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# Regularised Homotopy Continuation Method for the Motion Planning Problem

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**Abstract**—We address the motion planning problem for smooth control-nonlinear systems by presenting a regularised version of the well-established homotopy continuation method. The proposed approach (inspired by Tikhonov regularisation in the Moore-Penrose pseudo-inverse theory) deals with the two issues of the classical method: the singularities of the endpoint map of the nonlinear control system and the global existence of solutions of a differential equation posed on the control space, the path lifting equation (PLE). We analyse the properties of our method and show that it is always well-posed and that the regularised PLE always possesses a global solution. We additionally examine the effect of the regularisation and we give conditions under which the regularized solution converges to the initial solution. Finally, we illustrate the potential of our method through several numerical examples.

**Index Terms**—motion planning, nonlinear systems, singular control, continuation method, pseudo-inverse

## I. INTRODUCTION

In this paper, the motion planning problem (MPP) for control-nonlinear systems is addressed. We consider a control system  $\Xi : \dot{x} = F(x, u)$ , with the state  $x \in \mathcal{X}$  (an open subset of  $\mathbb{R}^n$ ) and the control  $u = (u_1, \dots, u_m)$  takes values in  $U$  an open subset of  $\mathbb{R}^m$  containing zero. The MPP is that of finding for every pair  $(x_0, x^*)$  of points of  $\mathcal{X}$  a control  $u^*$  that steers  $x_0$  to  $x^*$ . The MPP can be decomposed into two subproblems, which are both difficult to solve in their full generality. First is the accessibility/controllability one, i.e. describing the subset of  $\mathcal{X}$  that can be reached from  $x_0$  in time  $T$  using the controls of  $U$ . For some subclasses of control systems, this question may be answered by studying the structure of the Lie algebra generated by the dynamics [1]–[3]. Second is the problem of actually finding a control  $u^*$  that realises the transfer from  $x_0$  to  $x^*$ . Many different algorithmic methods have been developed in the literature, some of them are presented hereafter. Those two problems are related and they need to be solved in order. Indeed, given a control system we need first to establish the existence of a control  $u^*$  that steers  $x_0$  to  $x^*$ , and next we have to implement a numerical method to compute actually  $u^*$ . The algorithm that we present in this paper is devoted to solving the second subproblem of the MPP only.

Throughout the years, several methods have been developed to solve the MPP. However, there is still no general algorithm

that guarantees any control system to reach an accessible goal exactly. Moreover, most of the presented techniques apply to control-linear (driftless) systems only (and sometimes to some subclasses of control-linear systems), only the methods based on optimal control, the differential flatness, and the continuation method can argue to apply to any control-nonlinear system. We do not go into the details of each method, the interested reader will find supplementary informations in [4], [5], and references therein.

First, there is Lie bracket based methods, which apply to driftless systems. The basic idea is that Rashevsky-Chow theorem assumes that the vector fields  $g_1, \dots, g_m$  (supporting the control's direction) together with their iterated Lie brackets span the full tangent space. Therefore, if one can make the system follow the directions of the Lie brackets, then one can go in anywhere. This is achieved following iteratively the flow of the vector fields  $g_i$  for small times and using the Baker-Campbell-Hausdorff formula. In particular, the Lafferriere-Sussmann method [6] applies to any control-linear systems and gives an exact solution to the MPP in the case where the generated Lie algebra is nilpotent and furnishes an approximative method otherwise. However, this method requires some change of coordinates that, when numerically performed, induce instability and quadrature errors. Moreover the Lie algebra basis used by the method is defined locally only; making this method global appears to be a difficult task in practice. Another method is to use oscillatory controls for chained-systems, see [7]. This method constructs a sequence of highly oscillatory controls that steers the system to the target configuration  $x^*$  coordinate by coordinate. The interest of using oscillatory controls is to move each coordinate of the system without affecting the others but numerically this method does not behave well. Finally, the Liu-Sussmann method [8] combines ideas of the previous two methodology.

Second, there are more general methods that apply to control-nonlinear systems. The differential flatness property [9] is satisfied by many systems encountered in practice and is very useful and powerful. It gives the state and controls of a system as functions of the so-called flat outputs. Thus, to solve the MPP, one has to simply find a curve joining the initial point  $x_0$  and the terminal condition  $x^*$  in the flat output space. Then, we deduce the value of the open-loop controls through the equations of flatness. The main drawback of this

method is that there does not exist a general characterisation of flat systems and, even if a system is flat, then there does not exist an explicit method to find the flat outputs. Optimal control is one of the most important topic in the study of control-nonlinear systems. Indeed, the idea of associating a cost to each admissible trajectory and trying to find the trajectory minimising that cost seems to solve two problems at once. Usually, there are two types of numerical methods used to solve optimal control problems, namely, direct and indirect methods. Direct methods consist in discretising the equations, which yields a problem of nonlinear optimisation under constraints. Indirect methods use a shooting algorithm on a problem obtained from the application of the Pontryagin Maximum Principle (see [10]). Those methods are costly as soon as the dimension of the state increases, moreover, indirect methods may have a small convergence domain (they rely on Newton's algorithm). Nevertheless, using optimal control to solve the MPP is probably the best method that we have at our disposal. The continuation method, whose presentation will be extended in the next section, solves the MPP step by step by a parameter deformation. That approach requires first to characterise the singularities of the endpoint mapping and next to prove the global existence of a highly nonlinear differential equation on the space of controls. We stress that this method has the advantage that it applies to any kind of control system and that it is easy to implement, but it is difficult to check its assumptions in general.

There exists many other approaches to solve the motion planning problem, such as probabilistic, multi-level, optimisation, etc, see [4] for further details.

The paper is organised as follows. In the next section we will introduce the motion planning problem for control-affine systems, we will define and discuss some properties of the endpoint mapping and we will present the continuation method. Next, we will introduce our novel algorithm : the regularised continuation method and study its properties. Finally, we will illustrate the use of our method through some numerical examples.

## II. PRELIMINARIES

In this section, we precise the context that we are dealing with and introduce all the necessary tools to present our method. In particular, the end of this section is devoted to the presentation of the classical continuation method, which is our main inspiration for this work. Throughout this paper, the word *smooth* will always mean  $C^\infty$ -smooth and all objects (vector fields, functions) are considered smooth. Let  $\mathcal{X}$  be an open subset of  $\mathbb{R}^n$  and consider a control-affine system, with state  $x \in \mathcal{X}$  and  $m \geq 1$  controls, given in local coordinates by

$$\Sigma : \dot{x} = f(x) + \sum_{i=1}^m u_i g_i(x), \quad \text{and} \quad x(0) = x_0.$$

The control  $u = (u_1, \dots, u_m)$  takes values in  $U$ , an open connected subset of  $\mathbb{R}^m$  containing 0, and  $f, g_1, \dots, g_m$  are smooth vector fields on  $\mathcal{X}$ . Throughout the paper, we assume

that the initial state  $x_0 \in \mathcal{X}$  and the final time  $T > 0$  are fixed. We denote by  $\gamma_u$  the trajectory of  $\Sigma$  generated by  $u(\cdot)$  and starting from  $\gamma(0) = x_0$ . We set  $\mathcal{U}$  the class of admissible inputs, that is the set of  $U$ -valued functions defined on  $[0, T]$  such that the associated trajectory  $\gamma_u$  exists for all  $t \in [0, T]$ . We endow  $\mathcal{U}$  with the  $L^2$ -norm topology. Now, we formulate the problem of motion planning (MPP).

**Problem** (Motion Planning Problem). Consider a control-affine system  $\Sigma$  and fix an initial condition  $x_0 \in \mathcal{X}$  and a final time  $T > 0$ . For any point  $x^* \in \mathcal{X}$ , the MPP is to find a control  $u^* \in \mathcal{U}$  such that  $\gamma_{u^*}(0) = x_0$  and  $\gamma_{u^*}(T) = x^*$ .

As indicated in the introduction, the MPP contains the controllability/accessibility problem (existence of  $u^*$ ) and the numerical determination of  $u^*$  problem. In this work, we assume that  $u^*$  exists and we focus on its determination.

**Remark** (Extension to control-nonlinear systems). Notice that the MPP can also be considered for general control-nonlinear systems  $\Xi : \dot{x} = F(x, w)$ , where  $F$  is a smooth map and  $w$  is the control. But for three reasons it is convenient to consider control-affine systems only. Firstly, any control-nonlinear system  $\Xi$  can be prolonged to a control-affine system by augmenting its state space with the controls  $w$  and introducing new controls  $u_i = \dot{w}_i$ . The second reason is that using control-affine systems allows to endow the set of controls with the  $L^2$ -norm topology and therefore we benefit from its hilbertian structure. And, thirdly, for many applications (for instance, in robotics) the control systems that we consider are control-affine, or even control-linear systems.  $\blacklozenge$

a) *Endpoint mapping*: For a control-affine system  $\Sigma$ , we define the endpoint map as the application

$$E : \mathcal{U} \longrightarrow \mathcal{X} \\ u \longmapsto \gamma_u(T),$$

which associates to each control the terminal point of the corresponding trajectory. Let  $u$  and  $(u_n)_{n \in \mathbb{N}}$  be elements of  $L^2([0, T], U)$ , we denote by  $u_n \rightharpoonup u$  the weak convergence of the sequence  $u_n$  to  $u$  in  $L^2$ . To give the differential of  $E$ , we consider the linearised system along a trajectory  $\gamma_u$ :

$$\dot{y}_v(t) = A_u(t)y_v + B_u(t)v, \quad v \in L^2, \quad (1)$$

where  $v$  is a new control,  $A_u = \frac{\partial f}{\partial x}(\gamma_u) + \sum_{i=1}^m u_i \frac{\partial g_i}{\partial x}(\gamma_u)$ , and  $B_u = (g_1(\gamma_u), \dots, g_m(\gamma_u))$ . We have the following properties of the endpoint mapping.

**Proposition 1** ([11]). *The endpoint mapping  $E$  of a smooth control-affine system  $\Sigma$  satisfies the following properties:*

- 1) *The domain  $\mathcal{U}$  of  $E$  is open in  $L^2([0, T], U)$ ;*
- 2) *The endpoint map is  $L^2$ -Fréchet differentiable, and for all  $v \in \mathcal{U}$  we have  $dE(u) \cdot v = y_v(T)$*
- 3) *If  $u_n \rightharpoonup u$ , then  $dE(u_n) \rightarrow dE(u)$ .*

A control  $u_s$  (or its associated trajectory  $\gamma_{u_s}(t)$ ) is said to be *singular* on  $[0, T]$  if  $u_s$  is a singular point of the endpoint map, i.e. the Fréchet differential of  $E$  is not surjective at  $u_s$ . Otherwise we say that  $u$  is regular. A control  $u$  in the interior

of  $\mathcal{U}$  is singular if and only if the linearised system (1) along the trajectory  $\gamma_u$  is not controllable (see [12]). We denote by  $\mathcal{S} \subset \mathcal{U}$  the set of singular controls. It is a classical fact that singular controls are solutions of the time optimal problem and, thus, benefit from a Hamiltonian characterization. That characterisation relies on the existence of a non trivial adjoint vector, which is orthogonal to the image of  $dE_t(u_s)$ <sup>1</sup>. In particular,  $\text{im}(dE_t(u_s))$  is a subspace of the tangent space of codimension greater or equal than 1.

b) *Moore-Penrose Pseudo-Inverse*: We recall some properties of the Moore-Penrose pseudo-inverse (MPPI) which plays an important role in the continuation method (both for the classical and our regularised version); see [13]. Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be two Hilbert spaces, with  $\dim \mathcal{H}_2 < \infty$ . Let  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a continuous linear mapping and let  $A^*$  be its adjoint. We define the MPPI of  $A$  as the unique operator  $A^+ : \mathcal{H}_2 \rightarrow \mathcal{H}_1$  satisfying  $AA^+A = A$ ,  $A^+AA^+ = A^+$ ,  $AA^+$  and  $A^+A$  are self-adjoint. It can be obtained via

$$A^+ = \lim_{\lambda \rightarrow 0} A^*(AA^* + \lambda \text{Id}_{\mathcal{H}_2})^{-1}. \quad (2)$$

Finally, we define the operator  $P = \text{Id}_{\mathcal{H}_2} - AA^+$  from  $\mathcal{H}_2$  to  $\mathcal{H}_2$ , which is an orthogonal projector (i.e.  $P^* = P$  and  $P^2 = P$ ) satisfying

$$\ker P = \text{im } A, \quad \text{and} \quad \text{im } P = \ker A^+, \quad (3)$$

in particular, if  $A$  is onto then  $P = 0$ .

c) *Presentation of the classical continuation method*:

We now present the continuation method in the context of the motion planning problem. This method has been introduced by Sussmann [5] and developed by Chitour [14]; see also [15] for a practical implementation of that method applied to the rolling body problem.

Consider the endpoint map  $E : \mathcal{U} \rightarrow \mathbb{R}^n$  associated with a control-affine system and fix  $x^* \in \mathbb{R}^n$ . The purpose of the continuation method is to determine a control  $u^* \in \mathcal{U}$  such that  $E(u^*) = x^*$ , i.e. to find a preimage of  $x^*$ . We choose an arbitrary control  $u^0 \in \mathcal{U}$  and we set  $x^0 = E(u^0)$ , then we construct a smooth path  $\pi(s) \in \mathbb{R}^n$  between  $x^0$  and  $x^*$ , that is  $\pi : [0, 1] \rightarrow \mathbb{R}^n$  such that  $\pi(0) = x^0$  and  $\pi(1) = x^*$ . The key step in the method is the “lifting” of  $\pi$  into a path  $\Pi : [0, 1] \rightarrow \mathcal{U}$  satisfying

$$\forall s \in [0, 1], \quad E(\Pi(s)) = \pi(s). \quad (4)$$

In particular, notice that  $\Pi(0) = u^0$  satisfies that relation at  $s = 0$ . Since  $\Pi$  is defined implicitly, one usually proceeds by differentiating (4) with respect to  $s$ , which yields the following ordinary differential equation on  $\mathcal{U}$ :

$$dE(\Pi(s)) \frac{d\Pi}{ds}(s) = \frac{d\pi}{ds}(s), \quad s \in [0, 1], \quad \Pi(0) = u^0. \quad (5)$$

If this equation admits a global solution on  $[0, 1]$ , then  $u^* = \Pi(1)$  is a solution of the MPP. As long as  $\pi(s) \notin E(\mathcal{S})$ , the differential  $dE(\Pi(s))$  is right invertible, consequently we use

<sup>1</sup> $E_t$  is the endpoint map at time  $t > 0$ .

its Moore-Penrose pseudo-inverse  $P(\Pi(s)) : T_{E(\Pi(s))}\mathbb{R}^n \rightarrow \mathcal{U}$  defined by:

$$\forall u \in \mathcal{U} \setminus \mathcal{S}, \quad P(u) = dE(u)^* [dE(u)dE(u)^*]^{-1}.$$

And, to satisfy equation (5), we choose  $\Pi'(s)$  as

$$\Pi'(s) = P(\Pi(s))\pi'(s), \quad s \in [0, 1], \quad \Pi(0) = u^0, \quad (6)$$

We call this equation the *path lifting equation (PLE)*. If we can find a solution of (6) defined on the whole interval  $[0, 1]$  and with the initial condition  $\Pi(0) = u^0$ , then  $\Pi(1)$  is a solution of the MPP. Therefore, the two issues of the continuation method are the following:

- 1) **non-degeneracy**: it is necessary that  $dE(\Pi(s))$  has full rank at all  $s \in [0, 1]$ , that is  $\pi(s) \notin E(\mathcal{S})$ , where  $\mathcal{S}$  is the singular set of  $E$ .
- 2) **non-explosion**: equation (6) must have a global solution on  $[0, 1]$  with  $E(\Pi(0)) = \pi(0)$ .

If one of the above conditions fails to hold, then the continuation method may not work. To be efficient, the classical homotopy method needs therefore two strong properties: first, one should be able to locate the singularities of the endpoint map (still an open problem) and should prove that the PLE possesses a global solution. In [14] this has been proven for control-linear systems under some strong controllability conditions. Our regularised method introduced below solves this two issues. See Figure 1 for a visual summary of the method.

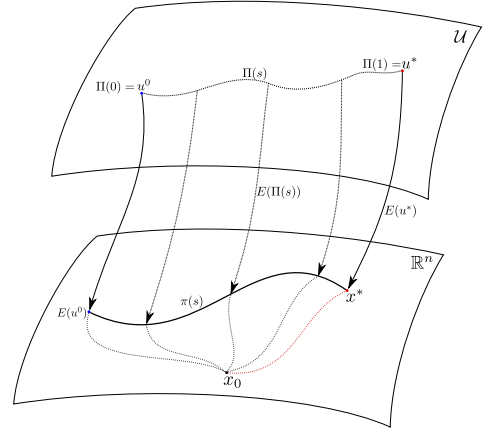


Figure 1: Representation of the continuation method applied to motion planning. The top, resp. bottom, shape represents the space of controls  $\mathcal{U}$ , resp. the state space  $\mathbb{R}^n$ . The arrows from top to bottom represents the application of the endpoint map  $E$ , the thick solid black line is the path  $\pi(s)$  between  $x^0 = E(u^0)$  and  $x^*$ , and the dotted black line on  $\mathcal{U}$  is the constructed path  $\Pi(s)$ .

### III. REGULARISATION OF THE CONTINUATION METHOD

We now introduce our novel method : a regularised version of the continuation method. We present our approach in the context of the MPP, nevertheless we emphasize that this method can be carried out on general applications of the

continuation method. Our proposed method simultaneously tackles the non-degeneracy and the non-explosion conditions of the PLE. However, its cost is the introduction of a family of problems whose solutions may not converge to a solution of the original problem.

The path lifting equation (6) is given in terms of the Moore-Penrose pseudo-inverse (MPPI), when the endpoint map is surjective. Equation (2) shows that the MPPI can be approximated by a family of linear operators, thus it suggests to introduce the following equation parametrised by  $\lambda > 0$

$$\Pi'_\lambda(s) = dE(\Pi_\lambda(s))^* [dE(\Pi_\lambda(s))dE(\Pi_\lambda(s) + \lambda \text{Id}_n)]^{-1} \pi'(s), \quad (7)$$

with the initial condition  $\Pi_\lambda(0) = u^0$ . We call this equation the *regularised path lifting equation* (R-PLE, shortly). The following theorem shows that the R-PLE is well posed and that it takes care of the issues of degeneracy and of explosion of the classical PLE.

**Theorem 1** (Existence and global solution of the R-PLE). *Assume that  $\pi(s)$  is a  $C^1$ -curve on  $\mathbb{R}^n$  and that  $0 \in \mathcal{U}$ . Then, for any  $\lambda > 0$  and any  $u^0 \in \mathcal{U}$  the regularised path lifting equation, with initial condition  $\Pi_\lambda(0) = u^0$ , admits a global solution on  $[0, 1]$ .*

The assumption that  $0 \in \mathcal{U}$  means that the trajectory associated with the zero control is well defined on  $[0, T]$ ; for control-linear systems this assumption is automatically satisfied and for control-affine systems, it corresponds to the fact that the trajectory of the dynamical system  $\dot{x}(t) = f(x(t))$  exists for all  $t \in [0, T]$ .

*Proof.* Using basic properties of the MPPI, we deduce that the R-PLE can equivalently be formulated as

$$\Pi'_\lambda(s) = \arg \min_{v \in \mathcal{U}} \|dE(\Pi_\lambda(s))v - \pi'(s)\|_{\mathbb{R}^n}^2 + \lambda \|v\|_{\mathcal{U}}^2.$$

Hence, for all  $v \in \mathcal{U}$  we have

$$\|dE(\Pi_\lambda(s))\Pi'_\lambda(s) - \pi'(s)\|_{\mathbb{R}^n}^2 + \lambda \|\Pi'_\lambda(s)\|_{\mathcal{U}}^2 \leq \|dE(\Pi_\lambda(s))v - \pi'(s)\|_{\mathbb{R}^n}^2 + \lambda \|v\|_{\mathcal{U}}^2$$

implying that

$$\lambda \|\Pi'_\lambda(s)\|_{\mathcal{U}}^2 \leq \|dE(\Pi_\lambda(s))v - \pi'(s)\|_{\mathbb{R}^n}^2 + \lambda \|v\|_{\mathcal{U}}^2,$$

In particular for  $v = 0$ , which belongs to  $\mathcal{U}$ , we get  $\lambda \|\Pi'_\lambda(s)\|_{\mathcal{U}}^2 \leq \|\pi'(s)\|_{\mathbb{R}^n}^2$ , concluding the proof. ■

Observe that Theorem 1 does not require that the path  $\pi(s)$  avoids the singularities of  $E$ . Hence, our regularised method avoids the main difficulty of the classical continuation method. The process of the regularised continuation method is summarised in Figure 2 below. The natural question that we will answer next is the convergence of the global solution of the R-PLE to a solution of the PLE as  $\lambda \rightarrow 0$ . Given the solution  $\Pi_\lambda(s)$  of the R-PLE, we construct, on the state space  $\mathbb{R}^n$ , the path

$$\pi_\lambda(s) := E(\Pi_\lambda(s)), \quad \forall s \in [0, 1],$$

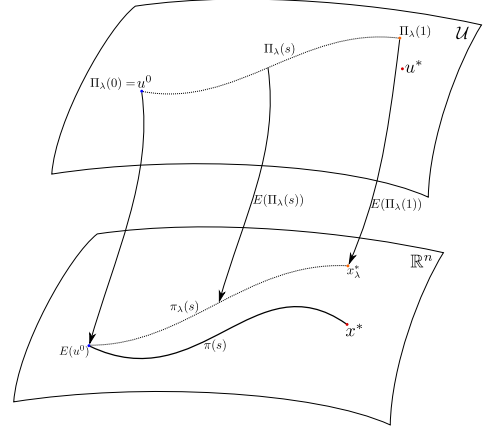


Figure 2: Representation of the regularised continuation method applied to motion planning. In addition to Figure 1, the dotted black line on  $\mathbb{R}^n$  represents the image  $\pi_\lambda(s)$  of the regularised solution  $\Pi_\lambda(s)$  by the endpoint map  $E$ .

which is different from  $\pi(s)$  except for  $s = 0$  (see Figure 2). We give a necessary condition for the path  $\pi(s)$  assuring that  $\lim_{\lambda \rightarrow 0} \pi_\lambda(1) = x^*$ . To this end, we define the error due to the regularisation by

$$e_\lambda(s) = \pi(s) - \pi_\lambda(s).$$

Observe that for all  $\lambda$ , we have  $e_\lambda(0) = 0$ , therefore if  $e'_\lambda(s)$  is close to zero (or even vanishes for almost every  $s$ ), then the error at  $s = 1$  should be small. We differentiate  $e_\lambda$  with respect to  $s$  and we get

$$\begin{aligned} e'_\lambda(s) &= \pi'(s) - \frac{d}{ds} (E(\Pi_\lambda(s))), \\ &= \pi'(s) - dE(\Pi_\lambda(s))\Pi'_\lambda(s), \\ &= [\text{Id} - dE(\Pi_\lambda(s))dE(\Pi_\lambda(s))^* \\ &\quad (dE(\Pi_\lambda(s))dE(\Pi_\lambda(s))^* + \lambda \text{Id})^{-1}] \pi'(s). \end{aligned}$$

For a general linear map  $A$ , we have  $\lim_{\lambda \rightarrow 0} AA^*(AA^* + \lambda \text{Id})^{-1} = AA^+$  and recall from (3) that the operator  $P = \text{Id} - AA^+$  is an orthogonal projector such that  $\ker P = \text{im } A$  and  $\text{im } P = \ker A^+$ . With our analysis, we conclude:

**Proposition 2.** *Suppose that for all  $s \in [0, 1]$ , there exists  $\Pi_0(s) \in \mathcal{U}$  such that  $\Pi_\lambda(s) \xrightarrow{\lambda \rightarrow 0} \Pi_0(s)$ . If*

$$\pi'(s) \in \text{im } dE(\Pi_0(s)), \quad \text{for almost every } s \in [0, 1], \quad (8)$$

*then the global solution  $\Pi_\lambda(s)$  of the regularised path lifting equation converges to a solution of the motion planning problem when  $\lambda$  goes to zero.*

*Proof.* Using Proposition 1 item 3 we obtain that  $dE(\Pi_\lambda) \rightarrow dE(\Pi_0)$ , so the analysis performed above the proposition holds and we get  $\lim_{\lambda \rightarrow 0} e'_\lambda(s) = 0$ . Hence,  $\lim_{\lambda \rightarrow 0} e_\lambda(1) = 0$  implying that  $\lim_{\lambda \rightarrow 0} \pi_\lambda(1) = \pi(1) = x^*$ . ■

To summarise, a necessary condition for the convergence of the regularised continuation method is  $\pi'(s) \in \text{im } dE(\Pi_0(s))$ ,

which implies that we can construct a path  $\pi(s)$  crossing the singular set  $\mathcal{S}$  but, if so, it should do it transversally. If, for every  $s$ , the control  $\Pi_0(s)$  is not singular then  $dE(\Pi_0(s))$  has full rank and condition (8) is automatically fulfilled. However, if for some  $\bar{s}$  the control  $\Pi_0(\bar{s})$  is singular, then the image of  $dE(\Pi_0(\bar{s}))$  is a subspace of  $\mathbb{R}^n$  with codimension at least 1. In that case, condition (8) is non-trivial, and  $\pi(s)$  has to be chosen accordingly.

Finally our strategy to solve the motion planning problem is given in the following procedure, whose main difficulty is to solve the R-PLE, which is an ordinary differential equation defined on the infinite dimensional control space  $\mathcal{U}$  (details about the implementation can be found in [16]).

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**Algorithm 1** Regularised continuation method for the MPP

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Choose  $\lambda > 0$ ;  
 Choose an arbitrary control  $u^0 \in \mathcal{U}$  and set  $x^0 = E(u^0)$ ;  
 Define a curve  $\pi : [0, 1] \rightarrow \mathcal{X}$  such that  $\pi(0) = x^0$  and  $\pi(1) = x^*$ ;  
 Solve numerically the R-PLE (7) with  $\Pi_\lambda(0) = u^0$ ;  
 Set  $u_\lambda^* = \Pi_\lambda(1)$ ;

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#### IV. NUMERICAL EXPERIMENT

In this section we illustrate the potential of the regularised continuation method with a numerical example. The following parameters are set for the two experiments described below. We fix the terminal time  $T = 1$ , we discretise the time interval  $[0, 1]$  with 257 equally spaced points, the interval  $[0, 1]$ , for the variable  $s$ , of the regularised path lifting equation is discretised with 513 equally spaced points, and the space of controls is approximated by continuous piecewise linear functions on 129 equally spaced points. We consider the Martinet system given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 1 \\ 0 \\ \frac{y^2}{2} \end{pmatrix} u_2,$$

whose nontrivial abnormal extremals, i.e. singularities of the endpoint map  $E$ , are the straight lines  $z = z_0$  contained in the plane  $\{y = 0\}$ , called the Martinet plane. We test algorithm 1 to find a control that steers the system from  $x_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$  to  $x^* = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ . Therefore, observe that the path  $\pi(s)$  will necessarily cross the Martinet plane and thus the classical regularised continuation method can not be applied on this example. We perform two experiments, which both start with the initial control  $u^0 = 0$ , hence  $x^0 = E(u_0) = x_0$ , and have for respective path  $\pi(s)$ :

- 1) A straight line  $\pi(s) = (1 - s)x^0 + sx^*$
- 2) For  $s \in [0, 0.3]$ , we go straight from  $x^0 = (1, -1, 1)^t$  to  $x^{0,1} = (1, 0, -1)^t$ ; second, for  $s \in ]0.3, 0.6]$ , we go straight from  $x^{0,1}$  to  $x^{0,2} = (2, 0, -1)^t$ , i.e. we follow a singular trajectory; and third, for  $s \in ]0, 1]$ , we go straight from  $x^{0,2}$  to  $x^* = (1, 1, -1)^t$ .

In both cases, the path  $\pi(s)$  satisfy condition (8) for all  $s$ . However, observe that the second experiment is more challenging on the crossing of the singular plane. Indeed, in that experiment,  $\pi(s)$  belongs to the singular plane  $\{y = 0\}$  for  $s$  in an interval, so clearly, the classical continuation method would not be applicable here.

Figure 3 shows the results of our implementation of algorithm 1 for  $\lambda = 10^{-3}$  on the Martinet system. First, we observe in Figures 3a to 3c that  $\pi_\lambda(s) = E(\Pi_\lambda(s))$  follows accurately the path  $\pi(s)$ , it deviates from that reference with an error of magnitude 0.01 (as illustrated in Figure 3d). Moreover notice that no special behaviour occurs when  $\pi_\lambda(s)$  reaches the singular plane (at  $s = 0.5$  for the first experiment, in blue, and for  $s \in [0.3, 0.6]$  for the second one, in red). Thus, we observe that, even in the presence of nontrivial singularities, the regularised path lifting equation is well-posed and possesses a global solution on  $[0, 1]$ , illustrating Theorem 1. Second, Figure 3d displays the norm of the difference  $\pi(s) - \pi_\lambda(s)$ , i.e. the difference between the reference path and the actually constructed one. For both experiment, by definition, we have  $e_\lambda(0) = 0$  and  $\|e_\lambda(s)\|$  is strictly increasing. We note that the error grows fast at the beginning of the path lifting equation and quickly stabilizes in ball of radius 0.04 around the target configuration  $x^*$ . This suggests the following improvement of the method: one could use a ‘‘warm-restart’’ of the method, that is stopping the procedure at  $s = 0.2$  and use the result of this truncated calculation as a new starting point for our algorithm (we simply need to compute a new path  $\pi$ ).

Next, we study the convergence of the solution when  $\lambda$  tends to zero. Figure 4 illustrates that when  $\lambda$  goes to zero, then the terminal error, i.e.  $\|\pi(1) - \pi_\lambda(1)\|$ , tends to zero. This demonstrate that, when (8) is satisfied, the regularised solution converges to a solution of the classical PLE (and thus of the MPP). We also remark a linear relation between the error and  $\lambda$ , which should be further investigated.

To conclude, the presented numerical experiments illustrate that our method can be successfully applied to cases, where  $\pi(s)$  passes through singularities for  $s$  in an interval (provided that  $\pi'(s)$  is suitably chosen).

#### V. CONCLUSION AND PERSPECTIVES

In this paper, we presented a regularised version of the continuation method, introducing the regularised path lifting equation. We showed that the R-PLE is always well posed and possesses a global solution on  $[0, 1]$ . Under the necessary condition that the constructed path is transversal to the singular set, we showed the convergence of the regularised solution to a solution of the classical continuation method as the regularisation parameter tends to zero. Finally, we illustrated the potential of the proposed approach via two numerical examples. In the near future, we plan to study the effect of other regularising term in the PLE, such as the introduction of a  $L^1$  penalty term to enforce sparse changes in the control along the path  $\Pi_\lambda(s)$ . We will also study the possibility of introducing constraints in the control space to account for physical limitations of the control actions.

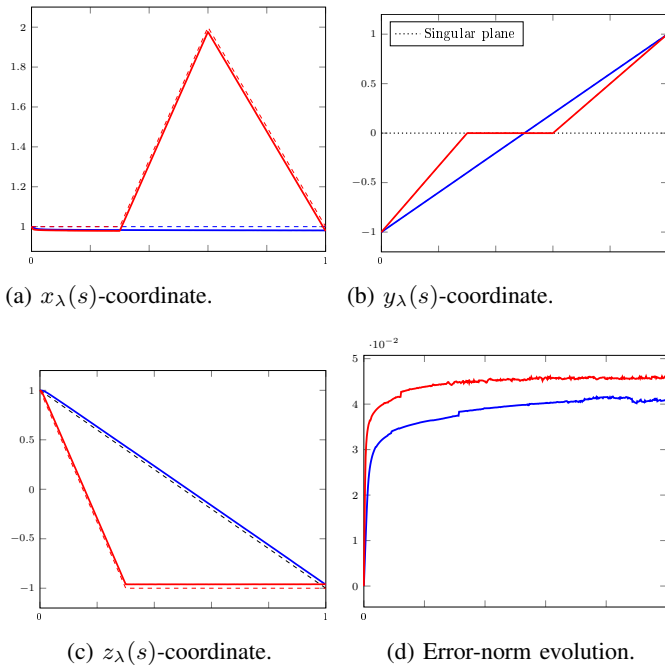


Figure 3: Results of the regularised continuation method for  $\lambda = 10^{-3}$ . The blue color corresponds to the first experiment and the red one to the second experiment. Figures 3a to 3c show the evolution for  $s \in [0, 1]$  of the coordinates  $(x_\lambda, y_\lambda, z_\lambda)$  of  $E(\Pi_\lambda(s))$ , solid line, and the reference path  $\pi(s)$ , dashed line; moreover for the  $y$ -component we display the singular plane as a dotted line. Figure 3d shows the evolution of the error norm  $\|\pi(s) - \pi_\lambda(s)\|$  for  $s \in [0, 1]$ .

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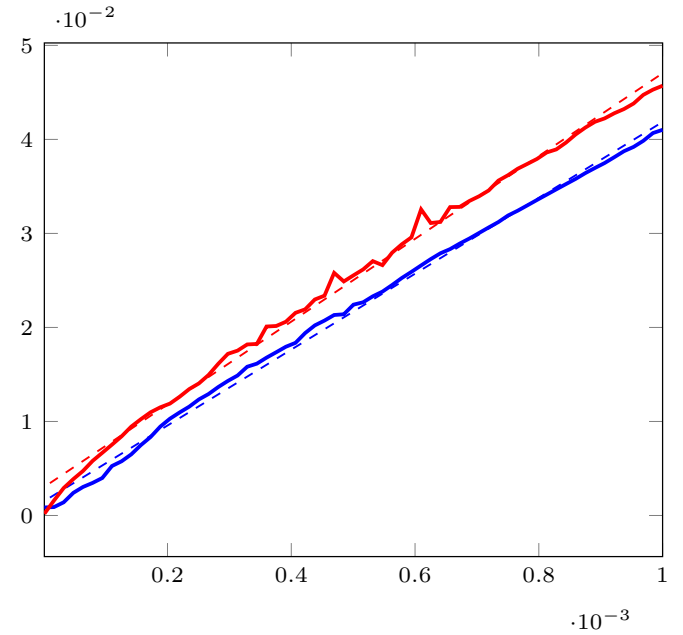


Figure 4: Convergence  $\lambda \rightarrow 0$  of the regularised continuation method. The blue color corresponds to the first experiment and the red one to the second experiment. It shows with a solid line the evolution of the terminal error norm  $\|e_\lambda(1)\|$  against  $\lambda$ , and the dashed line shows the best linear approximation.

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