped to the exponentially stable system (28) in  $H^{m+r}(\mathbb{T})$  for any  $r \in (-3/2 + \gamma, 3/2 - \gamma)$ .*

Our Theorem 3.3 also allow to generalize this to the classical diffusion equation:

$$(31) \quad \partial_t u = \partial_x(a\partial_x u) + bu + \phi K(u), \quad \text{on } \in [0, +\infty) \times [0, L],$$

$$(32) \quad \begin{aligned} c_1 u(t, 0) + c_2 \partial_x u(t, 0) &= 0 \\ c_3 u(t, L) + c_4 \partial_x u(t, L) &= 0, \end{aligned}$$

where  $a \in C^2([0, L]; \mathbb{R}_+^*)$ ,  $b \in L^2(0, L)$ , and  $c_1^2 + c_2^2 > 0$ ,  $c_3^2 + c_4^2 > 0$ . In particular, the following holds

**COROLLARY 3.8.** *If there exist  $c_1, c_2 > 0$  and  $\gamma \in (0, 1/2)$  such that  $\phi$  satisfies*

$$(33) \quad c_1 \leq \left| \int_0^L \phi(x) u_n(x) dx \right| \leq c_2 n^\gamma, \quad \forall n \in \mathbb{N}^*,$$

*where  $(u_n)_{n \in \mathbb{N}^*}$  is the sequence of eigenvectors of the operator  $\mathcal{A} := \partial_x(a\partial_x) + b$ , then for any  $\lambda \in (0, +\infty)$ , there exists a bounded linear feedback  $K \in \mathcal{L}(H^{1/2+\varepsilon}(0, L), \mathbb{C}^2)$  for any  $\varepsilon > 0$ , such*

that the diffusion equation (31)–(32) is exponentially stable in  $H^r(0, L)$  with decay rate  $\lambda$ , for any  $r \in (-3/2 + \gamma, 3/2 - \gamma)$ .

#### 4. STRATEGY AND OUTLINE

To prove our main result, the main challenge is to show the  $F$ -equivalence in each of the spaces  $H_i$  separately, more precisely:

**PROPOSITION 4.1.** *Let  $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{R}^m$  and  $\gamma = (\gamma_1, \dots, \gamma_m) \in [0, (\alpha_1 - 1)/2) \times \dots \times [0, (\alpha_m - 1)/2)$ . For any  $i \in \{1, \dots, m\}$ , assume that  $B_i \in \mathcal{H}_i^{\beta_i - \frac{\alpha_i}{2}}$  satisfies (11). Then for any  $\lambda \in \mathbb{R}_+ \setminus \mathcal{N}$ , where  $\mathcal{N} = \{\lambda_n^i - \lambda_p^i \mid (n, p) \in \mathbb{N}^*, i \in \{1, \dots, m\}\}$ , there exists a bounded linear operator  $K_i \in \mathcal{L}(\mathcal{H}_i^{\beta_i + \frac{\alpha_i}{2}}; \mathbb{C})$  such that  $K_i \in \mathcal{L}(\mathcal{H}_i^{\beta_i + \frac{1}{2} + \varepsilon}; \mathbb{C})$  for any  $\varepsilon > 0$  and a linear mapping  $T_i \in \mathcal{L}(\mathcal{H}_i^r)$  which is an isomorphism from  $\mathcal{H}_i^r$  to itself for any  $r \in (\beta_i + 1/2 - \alpha_i + \gamma_i, \beta_i + \alpha_i - 1/2 - \gamma_i)$  and maps the system,*

$$(34) \quad \partial_t u = \mathcal{A}u + B_i K_i(u), \quad u \in \mathcal{H}_i^r,$$

to the system,

$$(35) \quad \partial_t v = \mathcal{A}v - \lambda v, \quad v \in \mathcal{H}_i^r.$$

The idea is that if Proposition 4.1 holds, then a candidate feedback to obtain the exponential stability with decay rate  $\lambda$  for the total system is  $K = (K_1, \dots, K_m)^T$ . To do so, the main step is to show that the system with feedback  $K = (K_1, \dots, K_m)^T$  is well-posed, that is

**PROPOSITION 4.2.** *Under the assumption of Theorem 3.3, and with  $K = (K_1, \dots, K_m)^T$  where the  $K_i$  are given by Proposition 4.1, for any  $\vec{r}$  satisfying (13),  $\mathcal{A} + BK$  generates a semigroup on  $\mathcal{H}^{\vec{r}}$  and in particular the system (9) has a unique solution  $u \in C^0([0, +\infty); \mathcal{H}^{\vec{r}})$ .*

If Propositions 4.1 and 4.2 hold, then Theorem 3.1 follows by choosing  $T = T_1 + \dots + T_m$ . This is detailed in Section 5.8, while the proof of Propositions 4.1 and 4.2 are done respectively in Sections 5.1–5.6 and Section 5.7. In the remaining of this section, we present the strategy to show the main Proposition 4.1 and the spirit of the backstepping approach.

**Principle of the approach.** Formally, having  $T_i$  mapping the system (34) to the system (35) with  $v = T_i u$  means that  $T_i$  is an isomorphism with satisfies the following operator equality

$$(36) \quad T_i(\mathcal{A} + B_i K_i) = (\mathcal{A} - \lambda)T_i,$$

in some sense. Indeed, still formally, if (36) holds and  $v = T_i u$ , then

$$(37) \quad \partial_t v = T_i \partial_t u = T_i(\mathcal{A} + B_i K_i)u = (\mathcal{A} - \lambda)T_i u = (\mathcal{A} - \lambda)v,$$

and conversely, since  $T_i$  is an isomorphism. Because the operator equality (36) has no uniqueness in the solutions  $(T, K)$  –note that if  $(T, K)$  is a solution, then  $(aT, K)$  is still a solution for any  $a \neq 0$ – it is tempting to add a kind of normalization on  $T$  which would simplify (36). In this regards, a good approach is often to formally add a condition of the form

$$(38) \quad TB_i = B_i,$$

again in a sense to be defined (note that since  $B_i$  can be unbounded and not belong to the spaces  $\mathcal{H}_i^r$  on which we consider  $T_i$ , this inequality may have to be considered in a weak sense). With (38), the operator equality (36) becomes (formally)

$$(39) \quad T_i \mathcal{A} + B_i K_i = (\mathcal{A} - \lambda)T_i.$$

This operator equation is easier to solve than (36) since it is linear in  $(T, K)$ . When  $H$  is finite dimensional there exists a unique solution to (38)–(39) if  $(\mathcal{A}, B_i)$  is controllable in  $H_i$  (see [12, 15]) which is an additional motivation to adding the condition (38). When  $H$  is infinite dimensional, the situation is much less clear, but (39) can be translated in terms of actions of  $T$  on  $(\varphi_n^i)_{i \in \mathbb{N}}$ . Indeed, projecting the operator equality (39) on the eigenvector  $\varphi_n^i$  of  $\mathcal{A}$ , we get

$$(40) \quad \lambda_n^i T_i \varphi_n^i + B_i K_i(\varphi_n^i) = (\mathcal{A} - \lambda I) T_i \varphi_n^i.$$

We set  $h_n^i := T_i \varphi_n^i$ ;  $K_n^i := K_i(\varphi_n^i)$  and we project (40) on a bi-orthogonal family  $(\widetilde{\varphi}_p^i)_{p \in \mathbb{N}^*}$  associated to  $(\varphi_n^i)_{n \in \mathbb{N}^*}$  to obtain

$$\begin{aligned} \lambda_n^i \langle h_n^i, \widetilde{\varphi}_p^i \rangle + K_n^i \langle B_i, \widetilde{\varphi}_p^i \rangle &= \langle \mathcal{A} h_n^i, \widetilde{\varphi}_p^i \rangle - \lambda \langle h_n^i, \widetilde{\varphi}_p^i \rangle \\ &= \lambda_p^i \langle h_n^i, \widetilde{\varphi}_p^i \rangle - \lambda \langle h_n^i, \widetilde{\varphi}_p^i \rangle \end{aligned}$$

So we obtain, assuming that  $\lambda$  is chosen such that  $(\lambda_n^i - \lambda_p^i + \lambda) \neq 0$ , for any  $(n, p) \in \mathbb{N}^*$ ,

$$\langle h_n^i, \widetilde{\varphi}_p^i \rangle = -K_n^i \frac{\langle B_i, \widetilde{\varphi}_p^i \rangle}{\lambda_n^i - \lambda_p^i + \lambda}.$$

This leads to the following

$$(41) \quad T_i \varphi_n^i = h_n^i := \sum_{p \in \mathbb{N}^*} \langle h_n^i, \widetilde{\varphi}_p^i \rangle \varphi_p^i = -K_n^i \sum_{p \in \mathbb{N}^*} \frac{\langle B_i, \widetilde{\varphi}_p^i \rangle}{\lambda_n^i - \lambda_p^i + \lambda} \varphi_p^i.$$

Hence, if  $T_i$  is a solution to (39), it has to have the decomposition (41). In particular, as soon as  $K_i$  is chosen, the candidate transform  $T_i$  is fixed and we aim to show that it is indeed an isomorphism from  $\mathcal{H}_i^r$  to itself and maps the system (34) to the system (35).

### Outline of the proof of Proposition 4.1

The general strategy is the following

- (1) Show that the operator

$$(42) \quad S_i : \varphi_n^i \rightarrow \sum_{p \in \mathbb{N}^*} \frac{\varphi_p^i}{\lambda_n^i - \lambda_p^i + \lambda}$$

is a Fredholm operator from  $\mathcal{H}_i^r$  into itself for any  $r \in (1/2 - \alpha_i + \gamma_i, \alpha_i - 1/2)$ .

- (2) Show that  $\ker(S_i) = \ker(S_i^*) = \{0\}$  and consequently  $S_i$  is an isomorphism from  $H_i$  into itself.
- (3) Show that  $S_i$  is an isomorphism from  $\mathcal{H}_i^r$  into itself for any  $r \in (1/2 - \alpha_i + \gamma_i, \alpha_i - 1/2)$
- (4) Find an explicit candidate  $K_i \in \mathcal{L}(\mathcal{H}^{1/2+\varepsilon}; \mathbb{C})$  for any  $\varepsilon > 0$ , such that  $T_i B_i = B_i$  holds in  $\mathcal{H}_i^{-\alpha_i/2}$  with

$$(43) \quad T_i : \varphi_n^i \rightarrow -K_i(\varphi_n^i) \sum_{p \in \mathbb{N}^*} \frac{\langle B_i, \widetilde{\varphi}_p^i \rangle \varphi_p^i}{\lambda_n^i - \lambda_p^i + \lambda}$$

- (5) Show that the linear operator  $T_i$  is bounded from  $\mathcal{H}_i^{r_i}$  into itself for  $r_i \in (1/2 - \alpha + \gamma, \alpha - 1/2)$  and satisfies the operator equality (39) in  $\mathcal{L}(\mathcal{H}_i^{\alpha/2+s}; \mathcal{H}_i^{-\alpha/2+s})$  for any  $s \in (-\alpha - 1/2 + \gamma, (\alpha - 1/2 - \gamma))$ .
- (6) Show that  $T_i$  is an isomorphism from  $\mathcal{H}_i^r$  into itself for any  $r \in (1/2 - \alpha_i + \gamma_i, \alpha_i - 1/2)$ .

Step 1, 4 and 5 are similar to [19], the main difference with the duality compactness method introduced in [19] lies in the steps 2 and 3 where the duality argument used in [19] leverages the skew-adjoint properties and does not hold anymore and in the Step 5 and 6, where the boundedness of  $T_i$  is not straightforward when  $\gamma_i \neq 0$  while showing directly that  $T_i$  is an isomorphism from  $\mathcal{H}_i^{-\alpha/2}$  into itself as in [19] would be challenge. These steps are shown in Section 5.1–5.6.

### 5. PROOF OF THEOREM 3.1

We first prove Proposition 4.1, following the outline described in Section 4 and we then deduce Theorem 3.1 (see Section 5.8). Since several of the arguments of the proof (namely steps 1, 4, 5) are similar to [19], we just explicit the novelty and refer the reader to [19] for the remaining.

Since Proposition 4.1 consist in showing the  $F$ -equivalence on each of the spaces  $H_i$ , we start by fixing  $i \in \{1, \dots, m\}$  and we will work in  $H_i$  (resp.  $\mathcal{H}_i^s$ ). We also adopt the following convention: when an operator on  $H_i$  (resp.  $\mathcal{H}_i^s$ ) is defined by its action on  $(\varphi_n^i)_{n \in \mathbb{N}}$ , we extend it to  $H$  (resp.  $\mathcal{H}^s$ ) by setting its image to  $\{0\}$  on  $(\varphi_n^j)_{n \in \mathbb{N}}$  for any  $j \neq i$ . With this in mind, in the following, we will drop the dependency in  $i$  for the different quantities  $B_i, K_i, T_i, \beta_i, \alpha_i, \gamma_i, (\varphi_n^i)_{n \in \mathbb{N}}, (\lambda_n^i)_{n \in \mathbb{N}}$ , for the reader's convenience. We will also assume in the following that  $\beta = 0$ , the extension to the case  $\beta \neq 0$  is given in in Appendix A.5

Let us set, for  $n \in \mathbb{N}^*$ ,

$$(44) \quad q_n := \sum_{p \in \mathbb{N}^*} \frac{\varphi_p}{\lambda_n - \lambda_p + \lambda} = S\varphi_n,$$

**5.1. Step 1:  $S$  is a Fredholm operator.** We are going to show the following:

**PROPOSITION 5.1.** *For any  $r \in (1/2 - \alpha, \alpha - 1/2)$ , the operator  $S \in \mathcal{L}(\mathcal{H}^r)$  is a Fredholm operator of index 0. More precisely there exists a compact operator  $S_c \in \mathcal{L}(\mathcal{H}^r)$  such that*

$$(45) \quad S = \lambda^{-1}Id + S_c.$$

*Proof of Proposition 5.1.* □

Let  $r \in (1/2 - \alpha, \alpha - 1/2)$ . Showing that  $S$  is a Fredholm operator of order 0 in  $\mathcal{H}^r$  can be done essentially as in [19]. The first thing to observe is the following

**LEMMA 5.2.** *For any  $\lambda \in \mathbb{R}_+ \setminus \mathcal{N}$ , there exists  $C(\lambda) > 0$  such that  $|\lambda_n - \lambda_p + \lambda| \geq C(\lambda)|\lambda_n - \lambda_p|$ .*

*Proof.* For  $n = p$  the inequality is true for any constant  $C(\lambda)$ . For  $n \neq p$

$$(46) \quad \frac{|\lambda_n - \lambda_p \pm \lambda|}{|\lambda_n - \lambda_p|} \geq 1 - \frac{\lambda}{|\lambda_n - \lambda_p|}.$$

From (7)

$$\lim_{n^2+p^2 \rightarrow +\infty} \frac{\lambda}{|\lambda_n - \lambda_p|} \leq \lim_{n^2+p^2 \rightarrow +\infty} \frac{\lambda}{C \max(n, p)^{\alpha-1}} = 0.$$

Since  $\lambda_n - \lambda_p + \lambda \neq 0$  for any  $(n, p) \in \mathbb{N}^*$  by definition of  $\mathcal{N}$ , there exists  $C(\lambda) > 0$  such that

$$|\lambda_n - \lambda_p + \lambda| \geq C(\lambda)|\lambda_n - \lambda_p|$$

□

With this in mind, the key of this step is to notice that [19, Lemma 4.6] can still apply despite  $\mathcal{A}$  not being skew-adjoint. Indeed, we have

**LEMMA 5.3.** *For any  $s < \alpha - 1$  we have*

$$(47) \quad \sum_{n \in \mathbb{N}^* \setminus \{p\}} \frac{n^s}{|\lambda_n - \lambda_p + \lambda|} \lesssim p^{1-\alpha+s} \log(p) + p^{-\alpha}, \quad \forall p \in \mathbb{N}^*,$$

where  $\lesssim$  means lower or equal up to a multiplicative constant that does not depend on  $p$  or  $n$ .

*Proof.* From Lemma 5.2 it is enough to show that

$$(48) \quad \sum_{n \in \mathbb{N}^* \setminus \{p\}} \frac{n^s}{|\lambda_n - \lambda_p|} \lesssim p^{1-\alpha+s} \log(p) + p^{-\alpha}, \quad \forall p \in \mathbb{N}^*.$$

Since condition (6) and (7) hold, this is exactly given by [19, Lemma 4.2].  $\square$

We can now perform as in [19, Lemma 4.7] to show that

$$(49) \quad S_c : \sum_{n \in \mathbb{N}^*} a_n n^{-r} \varphi_n \mapsto \sum_{n \in \mathbb{N}^*} a_n n^{-r} \left( \sum_{p \in \mathbb{N}^* \setminus \{n\}} \frac{\varphi_p}{\lambda_n - \lambda_p + \lambda} \right),$$

is a compact operator from  $\mathcal{H}^r$  to itself:

let  $\varepsilon > 0$  to be chosen, by using Fubini's Theorem we have, for any  $a = \sum_{n \in \mathbb{N}^*} a_n n^{-r} \varphi_n \in \mathcal{H}^r$ ,

$$S_c a = \sum_{n \in \mathbb{N}^*} a_n n^{-r} \left( \sum_{p \in \mathbb{N}^* \setminus \{n\}} \frac{\varphi_p}{\lambda_n - \lambda_p + \lambda} \right) = \sum_{p \in \mathbb{N}^*} \varphi_p \left( \sum_{n \in \mathbb{N}^* \setminus \{p\}} \frac{a_n n^{-r}}{\lambda_n - \lambda_p + \lambda} \right).$$

Thus,

$$(50) \quad \begin{aligned} \|S_c a\|_{\mathcal{H}^{r+\varepsilon}}^2 &= \left\| \sum_{p \in \mathbb{N}^*} \varphi_p \left( \sum_{n \in \mathbb{N}^* \setminus \{p\}} \frac{a_n n^{-r}}{\lambda_n - \lambda_p + \lambda} \right) \right\|_{\mathcal{H}^{r+\varepsilon}}^2 \\ &= \sum_{p \in \mathbb{N}^*} p^{2r+2\varepsilon} \left| \sum_{n \in \mathbb{N}^* \setminus \{p\}} \frac{a_n n^{-r}}{\lambda_n - \lambda_p + \lambda} \right|^2. \end{aligned}$$

Hence, we can use Lemma 5.3 and perform exactly as in [19, Lemma 4.7] to get the following

**LEMMA 5.4.** *Consider  $r \in (1/2 - \alpha, \alpha - 1/2)$ . For any  $\varepsilon \in (0, \min\{\frac{\alpha-1}{2}, \alpha + r - \frac{1}{2}\})$ , it holds that*

$$\sum_{p \in \mathbb{N}^*} p^{2r+2\varepsilon} \left| \sum_{n \in \mathbb{N}^* \setminus \{p\}} \frac{a_n n^{-r}}{\lambda_n - \lambda_p + \lambda} \right|^2 \lesssim \sum_{n \in \mathbb{N}^*} |a_n|^2.$$

Using any  $\varepsilon$  provided by this Lemma, the following holds:

$$(51) \quad \|S_c a\|_{\mathcal{H}^{r+\varepsilon}}^2 \lesssim \sum_{n \in \mathbb{N}^*} |a_n|^2.$$

Since

$$(52) \quad \sum_{n \in \mathbb{N}^*} |a_n|^2 = \left\| \sum_{n \in \mathbb{N}^*} a_n (n^{-r} \varphi_n) \right\|_{\mathcal{H}^r}^2 = \|a\|_{\mathcal{H}^r}^2,$$

we conclude that

$$(53) \quad \|S_c a\|_{\mathcal{H}^{r+\varepsilon}} \lesssim \|a\|_{\mathcal{H}^r}$$

and from the compact embedding of  $\mathcal{H}^{r+\varepsilon}$  in  $\mathcal{H}^r$  (see Lemma 2.2), this implies in particular that  $S_c$  is a compact operator from  $\mathcal{H}^r$  to itself. Finally, noting that

$$(54) \quad S = \lambda^{-1}Id + S_c,$$

concludes the proof.

**5.2. Step 2:  $S$  is an isomorphism from  $\mathcal{H}^0$  to itself.** Using the results of Step 1 for  $r = 0$ , we deduce that  $S$  is a Fredholm operator of index 0 from  $\mathcal{H}^0$  to itself. Then,  $S$  is an isomorphism if and only if  $\ker S = \{0\}$ . Since it is of index 0, this is also equivalent to (see [24, Chapter 4])

$$(55) \quad \ker \tilde{S} = \{0\} \text{ where } \tilde{S} \text{ is the adjoint of } S.$$

Let us introduce

$$(56) \quad \tilde{q}_n := \sum_{p \in \mathbb{N}^*} \frac{\varphi_p}{\lambda_p - \lambda_n + \lambda}.$$

We first claim the following

**LEMMA 5.5.** *The adjoint of the operator  $S$  in  $\mathcal{H}^0$  is*

$$(57) \quad \tilde{S} : \varphi_n \mapsto \tilde{q}_n.$$

*Proof.* Indeed, consider any  $f = \sum_{n \in \mathbb{N}^*} f_n \varphi_n \in \mathcal{H}^0$  and  $g = \sum_{n \in \mathbb{N}^*} g_n \varphi_n \in \mathcal{H}^0$ . The following holds

$$\begin{aligned} \langle Sf, g \rangle_{\mathcal{H}^0} &= \left\langle \sum_{n \in \mathbb{N}^*} f_n q_n, \sum_{n \in \mathbb{N}^*} g_n \varphi_n \right\rangle_{\mathcal{H}^0} \\ &= \left\langle \sum_{n \in \mathbb{N}^*} f_n \sum_{p \in \mathbb{N}^*} \frac{\varphi_p}{\lambda_n - \lambda_p + \lambda}, \sum_{n \in \mathbb{N}^*} g_n \varphi_n \right\rangle_{\mathcal{H}^0} \\ &= \left\langle \sum_{p \in \mathbb{N}^*} \varphi_p \sum_{n \in \mathbb{N}^*} \frac{f_n}{\lambda_n - \lambda_p + \lambda}, \sum_{n \in \mathbb{N}^*} g_n \varphi_n \right\rangle_{\mathcal{H}^0} \\ &= \sum_{p \in \mathbb{N}^*} \sum_{n \in \mathbb{N}^*} \frac{\overline{g_p} f_n}{\lambda_n - \lambda_p + \lambda}. \end{aligned}$$

In the same way, we have the following

$$\begin{aligned} \langle f, \tilde{S}g \rangle_{\mathcal{H}^0} &= \left\langle \sum_{n \in \mathbb{N}^*} f_n \varphi_n, \sum_{n \in \mathbb{N}^*} g_n \tilde{q}_n \right\rangle_{\mathcal{H}^0} \\ &= \left\langle \sum_{n \in \mathbb{N}^*} f_n \varphi_n, \sum_{p \in \mathbb{N}^*} \varphi_p \sum_{n \in \mathbb{N}^*} \frac{g_n}{\lambda_p - \lambda_n + \lambda} \right\rangle_{\mathcal{H}^0} \\ &= \sum_{p \in \mathbb{N}^*} \sum_{n \in \mathbb{N}^*} \frac{f_p \overline{g_n}}{\lambda_p - \lambda_n + \lambda} \end{aligned}$$

Thus, we have  $\langle Sf, g \rangle = \langle f, \tilde{S}g \rangle$  for all  $f, g \in \mathcal{H}^0$ . □

In view of (54), (55),  $S$  is an isomorphism is equivalent to  $\ker S = \{0\}$  or  $\ker \tilde{S} = \{0\}$ , which in turn is equivalent to:

$$\forall (f_n)_n \in l^2, \sum_{n \in \mathbb{N}^*} f_n q_n = 0 \Leftrightarrow f_n = 0 \quad \forall n \in \mathbb{N}^*, \text{ or } \forall (f_n)_n \in l^2 \sum_{n \in \mathbb{N}^*} f_n \tilde{q}_n = 0 \Leftrightarrow f_n = 0 \quad \forall n \in \mathbb{N}^*.$$

This means that  $S$  is an isomorphism from  $\mathcal{H}^0$  to itself is equivalent to  $(q_n)_{n \in \mathbb{N}^*}$  is  $\omega$ -independent in  $\mathcal{H}^0$  or  $(\tilde{q}_n)_{n \in \mathbb{N}^*}$  is  $\omega$ -independent in  $\mathcal{H}^0$ . With a classical argument, one can show that  $(q_n)_{n \in \mathbb{N}^*}$

is either  $\omega$ -independent in  $\mathcal{H}^0$  or  $\mathcal{H}^0$ -dense for any  $\lambda \in (0, +\infty)$ . This is done in Section A.2 in Appendix. So, the conclusion follows from the following Lemma:

**LEMMA 5.6.**  $(q_n)_{n \in \mathbb{N}^*}$  is  $\mathcal{H}^0$ -dense  $\Leftrightarrow (\tilde{q}_n)_{n \in \mathbb{N}^*}$  is  $\omega$ -independent in  $\mathcal{H}^0$ .

*Proof.* Assume that  $(q_n)_{n \in \mathbb{N}^*}$  is not  $\mathcal{H}^0$ -dense. Then there exists a non trivial  $(f_n)_{n \in \mathbb{N}^*} \in l^2$  such that for any  $n \in \mathbb{N}^*$   $\langle q_n, f \rangle_{\mathcal{H}^0} = 0$  where  $f = \sum_{n \in \mathbb{N}^*} f_n \varphi_n$ .

$$\begin{aligned} \langle q_n, f \rangle_{\mathcal{H}^0} = 0 &\Leftrightarrow \sum_{p \in \mathbb{N}^*} \frac{\overline{f_p}}{\lambda_n - \lambda_p + \lambda} = 0 \\ &\Leftrightarrow \sum_{p \in \mathbb{N}^*} \frac{f_p}{\lambda_n - \lambda_p + \lambda} = 0 \end{aligned}$$

Thus,  $\sum_{p \in \mathbb{N}^*} f_p \tilde{q}_p = \sum_{p \in \mathbb{N}^*} f_p \sum_{n \in \mathbb{N}^*} \frac{\varphi_n}{\lambda_n - \lambda_p + \lambda} = \sum_{n \in \mathbb{N}^*} \varphi_n \sum_{p \in \mathbb{N}^*} \frac{f_p}{\lambda_n - \lambda_p + \lambda} = 0$ , showing that  $(\tilde{q}_n)_{n \in \mathbb{N}^*}$  is not  $\omega$ -independent. The converse is straightforward: let us assume that  $(\tilde{q}_n)_{n \in \mathbb{N}^*}$  is not  $\omega$ -independent. Then, there exists non trivial  $(f_n)_{n \in \mathbb{N}^*} \in l^2$  such that  $\sum_{p \in \mathbb{N}^*} f_p \tilde{q}_p = 0$ .

$$\begin{aligned} \sum_{p \in \mathbb{N}^*} f_p \tilde{q}_p = 0 &\Leftrightarrow \sum_{n \in \mathbb{N}^*} \varphi_n \sum_{p \in \mathbb{N}^*} \frac{f_p}{\lambda_n - \lambda_p + \lambda} = 0 \\ &\Leftrightarrow \sum_{p \in \mathbb{N}^*} \frac{f_p}{\lambda_n - \lambda_p + \lambda} = 0 \end{aligned}$$

Thus by setting  $f = \sum_{n \in \mathbb{N}^*} f_n \varphi_n \in \mathcal{H}^0$  it holds that  $\langle f, q_n \rangle_{\mathcal{H}^0} = \sum_{p \in \mathbb{N}^*} \frac{f_p}{\lambda_n - \lambda_p + \lambda} = 0$ , meaning that  $(q_n)_{n \in \mathbb{N}^*}$  is not  $\mathcal{H}^0$ -dense.  $\square$

**5.3. Step 3:  $S$  is an isomorphism from  $\mathcal{H}^r$  to itself for any  $r \in (1/2 - \alpha, \alpha - 1/2)$ .** Let us now look the operators  $S, \tilde{S}$  as their extension on  $\mathcal{H}^r$  defined as

$$\begin{aligned} S &: n^{-r} \varphi_n \mapsto n^{-r} q_n, \\ \tilde{S} &: n^{-r} \varphi_n \mapsto n^{-r} \tilde{q}_n. \end{aligned}$$

Then,  $(n^{-r} q_n)_{n \in \mathbb{N}^*}$  is a Riesz basis in  $\mathcal{H}^r$  if and only if  $S$  is an isomorphism from  $\mathcal{H}^r$  to itself. Since  $S$  is a Fredholm operator of index 0, this is equivalent to  $\ker S = \{0\}$ . As in the case  $r = 0$  this is equivalent to show that for any  $(f_n)_{n \in \mathbb{N}^*} \in l^2$

$$(58) \quad \sum_{n \in \mathbb{N}^*} f_n n^{-r} q_n = 0 \Leftrightarrow f_n = 0 \quad \forall n \in \mathbb{N}^*.$$

Let us look at the different cases:

- $r \in [0, \alpha - 1/2)$ ,  
Then  $(n^{-r} f_n)_{n \in \mathbb{N}^*} \in l^2$ . And, as  $(q_n)_{n \in \mathbb{N}^*}$  is a Riesz basis in  $\mathcal{H}^0$ , it holds that

$$\sum_{n \in \mathbb{N}^*} f_n n^{-r} q_n = 0 \Leftrightarrow n^{-r} f_n = 0 \quad \forall n \in \mathbb{N}^* \Leftrightarrow f_n = 0 \quad \forall n \in \mathbb{N}^*,$$

thus  $S$  is indeed an isomorphism from  $\mathcal{H}^r$  to itself.

- $r \in (1/2 - \alpha, 0)$  Showing the property (58) is equivalent to show that  $(n^{-r}q_n)_{n \in \mathbb{N}^*}$  is  $\omega$ -independent in  $\mathcal{H}^r$ . We assume here by contradiction that  $(n^{-r}q_n)_{n \in \mathbb{N}^*}$  is not  $\omega$ -independent in  $\mathcal{H}^r$ . Then, there exists a non trivial  $(f_n)_{n \in \mathbb{N}^*} \in l^2$  such that  $\sum_{n \in \mathbb{N}^*} f_n n^{-r} q_n = 0$ . Projecting this on the vector  $m^{-r} \varphi_m$ , we get

$$\begin{aligned}
 0 &= \left\langle \sum_{n \in \mathbb{N}^*} f_n n^{-r} q_n, m^{-r} \varphi_m \right\rangle_{\mathcal{H}^r} \\
 &= \left\langle \sum_{p \in \mathbb{N}^*} \varphi_p \sum_{n \in \mathbb{N}^*} \frac{f_n n^{-r}}{\lambda_n - \lambda_p + \lambda}, m^{-r} \varphi_m \right\rangle_{\mathcal{H}^r} \\
 (59) \quad &= \sum_{n \in \mathbb{N}^*} \frac{f_n n^{-r} m^r}{\lambda_n - \lambda_m + \lambda}
 \end{aligned}$$

Let us set  $f := \sum_{n \in \mathbb{N}^*} f_n n^r \varphi_n$ . As  $(f_n)_{n \in \mathbb{N}^*} \in l^2$ , it holds that  $f \in \mathcal{H}^{-r}$ . Since  $(n^r \varphi_n)_{n \in \mathbb{N}^*}$  is a Riesz basis of  $\mathcal{H}^{-r}$  and  $(f_n)_{n \in \mathbb{N}^*}$  is non trivial, we have that  $f \neq 0$ . Since we showed in the previous case that  $S$  is an isomorphism on  $\mathcal{H}^r$  for  $r \in [0, \alpha - 1/2)$ , then  $\tilde{S}$  is also an isomorphism from  $\mathcal{H}^r$  to itself for  $r \in [0, \alpha - 1/2)$ , as the adjoint of  $S$ . Thus,  $(n^{-r} \tilde{q}_n)_{n \in \mathbb{N}^*}$  is a Riesz basis of  $\mathcal{H}^r$  for  $r \in [0, \alpha - 1/2)$ . Then for  $r \in (1/2 - \alpha, 0)$ ,  $(n^r \tilde{q}_n)_{n \in \mathbb{N}^*}$  is a Riesz basis of  $\mathcal{H}^{-r}$ . In particular  $(n^r \tilde{q}_n)_{n \in \mathbb{N}^*}$  is  $\mathcal{H}^{-r}$ -dense. This ensures that there exists  $m_0 \in \mathbb{N}^*$  such that  $\langle f, m_0^r \tilde{q}_{m_0} \rangle_{\mathcal{H}^{-r}} \neq 0$ . It follows that

$$\langle f, m_0^r \tilde{q}_{m_0} \rangle_{\mathcal{H}^{-r}} = \sum_{n \in \mathbb{N}^*} \frac{f_n n^{-r} m_0^r}{\lambda_n - \lambda_{m_0} + \lambda} \neq 0$$

which contradicts (59). So  $(n^{-r}q_n)_{n \in \mathbb{N}^*}$  is  $\omega$ -independent in  $\mathcal{H}^r$  and we conclude that  $S$  is an isomorphism from  $\mathcal{H}^r$  to itself for any  $r \in (1/2 - \alpha, 0)$ .

We conclude that  $S$  is an isomorphism from  $\mathcal{H}^r$  to itself for any  $r \in (1/2 - \alpha, \alpha - 1/2)$ .

**5.4. Step 4: an explicit candidate for  $K$ .** Consider  $r \in (1/2 - \alpha, \alpha - 1/2)$ . Let us set

$$(60) \quad \tau : n^{-r} \varphi_n \mapsto b_n n^{-r} \varphi_n,$$

and

$$K : \varphi_n \mapsto K(\varphi_n) = K_n.$$

For any  $r \in (1/2 - \alpha, \alpha - 1/2)$ , thanks to (8), we can see that the operator  $\tau$  is an isomorphism from  $\mathcal{H}^r$  to  $\tau(\mathcal{H}^r)$ . And notice that this last space is a priori not  $\mathcal{H}^r$  when  $\gamma \neq 0$ . Thus, by seeing the expression of the operator  $T$  as follow

$$(61) \quad T : n^{-r} \varphi_n \mapsto -K_n n^{-r} \tau q_n,$$

the boundedness by above and below of  $K_n$  does not suffice to conclude that  $T$  is an isomorphism from  $\mathcal{H}^r$  to itself as done in [19]. Here, we rather see the operator as follow

$$T : n^{-r} \varphi_n \mapsto -K_n b_n \left( \frac{1}{b_n} n^{-r} \tau q_n \right).$$

In Section 5.6, we will show that, under our assumption on  $\gamma$ , the operator  $n^{-r} \varphi_n \mapsto b_n^{-1} n^{-r} \tau q_n$  is an isomorphism from  $\mathcal{H}^r$  to itself and then  $(b_n^{-1} n^{-r} \tau q_n)_{n \in \mathbb{N}^*}$  forms a Riesz basis of  $\mathcal{H}^r$ . Note that the latter is not obvious *a priori*, since  $\tau$  and  $S$  do not commute thus  $b_n^{-1} n^{-r} \tau q_n \leq n^{-r} q_n$ . From this we will be able to deduce that the operator  $T$  is an isomorphism from  $\mathcal{H}^r$  to itself if  $(K_n b_n)_{n \in \mathbb{N}^*}$  is uniformly bounded by above and below.



Recall that our goal is to have  $T$  that is a solution to (36) and (38). In this section we would like to construct  $K_n$  such that the normalization (38) holds and such that  $(K_n b_n)_{n \in \mathbb{N}}$  is bounded by above. The question of whether  $(K_n b_n)_{n \in \mathbb{N}}$  is bounded by below is the object of Steps (5)-(6). We first state the following Lemma

**LEMMA 5.7.** *There exists a unique sequence  $(-K_n)_{n \in \mathbb{N}^*}$  such that for any  $\varepsilon \in (0, \alpha - 1)$  the condition (38) holds in  $\tau(\mathcal{H}^{-\frac{1}{2}-\varepsilon})$  and  $(-K_n b_n n^{-(\frac{1}{2}+\varepsilon)})_{n \in \mathbb{N}^*} \in l^2$ .*

*Proof.* Noticing that  $B = \sum_{n \in \mathbb{N}^*} b_n \varphi_n \in \mathcal{H}^{-\alpha/2}$  with  $c_1 \leq b_n \leq c_2 n^\gamma$ , (recall that we assume  $\beta = 0$  to ease the notation, see Appendix A.5 for  $\beta \neq 0$ ) it holds from definition (60) that  $\tau$  is an isomorphism from  $\mathcal{H}^r$  to  $\tau(\mathcal{H}^r)$  for any  $r \in (1/2 - \alpha, \alpha - 1/2)$ . Thus for any  $r \in (-1/2 - \alpha, \alpha - 1/2)$ ,  $\tau(n^{-r} \varphi_n)_{n \in \mathbb{N}^*}$  forms a Riesz basis of  $\tau(\mathcal{H}^r)$ . Then the expression

$$(62) \quad B = \sum_{n \in \mathbb{N}^*} n^{-\frac{1}{2}-\varepsilon} \tau(n^{\frac{1}{2}+\varepsilon} \varphi_n)$$

makes sense in  $\tau(\mathcal{H}^{-\frac{1}{2}-\varepsilon})$  for any  $\varepsilon > 0$ , since  $(n^{-\frac{1}{2}-\varepsilon})_{n \in \mathbb{N}^*} \in l^2$ . In particular, the equation (62) shows that  $B \in \tau(\mathcal{H}^{-\frac{1}{2}-\varepsilon})$  for all  $\varepsilon > 0$ . The condition (38) can be expressed in a weak sense as

$$(63) \quad \sum_{n \in \mathbb{N}^*} -K_n b_n \tau q_n = \sum_{n \in \mathbb{N}^*} b_n \varphi_n,$$

and we yet know that its right side belongs to  $\tau(\mathcal{H}^{-\frac{1}{2}-\varepsilon})$  for all  $\varepsilon > 0$ . This means, in particular, that the left-hand side does too. As a consequence  $(-K_n b_n n^{-(\frac{1}{2}+\varepsilon)})_{n \in \mathbb{N}^*} \in l^2$  for all  $\varepsilon \in (0, \alpha - 1)$ . Indeed, in view of step 5.2,  $(n^{\frac{1}{2}+\varepsilon} q_n)$  is a Riesz basis of  $\mathcal{H}^{-\frac{1}{2}-\varepsilon}$  for any  $\varepsilon \in (0, \alpha - 1)$ , and since  $\tau$  is an isomorphism from  $\mathcal{H}^{-\frac{1}{2}-\varepsilon}$  to  $\tau(\mathcal{H}^{-\frac{1}{2}-\varepsilon})$ , then  $\tau(n^{\frac{1}{2}+\varepsilon} q_n)_n$  is a Riesz basis of  $\tau(\mathcal{H}^{-\frac{1}{2}-\varepsilon})$ .  $\square$

The regularity given by Lemma 5.7 is not enough *a priori*. To tackle this, we proceed as in [19] and exploit the special structure of  $(q_n)_{n \in \mathbb{N}}$  and the fact that  $(q_n)_{n \in \mathbb{N}} - \lambda^{-1}$  is more regular than  $(q_n)_{n \in \mathbb{N}}$  (see Lemma 5.3). We set

$$(64) \quad k_n := -(K_n b_n + \lambda) \quad \forall n \in \mathbb{N}^*.$$

**LEMMA 5.8.** *Consider  $r \in (1/2 - \alpha, \alpha - 1/2)$ . There exists  $\bar{\varepsilon} > 0$  such that the sequence  $(k_n n^\varepsilon)_{n \in \mathbb{N}^*}$  is uniformly bounded for any  $\varepsilon \in (0, \bar{\varepsilon})$ . In particular, the sequence  $(K_n b_n)_{n \in \mathbb{N}^*}$  defined by (64) is uniformly bounded.*

*Proof of Lemma 5.8.* For any  $\varepsilon > 0$ , it holds that  $(n^{-\frac{1}{2}-\varepsilon} \lambda)_{n \in \mathbb{N}^*} \in l^2$ , thus using Lemma 5.7 and (64), we have for all  $\varepsilon \in (0, \alpha - 1)$  that  $(k_n n^{-\frac{1}{2}-\varepsilon})_{n \in \mathbb{N}^*} \in l^2$ , and in particular

$$(65) \quad (k_n n^{-r})_{n \in \mathbb{N}^*} \in l^2, \quad \forall r \in \left(\frac{1}{2}, \alpha - \frac{1}{2}\right).$$

Still using Lemma 5.7, the equation (63) makes sense in  $\tau(\mathcal{H}^{-\frac{1}{2}-\varepsilon})$ , and by expressing  $k_n$  and  $\tau q_n$  it becomes

$$\sum_{n \in \mathbb{N}^*} \lambda \sum_{p \in \mathbb{N}^*} \frac{b_p \varphi_p}{\lambda_n - \lambda_p + \lambda} + \sum_{n \in \mathbb{N}^*} k_n \tau q_n = \sum_{n \in \mathbb{N}^*} b_n \varphi_n.$$

This yields to the following

$$(66) \quad \sum_{n \in \mathbb{N}^*} \lambda \sum_{p \in \mathbb{N}^* \setminus \{n\}} \frac{b_p \varphi_p}{\lambda_n - \lambda_p + \lambda} = - \sum_{n \in \mathbb{N}^*} k_n \tau q_n$$

which holds *a priori* in  $\tau(\mathcal{H}^{-\frac{1}{2}-\varepsilon})$  for any  $\varepsilon \in (0, \alpha - 1)$ . Actually, the equality (66) is more regular than  $\tau(\mathcal{H}^{-\frac{1}{2}-\varepsilon})$ . Indeed, if  $\alpha > 3/2$ , we can give a sense to (66) in  $\tau(\mathcal{H}^\varepsilon)$  for  $0 \leq \varepsilon \leq \alpha - 3/2$ . To prove that, let us first notice by Fubini's Theorem in  $\tau(\mathcal{H}^{-\frac{1}{2}-\varepsilon})$  that

$$\sum_{n \in \mathbb{N}^*} \lambda \sum_{p \in \mathbb{N}^* \setminus \{n\}} \frac{b_p \varphi_p}{\lambda_n - \lambda_p + \lambda} = \sum_{p \in \mathbb{N}^*} b_p \varphi_p \lambda \sum_{n \in \mathbb{N}^* \setminus \{p\}} \frac{1}{\lambda_n - \lambda_p + \lambda}.$$

And it holds that

$$\sum_{p \in \mathbb{N}^*} b_p \varphi_p \lambda \sum_{n \in \mathbb{N}^* \setminus \{p\}} \frac{1}{\lambda_n - \lambda_p + \lambda} = \sum_{p \in \mathbb{N}^*} \tau(p^{-\varepsilon} \varphi_p) \left( p^\varepsilon \lambda \sum_{n \in \mathbb{N}^* \setminus \{p\}} \frac{1}{\lambda_n - \lambda_p + \lambda} \right).$$

Since  $\tau(p^{-\varepsilon} \varphi_p)_{p \in \mathbb{N}^*}$  is a Riesz basis of  $\tau(\mathcal{H}^\varepsilon)$ , it suffices to show that

$$\left( p^\varepsilon \lambda \sum_{n \in \mathbb{N}^* \setminus \{p\}} \frac{1}{\lambda_n - \lambda_p + \lambda} \right)_{p \in \mathbb{N}^*} \in l^2$$

to conclude that the left hand side of (66) belongs to  $\tau(\mathcal{H}^\varepsilon)$ . Using Lemma 5.2 and Lemma 5.3, we have that

$$\begin{aligned} \left\| \left( p^\varepsilon \lambda \sum_{n \in \mathbb{N}^* \setminus \{p\}} \frac{1}{\lambda_n - \lambda_p + \lambda} \right)_{p \in \mathbb{N}^*} \right\|_{l^2}^2 &= \sum_{p \in \mathbb{N}^*} p^{2\varepsilon} \lambda^2 \left| \sum_{n \in \mathbb{N}^* \setminus \{p\}} \frac{1}{\lambda_n - \lambda_p + \lambda} \right|^2 \\ &\leq \frac{\lambda^2}{C(\lambda)} \sum_{p \in \mathbb{N}^*} p^{2\varepsilon} \left( \sum_{n \in \mathbb{N}^* \setminus \{p\}} \frac{1}{|\lambda_n - \lambda_p|} \right)^2 \\ &\lesssim \sum_{p \in \mathbb{N}^*} p^{2\varepsilon + 2(1-\alpha)} \log^2(p), \end{aligned}$$

and this converges for  $\varepsilon \in (0, \alpha - 3/2)$ . So if  $\alpha > 3/2$ , the left hand side of (66) holds in  $\tau(\mathcal{H}^\varepsilon)$  for  $\varepsilon \in (0, \alpha - 3/2)$ . Consider now the right hand side of (66). We have that

$$(67) \quad \sum_{n \in \mathbb{N}^*} k_n \tau q_n = \sum_{n \in \mathbb{N}^*} k_n n^\varepsilon \tau(n^{-\varepsilon} q_n).$$

Recalling that  $(n^{-\varepsilon} q_n)_{n \in \mathbb{N}^*}$  is a Riesz basis of  $\mathcal{H}^\varepsilon$  for  $\varepsilon \in (0, \alpha - 3/2)$ , it holds by the isomorphism property of  $\tau$  that  $\tau(n^{-\varepsilon} q_n)_{n \in \mathbb{N}^*}$  is a Riesz basis of  $\tau(\mathcal{H}^\varepsilon)$ . Thus in view of (66) and (67), for any  $\varepsilon \in (0, \alpha - 3/2)$ ,

$$(k_n n^\varepsilon)_{n \in \mathbb{N}^*} \in l^2.$$

In particular, we conclude that if  $\alpha > 3/2$ ,  $(k_n)_{n \in \mathbb{N}^*} \in l^\infty$ . In view of (64), we have  $K_n b_n = -(\lambda + k_n)$  and since  $(k_n)_{n \in \mathbb{N}^*}$  belongs to  $l^\infty$ , we get that

$$(68) \quad (K_n b_n)_{n \in \mathbb{N}^*} \in l^\infty \quad \forall \alpha > 3/2.$$

Consider now  $1 < \alpha \leq 3/2$ . In this case the gain of regularity in (66) is not enough to deduce the boundedness of  $K_n b_n$ . However we are able to show that the equality (66) holds in  $\tau(\mathcal{H}^{-\varepsilon})$ , for  $\varepsilon > 3/2 - \alpha$ . Indeed, similarly to the previous case, the following holds

$$\left\| \left( p^{-\varepsilon} \lambda \sum_{n \in \mathbb{N}^* \setminus \{p\}} \frac{1}{\lambda_n - \lambda_p + \lambda} \right)_{p \in \mathbb{N}^*} \right\|_{l^2}^2 = \sum_{p \in \mathbb{N}^*} p^{-2\varepsilon} \lambda^2 \left| \sum_{n \in \mathbb{N}^* \setminus \{p\}} \frac{1}{\lambda_n - \lambda_p + \lambda} \right|^2$$

$$\begin{aligned} &\leq \frac{\lambda^2}{C(\lambda)} \sum_{p \in \mathbb{N}^*} p^{-2\varepsilon} \left( \sum_{n \in \mathbb{N}^* \setminus \{p\}} \frac{1}{|\lambda_n - \lambda_p|} \right)^2 \\ &\lesssim \sum_{p \in \mathbb{N}^*} p^{-2\varepsilon + 2(1-\alpha)} \log^2(p). \end{aligned}$$

This converges for  $\varepsilon > 3/2 - \alpha$  and then the left hand side of (66) belongs to  $\tau(\mathcal{H}^{-\varepsilon})$ . Similarly as before and in view of (66),

$$(69) \quad (k_n n^{-\varepsilon})_{n \in \mathbb{N}^*} \in l^2 \quad \forall \varepsilon \in \left( \frac{3}{2} - \alpha, \alpha - \frac{1}{2} \right).$$

We can see that we obtain  $(\alpha - 1)$  gain of regularity between (69) and (65). So in the following, we will make an iterative principle to gain at each order this  $(\alpha - 1)$  regularity in order to prove Lemma 5.8. So let us go back to the equation (63) and expressing  $\tau q_n$

$$\sum_{n \in \mathbb{N}^*} -K_n b_n \left( \frac{b_n \varphi_n}{\lambda} + \sum_{p \in \mathbb{N}^* \setminus \{n\}} \frac{b_p \varphi_p}{\lambda_n - \lambda_p + \lambda} \right) = \sum_{n \in \mathbb{N}^*} b_n \varphi_n.$$

Replacing  $-K_n b_n$  by  $(\lambda + k_n)$  in the first term only and using Fubini's Theorem in  $\tau(\mathcal{H}^{-\frac{1}{2}-\varepsilon})$  we get:

$$\sum_{n \in \mathbb{N}^*} \frac{k_n b_n}{\lambda} \varphi_n - \sum_{p \in \mathbb{N}^*} b_p \varphi_p \sum_{n \in \mathbb{N}^* \setminus \{p\}} \frac{K_n b_n}{\lambda_n - \lambda_p + \lambda} = 0.$$

Since  $(\varphi_n)_{n \in \mathbb{N}^*}$  is a Riesz basis, we have by identification that

$$\frac{k_m b_m}{\lambda} = b_m \sum_{n \in \mathbb{N}^* \setminus \{m\}} \frac{K_n b_n}{\lambda_n - \lambda_m + \lambda} \quad \forall m \in \mathbb{N}^*.$$

Thanks to the fact that  $b_m \neq 0$  for all  $m \in \mathbb{N}^*$  and  $K_n b_n = -(\lambda + k_n)$ , this implies that

$$k_m = -\lambda \sum_{n \in \mathbb{N}^* \setminus \{m\}} \frac{\lambda + k_n}{\lambda_n - \lambda_m + \lambda} \quad \forall m \in \mathbb{N}^*.$$

Let us now set  $\lambda = e_n^0$  and  $k_n = k_n^0$ . It holds that

$$(70) \quad k_m = -\lambda \sum_{n \in \mathbb{N}^* \setminus \{m\}} \frac{e_n^0 + k_n^0}{\lambda_n - \lambda_m + \lambda} \quad \forall m \in \mathbb{N}^*.$$

Thus  $k_m$  can be rewritten as  $k_m = e_m^1 + k_m^1$  where

$$e_m^1 = -\lambda \sum_{n \in \mathbb{N}^* \setminus \{m\}} \frac{e_n^0}{\lambda_n - \lambda_m + \lambda}, \quad k_m^1 = -\lambda \sum_{n \in \mathbb{N}^* \setminus \{m\}} \frac{k_n^0}{\lambda_n - \lambda_m + \lambda} \quad \forall m \in \mathbb{N}^*.$$

And using Lemma 5.3, we have that

$$(71) \quad |e_m^1| \lesssim m^{1-\alpha} \log m + m^{-\alpha} \lesssim 1.$$

So, we focus on the regularity of  $k_n^1$ . We define  $\varepsilon_0 := \frac{3}{2} - \alpha$  and we have

$$\|(m^{-\varepsilon} k_m^1)_m\|_{l^2} \lesssim \sum_{m \in \mathbb{N}^*} m^{-2\varepsilon} \left( \sum_{n \in \mathbb{N}^* \setminus \{m\}} \frac{k_n^0}{\lambda_n - \lambda_m + \lambda} \right)^2.$$

Using Cauchy-Schwartz inequality and Lemma 5.2, we get

$$\|(m^{-\varepsilon} k_m^1)_m\|_{l^2} \lesssim \sum_{m \in \mathbb{N}^*} m^{-2\varepsilon} \left( \sum_{n \in \mathbb{N}^* \setminus \{m\}} \frac{|k_n^0|^2}{|\lambda_n - \lambda_m|} \right) \left( \sum_{n \in \mathbb{N}^* \setminus \{m\}} \frac{1}{|\lambda_n - \lambda_m|} \right).$$

Then applying Lemma (5.3) and Fubini's equality, we have

$$\begin{aligned} \|(m^{-\varepsilon} k_m^1)_m\|_{l^2} &\lesssim \sum_{m \in \mathbb{N}^*} m^{-2\varepsilon+1-\alpha} \log m \left( \sum_{n \in \mathbb{N}^* \setminus \{m\}} \frac{|k_n^0|^2}{|\lambda_n - \lambda_m|} \right) \\ &\lesssim \sum_{n \in \mathbb{N}^*} |k_n^0|^2 \left( \sum_{m \in \mathbb{N}^* \setminus \{n\}} \frac{m^{-2\varepsilon+1-\alpha} \log m}{|\lambda_n - \lambda_m|} \right) \\ &\lesssim \sum_{n \in \mathbb{N}^*} |k_n^0|^2 n^{-2(\varepsilon-1+\alpha)+\delta}, \end{aligned}$$

for any  $\delta > 0$ . As  $k_n^0 = k_n$  and in view of (69), for any  $\varepsilon - (1 - \alpha) \in (3/2 - \alpha, \alpha - 1/2)$  (i.e.  $\varepsilon \in (5/2 - 2\alpha, 1/2)$ ) there exists  $\delta > 0$  such that this converges. Thus for  $\alpha \in (1, 3/2]$ , the following holds

$$(72) \quad (n^{-\varepsilon} k_n^1)_{n \in \mathbb{N}^*} \in l^2, \quad \forall \varepsilon \in \left( \frac{5}{2} - 2\alpha, \frac{1}{2} \right).$$

If  $\alpha > \frac{5}{4}$ , we have  $\varepsilon_1 = 2\alpha - \frac{5}{2} > 0$  and then  $(n^\delta k_n^1)_{n \in \mathbb{N}^*} \in l^2$  for all  $\delta \in [0, \varepsilon_1)$ . This ensures that  $(k_n^1)_{n \in \mathbb{N}^*}$  is uniformly bounded. And combining with (71), it ensures that  $(k_n)_{n \in \mathbb{N}^*}$  also is uniformly bounded. Therefore  $(K_n b_n)_{n \in \mathbb{N}^*}$  is uniformly bounded because  $K_n b_n = -(\lambda + k_n)$ . If  $1 < \alpha \leq \frac{5}{4}$ , the uniform boundedness of  $K_n b_n$  cannot be immediately deduced, but we need to iterate again. So using the definition (70), we have from  $k_n^0 = k_n$ , that

$$\begin{aligned} k_m &= -\lambda \sum_{n \in \mathbb{N}^* \setminus \{m\}} \frac{e_n^0 + k_n^0}{\lambda_n - \lambda_m + \lambda} \\ &= -\lambda \sum_{n \in \mathbb{N}^* \setminus \{m\}} \frac{e_n^0}{\lambda_n - \lambda_m + \lambda} - \lambda \sum_{n \in \mathbb{N}^* \setminus \{m\}} \frac{k_n}{\lambda_n - \lambda_m + \lambda} \\ &= -\lambda \sum_{n \in \mathbb{N}^* \setminus \{m\}} \frac{e_n^0}{\lambda_n - \lambda_m + \lambda} - \lambda \sum_{n \in \mathbb{N}^* \setminus \{m\}} \frac{1}{\lambda_n - \lambda_m + \lambda} \left( -\lambda \sum_{p \in \mathbb{N}^* \setminus \{n\}} \frac{e_p^0 + k_p^0}{\lambda_p - \lambda_n + \lambda} \right). \end{aligned}$$

Referring to the definitions of  $e_n^1$  and  $k_n^1$ , it holds that

$$(73) \quad \begin{aligned} k_m &= e_m^1 - \lambda \sum_{n \in \mathbb{N}^* \setminus \{m\}} \frac{e_n^1}{\lambda_n - \lambda_m + \lambda} - \lambda \sum_{n \in \mathbb{N}^* \setminus \{m\}} \frac{k_n^1}{\lambda_n - \lambda_m + \lambda} \\ &= e_m^1 + e_m^2 + k_m^2, \end{aligned}$$

where

$$e_m^2 = -\lambda \sum_{n \in \mathbb{N}^* \setminus \{m\}} \frac{e_n^1}{\lambda_n - \lambda_m + \lambda}, \quad k_m^2 = -\lambda \sum_{n \in \mathbb{N}^* \setminus \{m\}} \frac{k_n^1}{\lambda_n - \lambda_m + \lambda} \quad \forall m \in \mathbb{N}^*.$$

Since  $k_m = e_m^1 + k_m^1$ , we have in view of (73) that  $k_m^1 = e_m^2 + k_m^2$ . This means that if  $(e_m^2)_{m \in \mathbb{N}}$  and  $(k_m^2)_{m \in \mathbb{N}}$  are uniformly bounded, then  $(k_m^1)_{m \in \mathbb{N}}$  is and consequently  $(k_m)_{m \in \mathbb{N}}$ . Let us first notice that combining (71) and Lemma 5.3, we have

$$|e_m^2| \lesssim m^{1-\alpha} \log n + m^{-\alpha} \lesssim 1.$$

For the  $(k_m)_{m \in \mathbb{N}}$ , we use again Lemma (5.3) and Fubini's Theorem to have, similarly as previously:

$$\begin{aligned} \|(m^{-\varepsilon} k_m^2)_m\|_{l^2} &\lesssim \sum_{m \in \mathbb{N}^*} m^{-2\varepsilon} \left( \sum_{n \in \mathbb{N}^* \setminus \{m\}} \frac{k_n^1}{\lambda_n - \lambda_m + \lambda} \right)^2 \\ &\lesssim \sum_{n \in \mathbb{N}^*} |k_n^1|^2 n^{-2(\varepsilon-1+\alpha)+\delta}, \end{aligned}$$

for any  $\delta > 0$ . In view of (72), there exists  $\delta > 0$  such that this converges for any  $\varepsilon - (1 - \alpha) \in (\frac{5}{2} - 2\alpha, \frac{1}{2})$  meaning for  $\varepsilon \in (\frac{7}{2} - 3\alpha, \frac{3}{2} - \alpha)$ . Thus, for  $\alpha \in (1, 3/2]$ ,

$$(n^{-\varepsilon} k_n^2)_{n \in \mathbb{N}^*} \in l^2, \quad \forall \varepsilon \in \left( \frac{7}{2} - 3\alpha, \frac{3}{2} - \alpha \right).$$

If  $\alpha > 7/6$ , then  $\varepsilon_2 = 3\alpha - \frac{7}{2} > 0$  and  $(n^\delta k_n^2)_{n \in \mathbb{N}^*} \in l^2$ , for any  $\delta \in [0, \varepsilon_2)$ . This ensures that  $(k_n^2)_{n \in \mathbb{N}^*}$  is uniformly bounded and then  $(k_n^2)_{n \in \mathbb{N}^*}$  is. Thus,  $(K_n b_n = -(e_n^0 + e_n^1 + e_n^2 + k_n^2))_{n \in \mathbb{N}^*}$  is uniformly bounded.

If  $1 < \alpha \leq \frac{7}{6}$ , we can continue the induction and have  $k_n^i = e_n^{i+1} + k_n^{i+1}$  for all  $i \in \mathbb{N}$  where

$$e_n^{i+1} = -\lambda \sum_{m \in \mathbb{N}^* \setminus \{n\}} \frac{e_m^i}{\lambda_m - \lambda_n + \lambda}, \quad k_n^{i+1} = -\lambda \sum_{m \in \mathbb{N}^* \setminus \{n\}} \frac{k_m^i}{\lambda_m - \lambda_n + \lambda} \quad \forall n \in \mathbb{N}^*, i \in \mathbb{N}.$$

And for any  $n \in \mathbb{N}^*$ ,

$$K_n b_n = - \left( k_n^i + \sum_{j=0}^i e_n^j \right), \quad \forall i \in \mathbb{N}$$

Thus, based on the previous computations, we obtain by induction the following:

$$(74) \quad |e_n^i| \lesssim n^{1-\alpha} \log n + n^{-\alpha} \lesssim 1,$$

$$(75) \quad \|(n^{-\varepsilon} k_n^{i+1})_{n \in \mathbb{N}^*}\|_{l^2} \lesssim \sum_{n \in \mathbb{N}^*} |k_n^i|^2 n^{-2(\varepsilon-1+\alpha)+\delta} \log^2 n$$

for any  $\delta > 0$  and (75) converges for  $\varepsilon \in (-\varepsilon_i, -\varepsilon_{i-2})$  with  $\varepsilon_i = (\alpha - 1)i - \varepsilon_0$  for all  $i \geq 2$ . It is worth stressing that there exists a finite  $i_0 \in \mathbb{N}$  such that  $\varepsilon_{i_0} > 0$ . In this case we have  $(n^\delta k_n^{i_0+1})_{n \in \mathbb{N}^*} \in l^2$  for any  $\delta \in [0, \varepsilon_{i_0})$  meaning that  $(k_n^{i_0+1})_{n \in \mathbb{N}^*} \in l^\infty$ . Combining this with (74) we conclude that

$$(76) \quad \left( K_n b_n = - \left( k_n^{i_0} + \sum_{j=0}^{i_0} e_n^j \right) \right)_{n \in \mathbb{N}^*} \in l^\infty \quad \forall \alpha \in (1, 3/2].$$

Therefore we conclude Lemma 5.8 from (68) and (76).  $\square$

**5.5. Step 5:  $T$  is bounded from  $\mathcal{H}^r$  to  $\mathcal{H}^r$  and satisfies the operator equality.** In this section, we would like to prove that the backstepping transformation  $T$  is bounded and the operator equality (36) holds at least in some  $\mathcal{H}^s$  space. More precisely, we prove the following Lemma:

**LEMMA 5.9.** *The operator  $T$  given by (43) is a bounded operator from  $\mathcal{H}^r$  to  $\mathcal{H}^r$  for any  $r \in (1/2 - \alpha + \gamma, \alpha - 1/2)$ . Moreover, we have the following operator equality,*

$$(77) \quad T(\mathcal{A} + BK) = (\mathcal{A} - \lambda I)T \quad \text{in } \mathcal{L}(\mathcal{H}^{\alpha/2+s}, \mathcal{H}^{-\alpha/2+s}), \quad \forall s \in \left(-\frac{\alpha-1}{2} + \gamma, \frac{\alpha-1}{2} - \gamma\right).$$

*Proof.* We recall here that for any  $r \in (\frac{1}{2} - \alpha, \alpha - \frac{1}{2})$ —and therefore for any  $r \in (\frac{1}{2} - \alpha + \gamma, \alpha - \frac{1}{2})$ —the operator  $T$  is defined as  $T : n^{-r}\varphi_n \mapsto -n^{-r}K_n\tau q_n$  with  $\tau$  an isomorphism from  $\mathcal{H}^r$  to  $\tau(\mathcal{H}^r)$  and  $(K_n)_{n \in \mathbb{N}^*}$  uniformly bounded. Since the space  $\tau(\mathcal{H}^r)$  is neither equal nor included in  $\mathcal{H}^r$  a priori, what we are able to say is that  $T$  is defined on  $\mathcal{H}^r$  to  $\tau(\mathcal{H}^r)$ . Here we will exploit the regularity on  $K_nb_n$  to first show that  $T$  is well defined from  $\mathcal{H}^r$  to itself and then deduce that it is bounded.

Let  $r \in (\frac{1}{2} - \alpha + \gamma, \alpha - \frac{1}{2})$  and  $a = \sum_{n \in \mathbb{N}^*} a_n n^{-r} \varphi_n \in \mathcal{H}^r$ , the following holds

$$\begin{aligned} Ta &= - \sum_{n \in \mathbb{N}^*} a_n K_n n^{-r} \tau q_n \\ &= - \sum_{n \in \mathbb{N}^*} a_n K_n n^{-r} \sum_{p \in \mathbb{N}^*} \frac{b_p \varphi_p}{\lambda_n - \lambda_p + \lambda} \\ &= -\frac{1}{\lambda} \sum_{n \in \mathbb{N}^*} a_n K_n n^{-r} b_n \varphi_n - \sum_{n \in \mathbb{N}^*} a_n K_n n^{-r} \sum_{p \in \mathbb{N}^* \setminus \{n\}} \frac{b_p \varphi_p}{\lambda_n - \lambda_p + \lambda} \\ &= -\frac{1}{\lambda} \sum_{n \in \mathbb{N}^*} a_n K_n b_n n^{-r} \varphi_n - \sum_{n \in \mathbb{N}^*} a_n K_n n^{-r} \tau S_c(\varphi_n). \end{aligned}$$

By setting

$$T_1 a = -\frac{1}{\lambda} \sum_{n \in \mathbb{N}^*} a_n K_n b_n n^{-r} \varphi_n, \quad T_2 a = - \sum_{n \in \mathbb{N}^*} a_n K_n n^{-r} \tau S_c(\varphi_n),$$

one has  $Ta = T_1 a + T_2 a$ . Notice that since  $(a_n)_{n \in \mathbb{N}^*} \in l^2$  and  $(K_n b_n)_{n \in \mathbb{N}^*} \in l^\infty$ , the term  $T_1 a$  belongs to  $\mathcal{H}^r$  and in particular we have:

$$(78) \quad \|T_1 a\|_{\mathcal{H}^r} \lesssim \|f\|_{\mathcal{H}^r}.$$

So, it suffices to show that the second term  $T_2 a$  belongs to  $\mathcal{H}^r$  to conclude that  $Ta \in \mathcal{H}^r$ . We have the following claim.

**CLAIM 5.10.** *For any  $r \in (\frac{1}{2} - \alpha, \alpha - \frac{1}{2})$  and  $s \geq \gamma$ , the space  $\tau(\mathcal{H}^{r+s})$  endowed with the norm*

$$(79) \quad \|f\|_{\tau(\mathcal{H}^{r+s})} := \|\tau^{-1}f\|_{\mathcal{H}^{r+s}}$$

*is a Hilbert space, continuously embedded in  $\mathcal{H}^r$ .*

*Proof.* For any  $q \in \mathbb{R}$ , since  $\tau$  is an isomorphism from  $\mathcal{H}^q$  into  $\tau(\mathcal{H}^q)$ , the space  $\mathcal{H}^q$  is a Hilbert space with the norm (79). Consider  $g \in \tau(\mathcal{H}^{r+s})$ , there exists  $f = \sum_{n \in \mathbb{N}^*} f_n n^{-r-s} \varphi_n \in \mathcal{H}^{r+s}$  such

that  $g = \tau(f)$ , thus

$$g = \sum_{n \in \mathbb{N}^*} (f_n b_n n^{-s}) n^{-r} \varphi_n.$$

From (11) (with  $\beta = 0$ ) and the fact that  $\gamma \leq s$ , we have  $b_n \leq c_2 n^s$ , and we observe that

$$\|g\|_{\mathcal{H}^r}^2 = \|(f_n b_n n^{-s})_{n \in \mathbb{N}^*}\|_{l^2}^2 = \sum_{n \in \mathbb{N}^*} |f_n b_n n^{-s}|^2 \lesssim \sum_{n \in \mathbb{N}^*} |f_n|^2 = \|g\|_{\tau(\mathcal{H}^{r+s})}^2.$$

and this concludes the Claim 5.10.  $\square$

Going back to the norm of  $T_2 a$  now, we have based on Claim 5.10,

$$\|T_2 a\|_{\mathcal{H}^r} = \left\| \sum_{n \in \mathbb{N}^*} a_n K_n n^{-r} \tau S_c(\varphi_n) \right\|_{\mathcal{H}^r} \lesssim \left\| \sum_{n \in \mathbb{N}^*} a_n K_n \tau S_c(n^{-r} \varphi_n) \right\|_{\tau(\mathcal{H}^{r+\gamma})}.$$

From (49) and Lemma 5.4,  $S_c$  is a continuous operator from  $\mathcal{H}^r$  to  $\mathcal{H}^{r+\varepsilon}$  for any  $\varepsilon \in [0, \min\{(\alpha - 1)/2, \alpha + r - 1/2\}]$ . Since  $r \in (1/2 - \alpha + \gamma, \alpha - 1/2)$  and  $\gamma \in (0, (\alpha - 1)/2)$  by assumption,  $\gamma$  belongs to  $[0, \min\{(\alpha - 1)/2, \alpha + r - 1/2\})$  and  $S_c$  is a continuous operator  $\mathcal{H}^r$  to  $\mathcal{H}^{r+\gamma}$  and, using the isomorphism property of  $\tau$ , one has

$$\left\| \sum_{n \in \mathbb{N}^*} a_n K_n \tau S_c(n^{-r} \varphi_n) \right\|_{\tau(\mathcal{H}^{r+\gamma})} \lesssim \left\| \sum_{n \in \mathbb{N}^*} a_n K_n \tau(n^{-r} \varphi_n) \right\|_{\tau(\mathcal{H}^r)}.$$

Using that  $\tau(n^{-r} \varphi_n)_{n \in \mathbb{N}^*}$  is a Riesz basis in  $\tau(\mathcal{H}^r)$  and  $(K_n)_{n \in \mathbb{N}^*}$  is uniformly bounded, we get

$$\left\| \sum_{n \in \mathbb{N}^*} a_n K_n \tau(n^{-r} \varphi_n) \right\|_{\tau(\mathcal{H}^r)} \lesssim \sum_{n \in \mathbb{N}^*} |a_n K_n|^2 \lesssim \sum_{n \in \mathbb{N}^*} |a_n|^2 = \|a\|_{\mathcal{H}^r}.$$

and combining this with (78), it holds that,

$$\|T a\|_{\mathcal{H}^r} \lesssim \|a\|_{\mathcal{H}^r},$$

and thus  $T$  is a bounded operator from  $\mathcal{H}^r$  to itself.

It remains now to prove the operator equality (77) to conclude the lemma. As, from Lemma 5.7, the equality (38) holds in  $\tau(\mathcal{H}^{-\alpha/2+s+\gamma})$  for any  $s \in (-(\alpha - 1)/2, (\alpha - 1)/2 - \gamma)$ , and in particular, from Claim 5.10, it holds in  $\mathcal{H}^{-\alpha/2+s}$ . Thus proving (77) amounts to show that

$$(80) \quad T\mathcal{A} + BK = (\mathcal{A} - \lambda I)T \quad \text{in } \mathcal{L}(\mathcal{H}^{\alpha/2+s}, \mathcal{H}^{-\alpha/2+s}), \quad \forall s \in \left(-\frac{\alpha-1}{2} + \gamma, \frac{\alpha-1}{2} - \gamma\right).$$

Observe first that all terms make sense : indeed,  $K : n^{-(\alpha/2+s)} \varphi_n \mapsto n^{-(\alpha/2+s)} K_n$  is a bounded operator from  $\mathcal{H}^{\alpha/2+s}$  to  $\mathbb{C}$  for any  $s > -(\alpha - 1)/2$ . On the other hand,  $B \in \mathcal{H}^{-\alpha/2}$  by assumption and in fact in  $\mathcal{H}^{-\alpha/2+s}$  for any  $s < (\alpha - 1)/2 - \gamma$  from (11) So  $B$  can be formally seen as an operator from  $\mathbb{C}$  to  $\mathcal{H}^{-\alpha/2+s}$  for any  $s \in (-(\alpha - 1)/2 + \gamma, (\alpha - 1)/2 - \gamma)$ . Thus  $BK$  is a bounded operator from  $\mathcal{H}^{\alpha/2+s}$  to  $\mathcal{H}^{-\alpha/2+s}$  for any  $s \in (-(\alpha - 1)/2, (\alpha - 1)/2)$ . Similarly, one can show that  $\mathcal{A}T$  and  $T\mathcal{A}$  are bounded operators from  $\mathcal{H}^{\alpha/2+s}$  to  $\mathcal{H}^{-\alpha/2+s}$  for any  $s \in (-(\alpha - 1)/2 + \gamma, (\alpha - 1)/2 - \gamma)$ , since  $T$  is a bounded operator from  $\mathcal{H}^r$  into itself for any  $r \in (1/2 - \alpha + \gamma, \alpha - 1/2)$ .

To show (80), it suffices to check that it holds against  $n^{-(\alpha/2+s)}\varphi_n$  for any  $n \in \mathbb{N}^*$  and  $s \in (-\alpha-1)/2+\gamma, (\alpha-1)/2-\gamma)$  which in turn amounts to show that it holds against any  $\varphi_n$  as the operators are linear. From the definition of  $T$  (see (43)) we have (in  $\mathcal{H}^{-(\alpha/2+s)}$ )

$$\begin{aligned} [T\mathcal{A} + BK - (\mathcal{A} - \lambda I)]\varphi_n &= \lambda_n T\varphi_n + BK_n - (\mathcal{A} - \lambda I)T\varphi_n \\ &= \lambda_n(-K_n) \sum_{p \in \mathbb{N}^*} \frac{b_p \varphi_p}{\lambda_n - \lambda_p + \lambda} + BK_n - (-K_n) \sum_{p \in \mathbb{N}^*} \frac{b_p(\lambda_p - \lambda)\varphi_p}{\lambda_n - \lambda_p + \lambda} \\ &= (-K_n) \sum_{p \in \mathbb{N}^*} b_p \varphi_p + BK_n \\ &= 0. \end{aligned}$$

□

**5.6. Step 6:  $T$  is an isomorphism from  $\mathcal{H}^r$  to itself.** To show that  $T$  is an isomorphism from  $\mathcal{H}^r$  to itself, we first show that  $\ker T = \{0\}$  in  $\mathcal{H}^{-\alpha/2}$ , then we deduce that  $(K_n b_n)_{n \in \mathbb{N}}$  is bounded by below and we deduce that an intermediary operator ( $\tilde{\tau}$  given below in Lemma 5.13) is an isomorphism and further that  $T$  is an isomorphism from  $\mathcal{H}^r$  into itself. In other words, we first show the following Lemma

**LEMMA 5.11.**  $\ker T^* = \{0\}$  in  $\mathcal{H}^{-\alpha/2}$ .

*Proof.* For that aim, we will perform the proof originally used in [11] for the Schrodinger equation and more recently in [20, 19]. The idea is to prove the following steps

- There exists  $\rho \in \mathbb{C}$  such that  $\mathcal{A} + BK + \lambda I + \rho I$  and  $\mathcal{A} + \rho I$  are invertible operator from  $\mathcal{H}^{\alpha/2}$  to  $\mathcal{H}^{-\alpha/2}$ .
- For such  $\rho$ , if  $\ker T^* \neq \{0\}$ , then  $(\mathcal{A} + \rho I)^{-1}$  has an eigenvector  $h$  which belongs to  $\ker T^*$ .
- No eigenvector of  $(\mathcal{A} + \rho I)^{-1}$  belong to  $\ker T^*$ .

From the two last steps, it holds that  $\ker T^* = \{0\}$  in  $\mathcal{H}^{-\alpha/2}$ . Although we have not consider  $\mathcal{A}$  to be decomposed in orthonormal eigenvector basis here, the proof of these three steps is very similar to what is done in [11, 20, 19] and the rigorous proof is provided in Appendix A.3. □

Now we want to show the following Lemma:

**LEMMA 5.12.** For any  $r \in (1/2 - \alpha + \gamma, \alpha - 1/2)$ ,  $T$  is an isomorphism from  $\mathcal{H}^r$  to itself.

Before proving this, let us first show the following:

**LEMMA 5.13.** For any  $r \in (1/2 - \alpha + \gamma, \alpha - 1/2)$ , the operator  $\tilde{\tau} : n^{-r}\varphi_n \mapsto \frac{1}{b_n}n^{-r}\tau q_n$  is a Fredholm operator of index 0 and an isomorphism from  $\mathcal{H}^r$  to  $\mathcal{H}^r$ .

*Proof of Lemma 5.13.* Let  $r \in (1/2 - \alpha + \gamma, \alpha - 1/2)$ , and  $f = \sum_{n \in \mathbb{N}^*} f_n n^{-r} \varphi_n \in \mathcal{H}^r$ . It holds that

$$\begin{aligned} \tilde{\tau}(f) &= \sum_{n \in \mathbb{N}^*} \frac{f_n}{b_n} n^{-r} \tau q_n \\ &= \sum_{n \in \mathbb{N}^*} \frac{f_n}{b_n} n^{-r} \sum_{p \in \mathbb{N}^*} \frac{b_p \varphi_p}{\lambda_n - \lambda_p + \lambda} \\ &= \frac{1}{\lambda} \sum_{n \in \mathbb{N}^*} f_n n^{-r} \varphi_n + \sum_{n \in \mathbb{N}^*} \frac{f_n}{b_n} n^{-r} \sum_{p \in \mathbb{N}^* \setminus \{n\}} \frac{b_p \varphi_p}{\lambda_n - \lambda_p + \lambda} \end{aligned}$$



$$= \frac{1}{\lambda} f + \tilde{\tau}_c(f)$$

where

$$\tilde{\tau}_c(f) = \sum_{n \in \mathbb{N}^*} \frac{f_n}{b_n} n^{-r} \sum_{p \in \mathbb{N}^* \setminus \{n\}} \frac{b_p \varphi_p}{\lambda_n - \lambda_p + \lambda}.$$

We first show that  $\tilde{\tau}_c$  is a compact operator on  $\mathcal{H}^r$ . Using Fubini's Theorem it holds that

$$\sum_{n \in \mathbb{N}^*} \frac{f_n}{b_n} n^{-r} \sum_{p \in \mathbb{N}^* \setminus \{n\}} \frac{b_p \varphi_p}{\lambda_n - \lambda_p + \lambda} = \sum_{p \in \mathbb{N}^*} \varphi_p \left( \sum_{n \in \mathbb{N}^* \setminus \{p\}} \frac{f_n}{b_n} n^{-r} \frac{b_p}{\lambda_n - \lambda_p + \lambda} \right).$$

Let  $\varepsilon > 0$  to be selected. So we have

$$\begin{aligned} \|\tilde{\tau}_c(f)\|_{\mathcal{H}^{r+\varepsilon}}^2 &= \left\| \sum_{p \in \mathbb{N}^*} \varphi_p \left( \sum_{n \in \mathbb{N}^* \setminus \{p\}} \frac{f_n}{b_n} n^{-r} \frac{b_p}{\lambda_n - \lambda_p + \lambda} \right) \right\|_{\mathcal{H}^{r+\varepsilon}}^2 \\ &= \sum_{p \in \mathbb{N}^*} p^{2r+2\varepsilon} |b_p|^2 \left| \sum_{n \in \mathbb{N}^* \setminus \{p\}} \frac{f_n}{b_n} n^{-r} \frac{1}{\lambda_n - \lambda_p + \lambda} \right|^2. \end{aligned}$$

Since  $(1/b_n)_{n \in \mathbb{N}^*}$  is uniformly bounded from (11) (recall that here  $\beta = 0$ ), we get that

$$\|\tilde{\tau}_c(f)\|_{\mathcal{H}^{r+\varepsilon}} \lesssim \sum_{p \in \mathbb{N}^*} p^{2r+2\varepsilon} |b_p|^2 \left| \sum_{n \in \mathbb{N}^* \setminus \{p\}} \frac{f_n n^{-r}}{\lambda_n - \lambda_p + \lambda} \right|^2.$$

As  $|b_p| \leq c_2 n^\gamma$ , still from (11) this yields

$$\|\tilde{\tau}_c(f)\|_{\mathcal{H}^{r+\varepsilon}} \lesssim \sum_{p \in \mathbb{N}^*} p^{2r+2\varepsilon+2\gamma} \left| \sum_{n \in \mathbb{N}^* \setminus \{p\}} \frac{f_n n^{-r}}{\lambda_n - \lambda_p + \lambda} \right|^2.$$

Since  $r \in (1/2 - \alpha + \gamma, \alpha - 1/2)$ , we have that  $\gamma < \alpha + r - 1/2$  and consequently  $\gamma < \varepsilon_r := \min\{\frac{\alpha-1}{2}, \alpha + r - \frac{1}{2}\}$  as  $\gamma < (\alpha - 1)/2$  by assumption. So, considering for instance  $\varepsilon = (\varepsilon_r - \gamma)/2$ , we obtain from Lemma 5.4 that

$$\sum_{p \in \mathbb{N}^*} p^{2r+2\varepsilon+2\gamma} \left| \sum_{n \in \mathbb{N}^* \setminus \{p\}} \frac{f_n n^{-r}}{\lambda_n - \lambda_p + \lambda} \right|^2 \lesssim \sum_{n \in \mathbb{N}^*} |f_n|^2.$$

This ensures that

$$\|\tilde{\tau}_c(f)\|_{\mathcal{H}^{r+\varepsilon}} \lesssim \|f\|_{\mathcal{H}^r},$$

which in view of Lemma 2.2 concludes that  $\tilde{\tau}_c$  is compact from  $\mathcal{H}^r$  to  $\mathcal{H}^r$ . Consequently the operator  $\tilde{\tau}$  is a Fredholm operator of index 0. Thus,  $\tilde{\tau}$  is an isomorphism from  $\mathcal{H}^r$  to  $\mathcal{H}^r$  if and only if  $\ker \tilde{\tau} = \{0\}$ . From the expression of  $\tilde{\tau}$ ,

$$f = \sum_{n \in \mathbb{N}^*} f_n n^{-r} \varphi_n \in \ker \tilde{\tau} \Leftrightarrow \sum_{n \in \mathbb{N}^*} \frac{f_n}{b_n} \tau(n^{-r} q_n) = 0.$$

The equality at the right hand side holds in  $\mathcal{H}^r$  and in particular in  $\tau(\mathcal{H}^r)$  since  $\mathcal{H}^r \subset \tau(\mathcal{H}^r)$ . So using the fact that  $(\tau(n^{-r}q_n))_{n \in \mathbb{N}^*}$  is a Riesz basis of  $\tau(\mathcal{H}^r)$ , we have

$$\begin{aligned} \sum_{n \in \mathbb{N}^*} \frac{f_n}{b_n} \tau(n^{-r}q_n) = 0 &\Leftrightarrow \frac{f_n}{b_n} = 0, \quad \forall n \in \mathbb{N}^* \\ &\Leftrightarrow f_n = 0 \quad \forall n \in \mathbb{N}^* \end{aligned}$$

Thus, we have shown that  $\ker \tilde{\tau} = \{0\}$ , hence,  $\tilde{\tau}$  is an isomorphism from  $\mathcal{H}^r$  to itself for any  $r \in (1/2 - \alpha + \gamma, \alpha - 1/2)$ .  $\square$

We are able to state the proof of Lemma 5.12 now.

*Proof of Lemma 5.12.* In view of (43) and Lemma 5.13, we may rewrite now  $T$  as follows

$$T : n^{-r}\varphi_n \mapsto -K_n b_n \tilde{\tau}(n^{-r}\varphi_n).$$

Based on Lemma 5.13,  $\tilde{\tau}$  is an isomorphism from  $\mathcal{H}^r$  to  $\mathcal{H}^r$ . This ensures that  $(\tilde{\tau}(n^{-r}\varphi_n))_{n \in \mathbb{N}^*}$  is a Riesz basis in  $\mathcal{H}^r$ . Since  $(n^{-r}\varphi_n)_{n \in \mathbb{N}^*}$  is also a Riesz basis in  $\mathcal{H}^r$ , and  $(K_n b_n)_{n \in \mathbb{N}} \in l^\infty$  from Lemma 5.8, we just have to prove that  $(K_n b_n)_{n \in \mathbb{N}}$  is uniformly bounded from below to conclude that  $T$  is an isomorphism in  $\mathcal{H}^r$ . From Lemma 5.11,  $\ker T^* = \{0\}$  in  $\mathcal{H}^{-\alpha/2}$ . In fact, from the decomposition (64) and the fact that  $\tilde{\tau}$  is a Fredholm operator of order 0, we can deduce that  $T$  is a Fredholm operator of order 0 from  $\mathcal{H}^{-\alpha/2}$  into itself (see Lemma A.1 in Appendix A.3). Hence,  $\ker T = \{0\}$  in  $\mathcal{H}^{-\alpha/2}$ . Let us assume by contradiction that there exists  $n_0 \in \mathbb{N}^*$  such that  $K_{n_0} b_{n_0} = 0$ . We have

$$T(n_0^{-r}\varphi_{n_0}) = K_{n_0} b_{n_0} \tilde{\tau}(n_0^{-r}\varphi_{n_0}) = 0.$$

This means that  $n_0^{-r}\varphi_{n_0} \in \ker T$ , and since  $\ker T = \{0\}$ ,  $n_0^{-r}\varphi_{n_0} = 0$  which is a contradiction. Therefore for any  $n \in \mathbb{N}^*$

$$(81) \quad K_n b_n \neq 0.$$

Recall that from Lemma (5.8),

$$|K_n b_n| = |\lambda + k_n|,$$

and  $(k_n n^\varepsilon)_{n \in \mathbb{N}^*} \in l^\infty$  for some  $\varepsilon > 0$ , which means that  $k_n \rightarrow 0$  when  $n \rightarrow +\infty$ . Then, there exists  $N > 0$  such that for any  $n \geq N$ ,

$$(82) \quad |K_n b_n| \geq \frac{\lambda}{2} > 0.$$

Combining (81) with (82) it holds that for any  $n \in \mathbb{N}^*$ ,

$$|K_n b_n| \geq \min \left\{ \frac{\lambda}{2}, \min_{n \leq N} |K_n b_n| \right\} > 0.$$

Therefore,  $K_n b_n$  is bounded from below and this concludes the proof of Lemma 5.12.  $\square$

**5.7. Step7: Well-posedness and proof of Proposition 4.2.** Since, we have built the isomorphism  $T$  and the feedback law  $K$ , for which (38) and (39) hold, we need to show the well-posedness of the closed loop system (9). We will adapt the method given in [15] for the well-posedness of closed loop water tank or in [19] for the well-posedness of closed loop systems described by skew-adjoint operators.

Since, this method relies on the semigroup theory of the operator  $\mathcal{A} + BK$ , we first provide a description and some properties about its domain:

**LEMMA 5.14.** *For any  $r \in (1/2 - \alpha + \gamma, \alpha - 1/2 - \gamma)$ , the domain  $D_r(\mathcal{A} + BK)$  of  $\mathcal{A} + BK$  is a Hilbert space, dense in  $\mathcal{H}^r$  and*

$$(83) \quad D_r(\mathcal{A} + BK) = T^{-1}(\mathcal{H}^{r+\alpha}).$$

*Proof.* Recall that, what we call  $D_r(\mathcal{A} + BK)$  for  $r \in (1/2 - \alpha + \gamma, \alpha - 1/2 - \gamma)$  is the following:

$$D_r(\mathcal{A} + BK) := \{f \in \mathcal{H}^r, (\mathcal{A} + BK)f \in \mathcal{H}^r\}.$$

We start by showing (83). For any  $f \in D_r(\mathcal{A} + BK) \subset \mathcal{H}^r$ , we have  $\mathcal{A}f \in \mathcal{H}^{r-\alpha}$  and then by definition of  $D_r(\mathcal{A} + BK)$  this implies that  $BKf \in \mathcal{H}^{r-\alpha}$ . As  $r < \alpha - 1/2 - \gamma$ , there exist  $\varepsilon > 0$  such that  $BKf \in \mathcal{H}^{-1/2-\gamma-\varepsilon}$ . This shows that  $Kf \in \mathbb{C}$  because  $B \in \mathcal{H}^{-1/2-\gamma-\varepsilon}$  for any  $\varepsilon > 0$ . Since  $Kf \in \mathbb{C}$ , it holds that  $BKf \in \mathcal{H}^{-1/2-\gamma-\varepsilon}$  for any  $\varepsilon > 0$ . Combining this with the definition of  $D_r(\mathcal{A} + BK)$  ensures that  $\mathcal{A}f \in \mathcal{H}^{-1/2-\gamma-\varepsilon}$ , for any  $\varepsilon > 0$ . So we get  $f = \mathcal{A}^{-1}\mathcal{A}f \in \mathcal{H}^{\alpha-1/2-\gamma-\varepsilon}$  for any  $\varepsilon > 0$ . This shows that  $D_r(\mathcal{A} + BK) \subset \mathcal{H}^{\alpha-1/2-\gamma-\varepsilon}$ . So in view of the operator equality (77), for any  $\varepsilon \in (0, (\alpha - 1) - 2\gamma)$ ,  $\alpha/2 - 1/2 - \gamma - \varepsilon \in (\gamma - (\alpha - 1)/2, (\alpha - 1)/2 - \gamma)$  and therefore it holds that

$$T(\mathcal{A} + BK)f = (\mathcal{A} - \lambda I)Tf \in \mathcal{H}^{-1/2-\gamma-\varepsilon}, \quad \forall f \in D_r(\mathcal{A} + BK).$$

Notice that by the property of  $T$  and the definition of  $D_r(\mathcal{A} + BK)$ ,  $T(\mathcal{A} + BK)f \in \mathcal{H}^r$  for any  $f \in D_r(\mathcal{A} + BK)$ , and hence  $(\mathcal{A} - \lambda I)Tf \in \mathcal{H}^r$ . So the operator equality holds in the following appropriate setting: for any  $r \in (1/2 - \alpha + \gamma, \alpha - 1/2 - \gamma)$ ,

$$(84) \quad T(\mathcal{A} + BK)f = (\mathcal{A} - \lambda I)Tf \in \mathcal{H}^r, \quad \forall f \in D_r(\mathcal{A} + BK).$$

Consider now  $f \in D_r(\mathcal{A} + BK)$ , we get from (84) that  $(\mathcal{A} - \lambda)Tf \in \mathcal{H}^r$ , and then  $Tf \in \mathcal{H}^{r+\alpha}$ . This implies that  $f \in T^{-1}(\mathcal{H}^{r+\alpha})$ , and thus show that  $D_r(\mathcal{A} + BK) \subset T^{-1}(\mathcal{H}^{r+\alpha})$ . Conversely, let us consider  $f \in T^{-1}(\mathcal{H}^{r+\alpha}) \subset \mathcal{H}^r$ . Then, since  $r < \alpha - 1/2 - \gamma$ , there exists  $\bar{\varepsilon} > 0$  such that for any  $\varepsilon \in (0, \bar{\varepsilon})$ ,  $f \in \mathcal{H}^{\alpha-1/2-\varepsilon-\gamma}$ . So by considering  $\varepsilon$  small enough we can apply the operator equality (77) to  $f$  and get

$$(85) \quad T(\mathcal{A} + BK)f = (\mathcal{A} - \lambda I)Tf.$$

Notice that  $f \in T^{-1}(\mathcal{H}^{r+\alpha})$  implies that  $Tf \in \mathcal{H}^{r+\alpha}$ , and thus  $(\mathcal{A} - \lambda I)Tf \in \mathcal{H}^r$ . This ensures from (85) that  $T(\mathcal{A} + BK)f \in \mathcal{H}^r$ . Since  $T$  is an isomorphism from  $\mathcal{H}^r$  to itself, it then holds that  $(\mathcal{A} + BK)f \in \mathcal{H}^r$ . Hence  $f \in D_r(\mathcal{A} + BK)$ , meaning that  $T^{-1}(\mathcal{H}^{r+\alpha}) \subset D_r(\mathcal{A} + BK)$ . This concludes the proof of (83).

Since  $T$  is an isomorphism from  $\mathcal{H}^r$  to itself, and from the density of  $\mathcal{H}^{r+\alpha}$  in  $\mathcal{H}^r$ , it immediately holds based on (83) that  $D_r(\mathcal{A} + BK)$  is dense in  $\mathcal{H}^r$ .

It remains now to prove that  $D_r(\mathcal{A} + BK)$  is a Hilbert space to conclude Lemma 5.14. For that, we just need to prove that  $D_r(\mathcal{A} + BK)$  is complete since  $D_r(\mathcal{A} + BK) \subset \mathcal{H}^r$  which is a Hilbert space. So, let consider any Cauchy sequence  $(f_n)_n \subset D_r(\mathcal{A} + BK)$ . It holds that

$$\|f_n - f_p\|_{D_r(\mathcal{A} + BK)} \xrightarrow{n, p \rightarrow +\infty} 0.$$

Recall that  $\|f_n - f_p\|_{D_r(\mathcal{A}+BK)} := \|f_n - f_p\|_{\mathcal{H}^r} + \|(\mathcal{A} + BK)(f_n - f_p)\|_{\mathcal{H}^r}$ , we have

$$(86) \quad \|f_n - f_p\|_{\mathcal{H}^r} + \|(\mathcal{A} + BK)(f_n - f_p)\|_{\mathcal{H}^r} \xrightarrow{n,p \rightarrow +\infty} 0.$$

Since  $(f_n)_n \subset D_r(\mathcal{A} + BK)$ , by using the operator equality (84), this implies that

$$\|(\mathcal{A} - \lambda I)(Tf_n - Tf_p)\|_{\mathcal{H}^r} = \|T(\mathcal{A} + BK)(f_n - f_p)\|_{\mathcal{H}^r} \lesssim \|(\mathcal{A} + BK)(f_n - f_p)\|_{\mathcal{H}^r} \xrightarrow{n,p \rightarrow +\infty} 0.$$

This combining with (86), ensures that

$$\|\mathcal{A}(Tf_n - Tf_p)\|_{\mathcal{H}^r} \xrightarrow{n,p \rightarrow +\infty} 0, \quad \text{and} \quad \|Tf_n - Tf_p\|_{\mathcal{H}^r} \xrightarrow{n,p \rightarrow +\infty} 0.$$

Thus, we obtain

$$\|Tf_n - Tf_p\|_{\mathcal{H}^{r+\alpha}} \xrightarrow{n,p \rightarrow +\infty} 0.$$

This shows that  $(Tf_n)_n$  is a Cauchy sequence in  $\mathcal{H}^{r+\alpha}$ . Then, since  $\mathcal{H}^{r+\alpha}$  is complete, there exists  $g \in \mathcal{H}^{r+\alpha}$  such that  $Tf_n \rightarrow g$  in  $\mathcal{H}^{r+\alpha}$ . As  $T$  is an isomorphism on  $\mathcal{H}^r$ , and  $\mathcal{H}^{r+\alpha} \subset \mathcal{H}^r$  there exists a unique  $f \in \mathcal{H}^r$  such that  $Tf = g$  and  $\|Tf_n - Tf\|_{\mathcal{H}^{r+\alpha}} \rightarrow 0$ , which in turn implies, from the compact embedding of  $\mathcal{H}^{r+\alpha}$  in  $\mathcal{H}^r$  and the isomorphism property of  $T$  in  $\mathcal{H}^r$

$$(87) \quad \|f_n - f\|_{\mathcal{H}^r} \xrightarrow{n \rightarrow +\infty} 0.$$

We also have

$$(88) \quad \begin{aligned} \|(\mathcal{A} + BK)(f_n - f)\|_{\mathcal{H}^r} &\lesssim \|T(\mathcal{A} + BK)(f_n - f)\|_{\mathcal{H}^r} \\ &= \|(\mathcal{A} - \lambda I)T(f_n - f)\|_{\mathcal{H}^r} \lesssim \|Tf_n - Tf\|_{\mathcal{H}^{r+\alpha}} \xrightarrow{n \rightarrow +\infty} 0. \end{aligned}$$

Combining (87) and (88), it holds that:

$$\|f_n - f\|_{D_r(\mathcal{A}+BK)} = \|f_n - f\|_{\mathcal{H}^r} + \|(\mathcal{A} + BK)(f_n - f)\|_{\mathcal{H}^r} \xrightarrow{n \rightarrow +\infty} 0.$$

Hence,  $D_r(\mathcal{A} + BK)$  is complete and therefore a Hilbert space.  $\square$

As we know now that the domain  $D_r(\mathcal{A} + BK)$  is a Hilbert space and it is not an empty set, we are able to state the following lemma:

**LEMMA 5.15.** *The operator  $\mathcal{A} + BK$  with domain  $D_r(\mathcal{A} + BK)$  generates a  $C^0$  semi-group on  $\mathcal{H}^r$ .*

*Proof.* Let us first notice that for any  $r \in (1/2 - \alpha + \gamma, \alpha - 1/2 - \gamma)$ ,  $D_r(\mathcal{A} - \lambda I) := \{f \in \mathcal{H}^r, (\mathcal{A} - \lambda I)f \in \mathcal{H}^r\} = \mathcal{H}^{r+\alpha}$ . Since  $\mathcal{A}$  generates a  $C^0$  semigroup on  $\mathcal{H}^r$ , then  $\mathcal{A} - \lambda I$  generates also a  $C^0$  semigroup which we denote by  $e^{t(\mathcal{A} - \lambda I)}$ . Let us now define the following  $C^0$  semigroup on  $\mathcal{H}^r$ ,

$$(89) \quad S(t) := T^{-1}e^{t(\mathcal{A} - \lambda I)}T, \quad \forall t \geq 0.$$

Since  $T$  is an isomorphism,  $D_r(\mathcal{A} - \lambda I) = \mathcal{H}^{r+\alpha}$ , and Lemma 5.14 it holds that

$$\begin{aligned} \left\{ f \in \mathcal{H}^r, \lim_{t \rightarrow +\infty} \frac{S(t)f - f}{t} \text{ exists in } \mathcal{H}^r \right\} &= \left\{ f \in \mathcal{H}^r, \lim_{t \rightarrow +\infty} T^{-1} \left( \frac{e^{t(\mathcal{A} - \lambda I)}Tf - Tf}{t} \right) \text{ exists in } \mathcal{H}^r \right\} \\ &= \left\{ f = T^{-1}g \in \mathcal{H}^r, \lim_{t \rightarrow +\infty} T^{-1} \left( \frac{e^{t(\mathcal{A} - \lambda I)}g - g}{t} \right) \text{ exists in } \mathcal{H}^r \right\} \\ &= \left\{ f = T^{-1}g \in \mathcal{H}^r, \lim_{t \rightarrow +\infty} \left( \frac{e^{t(\mathcal{A} - \lambda I)}g - g}{t} \right) \text{ exists in } \mathcal{H}^r \right\} \end{aligned}$$

$$\begin{aligned}
&= \{f = T^{-1}g, g \in D_r(A - \lambda I) = \mathcal{H}^{r+\alpha}\} \\
&= T^{-1}(\mathcal{H}^{r+\alpha}) = D_r(A + BK).
\end{aligned}$$

This shows that the domain of the operator which generates  $S(t)$  is  $D_r(A + BK)$ .

Moreover, for any  $f \in D_r(A + BK)$ , we have  $Tf \in \mathcal{H}^{r+\alpha}$  and then it holds that

$$\begin{aligned}
\lim_{t \rightarrow +\infty} \frac{S(t)f - f}{t} &= \lim_{t \rightarrow +\infty} T^{-1} \left( \frac{e^{t(A-\lambda I)}Tf - Tf}{t} \right) \\
&= T^{-1}(A - \lambda I)Tf.
\end{aligned}$$

So using the operator equality (84) we get that

$$\lim_{t \rightarrow +\infty} \frac{S(t)f - f}{t} = (A + BK)f.$$

Hence, the operator  $A + BK$  with domain  $D_r(A + BK)$  is the generator of  $C^0$  semi-group  $S(t)$  and this concludes Lemma 5.15.  $\square$

*Proof of Proposition 4.2.* Based on Lemma 5.15, we know that for any  $i \in \{1, \dots, m\}$ , the operator  $A + B_i K_i$  generates a  $C^0$  semi-group  $S_i(t)$  on  $\mathcal{H}^{r_i}$  for any  $r_i \in (1/2 - \alpha_i + \gamma_i, \alpha_i - 1/2 - \gamma_i)$ . Now, we would like to show that the operator  $A + BK = A + \sum_{i=1}^m B_i K_i$  with domain  $D_{\vec{r}}(A + BK) :=$

$\sum_{i=1}^m D_{r_i}(A + B_i K_i)$  generates the  $C^0$  semi-group

$$(90) \quad S(t) := \sum_{i=1}^m S_i(t)$$

on  $\mathcal{H}^{\vec{r}} := \mathcal{H}^{r_1} + \dots + \mathcal{H}^{r_m}$ . Recall that  $S_i(t) = T_i^{-1}e^{t(A-\lambda I)}T_i$ , and since  $T_i\varphi_n^j = 0$  for any  $i \neq j$ , it holds that  $S_i(t)\varphi_n^j = 0$  for any  $i \neq j$ . So for any  $f^j \in \mathcal{H}^{r_j}$ , we have that  $S_i(t)f^j = 0$  for any  $i \neq j$ . This ensures in particular that  $S_i(t)S_j(s)f^k = 0$  for any  $i \neq j, k \in \{1, \dots, m\}$  and  $s, t \geq 0$ . Combining this with the fact that  $(S_i(t))_{t \geq 0}$  is a  $C^0$  semigroup on  $\mathcal{H}^{r_i}$ , it holds that  $S(t)$  is a  $C^0$  semi-group of  $\mathcal{H}^{\vec{r}}$ . Looking at its domain:

$$\begin{aligned}
\left\{ f = \sum_{j=1}^m f^j \in \mathcal{H}^{\vec{r}}, \lim_{t \rightarrow +\infty} \frac{S(t)f - f}{t} \text{ exists} \right\} &= \left\{ f = \sum_{j=1}^m f^j \in \mathcal{H}^{\vec{r}}, \lim_{t \rightarrow +\infty} \frac{\sum_{i=1}^m S_i(t)f - f}{t} \text{ exists} \right\} \\
&= \left\{ f = \sum_{j=1}^m f^j \in \mathcal{H}^{\vec{r}}, \lim_{t \rightarrow +\infty} \sum_{i=1}^m \frac{S_i(t)f^i - f^i}{t} \text{ exists} \right\} \\
&= \left\{ f = \sum_{j=1}^m f^j \in \mathcal{H}^{\vec{r}}, \lim_{t \rightarrow +\infty} \frac{S_i(t)f^i - f^i}{t} \text{ exists } \forall i \in \{1, \dots, m\} \right\} \\
&= \sum_{i=1}^m D_{r_i}(A + B_i K_i) = D_{\vec{r}}(A + BK).
\end{aligned}$$

This shows that  $D_{\vec{r}}(A + BK)$  is the domain of the corresponding generator to  $S(t)$ . Moreover, for any  $f \in D_{\vec{r}}(A + BK)$ , there exists  $(f^1, \dots, f^m) \in \mathcal{H}^{r_1} \times \dots \times \mathcal{H}^{r_m}$  such that  $f = \sum_{i=1}^m f^i$ ,

and

$$\begin{aligned}
 \lim_{t \rightarrow +\infty} \frac{S(t)f - f}{t} &= \lim_{t \rightarrow +\infty} \frac{\sum_{i=1}^m S_i(t)f^i - \sum_{i=1}^m f^i}{t} \\
 &= \sum_{i=1}^m \lim_{t \rightarrow +\infty} \frac{S_i(t)f^i - f^i}{t} \\
 &= \sum_{i=1}^m (\mathcal{A} + B_i K_i) f^i = (\mathcal{A} + BK)f
 \end{aligned}$$

Therefore, we conclude that the operator  $\mathcal{A} + BK$  generates a  $C^0$  semi-group on  $\mathcal{H}^{\vec{r}}$ . this concludes the proof of Proposition 4.2.  $\square$

**5.8. Proof of Theorem 3.1.** Theorem 3.1 follows from all the previous steps and the following Lemma, which is an immediate consequence from the fact that  $\mathcal{N}$  is countable.

**LEMMA 5.16.** *For any  $\lambda_0 > 0$  there exists  $\lambda > \lambda_0$  such that  $\lambda \in \mathbb{R}_+ \setminus \mathcal{N}$ .*

*Proof of Theorem 3.1.* Let  $\lambda_0 > 0$ , and  $\lambda > 0$  given by Lemma 5.2. From Proposition 4.1 there exists  $K_i \in \mathcal{L}(\mathcal{H}^{\beta_i + 1/2 + \varepsilon}; \mathbb{C})$  and  $T_i \in \mathcal{L}(\mathcal{H}^{r_i})$  such that  $T_i$  is an isomorphism from  $\mathcal{H}^{r_i}$  into itself for any  $r_i \in (\beta_i + 1/2 - \alpha_i + \gamma_i, \beta_i + \alpha_i - 1/2 - \gamma_i)$  and maps the system (34) to (35). Setting

$$(91) \quad K = (K_1, \dots, K_m)^T, \quad T = T_1 + \dots + T_m,$$

we know from Proposition 4.2 that the system (9) is well-posed in  $\mathcal{H}^{\vec{r}}$  (defined as (12)) for any  $r$  satisfying (13). It remains to show that  $T$  is an isomorphism from  $\mathcal{H}^{\vec{r}}$  to itself which maps the system (9) to (10) and that, as a consequence, the system (9) is exponentially stable.  $T$  is belongs clearly to  $\mathcal{L}(\mathcal{H}^{\vec{r}})$ . We define  $T^{-1} := (T_1^{-1} + \dots + T_m^{-1}) \in \mathcal{L}(\mathcal{H}^{\vec{r}})$ . Recall that  $T_i \varphi_n^j = 0$ ,  $T_i^{-1} \varphi_n^j = 0$  and  $K_i \varphi_n^j = 0$  for any  $j \neq i$  and  $n \in \mathbb{N}^*$ . Thus, for any  $f = \sum_{i=1}^m \sum_{n \in \mathbb{N}^*} f_n^i \varphi_n^i \in \mathcal{H}^{\vec{r}}$ ,

$$(92) \quad T^{-1}Tf = T^{-1} \left( \sum_{i=1}^m \sum_{n \in \mathbb{N}^*} f_n^i T_i \varphi_n^i \right) = \sum_{i=1}^m \sum_{n \in \mathbb{N}^*} f_n^i \varphi_n^i = f,$$

hence  $T$  is an isomorphism from  $\mathcal{H}^{\vec{r}}$  into itself with inverse  $T^{-1}$ . To show that  $T$  maps (9) to (10) it suffices to show that the operator equality (77) –which currently holds in  $\mathcal{L}(\mathcal{H}_i^{\alpha_i/2 + s_i}, \mathcal{H}_i^{-\alpha_i/2 + s_i})$  with  $T_i$ ,  $B_i$  and  $K_i$  for any  $i \in \{1, \dots, m\}$ – in fact also holds in  $\mathcal{L}(\mathcal{H}^{\alpha/2 + s}, \mathcal{H}^{-\alpha/2 + s})$  with  $T$ ,  $B$  and  $K$  where

$$(93) \quad \alpha = (\alpha_1, \dots, \alpha_m), \quad s = (s_1, \dots, s_m), \quad \gamma = (\gamma_1, \dots, \gamma_m) \quad s_i \in \left( -\frac{(\alpha_i - 1)}{2} + \gamma_i, \frac{\alpha_i - 1}{2} - \gamma_i \right).$$

This is a consequence of the fact that, by definition,  $T_i \varphi_n^j = 0$ ,  $K_i \varphi_n^j = 0$  for any  $i \neq j$  and any  $n \in \mathbb{N}^*$ : let  $f = \sum_{i=1}^m f^i$  with  $f^i = \sum_{n \in \mathbb{N}^*} f_n^i \varphi_n^i$  and consider its smooth approximation  $\tilde{f}^i = \sum_{n=1}^N f_n^i \varphi_n^i$ . Observe that  $K\tilde{f} = (K_1 \tilde{f}^1, \dots, K_m \tilde{f}^m)^T$  and  $BK\tilde{f} = \sum_{i=1}^m B_i K_i \tilde{f}^i$ , thus we have

$$(94) \quad T(\mathcal{A} + BK)\tilde{f} = T \left( \sum_{i=1}^m \mathcal{A} \tilde{f}^i + B_i K_i \tilde{f}^i \right) = \sum_{i=1}^m T_i \mathcal{A} \tilde{f}^i + B_i K_i \tilde{f}^i$$

where we used that, for any  $i \in \{1, \dots, m\}$  and  $s \in \mathbb{R}$ , we have  $\mathcal{A} \in \mathcal{L}(\mathcal{H}_i^{s+\alpha_i}, \mathcal{H}_i^s)$ . Thus using (77) for each  $i \in \{1, \dots, m\}$ ,

$$(95) \quad T(\mathcal{A} + BK)\tilde{f} = \sum_{i=1}^m T_i(\mathcal{A} + B_i K_i)\tilde{f}^i = \sum_{i=1}^m (\mathcal{A} - \lambda)T_i\tilde{f}^i = (\mathcal{A} - \lambda) \sum_{i=1}^m T_i\tilde{f}^i = (\mathcal{A} - \lambda I)T\tilde{f}.$$

Note that one can pass to the limit (in  $\mathcal{H}^{-\alpha/2+s}$ )  $N \rightarrow +\infty$  in each of this terms since  $f \in \mathcal{H}^{-\alpha/2+s}$  and the operator equality (77) holds in  $\mathcal{L}(\mathcal{H}_i^{-\alpha_i/2+s}, \mathcal{H}_i^{-\alpha_i/2+s})$  for any  $i \in \{1, \dots, m\}$ . As a consequence,

$$(96) \quad T(\mathcal{A} + BK) = (\mathcal{A} - \lambda I)T \text{ in } \mathcal{L}(\mathcal{H}^{\alpha/2+s}, \mathcal{H}^{-\alpha/2+s}) \text{ with (93).}$$

Finally it remains to show that for any  $\mu > 0$ ,  $\lambda_0$  can be chosen large enough such that the system (9) is exponentially stable with decay rate  $\mu$ . Since  $\mathcal{A}$  generates a  $C^0$  semigroup with finite growth bound on  $\mathcal{H}^{\vec{r}}$  (see Lemma A.2 in Appendix A.4), there exists  $M \geq 1$  and  $\omega \in \mathbb{R}$  such that for any  $f \in \mathcal{H}^{\vec{r}}$ ,

$$(97) \quad \|e^{t\mathcal{A}}f\|_{\mathcal{H}^{\vec{r}}} \leq M e^{\omega t} \|f\|_{\mathcal{H}^{\vec{r}}}, \quad \forall t \in [0, +\infty),$$

and in particular for  $\lambda > \lambda_0 > \mu + \omega$ ,

$$(98) \quad \|e^{t(\mathcal{A}-\lambda I)}f\|_{\mathcal{H}^{\vec{r}}} \leq M e^{-\mu t} \|f\|_{\mathcal{H}^{\vec{r}}}, \quad \forall t \in [0, +\infty).$$

Let  $u_0 \in \mathcal{H}^{\vec{r}}$ , the (unique) solution in  $C^0([0, +\infty); \mathcal{H}^{\vec{r}})$  to the closed loop system (9) is  $u : t \rightarrow S(t)u_0$  and, using (89)

$$(99) \quad \|S(t)u_0\|_{\mathcal{H}^{\vec{r}}} = \|T^{-1}e^{t(\mathcal{A}-\lambda I)}Tu_0\|_{\mathcal{H}^{\vec{r}}} \leq \|T^{-1}\|_{\mathcal{L}(\mathcal{H}^{\vec{r}})}\|T\|_{\mathcal{L}(\mathcal{H}^{\vec{r}})}Me^{-\mu t}\|u_0\|_{\mathcal{H}^{\vec{r}}}, \quad \forall t \in [0, +\infty).$$

This ends the proof of Theorem 3.1.  $\square$

## 6. CONCLUSION AND PERSPECTIVES

In this paper, we extended the  $F$ -equivalence approach (or generalized backstepping), to a wide class of linear systems. This includes systems that are not necessarily skew-adjoint or self-adjoint provided that the differential operator involved has a Riesz basis of eigenvectors (hence not necessarily orthonormal) and is of order larger than one. We illustrate this result with the rapid stabilization of the Schroedinger equation and the diffusion equations. This result makes the  $F$ -equivalence one step closer to a general theory. Nevertheless, it is still a young approach and there are many questions that remain open and that would be interesting to consider

- If  $\mathcal{A}$  only has a generalized basis of eigenvectors (and not a Riesz basis of eigenvectors which we assume here), is it still possible to extend this method? This is interesting in particular for systems of PDE, where  $\mathcal{A}$  does not always has a Riesz basis of eigenvectors.
- What happens if the eigenvalues  $\lambda_n$  are infinite dimensional? In particular, this could be useful for multidimensional systems. While Theorem 3.1 and 3.3 are not limited to 1D systems, in several multidimensional systems the eigenvalues  $\lambda_n$  may have infinite multiplicity (the heat equation for instance).
- Can these linear results be extended to nonlinear (quasilinear) systems? This is all the more important that for infinite dimensional system the stability of the linearized system does not necessarily imply the stability of the nonlinear system, even locally [17] when they are at least quasilinear, and there is so far no general method to stabilizing these systems.
- Is it possible to further relax the conditions on  $B$ ?

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## APPENDIX A. APPENDIX

A.1. **Proof of Lemma 2.1.** We start by proving that  $\|\cdot\|_{\mathcal{H}^r}$  is a norm on  $\mathcal{H}^r$ . Let  $f \in \mathcal{H}^r$ , by definition there exists  $(f_n)_{n \in \mathbb{N}^*} \in l^2$  such that  $f = \sum_{n \in \mathbb{N}^*} f_n \varphi_n$  and therefore  $\|f\|_{\mathcal{H}^r}$  is well-defined. Besides, since  $(\varphi_n)_{n \in \mathbb{N}}$  is a Riesz basis,  $f = 0$  is equivalent to  $f_n = 0$  for any  $n \in \mathbb{N}^*$  and consequently to  $\|f\|_{\mathcal{H}^r} = 0$ . Also for any  $\mu \in \mathbb{C}$

$$(100) \quad \|\mu f\|_{\mathcal{H}^r} = \left\| \sum_{n \in \mathbb{N}^*} \mu f_n \varphi_n \right\|_{\mathcal{H}^r} = |\mu| \left( \sum_{n \in \mathbb{N}^*} n^{2r} |f_n|^2 \right)^{1/2} = |\mu| \|f\|_{\mathcal{H}^r}.$$

Finally, since  $(\sum_n n^{2r} |f_n + g_n|^2)^{1/2} \leq (\sum_n n^{2r} |f_n|^2)^{1/2} + (\sum_n n^{2r} |g_n|^2)^{1/2}$  for any  $(g_n)_{n \in \mathbb{N}^*} \in l^2$ ,  $\|\cdot\|_{w, \mathcal{H}^s}$  is indeed a norm. One can easily check that it derives from the inner product given by (4). Since there is an isomorphism between  $l^2(\mathbb{N}^*)$  and  $\mathcal{H}^r$  given by

$$(101) \quad \iota : (f_n)_{n \in \mathbb{N}^*} \mapsto \sum_{n \in \mathbb{N}^*} f_n n^{-r} \varphi_n,$$

such that for any  $(f_n)_{n \in \mathbb{N}^*}$   $\|\iota(f_n)_{n \in \mathbb{N}^*}\|_{\mathcal{H}^r} = \|f_n\|_{l^2}$ , then  $\mathcal{H}^r$  is also complete and hence a Banach space. Finally, since  $\|\cdot\|_{\mathcal{H}^r}$  is associated to the inner product  $\langle \cdot, \cdot \rangle_{\mathcal{H}^r}$  it is a Hilbert space.

A.2.  $(q_n)_{n \in \mathbb{N}^*}$  is either  $\mathcal{H}^0$ -dense or  $\omega$  independent in  $\mathcal{H}^0$ . The following argument is classical. It was first provided in [11] for the Schroedinger equation, and recently in [20] and [19] for the heat equation and skew-adjoint operators respectively. We detail it here for completeness. We assume that  $\lambda_n \neq 0$ . The proof can be easily adapted to the case where  $\lambda_1 = 0$ , since the resolvent  $\mathcal{A}_\lambda$  defined below is well defined and invertible on  $\mathcal{H}^0$ .

Recall that  $q_n \in \mathcal{H}^{\alpha-1/2-\varepsilon}$  for any  $\varepsilon > 0$ . We define  $r_n := (\lambda_n + \lambda/2)^{-1}$  and we obtain by definition of  $q_n$  in (44) that

$$(\mathcal{A} - \lambda - \lambda_n)q_n = - \sum_{p \in \mathbb{N}^*} \varphi_p = h, \quad \text{in } \mathcal{H}^{\alpha-1/2-\varepsilon}$$

which becomes by assuming without loss of generality that  $\lambda/2$  belongs to the resolvent set of  $\mathcal{A}$ ,

$$\mathcal{A}_\lambda q_n = r_n q_n - r_n \mathcal{A}_\lambda h \quad \text{in } \mathcal{H}^{\alpha-1/2-\varepsilon}$$

where

$$\mathcal{A}_\lambda := (\mathcal{A} + \lambda/2)^{-1}.$$

Let us now assume that the family  $(q_n)_{n \in \mathbb{N}^*}$  is not  $\omega$  independent in  $\mathcal{H}^0$ . Then, there exists non trivial  $(c_n)_{n \in \mathbb{N}^*} \in l^2$  such that

$$\sum_{n \in \mathbb{N}^*} c_n q_n = 0 \quad \text{in } \mathcal{H}^0.$$

By applying  $\mathcal{A}_\lambda$  to this equation, we get

$$\sum_{n \in \mathbb{N}^*} c_n r_n q_n = \left( \sum_{n \in \mathbb{N}^*} c_n r_n \right) \mathcal{A}_\lambda h \quad \text{in } \mathcal{H}^0$$



which is well defined since  $(r_n)_n \in l^2$ . Applying again  $\mathcal{A}_\lambda$ , we get

$$\sum_{n \in \mathbb{N}^*} c_n r_n^2 q_n = \left( \sum_{n \in \mathbb{N}^*} c_n r_n^2 \right) \mathcal{A}_\lambda h + \left( \sum_{n \in \mathbb{N}^*} c_n r_n \right) \mathcal{A}_\lambda^2 h \quad \text{in } \mathcal{H}^0$$

By induction, it holds that

$$(102) \quad \sum_{n \in \mathbb{N}^*} c_n r_n^m q_n = \sum_{i=1}^m \left( \sum_{n \in \mathbb{N}^*} c_n r_n^{m+1-i} \right) \mathcal{A}_\lambda^i h = \sum_{i=1}^m C_{m+1-i} \mathcal{A}_\lambda^i h \quad \text{in } \mathcal{H}^0,$$

where

$$C_j := \sum_{n \in \mathbb{N}^*} c_n r_n^j < +\infty, \quad j \in \mathbb{N}^*.$$

Let us now distinguish two cases:

- The  $\{C_m\}$  are identically zero. Then we define the complex variable function

$$G(z) := \sum_{n \in \mathbb{N}^*} c_n r_n e^{r_n z}.$$

This function is holomorphic and since  $\{C_m\}$  are identically zero, we have

$$G^{(m)}(0) = 0, \quad \forall m \in \mathbb{N}.$$

Thus  $G \equiv 0$ , and therefore  $c_n = 0$  for all  $n \in \mathbb{N}^*$ . This is in contradiction with the definition of  $(c_n)_{n \in \mathbb{N}^*}$ . Hence,  $(q_n)_{n \in \mathbb{N}^*}$  is  $\omega$  independent in  $\mathcal{H}^0$ .

- The  $\{C_m\}$  are not identically zero. Let us denote  $m_0 := \inf\{n \in \mathbb{N}^*, C_n \neq 0\}$ . Then, starting with  $m = m_0$ , we have based on (102) by induction

$$(103) \quad \mathcal{A}_\lambda^m \in \text{span}\{q_n\}_{n \in \mathbb{N}^*}, \quad \forall m \in \mathbb{N}^*.$$

Suppose that the family  $(q_n)_{n \in \mathbb{N}^*}$  is not  $\mathcal{H}^0$ -dense. Then there exists a nonzero  $d = \sum_{n \in \mathbb{N}^*} d_n \varphi_n \in \mathcal{H}^0$  such that

$$\langle g, d \rangle_{\mathcal{H}^0} = 0, \quad \forall g \in \text{span}\{q_n\}_{n \in \mathbb{N}^*}$$

which in particular yields,

$$\langle \mathcal{A}_\lambda^m h, d \rangle_{\mathcal{H}^0} = 0, \quad \forall m \in \mathbb{N}^*.$$

Noticing that  $h = - \sum_{n \in \mathbb{N}^*} \varphi_n \in \mathcal{H}^{-1}$ , we get that

$$(104) \quad \sum_{n \in \mathbb{N}^*} \bar{d}_n r_n^m = 0, \quad \forall m \in \mathbb{N}^*.$$

By defining the complex variable function,

$$\tilde{G}(z) := \sum_{n \in \mathbb{N}^*} \bar{d}_n r_n e^{r_n z}.$$

This function is holomorphic and from (104),  $\tilde{G}^{(m)} = 0$  for all  $m \in \mathbb{N}^*$ . Thus  $\tilde{G} \equiv 0$ , and further  $d_n = 0$  for any  $n \in \mathbb{N}^*$  which is a contradiction. Therefore the family  $(q_n)_{n \in \mathbb{N}^*}$  is either  $\mathcal{H}^0$ -dense or  $\omega$  independent.

**A.3. Proof of Lemma 5.11.** We proceed as presented in Section 5.6. Before doing that let us first show that

**LEMMA A.1.** *For any  $r \in (1/2 - \alpha + \gamma, \alpha - 1/2)$ , the operator  $T$  is a Fredholm operator of index 0 from  $\mathcal{H}^r$  to itself.*

*Proof.* Looking at the definition of  $T$  given by (61) and (64), we have for any  $n \in \mathbb{N}^*$

$$\begin{aligned} T\varphi_n &= \frac{\lambda + k_n}{b_n} \sum_{p \in \mathbb{N}^*} \frac{b_p \varphi_p}{\lambda_n - \lambda_p + \lambda} \\ (105) \qquad &= \lambda \tilde{\tau} \varphi_n + k \circ \tilde{\tau} \varphi_n, \end{aligned}$$

where  $\tilde{\tau}$  is given by Lemma 5.13 and  $k$  is defined on  $\mathcal{H}^r$  by  $k : n^{-r} \tilde{\tau} \varphi_n \mapsto k_n n^{-r} \tilde{\tau} \varphi_n$ . Notice that the operator  $k$  is a compact operator. Indeed, let  $f \in \mathcal{H}^r$ . Since,  $\tilde{\tau}$  is an isomorphism from  $\mathcal{H}^r$  to itself, it holds that  $(n^{-r} \tilde{\tau} \varphi_n)_{n \in \mathbb{N}^*}$  is a Riesz basis of  $\mathcal{H}^r$  and in particular there exists  $(f_n)_{n \in \mathbb{N}^*} \in l^2$  such that  $f = \sum_{n \in \mathbb{N}^*} f_n n^{-r} \tilde{\tau} \varphi_n$ . Let  $\varepsilon > 0$  to be chosen,

$$k(f) = \sum_{n \in \mathbb{N}^*} k_n f_n n^{-r} \tilde{\tau} \varphi_n = \sum_{n \in \mathbb{N}^*} k_n n^\varepsilon f_n n^{-r-\varepsilon} \tilde{\tau} \varphi_n.$$

Since from Lemma 5.8, there exists  $\varepsilon > 0$  such that  $(k_n n^\varepsilon)_{n \in \mathbb{N}^*} \in l^\infty$ , we have that  $(f_n k_n n^\varepsilon)_{n \in \mathbb{N}^*} \in l^2$ . And this ensures that  $k(f) \in \mathcal{H}^{r+\varepsilon}$  and, in particular,

$$\|k(f)\|_{\mathcal{H}^{r+\varepsilon}} = \left\| \sum_{n \in \mathbb{N}^*} k_n n^\varepsilon f_n n^{-r-\varepsilon} \tilde{\tau} \varphi_n \right\|_{\mathcal{H}^{r+\varepsilon}} \lesssim \|k_n n^\varepsilon f_n\|_{l^2} \lesssim \|f_n\|_{l^2} \lesssim \|f\|_{\mathcal{H}^r}.$$

Thus,  $k$  is compact in  $\mathcal{H}^r$  from Lemma 2.2. Combining this with the fact that  $\tilde{\tau}$  is a Fredholm operator of index 0 (Lemma 5.13) and thanks to the expression (105),  $T$  is a Fredholm operator of index 0 in  $\mathcal{H}^r$  (see for instance [8, p. 169]).  $\square$

- Let us define  $z := \lambda + \rho$ , where  $\rho \in \mathbb{C}$ . We assume without loss of generality that the operator  $\mathcal{A}$  is invertible. Otherwise, since its spectrum is countable, there exists sufficiently small  $\delta \neq 0$  such that  $\mathcal{A} + \delta I$  is invertible. So showing that  $\mathcal{A} + BK + zI$  is invertible amounts to show that  $I + \mathcal{A}^{-1}BK + z\mathcal{A}^{-1}$  is invertible. We now consider the following two distinct cases:

- (a)  $K(\mathcal{A}^{-1}B) \neq -1$ .

We can check that for any  $f \in \mathcal{H}^{\alpha/2}$ , the function  $\varphi \in \mathcal{H}^{\alpha/2}$  defined by

$$\varphi := f - \frac{\mathcal{A}^{-1}B(Kf)}{1 + K(\mathcal{A}^{-1}B)}$$

is the unique solution to

$$(I + \mathcal{A}^{-1}BK)\varphi = f.$$

Thus, the operator  $I + \mathcal{A}^{-1}BK$  is invertible. Since  $\mathcal{A}$  is a differential operator, then  $\mathcal{A}^{-1}$  is a continuous operator. And then, thanks to the openness of invertible operators, there exists  $\varepsilon > 0$  such that for any  $|z| < \varepsilon$

$$(106) \qquad I + \mathcal{A}^{-1}BK + z\mathcal{A}^{-1}$$

is invertible in  $\mathcal{H}^{-\alpha/2}$ .

(b)  $K(\mathcal{A}^{-1}B) = -1$ .

For any  $v \in \mathcal{H}^{\alpha/2}$ , it holds that

$$(I + \mathcal{A}^{-1}BK)v = 0 \Leftrightarrow v = -\mathcal{A}^{-1}B(Kv) \in \text{span}\{\mathcal{A}^{-1}B\}.$$

This ensures that 0 is an eigenvalue of  $I + \mathcal{A}^{-1}BK$  with multiplicity 1 and the corresponding eigenspace is generated by  $\mathcal{A}^{-1}B$ . Thus, there exist small open neighborhoods  $\Omega$  and  $\tilde{\Omega}$  of 0 in  $\mathbb{C}$  satisfying:

$$(107) \quad (I + \mathcal{A}^{-1}BK + z\mathcal{A}^{-1})y(z) = \lambda(z)y(z)$$

$$(108) \quad y : z \mapsto y(z) \in \mathcal{H}^{\alpha/2}, \text{ is holomorphic}$$

$$(109) \quad \lambda : z \mapsto \lambda(z) \in \tilde{\Omega}, \text{ is holomorphic}$$

$$(110) \quad \lambda(0) = 0, \quad y_0 := y(0) = \mathcal{A}^{-1}B$$

in such fashion that for any  $z \in \Omega$ ,  $\lambda(z)$  is the unique eigenvalue inside  $\tilde{\Omega}$ . Since  $\lambda(0) = 0$ , either  $\lambda$  is identically 0 in  $\Omega$  or there exists small neighborhood  $\omega$  such that  $\lambda(z) \neq 0$  for any  $z \in \omega \setminus \{0\}$ . Let us show that  $\lambda$  is not identically 0. For that, we assume by contradiction that it is. From the holomorphy property of  $y$ , there exists a sequence  $(y_k)_{k \in \mathbb{N}^*} \in \mathcal{H}^{\alpha/2}$  such that

$$y(z) = \sum_{k \in \mathbb{N}^*} y_k z^k,$$

with  $y_0 = \mathcal{A}^{-1}B$ . Then, in view of (107) and from assumption that  $\lambda \equiv 0$ , it holds that

$$(I + \mathcal{A}^{-1}BK + z\mathcal{A}^{-1}) \sum_{k \in \mathbb{N}^*} y_k z^k = 0, \text{ in } \mathcal{H}^{\alpha/2}$$

which ensures by the unicity of the development in entire function that

$$(111) \quad (I + y_0 K)y_k + \mathcal{A}^{-1}y_{k-1} = 0, \text{ in } \mathcal{H}^{\alpha/2}, \forall k \in \mathbb{N}^*.$$

By applying  $K$  to this equation and noticing that  $Ky_0 = -1$ , we get

$$K(\mathcal{A}^{-1}y_{k-1}) = 0, \quad \forall k \in \mathbb{N}^* \Leftrightarrow K(\mathcal{A}^{-1}y_k) = 0, \quad \forall k \in \mathbb{N}$$

Applying now  $K\mathcal{A}^{-1}$  to the same equation (111), we get

$$K(\mathcal{A}^{-1}y_k) + K(\mathcal{A}^{-1}y_0)(Ky_k) + K(\mathcal{A}^{-2}y_{k-1}) = 0, \quad \forall k \in \mathbb{N}^*$$

and then

$$K(\mathcal{A}^{-2}y_{k-1}) = 0, \quad \forall k \in \mathbb{N}^* \Leftrightarrow K(\mathcal{A}^{-2}y_k) = 0, \quad \forall k \in \mathbb{N}$$

By induction, applying  $K\mathcal{A}^{-(n-1)}$  to (111), we get that

$$K(\mathcal{A}^{-n}y_k) = 0, \quad \forall k \in \mathbb{N}, n \in \mathbb{N}^*.$$

In particular

$$K(\mathcal{A}^{-n}y_0) = 0, \quad \forall n \in \mathbb{N}^*,$$

which implies that

$$\sum_{m \in \mathbb{N}^*} \frac{b_m K_m}{\lambda_m^l} = 0, \quad \forall l \geq 2.$$

Using the holomorphic function technique in Appendix A.2, we get that  $b_m K_m = 0$ , which is a contradiction. Therefore, there exists  $\varepsilon > 0$  such that  $\lambda(z) \neq 0$  for any

$z \in \Omega$  and  $|z| < \varepsilon$ . Since  $\lambda(z)$  is the unique eigenvalue inside  $\tilde{\Omega}$ ,  $I + \mathcal{A}^{-1}BK + z\mathcal{A}^{-1}$  is invertible.

In the two cases, there exists at least a sequence of  $(z_k)_{k \in \mathbb{N}}$  converging to 0 such that  $I + \mathcal{A}^{-1}BK + z_k\mathcal{A}^{-1}$  is invertible from  $\mathcal{H}^{\alpha/2}$  to  $\mathcal{H}^{-\alpha/2}$ . Since the spectrum of  $\mathcal{A} + \rho I$  is discrete, we can find  $\rho := z_k - \lambda$  such that both  $\mathcal{A} + \rho I$  and  $\mathcal{A} + BK + \lambda I + \rho I = \mathcal{A}(I + \mathcal{A}^{-1}BK + z_k\mathcal{A}^{-1})$  are invertible operators from  $\mathcal{H}^{\alpha/2}$  to  $\mathcal{H}^{-\alpha/2}$ .

- We assume here that  $\ker T^* \neq \{0\}$  and we want to show that  $(\mathcal{A}^* + \bar{\rho}I)^{-1}$  has an eigenvector  $h$  in  $\ker T^*$  with  $\rho$  defined above. Recall the operator equality

$$T(\mathcal{A} + BK) = (\mathcal{A} - \lambda I)T.$$

Based on Lemma 5.9, this equality holds at least where the operator are seen as acting on  $\mathcal{H}^{\alpha/2}$  to  $\mathcal{H}^{-\alpha/2}$ . Then, for any  $\rho$  as in above, the following holds:

$$T(\mathcal{A} + BK + \lambda I + \rho I) = (\mathcal{A} + \rho I)T.$$

Since,  $\mathcal{A} + BK + \lambda I + \rho I$  and  $\mathcal{A} + \rho I$  are invertible, we get

$$(112) \quad (\mathcal{A} + \rho I)^{-1}T = T(\mathcal{A} + BK + \lambda I + \rho I)^{-1}.$$

As  $\ker T^* \neq \{0\}$ , we can select  $h \neq 0$  such that  $h \in \ker T^*$  and  $h \in \mathcal{H}^{-\alpha/2}$ . So, from (112), for any  $\varphi \in \mathcal{H}^{-\alpha/2}$ , it holds that

$$\begin{aligned} 0 &= \langle (\mathcal{A} + \rho I)^{-1}T\varphi - T(\mathcal{A} + BK + \lambda I + \rho I)^{-1}\varphi, h \rangle_{\mathcal{H}^{-\alpha/2}} \\ &= \langle \varphi, T^*(\mathcal{A}^* + \bar{\rho}I)^{-1}h \rangle_{\mathcal{H}^{-\alpha/2}} - \langle (\mathcal{A} + BK + \lambda I + \rho I)^{-1}\varphi, T^*h \rangle_{\mathcal{H}^{-\alpha/2}} \\ &= \langle \varphi, T^*(\mathcal{A}^* + \bar{\rho}I)^{-1}h \rangle_{\mathcal{H}^{-\alpha/2}}. \end{aligned}$$

This ensures that  $T^*(\mathcal{A}^* + \bar{\rho}I)^{-1}h = 0$  in  $\mathcal{H}^{-\alpha/2}$ , and thus  $(\mathcal{A}^* + \bar{\rho}I)^{-1}h \in \ker T^*$ . We have then shown that for any  $f \in \ker T^*$ ,  $(\mathcal{A}^* + \bar{\rho}I)^{-1}f \in \ker T^*$ .

Because  $\ker T^*$  is of finite dimension (recall that  $T$  is Fredholm, hence  $T^*$  is) and not reduced to  $\{0\}$  therefore the restriction of  $(\mathcal{A}^* + \bar{\rho}I)^{-1}$  to  $\ker T^*$  belongs to  $\mathcal{L}(\ker T^*)$  and is an operator on a space of finite-dimension, therefore there exists an eigenfunction  $h \in \ker T^*$  of  $(\mathcal{A}^* + \bar{\rho}I)^{-1}$ , associated to an eigenvalue  $\mu \neq 0$  (since the operator  $(\mathcal{A} + \bar{\rho}I)^{-1}$  is invertible). Then we have

$$(\mathcal{A} + \bar{\rho}I)^{-1}h = \mu h,$$

which in particular implies that

$$(113) \quad \mathcal{A}^*h = \frac{1 - \bar{\rho}\mu}{\mu}h,$$

and  $h \in \mathcal{H}^{\alpha/2}$ . This ensures that  $h$  is an eigenfunction of  $\mathcal{A}^*$  associated to the eigenvalue  $(1 - \bar{\rho}\mu)/\mu$ . As the subspaces of  $\mathcal{H}^{-\alpha/2}$  which are the eigenspaces of  $\mathcal{A}^*$  have dimension 1, the dimension of eigenspace associated to  $(1 - \bar{\rho}\mu)/\mu$  is one. Since  $(n^{\alpha/2}\varphi_n)_{n \in \mathbb{N}^*}$  forms a Riesz basis of  $\mathcal{H}^{-\alpha/2}$ , there exists  $n_0 \in \mathbb{N}^*$  such that  $\text{Eig}((1 - \bar{\rho}\mu)/\mu) = \text{span}\{n_0^{\alpha/2}\varphi_{n_0}\}$ .

In view of (113),  $h \in \text{Eig}((1 - \bar{\rho}\mu)/\mu)$ , then there exists  $C > 0$  such that  $h = Cn_0^{\alpha/2}\varphi_{n_0}$ . So, we have finally shown that if  $\ker T^* \neq \{0\}$ , there exist  $n_0 \in \mathbb{N}^*, C > 0$  such that  $Cn_0^{\alpha/2}\varphi_{n_0} \in \ker T^*$  and  $Cn_0^{\alpha/2}\varphi_{n_0}$  is an eigenfunction of  $(\mathcal{A}^* + \bar{\rho}I)$ .

- From what is done above, if  $\ker T^* \neq \{0\}$ , there exist  $n_0 \in \mathbb{N}^*, C > 0$  such that  $Cn_0^{\alpha/2}\varphi_{n_0} \in \ker T^*$ . So for any  $\varphi \in \mathcal{H}^{-\alpha/2}$ ,

$$0 = \langle T\varphi, n_0^{\alpha/2}\varphi_{n_0} \rangle_{\mathcal{H}^{-\alpha/2}} = \langle \varphi, T^*(n_0^{\alpha/2}\varphi_{n_0}) \rangle_{\mathcal{H}^{-\alpha/2}}$$

In particular for  $\varphi = B = \sum_{n \in \mathbb{N}^*} b_n \varphi_n$  we have

$$0 = \langle TB, n_0^{\alpha/2} \varphi_{n_0} \rangle_{\mathcal{H}^{-\alpha/2}}$$

which combining with the fact that  $TB = B$  in  $\mathcal{H}^{-\alpha/2}$ , ensures that

$$0 = \langle B, n_0^{\alpha/2} \varphi_{n_0} \rangle_{\mathcal{H}^{-\alpha/2}} = \frac{b_{n_0}}{n_0^{\alpha/2}}.$$

This is a contradiction due to assumption (8). Hence,  $\ker T^* = \{0\}$ .

**A.4.  $\mathcal{A}$  generates a semigroup on  $\mathcal{H}^{\vec{r}}$ .** We show the following

**LEMMA A.2.** *Under the assumption of Theorem 3.1,  $\mathcal{A}$  generates a  $C^0$  semigroup on  $\mathcal{H}^{\vec{r}}$  for any  $r = (r_1, \dots, r_m) \in \mathbb{R}^m$ .*

This is due to the intrinsic link between the definition of the spaces  $\mathcal{H}^{\vec{r}}$  and  $\mathcal{A}$  (see Section 2). We explicit this below. Given the definition of  $\mathcal{H}^{\vec{r}}$  it suffices to show that for any  $i \in \{1, \dots, m\}$  and  $r \in \mathbb{R}$ ,  $\mathcal{A}$  generates a  $C^0$  semigroup on  $\mathcal{H}_i^r$ . We define for any  $t \geq 0$

$$(114) \quad \begin{aligned} \mathcal{H}_i^r &\rightarrow \mathcal{H}_i^r \\ \mathcal{S}_{i,r}(t) : \sum_{n \in \mathbb{N}^*} f_n \varphi_n^i &\mapsto \sum_{n \in \mathbb{N}^*} e^{\lambda_n^i t} f_n \varphi_n^i. \end{aligned}$$

Since  $\mathcal{A}$  generates a  $C^0$  semigroup with finite growth bound, there exists  $\omega > 0$  such that  $|e^{\lambda_n^i t}| \leq e^{\omega t}$  for any  $n \in \mathbb{N}^*$  and therefore  $\mathcal{S}_{i,r}(t) \in \mathcal{L}(\mathcal{H}^r)$  for any  $t \geq 0$ . Clearly,  $\mathcal{S}_{i,r}(0) = Id$  and for  $(t, s) \in \mathbb{R}_+$ ,  $\mathcal{S}_{i,r}(t+s) = \mathcal{S}_{i,r}(t)\mathcal{S}_{i,r}(s)$ . Finally, using the fact that  $\tau_r : \varphi_n^i \rightarrow n^{-r} \varphi_n^i$  is an isomorphism from  $\mathcal{H}_i^0$  to  $\mathcal{H}_i^r$  which commutes with  $\mathcal{A}$  and satisfies

$$(115) \quad \mathcal{S}_{i,r} \tau_r = \tau_r \mathcal{S}_{i,0},$$

and the fact that  $\mathcal{A}$  is the infinitesimal generator of  $\mathcal{S}_{i,0}$  on  $\mathcal{H}_i^0$ , we deduce directly that  $\mathcal{S}_{i,r}$  is a  $C^0$  semigroup and that  $\mathcal{A}$  is the infinitesimal generator of  $\mathcal{S}_{i,r}$  with domain  $D_r(\mathcal{A}) := \tau_r(D_0(\mathcal{A})) = \mathcal{H}_i^{r+\alpha_i}$ .

**A.5. Extension to the case  $\beta \neq 0$ .** Suppose that Proposition 4.1 holds for  $\beta = 0$ , then it also holds for  $\beta \neq 0$  exactly as in [19]. To do so, it suffices to define  $\tilde{B} = \sum_n n^\beta b_n \varphi_n$ , apply Proposition 4.1 with  $\tilde{B}$  to obtain a feedback operator  $\tilde{K}$  and an isomorphism  $\tilde{T}$ . Then Proposition 4.1 holds for  $\beta \neq 0$  with  $T := M^{-1} \tilde{T} M$ , and  $K = \tilde{K} M$  where  $M$  is the isomorphism from  $\mathcal{H}^{\beta+s}$  to  $\mathcal{H}^s$  for any  $s \in \mathbb{R}$  defined by

$$(116) \quad M : n^{-\beta} \varphi_n \mapsto \varphi_n.$$

We refer to [19] for more details.

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