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Principal eigenvalues for Fully Non Linear singular or degenerate operators in punctured balls

Françoise Demengel
Département de Mathématiques, CY Paris University

Abstract

This paper is devoted to the proof of the existence of the principal eigenvalue and related eigenfunctions for fully nonlinear degenerate or singular uniformly elliptic equations posed in a punctured ball, in presence of a singular potential. More precisely, we analyze existence, uniqueness and regularity of solutions $(\bar{\lambda}_\gamma, u_\gamma)$ of the equation

$$|\nabla u|^\alpha F(D^2 u_\gamma) + \bar{\lambda}_\gamma \frac{u_\gamma^{1+\alpha}}{r^\gamma} = 0 \text{ in } B(0, 1) \setminus \{0\}, \quad u_\gamma = 0 \text{ on } \partial B(0, 1)$$

where $u_\gamma > 0$ in $B(0, 1)$, $\alpha > -1$ and $\gamma > 0$. We prove existence of radial solutions which are continuous on $\overline{B(0, 1)}$ in the case $\gamma < 2 + \alpha$, and a non existence result for $\gamma > 2 + \alpha$. We also give the explicit value of $\bar{\lambda}_{2+\alpha}$ in the case of the Pucci's operators, which generalizes the Hardy–Sobolev constant for the Laplacian, and the previous results of [4].

1 Introduction

We look for “radial eigenvalues” and related positive eigenfunctions for the problem

$$|\nabla u|^\alpha F(D^2 u) + \mu r^{-\gamma} u^{1+\alpha} = 0 \text{ in } \overline{B(0, 1)} \setminus \{0\},$$

where $\gamma > 0$, and F is a second order, fully nonlinear, uniformly elliptic operator. α is supposed to be > -1 and the solutions are intended in the viscosity sense given by [2]. In the model cases, F will be one of Pucci's extremal operators. It is clear that, if $\alpha = 0$, as in the case of Laplace operator, $\gamma = 2$ is a critical case, being F a second order differential operator. This case $\alpha = 0$ is studied in [4] and the present paper extends most of the results in [4] to the case $\alpha \neq 0$. As in the case of the Laplacian, for $\gamma < 2 + \alpha$ not “too bad” solutions are expected to exist, whereas there are no solutions if $\gamma > 2 + \alpha$. In the borderline case $\gamma = 2 + \alpha$, we will prove existence of unbounded “eigenfunctions”.

The case of the Laplacian and $\gamma = 2$ is naturally linked to Hardy's inequality: let us suppose that $N > 2$ and let $u \in H_0^1(B(0, 1))$ (respectively, $u \in H^1(\mathbb{R}^N)$), then $\frac{u(x)}{|x|}$ belongs to $L^2(B(0, 1))$ (respectively $\frac{u(x)}{|x|} \in L^2(\mathbb{R}^N)$), and there exists a positive constant c such that

$$\int \left(\frac{|u(x)|}{|x|} \right)^2 \leq c \int |\nabla u|^2.$$

Furthermore, the best constant is $c = \frac{4}{(N-2)^2}$ and it is not achieved, in the sense that

$$\inf_{u \in H_0^1(B(0,1)), \int_{B(0,1)} \left(\frac{|u(x)|}{|x|}\right)^2 = 1} |\nabla u|^2 = \frac{(N-2)^2}{4}. \quad (1.1)$$

and there is no $u \in H_0^1$ which realizes the infimum. By obvious arguments, $B(0,1)$ can be replaced by any bounded regular open set of \mathbb{R}^N containing 0, and the optimal constant does not depend on the size of the open set. Note that the right hand side of (1.1) coincides with the variational characterization of the "first" or "principal" eigenvalue for the equation

$$-\Delta u = \lambda \frac{u}{|x|^2}.$$

For further knowledge on the Hardy–Sobolev inequality and for the case of the p -Laplacian, we refer to [9, 10].

On the other hand, if we replace in the constraint the power 2 by an exponent $\gamma < 2$, compactness from $H_0^1(B(0,1))$ into the weighted space $L^2(B(0,1), \frac{1}{r^\gamma})$ holds, and this allows to obtain existence of minima in $H_0^1(B(0,1))$ by standard arguments in calculus of variations. In that case, denoting

$$\bar{\lambda}_\gamma = \inf_{u \in H_0^1(B(0,1)), \int_{B(0,1)} \frac{|u(x)|^2}{|x|^\gamma} = 1} |\nabla u|^2 \quad (1.2)$$

one sees that $\bar{\lambda}_\gamma$ is also the first eigenvalue for the equation

$$\Delta u + \bar{\lambda}_\gamma \frac{u}{r^\gamma} = 0,$$

meaning that $\bar{\lambda}_\gamma$ is such that there exists $u > 0$ in $H_0^1(B(0,1))$ satisfying the equation. Note that, by its definition, $\bar{\lambda}_\gamma$ depends on the domain, since

$$\bar{\lambda}_\gamma(B(0,t)) = \frac{1}{t^{2-\gamma}} \bar{\lambda}_\gamma(B(0,1)).$$

If $\gamma > 2$ there is no embedding from $H_0^1(B(0,1))$ into $L^2(B(0,1), \frac{1}{r^\gamma})$. Indeed, as an example, the function

$$u(r) = r^{-\frac{N-2}{2}+\epsilon} (-\log r)$$

with $0 < \epsilon < \frac{\gamma-2}{2}$, belongs to $H_0^1(B(0,1))$ and satisfies

$$\int_{B(0,1)} \frac{u(|x|)^2}{|x|^\gamma} = +\infty.$$

Let us now focus on non variational cases. More precisely, let us suppose that F is fully nonlinear uniformly elliptic operator, that is F is a continuous function defined on the set \mathcal{S}_N of symmetric $N \times N$ matrices, and it satisfies, for positive constants $\Lambda \geq \lambda > 0$,

$$\lambda \operatorname{tr}(M') \leq F(M + M') - F(M) \leq \Lambda \operatorname{tr}(M'), \quad (1.3)$$

for all $M, M' \in \mathcal{S}_N$, with M' positive semidefinite.

Suppose also that F is rotationally invariant and positively homogeneous of degree 1, i.e.

$$\text{for any } M \in \mathcal{S} \text{ and } t > 0, F(tM) = tF(M). \quad (1.4)$$

In this case the principal eigenvalue for the equation

$$F(D^2u) + \lambda \frac{u}{|x|^\gamma} = 0 \quad \text{in } B(0, 1) \setminus \{0\}$$

can be defined on the model of [1], i.e. by the optimization formula

$$\bar{\lambda}_\gamma = \bar{\lambda}_\gamma(F, r^{-\gamma}) = \sup\{\mu : \exists u \in C(B(0, 1) \setminus \{0\}), u > 0 \text{ in } B(0, 1), F(D^2u) + \mu \frac{u}{r^\gamma} \leq 0\}, \quad (1.5)$$

where the differential inequality appearing above is understood in the viscosity sense. In the model cases, the operator F will be one of Pucci's extremal operators. Let us recall their definition: by decomposing each matrix $M \in \mathcal{S}_N$ as $M = M^+ - M^-$, where M^+ and M^- are positive semidefinite matrices satisfying $M^+M^- = O$, then Pucci's sup operator can be defined as

$$\mathcal{M}_{\lambda, \Lambda}^+(M) = \Lambda \text{tr}(M^+) - \lambda \text{tr}(M^-),$$

as well as Pucci's inf operator is given by

$$\mathcal{M}_{\lambda, \Lambda}^-(M) = \lambda \text{tr}(M^+) - \Lambda \text{tr}(M^-) = -\mathcal{M}_{\lambda, \Lambda}^+(-M).$$

In the sequel, we will omit in the notation the dependence on the ellipticity constants, which are fixed once for all.

We further recall that for a C^2 radial function $u(x) = u(|x|)$, one has

$$D^2u(x) = u''(r) \frac{x \otimes x}{r^2} + \frac{u'(r)}{r} \left(I - \frac{x \otimes x}{r^2} \right),$$

and, as a consequence,

$$\mathcal{M}^+(D^2u) = \Lambda(N-1) \left(\frac{u'(r)}{r} \right)^+ - \lambda(N-1) \left(\frac{u'(r)}{r} \right)^- + \Lambda(u''(r))^+ - \lambda(u''(r))^-,$$

$$\mathcal{M}^-(D^2u) = \lambda(N-1) \left(\frac{u'(r)}{r} \right)^+ - \Lambda(N-1) \left(\frac{u'(r)}{r} \right)^- + \lambda(u''(r))^+ - \Lambda(u''(r))^-.$$

Thus, the ODEs satisfied by radial solutions of Pucci's extremal equations have coefficients depending on the dimension like parameters, associated with \mathcal{M}^+ and \mathcal{M}^- respectively, defined as

$$N_+ = \frac{\lambda}{\Lambda}(N-1) + 1, \quad N_- = \frac{\Lambda}{\lambda}(N-1) + 1.$$

Note that one has always $N_- \geq N \geq N_+$, with equalities holding true if and only if $\Lambda = \lambda$. We will assume always that $N_+ > 2$. This will imply in particular that since $\gamma < \alpha + 2$, $(N_+ - 1)(1 + \alpha) + 1 - \gamma > 0$.

We recall the results obtained in [4].

Theorem 1.1. *Suppose that F satisfies (1.3), (1.4) and is rotationally invariant. Suppose that $\gamma < 2$. Then:*

(i) $\bar{\lambda}_\gamma$ defined in (1.5) is positive and there exists a function u , continuous in $\overline{B(0,1)}$, radial, strictly positive in $B(0,1)$, such that

$$\begin{cases} F(D^2u) + \bar{\lambda}_\gamma \frac{u}{r^\gamma} = 0 & \text{in } B(0,1) \setminus \{0\} \\ u = 0 & \text{on } \partial B(0,1) \end{cases}.$$

Furthermore u is $C^2(B(0,1) \setminus \{0\})$ and it can be extended on $B(0,1)$ as a Lipschitz continuous function if $\gamma \leq 1$, as a function of class $C^1(B(0,1))$ when $\gamma < 1$, and as an Hölder continuous function with exponent $2 - \gamma$ if $\gamma > 1$.

(ii) Assume that $\gamma = 2$. Then: For the operator \mathcal{M}^+ one has

$$\bar{\lambda}_2(\mathcal{M}^+) = \Lambda \frac{(N_+ - 2)^2}{4}$$

and the function $u(x) = r^{-\frac{N_+ - 2}{2}}(-\ln r)$ is an explicit solution of

$$\begin{cases} \mathcal{M}^+(D^2u) + \bar{\lambda}_\gamma \frac{u}{r^\gamma} = 0 & \text{in } B(0,1) \setminus \{0\} \\ u = 0 & \text{on } \partial B(0,1) \end{cases}$$

(iii) If $\gamma > 2$, then the eigenvalue $\bar{\lambda}'_\gamma$ defined by

$$\bar{\lambda}'_\gamma := \sup\{\mu : \exists u \in C^2(B(0,1) \setminus \{0\}), u > 0 \text{ in } B(0,1) \setminus \{0\}, u \text{ radial, } F(D^2u) + \mu \frac{u}{r^\gamma} \leq 0\}$$

satisfies $\bar{\lambda}'_\gamma = 0$.

We now describe the results here enclosed. As we said above, we look for positive "radial eigenvalues" for the equation

$$|\nabla u|^\alpha F(D^2u) + \mu r^{-\gamma} u^{1+\alpha} = 0 \text{ in } \overline{B(0,1)} \setminus \{0\},$$

where $\alpha > -1$, $0 < \gamma < \alpha + 2$. F is always a fully non linear positively homogeneous operator, ie, it satisfies (1.3), (1.4), and is rotationally invariant. For $\gamma > 0$ we define

$$\bar{\lambda}(F, r^{-\gamma}, B(0,1)) = \sup\{\mu : \exists u \in C(B(0,1) \setminus \{0\}), u > 0 \text{ in } B(0,1), \text{ radial, } |\nabla u|^\alpha F(D^2u) + \mu \frac{u^{1+\alpha}}{r^\gamma} \leq 0\}. \quad (1.6)$$

In the sequel we will simply denote it $\bar{\lambda}_\gamma$ when there is no ambiguity on the domain, nor on the operator.

Theorem 1.2. *Suppose that F satisfies (1.3), (1.4) and is rotationally invariant, and that $\alpha > -1$. Suppose that $0 < \gamma < 2 + \alpha$. Then: $\bar{\lambda}_\gamma$ defined in (1.6) is positive and there exists a function u , continuous in $\overline{B(0,1)}$, radial, strictly positive in $B(0,1)$, such that*

$$\begin{cases} |\nabla u|^\alpha F(D^2u) + \bar{\lambda}_\gamma \frac{u^{1+\alpha}}{r^\gamma} = 0 & \text{in } B(0,1) \setminus \{0\} \\ u = 0 & \text{on } \partial B(0,1) \end{cases}.$$

Furthermore u is $C^1(B(0,1) \setminus \{0\})$, $|u|^\alpha u' \in C^1(B(0,1) \setminus \{0\})$, u is Lipschitz continuous on $\overline{B(0,1)}$ in the case $\gamma \leq 1$, $C^1(B(0,1))$ when $\gamma < 1$, and Hölder continuous of exponent $\frac{2+\alpha-\gamma}{1+\alpha}$ if $\gamma > 1$.

Remark 1.3. *In fact we will prove that $u \in \mathcal{C}^2(B(0,1) \setminus \{0\})$.*

Statement (i) of the above theorem shows in particular that $\bar{\lambda}_\gamma$ is actually achieved on smooth radial eigenfunctions. Thus, if we define

$$\bar{\lambda}'_\gamma = \sup\{\mu : \exists u \in \mathcal{C}(B(0,1) \setminus \{0\}), u > 0 \text{ in } B(0,1), \text{ radial, } u \in \mathcal{C}^1(B(0,1) \setminus \{0\}), \\ |u'|^\alpha u' \in \mathcal{C}^1(B(0,1) \setminus \{0\}), |\nabla u|^\alpha F(D^2 u) + \mu \frac{u^{1+\alpha}}{r^\gamma} \leq 0\}, \quad (1.7)$$

it then follows that

$$\bar{\lambda}_\gamma = \bar{\lambda}'_\gamma.$$

Actually, we will work initially with the smooth eigenvalue $\bar{\lambda}'_\gamma$, and we will finally prove that it coincides with $\bar{\lambda}_\gamma$.

Theorem 1.4. *Suppose that $F = \mathcal{M}^+$ and that $\gamma = 2 + \alpha$. Then:*

(i)

$$\bar{\lambda}_{2+\alpha} = \frac{\Lambda}{\alpha + 1} \left(\frac{(N_+ - 2)(1 + \alpha)}{2 + \alpha} \right)^{2+\alpha}$$

A supersolution which satisfies the extremal inequality for $\bar{\lambda}_{2+\alpha}$ is $r^{-\tau}$ with $\tau = \frac{(N_+ - 2)(\alpha + 1)}{2 + \alpha}$.

(ii) $\bar{\lambda}_{2+\alpha}$ is stable under various regularization

$$\bar{\lambda}_{2+\alpha} = \lim_{\gamma \rightarrow 2+\alpha} \bar{\lambda}_\gamma$$

$$\bar{\lambda}_{2+\alpha} = \lim_{\delta \rightarrow 0} \bar{\lambda} \left(\mathcal{M}^+, \frac{1}{r^{2+\alpha}}, B(0,1) \setminus \overline{B(0,\delta)} \right)$$

$$\bar{\lambda}_{2+\alpha} = \lim_{\epsilon \rightarrow 0} \bar{\lambda} \left(\mathcal{M}^+, \frac{1}{(r^2 + \epsilon^2)^{\frac{\alpha+2}{2}}}, B(0,1) \right).$$

We will finally prove

Theorem 1.5. *Suppose that F satisfies (1.3), (1.4) and is rotationally invariant. Suppose that $\alpha > -1$. If $\gamma > \alpha + 2$, $\bar{\lambda}'_\gamma = 0$.*

2 The case $\gamma < \alpha + 2$

2.1 Maximum principles, existence and regularity results

For simplicity in the reading let us introduce the space invoked above

$$\mathcal{C}_{1,\alpha} = \{u \in \mathcal{C}^1(B(0,1) \setminus \{0\}), \text{ radial, } |u'|^\alpha u' \in \mathcal{C}^1(B(0,1) \setminus \{0\})\}. \quad (2.1)$$

Remark 2.1. Let us observe that the radial solutions of $|\nabla u|^\alpha F(D^2u) = \frac{f}{r^\gamma}$ satisfy $u \in C_{1,\alpha}$. Indeed, let us use the notations in [8], [7] :

$$\mathcal{F}(x, l, m) = F(lId + (m - l)\frac{x \otimes x}{r^2})$$

Then the equation above can be written

$$\mathcal{F}(x, r^{-1}|u'|^\alpha u', (|u'|^\alpha u')') = \frac{f}{r^\gamma}$$

and then using the Ellipticity of F , adapting Lemma 2.1 in [7], (see also [8]), there exists \mathcal{G} locally Lipschitz in \mathbb{R}^2 so that

$$(|u'|^\alpha u')' = \mathcal{G}(|u'|^\alpha u' r^{-1}, \frac{f(r)}{r^\gamma}).$$

As a consequence the solutions of such equations satisfy $u \in C_{1,\alpha}$. Furthermore if $u' \neq 0$, we get that $u'' \in C(B(0,1) \setminus \{0\})$, dividing the equation defining \mathcal{G} by $|u'|^\alpha$. This will be the case for an eigenfunction, since we will observe that it satisfies $u' < 0$ in $r > 0$.

The first results of the present section are two crucial technical lemmata .

Lemma 2.2. Let $f \in C(B(0,1) \setminus \{0\})$ be a radial, bounded and positive function and assume that $u \in C_{1,\alpha}$ is a radial, bounded function which satisfies

$$|\nabla u|^\alpha \mathcal{M}^+(D^2u) \geq fr^{-\gamma} \quad \text{in } B(0,1) \setminus \{0\}. \quad (2.2)$$

Then

(i) $u' \geq 0$ in a right neighborhood of 0;

(ii) $\lim_{r \rightarrow 0} u'(r)r^{N-1} = 0$ and in a right neighborhood of 0 one has

$$u'(r) \geq \left((1 + \alpha) \frac{\inf f}{\Lambda((N_- - 1)(1 + \alpha) + 1 - \gamma)} \right)^{\frac{1}{1+\alpha}} r^{\frac{1-\gamma}{1+\alpha}};$$

(iii) if, furthermore, $|\nabla u|^\alpha \mathcal{M}^+(D^2u) = fr^{-\gamma}$, then, in a right neighborhood of 0, one also has

$$u'(r) \leq \left((1 + \alpha) \frac{\sup f}{\lambda((N - 1)(1 + \alpha) + 1 - \gamma)} \right)^{\frac{1}{1+\alpha}} r^{\frac{1-\gamma}{1+\alpha}}.$$

In particular, there exists a constant $c > 0$ such that, for r sufficiently small,

$$|u'(r)| \leq cr^{\frac{1-\gamma}{1+\alpha}}$$

and then u can be extended as a Lipschitz continuous function on $\overline{B(0,1)}$ if $\gamma \leq 1$, it belongs to $C^1(\overline{B(0,1)})$ if $\gamma < 1$, and it is Hölder continuous in $B(0,1)$ with exponent $\frac{2+\alpha-\gamma}{1+\alpha}$ if $\gamma > 1$.

For the convenience of the reading we define $G_{\Lambda,\lambda}(p, q) = \Lambda(p^+ + q^+) - \lambda(p^- + q^-)$. Note that $G_{\Lambda,\lambda}$ is subadditive .

Proof. Note that when u is radial,

$$|\nabla u|^\alpha \mathcal{M}^+(D^2 u) = G_{\Lambda, \lambda}((1 + \alpha)^{-1}(|u'|^\alpha u')', (\frac{N-1}{r})|u'|^\alpha u')$$

Suppose that u' changes sign around 0, then, since u is \mathcal{C}^1 and $|u'|^\alpha u'$ is \mathcal{C}^1 , there exists a numerable set $\{r_n\}_n$ so that on $[r_{2n+1}, r_{2n}]$ $u' \leq 0$, and on $[r_{2n+2}, r_{2n+1}]$ $u' \geq 0$. Then there exists s_n in $]r_{2n+1}, r_{2n}[$ so that $(|u'|^\alpha u')'(s_n) = 0$, a contradiction with the inequation satisfied by u . Then u' does not change sign and if we had $u' \leq 0$ then we would have

$$\Lambda(|u'|^\alpha u')'(1 + \alpha)^{-1} + \lambda|u'|^\alpha u'(N-1)r^{-1} \geq fr^{-\gamma}.$$

which would imply that

$$(|u'|^\alpha u' r^{(N_+-1)(1+\alpha)})' \geq (1 + \alpha)\Lambda^{-1} fr^{(N_+-1)(1+\alpha)-\gamma}$$

In particular $|u'|^\alpha u' r^{(N_+-1)(1+\alpha)}$ has a limit when r goes to zero. Suppose that this limit is < 0 , then this would imply for some positive constant c

$$u' \leq -cr^{1-N_+}$$

near zero, which contradicts u bounded. We have obtained that $\lim |u'|^\alpha u' r^{N_+-1} = 0$. Then integrating the equation above one obtains

$$|u'|^\alpha u' r^{(N_+-1)(1+\alpha)} \geq 0$$

which yields a contradiction. We have obtained that $u' \geq 0$ and then whatever is the sign of u'' one has

$$(1 + \alpha)^{-1}(|u'|^\alpha u')' + \frac{\Lambda(N-1)}{r\lambda}|u'|^\alpha u' \geq fr^{-\gamma}\Lambda^{-1}$$

which implies that

$$(|u'|^\alpha u' r^{(N_--1)(1+\alpha)})' \geq (1 + \alpha)\Lambda^{-1} fr^{-\gamma+(N_--1)(1+\alpha)}$$

Even if it is not necessary for the sequel, let us remark in particular that $|u'|^\alpha u' r^{(N_--1)(1+\alpha)}$ has a limit when r goes to zero, and if the limit was $c > 0$ this would contradict once more that u is bounded. Then $\lim_{r \rightarrow 0} |u'|^\alpha u' r^{(N_--1)(1+\alpha)} = 0$. We obtain by integrating

$$|u'|^\alpha u' r^{(N_--1)(1+\alpha)} \geq (1 + \alpha)\Lambda^{-1} \inf fr^{(N_--1)(1+\alpha)+1-\gamma}$$

which yields the result.

item (iii) can be proved in the same fashion. □

Remark 2.3. *From the arguments used in the proof of item 1, $f > 0$ is sufficient to get $u' > 0$ near zero. This will be used in the proof of the uniqueness in Theorem 2.11. .*

Remark 2.4. By using the change of variable $v = -u$, one gets that if $f \in C(B(0,1) \setminus \{0\})$ is a radial, bounded and positive function and $u \in C_{1,\alpha}$ is a bounded radial function satisfying

$$|\nabla u|^\alpha \mathcal{M}^-(D^2 u) \leq -fr^{-\gamma} \quad \text{in } B(0,1) \setminus \{0\},$$

then, for r sufficiently small, $u'(r) \leq 0$, $\lim_{r \rightarrow 0} u'(r)r^{N-1} = 0$ and

$$u'(r) \leq -\frac{\inf f}{\Lambda((N_- - 1)(1 + \alpha) + 1 - \gamma)} r^{\frac{1-\gamma}{1+\alpha}}.$$

Moreover, if $|\nabla u|^\alpha \mathcal{M}^-(D^2 u) = -fr^{-\gamma}$ in $B(0,1) \setminus \{0\}$, then $|u'(r)| \leq cr^{\frac{1-\gamma}{1+\alpha}}$ for a positive constant c . Hence u is Lipschitz continuous on $\overline{B(0,1)}$ for $\gamma \leq 1$, it belongs to $C^1(B(0,1))$ if $\gamma < 1$, and it is Hölder continuous in $B(0,1)$ with exponent $\frac{2+\alpha-\gamma}{1+\alpha}$ for $\gamma > 1$.

Obviously, since $\mathcal{M}^- \leq \mathcal{M}^+$, one gets an analogous conclusion when

$$|\nabla u|^\alpha \mathcal{M}^+(D^2 u) \leq -fr^{-\gamma}.$$

Lemma 2.5. Let $f, g \in C(B(0,1) \setminus \{0\})$ be radial, bounded functions such that $f > g$ and assume that $u, v \in C_{1,\alpha} \cap C(B(0,1))$ are radial functions satisfying respectively

$$|\nabla u|^\alpha F(D^2 u) \geq fr^{-\gamma} \quad \text{in } B(0,1) \setminus \{0\} \quad (2.3)$$

and

$$|\nabla v|^\alpha F(D^2 v) \leq gr^{-\gamma} \quad \text{in } B(0,1) \setminus \{0\}. \quad (2.4)$$

Then

(i) $u' - v' \geq 0$ in a right neighborhood of 0;

(ii) In a right neighborhood of 0 one has

$$|u'|^\alpha u' - |v'|^\alpha v' \geq (1 + \alpha) \frac{\inf(f - g)}{\Lambda((N_- - 1)(1 + \alpha) + 1 - \gamma)} r^{1-\gamma};$$

Proof. Suppose first that $u' - v'$ has not a constant sign near zero. Then, there exists a decreasing sequence $\{r_n\}$ converging to 0, such that $(u' - v')(r_n) = 0$ for all n , and $u' - v' \leq 0$ in $]r_{2n+1}, r_{2n}[$, $u' - v' \geq 0$ in $]r_{2n+2}, r_{2n+1}[$. Since $(u' - v')(r_{2n}) = (u' - v')(r_{2n+1}) = 0$, there exists some $s_{2n} \in]r_{2n+1}, r_{2n}[$ such that $(|u'|^\alpha u' - |v'|^\alpha v')(s_{2n}) = 0$. Then we use the inequality

$$\begin{aligned} |\nabla u|^\alpha F(D^2 u) &- |\nabla v|^\alpha F(D^2 v) \\ &= F(|\nabla u|^\alpha D^2 u) - F(|\nabla v|^\alpha D^2 v) \\ &\leq \mathcal{M}^+(|\nabla u|^\alpha D^2 u - |\nabla v|^\alpha D^2 v) \\ &= G_{\Lambda,\lambda} \left(\frac{(|u'|^\alpha u' - |v'|^\alpha v')(N-1)}{r}, (1 + \alpha)^{-1} (|u'|^\alpha u' - |v'|^\alpha v')' \right) \end{aligned} \quad (2.5)$$

This yields a contradiction since $G_{\Lambda,\lambda} \left(\frac{(|u'|^\alpha u' - |v'|^\alpha v')(N-1)}{r}, (1 + \alpha)^{-1} (|u'|^\alpha u' - |v'|^\alpha v')' \right) (s_{2n}) \leq 0$. Suppose now that $u' - v' \leq 0$ in a neighborhood of zero. Using (2.5) above,

$$\Lambda(|u'|^\alpha u' - |v'|^\alpha v')' + \lambda(|u'|^\alpha u' - |v'|^\alpha v') \frac{(N-1)}{r} \geq (f - g)r^{-\gamma}.$$

Hence

$$((|u'|^\alpha u' - |v'|^\alpha v')r^{(1+\alpha)(N+1)})' \geq (1+\alpha)(f-g)\Lambda^{-1}r^{(1+\alpha)(N-1)-\gamma}$$

Since $f - g > 0$, $(|u'|^\alpha u' - |v'|^\alpha v')r^{(1+\alpha)(N+1)}$ has a limit when r goes to zero. We prove by contradiction that this limit is ≥ 0 .

In order to obtain this, we make an additional assumption which will be deleted in a second step

First step

Suppose that either $\limsup_{r \rightarrow 0} u' r^{N+1} \geq 0$, or $\liminf_{r \rightarrow 0} v' r^{N+1} \leq 0$. And assume by contradiction that

$$(|u'|^\alpha u' - |v'|^\alpha v')r^{(1+\alpha)(N+1)} \rightarrow -c < 0.$$

- if $\limsup_{r \rightarrow 0} u' r^{N+1} \geq 0$, one derives that $|v'|^\alpha v' \geq cr^{(-N+1)(1+\alpha)}$, and then $v' \geq cr^{(-N+1)}$ a contradiction with the boundedness of v .

- If $\liminf_{r \rightarrow 0} v' r^{N+1} \leq 0$,

$$|u'|^\alpha u' \leq -cr^{(-N+1)(1+\alpha)}$$

once more a contradiction with the boundedness of u . We have obtained that $(|u'|^\alpha u' - |v'|^\alpha v')r^{(1+\alpha)(N+1)} \geq 0$, hence $u' - v' \geq 0$ and whatever is the sign of $(|u'|^\alpha u')' - (|v'|^\alpha v')'$ one has

$$((|u'|^\alpha u' - |v'|^\alpha v')r^{(1+\alpha)(N-1)})' \geq (1+\alpha)(f-g)r^{(N-1)(1+\alpha)-\gamma}\Lambda^{-1}$$

This implies by integrating that

$$(|u'|^\alpha u' - |v'|^\alpha v')(r) \geq (1+\alpha)\Lambda^{-1} \frac{\inf(f-g)r^{1-\gamma}}{(N-1)(1+\alpha) + 1 - \gamma}.$$

Second step

We use the previous step to compare u with w defined as follows : For $0 < \tau_1 \leq \inf(\frac{\alpha+2-\gamma}{\alpha+1}, N_+)$, let us consider the radial function

$$w(r) = L(1 - r^{\tau_1}) + u(1) \tag{2.6}$$

where $L > 0$ is a constant to be suitably chosen. A direct computation shows that

$$\mathcal{M}^+(D^2 w) \leq L\tau_1\Lambda(|\tau_1 - 1| - (N_+ - 1))r^{\tau_1-2}.$$

so that

$$\mathcal{M}^+(D^2 w) \leq -CLr^{\tau_1-2} \quad \text{in } B(0, 1) \setminus \{0\}$$

with $C = \tau_1\Lambda(N_+ - 1 - |\tau_1 - 1|) > 0$.

Then

$$|\nabla w|^\alpha \mathcal{M}^+(D^2 w) \leq -C(L\tau)^\alpha Lr^{\tau-2+(\tau-1)(\alpha)} \leq -|f|_\infty r^{-\gamma}$$

choosing conveniently L . Note that since $\gamma < \alpha + 2$, $\tau_1 - 1 + N_+ - 1 > 0$, and then $\lim_{r \rightarrow 0} w' r^{N+1} = 0$. We can apply the first step and obtain that for some positive constant c

$$|u'|^\alpha u'(r) - |w'|^\alpha w'(r) \geq cr^{1-\gamma}$$

this yields

$$|u'|^\alpha u' \geq -\tau_1^{1+\alpha} L^{1+\alpha} r^{(\tau_1-1)(1+\alpha)} + cr^{1-\gamma}$$

In particular by the assumptions on τ_1 and γ one has $\limsup_{r \rightarrow 0} u'(r)r^{N+1} \geq 0$.

In the same manner if v satisfies

$$|\nabla v|^\alpha F(D^2 v) \leq gr^{-\gamma}$$

Taking $w = -L(1 - r^{\tau_1}) + v(1)$, as soon as L is large enough, $|\nabla w|^\alpha \mathcal{M}^-(D^2 w) \geq |g|_\infty r^{-\gamma}$. Using the first step one gets $|v'|^\alpha v' \leq |w'|^\alpha w' - cr^{1-\gamma}$, hence $\liminf_{r \rightarrow 0} v' r^{N+1} \leq 0$.

As a consequence the result in the first step extends to any u and v without the assumption $\limsup_{r \rightarrow 0} u'(r) r^{N+1} \geq 0$, or $\liminf_{r \rightarrow 0} v' r^{N+1} \leq 0$, and items (i) and (ii) are proved. \square

Theorem 2.6. *Let $f, g \in C(B(0, 1))$ be radial functions and assume that $u, v \in C(\overline{B(0, 1)}) \cap C_{1, \alpha}$, are radial functions satisfying in $B(0, 1) \setminus \{0\}$*

$$|\nabla u|^\alpha F(D^2 u) - \beta |u|^\alpha u(r) r^{-\gamma} \geq f(r) r^{-\gamma}$$

$$|\nabla v|^\alpha F(D^2 v) - \beta |v|^\alpha v(r) r^{-\gamma} \leq g(r) r^{-\gamma}$$

with $\beta \geq 0$ and $f \geq g$ in $B(0, 1)$, one of the inequalities being strict. Then $u \leq v$ on $\partial B(0, 1)$ implies $u \leq v$ in $B(0, 1)$.

Proof. We suppose by contradiction that

$$\max_{B(0, 1)} (u - v) > 0.$$

If the maximum is achieved at 0, then $(u - v)(0) > 0$ and, by the assumptions on f, g and β , there exist $\delta > 0$ and a neighborhood on the right of 0 on which

$$\beta(|u|^\alpha u(r) - |v|^\alpha v(r)) r^{-\gamma} + (f(r) - g(r)) r^{-\gamma} \geq \delta r^{-\gamma} > 0$$

Using Lemma 2.5, one gets that for some positive constant c , $|u'|^\alpha u' - |v'|^\alpha v' \geq cr^{1-\gamma}$, which contradicts the fact that $u - v$ attains its maximum at 0. Hence, there exists $0 < \bar{r} < 1$ such that $u(\bar{r}) - v(\bar{r}) = \max(u - v)$. Since the supremum cannot be achieved on 0, for $\epsilon > 0$ there exists $\delta < \bar{r}$ that we can take small enough in order that $u(r) < v(r) + \max(u - v) - \epsilon$, for any $r < \delta$. Then $w = v + \max(u - v)$ satisfies

$$|\nabla w|^\alpha F(D^2 w) - \beta |w|^\alpha w r^{-\gamma} < g(r) r^{-\gamma}$$

Applying the comparison principle in [2] in $r > \delta$ for regular coefficients, one gets that $u \leq w$ in $r > \delta$, and then $u - v \leq \sup(u - v) - \epsilon$ in the whole ball, a contradiction. \square

Next, we have the following existence, uniqueness and regularity result.

Theorem 2.7. *Let $f \in C(B(0, 1))$ be a radial, bounded function. Let further F be a rotationally invariant operator satisfying (1.4), (1.3). For $\beta \geq 0$ and $b \in \mathbb{R}$ there exists a (unique when $\beta > 0$ or when f has a sign and $\beta = 0$) bounded radial function $u \in C(\overline{B(0, 1)}) \cap C_{1, \alpha}$ satisfying*

$$\begin{cases} |\nabla u|^\alpha F(D^2 u) - \beta |u|^\alpha u r^{-\gamma} = r^{-\gamma} f(r) & \text{in } B(0, 1) \setminus \{0\} \\ u(1) = b. \end{cases} \quad (2.7)$$

Moreover one has: $u \in C^1(\overline{B(0, 1)})$ if $\gamma < 1$, u is Lipschitz continuous in $\overline{B(0, 1)}$ if $\gamma \leq 1$, u is Hölder continuous in $\overline{B(0, 1)}$ with exponent $\frac{2+\alpha-\gamma}{1+\alpha}$ if $\gamma > 1$.

Proof. For every $n \in \mathbb{N}$ let us introduce the regularized Dirichlet boundary value problem

$$\begin{cases} |\nabla u_n|^\alpha F(D^2 u_n) - \beta |u_n|^\alpha u_n (r^2 + 1/n)^{-\gamma/2} = (r^2 + 1/n)^{-\gamma/2} f(r) & \text{in } B(0, 1) \\ u_n(1) = b \end{cases} \quad (2.8)$$

which, by [2], has a unique radial solution $u_n \in C(\overline{B(0, 1)})$. For $0 < \tau_1 \leq 2 - \gamma$, let us consider the radial function in (2.6) replacing $w(1)$ by b where always $L > 0$ is a constant suitably chosen, and τ_1 satisfies $\tau_1 \leq \inf(\frac{\alpha+2-\gamma}{\alpha+1}, N_+)$. Then

$$|\nabla w|^\alpha \mathcal{M}^+(D^2 w) \leq -|f|_\infty r^{-\gamma} \leq -(r^2 + 1/n)^{-\gamma/2} f(r).$$

Then one also has

$$|\nabla w|^\alpha \mathcal{M}^+(D^2 w) - \beta |w|^\alpha w (r^2 + 1/n)^{-\gamma/2} \leq -|f|_\infty r^{-\gamma} \leq -(r^2 + 1/n)^{-\gamma/2} f(r).$$

Up to change the constant L one also has

$$|\nabla w|^\alpha \mathcal{M}^+(D^2(-w)) - \beta |w|^\alpha (-w) (r^2 + 1/n)^{-\gamma/2} \leq -|f|_\infty r^{-\gamma} \leq -(r^2 + 1/n)^{-\gamma/2} f(r).$$

The standard comparison principle in [2] implies that $-w \leq u_n \leq w$ and then implies that the sequence $\{u_n\}$ is uniformly bounded in $C(\overline{B(0, 1)})$. Hence, it is locally uniformly bounded in $C_{1,\alpha}$, in particular uniformly Hölder continuous, and, up to a subsequence, it is converging locally uniformly in $\overline{B(0, 1)} \setminus \{0\}$ to a radial solution u of problem (2.7), which is a globally continuous function belonging to $C_{1,\alpha}$.

Let us now show that the constructed bounded radial solution u satisfies the further regularity announced. We take now \bar{w} a solution of

$$\begin{cases} |\nabla \bar{w}|^\alpha \mathcal{M}^+(D^2 \bar{w}) = -(\|f\|_\infty + B + 1)r^{-\gamma} & \text{in } B(0, 1) \setminus \{0\} \\ \bar{w}(1) = u(1) \end{cases}$$

where $B > 0$ is a constant such that $\beta |u|^{1+\alpha} \leq B$ in $B(0, 1)$. Lemma 2.5 and Remark 2.4 applied to \bar{w} yield that

$$|\bar{w}'| \leq cr^{\frac{1-\gamma}{1+\alpha}}$$

for some $c > 0$. As a consequence, we have by Lemma 2.5 (i), in a right neighborhood of 0, $(u - \bar{w})'(r) \geq 0$. Hence,

$$u'(r) \geq -cr^{\frac{1-\gamma}{1+\alpha}}$$

for r small enough. Analogously, we have

$$\mathcal{M}^-(|\nabla u|^\alpha D^2 u + |\nabla \bar{w}|^\alpha D^2 \bar{w}) \leq |\nabla u|^\alpha F(D^2 u) + |\nabla \bar{w}|^\alpha \mathcal{M}^+(D^2(\bar{w})) \leq -r^{-\gamma} \quad \text{in } B(0, 1) \setminus \{0\}$$

which implies, by Remark 2.4,

$$u'(r) \leq -\bar{w}'(r) \leq cr^{\frac{1-\gamma}{1+\alpha}}$$

for r small enough. Arguing as in the proof of Lemma 2.5, we deduce from the estimate $|u'(r)| \leq cr^{\frac{1-\gamma}{1+\alpha}}$ for r sufficiently small, that u is Lipschitz continuous in $\overline{B(0, 1)}$ if $\gamma \leq 1$, it belongs to $C^1(\overline{B(0, 1)})$ if $\gamma < 1$, and it is Hölder continuous in $\overline{B(0, 1)}$ with exponent $\frac{2+\alpha-\gamma}{1+\alpha}$ if $\gamma > 1$.

The uniqueness is obvious by the comparison principle when $\beta > 0$. If f has a sign and $\beta = 0$ for example $f < 0$, then if u and v are two solutions of

$$\begin{cases} |\nabla u|^\alpha F(D^2 u) = r^{-\gamma} f(r) & \text{in } B(0, 1) \setminus \{0\} \\ u(1) = b \end{cases},$$

$u_\epsilon = (u - b)(1 + \epsilon) + b$ satisfies $u_\epsilon(1) = b$, and

$$|\nabla u_\epsilon|^\alpha F(D^2 u_\epsilon) = f(r)(1 + \epsilon)^{1+\alpha} r^{-\gamma} < \frac{f(r)}{r^\gamma}$$

and then $u_\epsilon \geq v$. By passing to the limit one obtains $u \geq v$. Exchanging the roles of u and v one gets the uniqueness. \square

The argument used in the above proof yields also the following compactness result :

Theorem 2.8. *Let $\{u_n\}_n$ be a uniformly bounded sequence of radial functions belonging to $C_{1,\alpha}$ and satisfying*

$$|\nabla u_n|^\alpha F(D^2 u_n) = f_n r^{-\gamma} \quad \text{in } B(0, 1) \setminus \{0\},$$

where the f_n are radial, bounded and continuous in $\overline{B(0, 1)}$. If $\{f_n\}_n$ is uniformly bounded, then $\{u_n\}_n$ is equicontinuous, thus uniformly converging in $B(0, 1)$ up to a subsequence. If f_n is uniformly converging to $f \in C(\overline{B(0, 1)})$, then, up to a subsequence, $\{u_n\}_n$ is uniformly converging to a radial solution $u \in C_{1,\alpha} \cap C(\overline{B(0, 1)})$ of

$$|\nabla u|^\alpha F(D^2 u) = r^{-\gamma} f \quad \text{in } B(0, 1) \setminus \{0\}.$$

In the next results, we prove several properties of the ‘‘smooth’’ eigenvalue defined in (1.7), that we recall here for the comfort of the reader

$$\bar{\lambda}'_\gamma := \sup\{\mu : \exists u \in \mathcal{C}_{1,\alpha}, u > 0 \text{ in } B(0, 1) \setminus \{0\}, u \text{ radial, } |\nabla u|^\alpha F(D^2 u) + \mu \frac{u^{1+\alpha}}{r^\gamma} \leq 0\}.$$

In Subsection 2.3 we will prove in fact that $\bar{\lambda}'_\gamma = \bar{\lambda}_\gamma$.

Let us start by proving the validity of the maximum principle below the value $\bar{\lambda}'_\gamma$.

Theorem 2.9. *Let $\mu < \bar{\lambda}'_\gamma$ and suppose that $u \in C(\overline{B(0, 1)}) \cap \mathcal{C}_{1,\alpha}$ is a radial function satisfying*

$$|\nabla u|^\alpha F(D^2 u) + \mu |u|^\alpha u r^{-\gamma} \geq 0 \quad \text{in } B(0, 1) \setminus \{0\}.$$

If $u(1) \leq 0$, then $u \leq 0$ in $\overline{B(0, 1)}$.

Proof. If $\mu < 0$, we just apply Theorem 2.6 with $v \equiv g \equiv f \equiv 0$. So, we can assume without loss of generality that $\mu \geq 0$. For $\mu' \in]\mu, \bar{\lambda}'_\gamma[$, let $v \in \mathcal{C}_{1,\alpha}$ be a radial function satisfying

$$|\nabla v|^\alpha F(D^2 v) + \mu' v^{1+\alpha} r^{-\gamma} \leq 0, \quad v > 0 \quad \text{in } B(0, 1) \setminus \{0\}.$$

We can assume without loss of generality that $v > 0$ on $\partial B(0, 1)$, e.g. by performing a dilation in r (this may change a little μ' but we can still suppose by continuity that $\mu' \in]\mu, \bar{\lambda}'_\gamma[$). Arguing as in the proof of Lemma 2.5, since $|\nabla v|^\alpha \mathcal{M}^-(D^2 v) + \mu' v^{1+\alpha} r^{-\gamma} \leq 0$, we easily obtain that v' has constant sign

near zero. Assuming by contradiction that $v'(r) \geq 0$ in a neighborhood of zero, then v is bounded in $B(0, 1) \setminus \{0\}$. Hence, Remark 2.4 applies and yields $v'(r) \leq 0$ for r small enough: a contradiction. This shows that $v'(r) \leq 0$ in a neighborhood of 0.

Then, there are two possible cases: either $\lim_{r \rightarrow 0} v(r) = +\infty$ or v can be extended as a continuous function on $\overline{B(0, 1)}$. In the first case, let $\epsilon > 0$ be given and let $\delta > 0$ be small in order that $\sup u \leq \epsilon \inf_{r < \delta} v$, then applying the comparison principle in [2] for regular coefficients in the domain $B(0, 1) \setminus \overline{B(0, \delta)}$, one gets $u \leq \epsilon v$ in $r > \delta$, hence $u \leq \epsilon v$ everywhere. Letting $\epsilon \rightarrow 0$, we get the conclusion. On the other hand, if v is bounded and continuous on $\overline{B(0, 1)}$, we can argue by contradiction : Let us assume that u is positive somewhere in $B(0, 1)$, so that $\frac{u}{v}$ has a positive maximum on $\overline{B(0, 1)}$, achieved at some point inside $B(0, 1)$. Up to a multiplicative constant for v , we can suppose that

$$\max_{\overline{B(0, 1)}} \frac{u}{v} = 1,$$

so that $u(r) \leq v(r)$. If the maximum is achieved at 0, then one has $u(0) = v(0) > 0$. By continuity, for r small enough one has

$$|\nabla u|^\alpha F(D^2 u) - |\nabla v|^\alpha F(D^2 v) \geq \frac{1}{2}(\mu' v^{1+\alpha}(0) - \mu u^{1+\alpha}(0))r^{-\gamma}.$$

Since $\mu' v^{1+\alpha}(0) - \mu u^{1+\alpha}(0) = (\mu' - \mu)v^{1+\alpha}(0) > 0$, we can use Lemma 2.5 (ii) and we get that $(u - v)'(r) > 0$ in a right neighborhood of 0. This is a contradiction to $u - v$ has a maximum point at zero.

Hence, we have that $1 = \max \frac{u}{v} > \frac{u(0)}{v(0)}$. Let us select $\eta < 1$ such that $\eta > \max\left\{\left(\frac{\mu}{\mu'}\right)^{\frac{1}{1+\alpha}}, \frac{u(0)}{v(0)}\right\}$. Then, the function $u - \eta v$ has a positive maximum achieved at some point $0 < \bar{r} < 1$. Since \bar{r} is an interior point, using the arguments in the comparison Theorem for regular coefficients in [2], or using remark 2.10 below, valid for functions depending on one variable, one has

$$|\nabla u|^\alpha F(D^2 u)(\bar{r}) \leq |\nabla(\eta v)|^\alpha F(D^2 \eta v)(\bar{r})$$

and then using (1.4), we get

$$-\mu v^{1+\alpha}(\bar{r})\bar{r}^{-\gamma} \leq -\mu u^{1+\alpha}(\bar{r})\bar{r}^{-\gamma} \leq |\nabla u|^\alpha F(D^2 u)(\bar{r}) \leq |\nabla \eta v|^\alpha F(D^2 \eta v)(\bar{r}) \leq -\mu' \eta^{1+\alpha} v^{1+\alpha}(\bar{r})\bar{r}^{-\gamma},$$

which gives the contradiction $\mu \geq \mu' \eta^{1+\alpha}$. □

Remark 2.10. Suppose that $\alpha > -1$, that u and v are defined $]\bar{r} - h, \bar{r} + h[$ are \mathcal{C}^1 and such that u and $v \in \mathcal{C}_{1, \alpha}$. Then if \bar{r} is a maximum point, one has

$$u'(\bar{r}) = v'(\bar{r}) \text{ and } (|u'|^\alpha u')'(\bar{r}) \leq (|v'|^\alpha v')'(\bar{r}).$$

Indeed, the result is obvious when $u'(\bar{r}) \neq 0$, since in that case the assumption $u'|^\alpha u', |v'|^\alpha v' \in \mathcal{C}^1$ implies that u and v are \mathcal{C}^2 on \bar{r} . Then

$$(|u'|^\alpha u')'(\bar{r}) = (1 + \alpha)|u'|^\alpha(\bar{r})u''(\bar{r}) \leq (1 + \alpha)|u'|^\alpha(\bar{r})v''(\bar{r}) = (1 + \alpha)|v'|^\alpha(\bar{r})v''(\bar{r}) = (|v'|^\alpha v')'(\bar{r})$$

If now $u'(\bar{r}) = v'(\bar{r}) = 0$, let us consider $u_\epsilon = u - \epsilon r$, $v_\epsilon = v - \epsilon r$, $u_\epsilon - v_\epsilon$ has its maximum also achieved on \bar{r} and $u'_\epsilon(\bar{r}) \neq 0$, hence by the first part

$$(|u'_\epsilon|^\alpha u'_\epsilon)'(\bar{r}) \leq (|v'_\epsilon|^\alpha v'_\epsilon)'(\bar{r})$$

and passing to the limit when ϵ tends to zero, one gets

$$(|u'|^\alpha u')'(\bar{r}) \leq (|v'|^\alpha v')'(\bar{r})$$

The next result provides the existence, uniqueness and regularity of solutions below the eigenvalue $\bar{\lambda}'_\gamma$.

Theorem 2.11. *Let f be a radial and continuous function in $\overline{B(0,1)}$ satisfying $f \leq 0$, with f not identically zero. Then, for every $\mu < \bar{\lambda}'_\gamma$ there exists a unique, positive, bounded, radial function $u \in \mathcal{C}_{1,\alpha}$ satisfying*

$$\begin{cases} |\nabla u|^\alpha F(D^2 u) + \mu u^{1+\alpha} r^{-\gamma} = f(r) r^{-\gamma} & \text{in } B(0,1) \setminus \{0\} \\ u(1) = 0. \end{cases}$$

Moreover, u can be extended as a strictly positive continuous function in $B(0,1)$, Lipschitz continuous in $\overline{B(0,1)}$ if $\gamma \leq 1$, \mathcal{C}^1 if $\gamma < 1$, $(\frac{2+\alpha-\gamma}{1+\alpha})$ -Hölder continuous in $\overline{B(0,1)}$ if $\gamma > 1$.

Proof. As in the proof of Theorem 2.9, we can assume without loss of generality that $\mu > 0$, otherwise the conclusion just follows from Theorem 2.7. The uniqueness of u follows from Theorem 2.9.

As far as existence is concerned, let us recursively define a sequence $\{u_n\}_{n \geq 0}$ as follows: we set $u_0 \equiv 0$, and by using Theorem 2.7, we define $u_{n+1} \in \mathcal{C}_{1,\alpha} \cap C(\overline{B(0,1)})$ as the unique bounded radial solution of

$$\begin{cases} |\nabla u_{n+1}|^\alpha F(D^2 u_{n+1}) = (f - \mu u_n^{1+\alpha}) r^{-\gamma} & \text{in } B(0,1) \setminus \{0\} \\ u_{n+1}(1) = 0. \end{cases}$$

By Theorem 2.6, we have that $u_{n+1} \geq 0$, hence it is strictly positive in $B(0,1) \setminus \{0\}$ by the standard strong maximum principle, since f is not identically zero. In particular, u_n is not identically zero for all $n \geq 1$. By applying the comparison principle in Theorem 2.6 again, we deduce also that $u_{n+1} \geq u_n$. Let us prove that $\{u_n\}_n$ is uniformly bounded. If not, by setting $v_n = \|u_n\|_\infty^{-1} u_n$ and $k_n = \|u_{n+1}\|_\infty^{-1} \|u_n\|_\infty \leq 1$, one gets that v_{n+1} satisfies

$$\begin{cases} |\nabla v_{n+1}|^\alpha F(D^2 v_{n+1}) = \left(\frac{f(r)}{\|u_{n+1}\|_\infty^{1+\alpha}} - \mu k_n^{1+\alpha} v_n^{1+\alpha}(r) \right) r^{-\gamma} & \text{in } B(0,1) \setminus \{0\} \\ v_{n+1}(1) = 0 \end{cases}$$

Since $\{v_n\}_n$ is uniformly bounded, by applying Theorem 2.8, we can extract a subsequence still denoted by $\{v_n\}_n$ uniformly converging to a function $v \geq 0$ satisfying

$$\begin{cases} |\nabla v|^\alpha F(D^2 v) + \mu k^{1+\alpha} v^{1+\alpha} r^{-\gamma} = 0 & \text{in } B(0,1) \setminus \{0\} \\ v(1) = 0 \end{cases}$$

where $k \leq 1$ is the limit of some converging subsequence of $\{k_n\}_n$. Since v is a radial solution, one has that $v \in \mathcal{C}_{1,\alpha}$ and, since $\mu k^{1+\alpha} \leq \mu < \bar{\lambda}'_\gamma$, Theorem 2.9 yields $v \leq 0$. Hence, we get $v \equiv 0$, a contradiction with $\|v\|_\infty = 1$.

We have obtained that $\{u_n\}_n$ is bounded, and using once more Theorem 2.8, we deduce that $\{u_n\}$ uniformly converges to some u , which satisfies the desired equation. By the strong maximum principle, we get that $u > 0$ in $B(0,1) \setminus \{0\}$. Moreover, by Remark 2.4, we have that $u'(r) \leq 0$ for $r > 0$ small enough, which implies $u(0) > 0$. Finally, the global regularity of u follows from Remark 2.1 and Theorem 2.7. In fact $u' < 0$ and then $u \in \mathcal{C}^2(B(0,1) \setminus \{0\})$.

We now prove the uniqueness. Suppose that u and v are two (positive) radial solutions of

$$|\nabla u|^\alpha F(D^2 u) + \mu u^{1+\alpha} r^{-\gamma} = f r^{-\gamma}, \quad u(1) = 0$$

If $f \equiv 0$, the maximum principle implies that $u \equiv 0$, so we can suppose that f is not identically 0. We prove that $u \leq v$, next, exchanging the roles of u and v one gets the uniqueness. Suppose by contradiction that $\sup \frac{u}{v} = \eta > 1$. (The supremum is well defined by Hopf principle in [2]). Then $u \leq \eta v$, and the supremum is achieved either on $\bar{r} > 0$, or "on the boundary" which means that $\lim_{r \rightarrow 1} \frac{u}{v} = \eta$, or on 0. Suppose that we are in one of the first situations, then since $u' < 0$ by the equation and by lemma 2.5 one gets using the strong comparison principle, proposition 4.4 in [3], that $u = \eta v$ everywhere in $\overline{B(0,1)} \setminus \{0\}$ and then by continuity $u = \eta v$ in $\overline{B(0,1)}$. This contradicts the equations since one would obtain then $f = \eta^{1+\alpha} f$, a contradiction. We are back to the case where $u < \eta v$ for any $r \neq 0$. Then one has

$$\mathcal{M}^+(|\nabla u|^\alpha D^2 u - |\nabla(\eta v)|^\alpha D^2(\eta v)) \geq ((1 - \eta^{1+\alpha})f + \mu((\eta v)^{1+\alpha} - u^{1+\alpha}))r^{-\gamma} > 0$$

and then using remark 2.3 one gets that $u' > \eta v'$ in a neighborhood of 0, which contradicts the fact that $u - \eta v$ has a supremum at zero. \square

We can now prove that the smooth eigenvalue $\bar{\lambda}'_\gamma$ is actually achieved on smooth eigenfunctions.

Theorem 2.12. *There exists $u \in C(\overline{B(0,1)}) \cap C^2(B(0,1) \setminus \{0\})$, radial, strictly positive in $B(0,1)$ and satisfying*

$$\begin{cases} |\nabla u|^\alpha F(D^2 u) + \bar{\lambda}'_\gamma u^{1+\alpha} r^{-\gamma} = 0 & \text{in } B(0,1) \setminus \{0\} \\ u(1) = 0 \end{cases}$$

Furthermore, in $\overline{B(0,1)}$, u is Lipschitz continuous when $\gamma \leq 1$, C^1 when $\gamma < 1$, and Hölder continuous with exponent $\frac{2+\alpha-\gamma}{1+\alpha}$ if $\gamma > 1$.

Proof. We consider a sequence $\{\lambda_n\}$, with $\lambda_n \rightarrow \bar{\lambda}'_\gamma$ and $\lambda_n < \bar{\lambda}'_\gamma$ and, for all n , the solution $u_n \in C(\overline{B(0,1)}) \cap \mathcal{C}_{1,\alpha}$, (even $C^2(B(0,1) \setminus \{0\})$ by remark. 2.1) provided by Theorem 2.11 of

$$|\nabla u_n|^\alpha F(D^2 u_n) + \lambda_n u_n^{1+\alpha} r^{-\gamma} = -r^{-\gamma}, \quad u_n(1) = 0.$$

We claim that the positive sequence $\{\|u_n\|_\infty\}_n$ is unbounded. Indeed, arguing by contradiction, if $\{u_n\}_n$ is uniformly bounded, then, by using Theorem 2.8 and considering a subsequence if necessary, we obtain that there exists a solution $u \in C(\overline{B(0,1)}) \cap \mathcal{C}_{1,\alpha}$, $u \geq 0$, of

$$|\nabla u|^\alpha F(D^2 u) + \bar{\lambda}'_\gamma u^{1+\alpha} r^{-\gamma} = -r^{-\gamma}, \quad u(1) = 0.$$

Then, arguing as in the proof of Theorem 2.11, we deduce that u is strictly positive in $B(0,1)$, and by taking $0 < \epsilon < \frac{1}{\|u\|_\infty^{1+\alpha}}$, we see that u satisfies

$$|\nabla u|^\alpha F(D^2 u) + (\bar{\lambda}'_\gamma + \epsilon) u^{1+\alpha} r^{-\gamma} \leq 0,$$

a contradiction to the definition of $\bar{\lambda}'_\gamma$. It then follows that the sequence $\{u_n\}_n$ is not uniformly bounded. Normalizing and considering a subsequence, letting $n \rightarrow \infty$ yields the existence of $u \in C(\overline{B(0,1)}) \cap \mathcal{C}_{1,\alpha}$ satisfying

$$|\nabla u|^\alpha F(D^2 u) + \bar{\lambda}'_\gamma u^{1+\alpha} r^{-\gamma} = 0, \quad \text{in } B(0,1) \setminus \{0\}, \quad u(1) = 0.$$

Finally, the strict positivity of u in $B(0, 1)$ and its global regularity in $\overline{B(0, 1)}$ follow by arguing as in the proof of Theorem 2.11. □

Proposition 2.13. *The first eigenvalue is simple.*

Proof. We have proved that there exists a positive bounded eigenfunction v . Take another eigenfunction $u > 0$. Let us define $\eta^{-1} = \sup \frac{v}{u}$ which is well defined by Hopf principle and is finite. Then $u - \eta v$ is nonnegative in $\overline{B(0, 1)} \setminus \{0\}$ and satisfies

$$\begin{aligned} \mathcal{M}^{-}(|\nabla u|^{\alpha} D^2 u - |\nabla(\eta v)|^{\alpha} D^2(\eta v)) &\leq F(|\nabla u|^{\alpha} D^2 u) - F(|\nabla(\eta v)|^{\alpha} D^2(\eta v)) \\ &= -\bar{\lambda} u^{1+\alpha} + \bar{\lambda}(\eta v)^{1+\alpha} \\ &\leq 0 \end{aligned}$$

If by contradiction $u - \eta v > 0$ on some point in $B(0, 1) \setminus \{0\}$, using the fact that u' and v' are < 0 and using the strong comparison principle, Proposition 4.4 in [3] one gets $u - \eta v > 0$ in the whole of $B(0, 1) \setminus \{0\}$. We now distinguish two cases

-If $u(0) = +\infty$, then necessarily the supremum is "achieved" on 1 say

$$\frac{1}{\eta} = \lim_{r \rightarrow 1} \frac{v}{u} = \frac{v'(1)}{u'(1)}$$

But this contradicts Hopf's part in Proposition 4.4 of lemma in [3].

- If $u(0) < \infty$, the maximum could be achieved on 0, but then we get from Lemma 2.5 that $(u - \eta v)' \leq 0$ for r sufficiently small, hence $u - \eta v$ cannot have a strict minimum at zero. Thus we have obtained that all the eigenfunctions are bounded and multiple to each others. We have also obtained that $\bar{\lambda}_{\gamma} = \bar{\lambda}'_{\gamma}$. □

2.2 The first eigenvalue inherited from \mathbb{R}^+

In all that section we assume that $F = \mathcal{M}^+$. The case of \mathcal{M}^- is easily derived by exchanging λ and Λ . We present an alternative proof of the existence of radial eigenfunctions related to the eigenvalue $\bar{\lambda}_{\gamma}$. Here, the idea is to show the existence of global solutions defined in $(0, +\infty)$ of the ODE associated with positive radial solutions of the equation

$$|\nabla u|^{\alpha} \mathcal{M}^+(D^2 u) = -\frac{u}{r^{\gamma}} \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

We prove next that such solution take the value 0 the first time on some $\bar{r} > 0$. Then considering $v(r) = u(\bar{r}r)$, v satisfies

$$|\nabla v|^{\alpha} \mathcal{M}^+(D^2 v) + \bar{r}^{2+\alpha} \frac{v}{(\bar{r}r)^{\gamma}} = 0$$

and then by the definition of $\bar{\lambda}$,

$$\bar{\lambda} \geq \bar{r}^{2+\alpha-\gamma}.$$

If furthermore one had $\bar{\lambda} > \bar{r}^{2+\alpha-\gamma}$, then the fact that $v > 0$ is positive and zero on $r = 1$ would be a contradiction with the maximum principle. We have obtained that $\bar{r}^{2+\alpha-\gamma}$ is the first eigenvalue and v some corresponding eigenfunction.

$$V_{r_o} = \{u \in C([0, r_o]) : |u(r) - 1| \leq \frac{1}{2}, u(0) = 1\}.$$

where r_o will be chosen later . For $\gamma < 1$ let us define T on $V(r_o)$ by

$$Tu(r) = 1 - \int_0^r \left(\frac{1+\alpha}{\lambda} \right)^{\frac{1}{1+\alpha}} s^{1-N} \left(\int_0^s u^{1+\alpha} t^{(N-1)(1+\alpha)-\gamma} dt \right)^{\frac{1}{1+\alpha}} ds$$

We prove below that for r_o small enough T is a contraction mapping and then it possesses a unique fixed point. Let r_o be defined by

$$r_o^{\frac{2+\alpha-\gamma}{1+\alpha}} < (2+\alpha-\gamma) 3^{-1-|\alpha|} (1+\alpha^+)^{\frac{-2-\alpha}{1+\alpha}} \lambda^{\frac{1}{1+\alpha}} ((N-1)(1+\alpha) + (1-\gamma))^{\frac{1}{1+\alpha}}. \quad (2.9)$$

Then T maps V_{r_o} into itself for r_o small enough. Indeed

$$|Tu(r) - 1| \leq \left(\frac{3}{2} \right) r^{\frac{2+\alpha-\gamma}{1+\alpha}} \frac{(1+\alpha)^{\frac{2+\alpha}{1+\alpha}}}{\lambda^{\frac{1}{1+\alpha}} (2+\alpha-\gamma) ((N-1)(1+\alpha) + (1-\gamma))^{\frac{1}{1+\alpha}}}$$

To prove that T is a contraction mapping on $V(r_o)$, we consider separately the cases $\alpha > 0$ and $\alpha < 0$.
- If $\alpha > 0$, let us define the space $\mathbf{L}^\alpha([0, r]) = L^{\alpha+1}([0, r], t^{(N-1)(1+\alpha)-\gamma})$ of functions with power $(\alpha+1)$ - integrable for the measure $t^{(N-1)(1+\alpha)-\gamma} dt$. Using Minkowski inequality one has

$$\left| |u|_{\mathbf{L}^\alpha([0, r])} - |v|_{\mathbf{L}^\alpha([0, r])} \right| \leq |u - v|_{\mathbf{L}^\alpha([0, r])}$$

and then

$$\begin{aligned} |Tu - Tv|(r) &\leq (1+\alpha)^{\frac{1}{1+\alpha}} \int_0^r \frac{1}{\lambda^{\frac{1}{1+\alpha}} s^{N-1}} |u - v|_{\mathbf{L}^\alpha([0, s])} ds \\ &\leq \frac{(1+\alpha)^{\frac{1}{1+\alpha}}}{\lambda^{\frac{1}{1+\alpha}} (N-1)(1+\alpha) + (1-\gamma))^{\frac{1}{1+\alpha}}} |u - v|_\infty \int_0^r t^{1-N} t^{N-1+\frac{1-\gamma}{1+\alpha}} dt \\ &\leq |u - v|_\infty r^{\frac{2+\alpha-\gamma}{1+\alpha}} \frac{(1+\alpha)^{\frac{2+\alpha}{1+\alpha}}}{(2+\alpha-\gamma) \lambda^{\frac{1}{1+\alpha}} ((N-1)(1+\alpha) + (1-\gamma))^{\frac{1}{1+\alpha}}} \\ &\leq \frac{1}{2} |u - v|_\infty. \end{aligned}$$

So T is a contraction mapping on V_{r_o} under the assumption (2.9).

If $\alpha < 0$ we prove that under the assumption (2.9), T is a contraction mapping . We use twice the mean value's Theorem :

$$\begin{aligned}
|Tu(r) - Tv(r)| &\leq \frac{(1+\alpha)^{\frac{1}{1+\alpha}}}{\lambda^{\frac{1}{1+\alpha}}} \int_0^r \frac{1}{s^{N-1}} \left| \left(\int_0^s u^{1+\alpha} t^{(N-1)(1+\alpha)-\gamma} dt \right)^{\frac{1}{1+\alpha}} - \left(\int_0^s v^{1+\alpha} t^{(N-1)(1+\alpha)-\gamma} dt \right)^{\frac{1}{1+\alpha}} \right| ds \\
&\leq \lambda^{\frac{1}{1+\alpha}} \frac{(1+\alpha)^{\frac{1}{1+\alpha}}}{1+\alpha} ((N-1)(1+\alpha) + (1-\gamma))^{\frac{\alpha}{1+\alpha}} \left(\frac{3}{2} \right)^{-\alpha} \times (1+\alpha) \\
&\quad \int_0^r s^{1-N} s^{-\alpha(N-1) - \frac{(1-\gamma)\alpha}{1+\alpha}} \left| \int_0^s (u^{1+\alpha} - v^{1+\alpha})(t) t^{(N-1)(1+\alpha)-\gamma} dt \right| ds \\
&\leq \lambda^{\frac{1}{1+\alpha}} (1+\alpha)^{\frac{1}{1+\alpha}} 2^{-\alpha} ((N-1)(1+\alpha) + (1-\gamma))^{\frac{\alpha}{1+\alpha}} \left(\frac{3}{2} \right)^{-\alpha} \int_0^r s^{-(1+\alpha)(N-1) - \frac{(1-\gamma)\alpha}{1+\alpha}} \int_0^s |u-v|_{\infty} t^{(N-1)(1+\alpha)-\gamma} dt ds \\
&\leq \lambda^{\frac{1}{1+\alpha}} 3^{-\alpha} (1+\alpha)^{\frac{1}{1+\alpha}} ((N-1)(1+\alpha) + (1-\gamma))^{\frac{-1}{1+\alpha}} |u-v|_{\infty} \int_0^r s^{-(1+\alpha)(N-1) - \frac{(1-\gamma)\alpha}{1+\alpha}} s^{(N-1)(1+\alpha)+1-\gamma} ds \\
&\leq \lambda^{\frac{1}{1+\alpha}} 3^{-\alpha} \frac{(1+\alpha)^{\frac{2+\alpha}{1+\alpha}}}{2+\alpha-\gamma} ((N-1)(1+\alpha) + (1-\gamma))^{\frac{-1}{1+\alpha}} |u-v|_{\infty} r^{\frac{2+\alpha-\gamma}{1+\alpha}} \\
&\leq \frac{1}{2} |u-v|_{\infty}.
\end{aligned}$$

by the choice of r_o .

We need to prove that the fixed point above is a solution of

$$|\nabla u|^{\alpha} \mathcal{M}^+(D^2 u) = -u^{1+\alpha} r^{-\gamma}$$

for that aim we prove that if $\gamma < 1$, $u' < 0$ and $(|u'|^{\alpha} u')' < 0$. The first assertion is immediate, and then since $u(0) = 1$, $u \leq 1$, one gets that

$$|u'|^{\alpha} u' \geq -\frac{1+\alpha}{\lambda r^{(N-1)(1+\alpha)}} \int_0^r t^{(N-1)(1+\alpha)-\gamma} dt \geq -\frac{1+\alpha}{\lambda} \frac{r^{1-\gamma}}{(N-1)(1+\alpha) + 1 - \gamma}$$

From this we derive

$$u(r) \geq 1 - \frac{((1+\alpha)\lambda^{-1})^{\frac{1}{1+\alpha}}}{2+\alpha-\gamma} \frac{(1+\alpha)r^{\frac{2+\alpha-\gamma}{1+\alpha}}}{((N-1)(1+\alpha) + 1 - \gamma)^{\frac{1}{1+\alpha}}}.$$

Then

$$|u|^{\alpha} u(r) \geq 1 - \frac{(1+\alpha)^{\frac{3+2\alpha}{1+\alpha}} \lambda^{\frac{-1}{1+\alpha}}}{2+\alpha-\gamma} \frac{r^{\frac{2+\alpha-\gamma}{1+\alpha}}}{((N-1)(1+\alpha) + 1 - \gamma)^{\frac{1}{1+\alpha}}} + o(r^{\frac{2+\alpha-\gamma}{1+\alpha}}).$$

Using the fact that u is a fixed point of T

$$\begin{aligned}
(|u'|^{\alpha} u')' &= -(1+\alpha)u^{1+\alpha} \lambda^{-1} r^{-\gamma} - (N-1)r^{-1}(1+\alpha)|u'|^{\alpha} u' \\
&\leq -r^{-\gamma}(1+\alpha)\lambda^{-1} + (N-1) \frac{(1+\alpha)^2}{\lambda((N-1)(1+\alpha) + 1 - \gamma)} r^{-\gamma} + O(r^{\frac{(1-\gamma)(2+\alpha)}{1+\alpha}}) \\
&= r^{-\gamma}(1+\alpha)\lambda^{-1} \frac{-1+\gamma}{(N-1)(1+\alpha) + 1 - \gamma} + o(r^{-\gamma}) \\
&< 0
\end{aligned}$$

We now consider the case $\gamma \geq 1$. Here we suppose

$$r_0^{\frac{2+\alpha-\gamma}{1+\alpha}} < (2+\alpha-\gamma)2^{\alpha^+}3^{-1-|\alpha|}(1+\alpha^+)^{\frac{-2-\alpha}{1+\alpha}}\Lambda^{\frac{1}{1+\alpha}}((N_+-1)(1+\alpha)+(1-\gamma))^{\frac{1}{1+\alpha}} \quad (2.10)$$

and as in the case $\gamma < 1$ $V_{r_0} = \{u \in C([0, r_0]) : |u(r) - 1| \leq \frac{1}{2}, u(0) = 1\}$. We define T by

$$Tu(r) = 1 - \int_0^r \left(\frac{1+\alpha}{\Lambda s^{(N_+-1)(1+\alpha)}} \int_0^s u^{1+\alpha}(t)t^{(N_+-1)(1+\alpha)-\gamma} dt \right)^{\frac{1}{1+\alpha}} ds$$

Following the same calculations as in the case $\gamma < 1$, T maps V_{r_0} into itself and is a contraction mapping. We need to prove that $|u'|^\alpha u'$ has a derivative ≥ 0 .

If $\gamma > 1$ we have for some positive constant c which can vary from one line to another, $|u'|^\alpha u' \geq -cr^{\frac{1-\gamma}{1+\alpha}}$ and then

$$u(r) \geq 1 - cr^{\frac{2+\alpha-\gamma}{1+\alpha}}$$

which in turn implies that

$$|u'|^\alpha u' \leq -\frac{1}{\Lambda} \frac{r^{1-\gamma}(1+\alpha)}{((N_+-1)(1+\alpha)+1-\gamma)} + o(r^{1-\gamma})$$

and then using once more $u < 1$,

$$\begin{aligned} (|u'|^\alpha u')' &\geq -(1+\alpha)r^{-\gamma}\Lambda^{-1} + (1+\alpha)\frac{(N_+-1)(1+\alpha)}{r} \frac{1}{\Lambda} \frac{r^{1-\gamma}}{(N_+-1)(1+\alpha)+1-\gamma} + o(r^{-\gamma}) \\ &= \frac{(1+\alpha)\Lambda^{-1}(\gamma-1)}{(N_+-1)(1+\alpha)+1-\gamma} r^{-\gamma} + o(r^{-\gamma}) > 0. \end{aligned}$$

There remains to do the case $\gamma = 1$. Here we need to go further in the second term of the limited development :

$$|u'|^\alpha u' = -\frac{(1+\alpha)}{\Lambda} r^{-(N_+-1)(1+\alpha)} \int_0^r u^{1+\alpha}(t)t^{(N_+-1)(1+\alpha)-1} dt \geq -\frac{1}{\Lambda(N_+-1)}$$

Then

$$u(r) \geq 1 - \left(\frac{1}{\Lambda(N_+-1)} \right)^{\frac{1}{1+\alpha}} r$$

hence

$$|u|^\alpha u \geq 1 - (1+\alpha) \left(\frac{1}{\Lambda(N_+-1)} \right)^{\frac{1}{1+\alpha}} r + O(r^2)$$

This in turn implies

$$|u'|^\alpha u' \leq -\frac{1}{\Lambda(N_+-1)} + \frac{(1+\alpha)^2}{\Lambda} \left(\frac{1}{\Lambda(N_+-1)} \right)^{\frac{1}{1+\alpha}} \frac{r}{(N_+-1)(1+\alpha)+1} + o(r)$$

From this we derive

$$u(r) \leq 1 - \left(\frac{1}{\Lambda(N_+-1)} \right)^{\frac{1}{1+\alpha}} r + O(r^2)$$

and

$$u^{1+\alpha} \leq 1 - (\alpha + 1) \left(\frac{1}{\Lambda(N_+ - 1)} \right)^{\frac{1}{1+\alpha}} r + O(r^2).$$

Reporting in the equation

$$\begin{aligned} (|u'|^\alpha u')' &\geq \frac{(1+\alpha)^2}{\Lambda} r^{-1} r \left(\frac{1}{\Lambda(N_+ - 1)} \right)^{\frac{1}{1+\alpha}} \\ &- \frac{(1+\alpha)^3(N_+ - 1)}{\Lambda} \left(\frac{1}{\Lambda(N_+ - 1)} \right)^{\frac{1}{1+\alpha}} ((N_+ - 1)(1+\alpha) + 1)^{-1} + O(r) \\ &= \left(\frac{1}{\Lambda(N_+ - 1)} \right)^{\frac{1}{1+\alpha}} \frac{(1+\alpha)^2}{\Lambda(N_+ - 1)(1+\alpha) + 1} + O(r) > 0. \end{aligned}$$

Let us prove that there exists \bar{r} so that $u(\bar{r}) = 0$. If not $u > 0$ and then from the equation u' remains < 0 . Let us define , inspired by [7], [6] :

$$y(r) = \frac{|u'|^\alpha u' r^{(N-1)(1+\alpha)}}{u^{1+\alpha}}$$

which is then always < 0 . y satisfies the inequality

$$y'(r) \leq -(\alpha + 1) \frac{r^{(N-1)(1+\alpha)-\gamma}}{\Lambda} - (\alpha + 1) |y|^{\frac{\alpha+2}{\alpha+1}} r^{1-N}$$

and then defining for some $r_1 > 0$

$$k(r) = \int_{r_1}^r \frac{|y|^{\frac{\alpha+2}{\alpha+1}}}{t^{N-1}} dt,$$

$$y(r) + (\alpha + 1)k(r) \leq y(r_1) - (\alpha + 1) \frac{r^{(N-1)(1+\alpha)+1-\gamma} - r_1^{(N-1)(1+\alpha)+1-\gamma}}{((N-1)(1+\alpha) + 1 - \gamma)\Lambda}$$

In particular for some constant $c > 0$ which can vary from one line to another

$$y(r) \leq -cr^{(N-1)(1+\alpha)+1-\gamma}$$

which implies that

$$k(r) \geq c \int_{r_1}^r \frac{t^{(N-1)(2+\alpha)+(1-\gamma)\frac{\alpha+2}{\alpha+1}}}{t^{N-1}} dt \geq cr^{(N-1)(1+\alpha)+(1-\gamma)\frac{\alpha+2}{\alpha+1}+1}.$$

On the other hand since $k \leq -(\alpha + 1)^{-1}y$ for r large one has

$$k' \geq ck^{\frac{\alpha+2}{\alpha+1}} r^{1-N}$$

which implies

$$\frac{k'}{k^{\frac{\alpha+2}{\alpha+1}}} \geq cr^{1-N}$$

and after integrating between r and $+\infty$, using $\lim_{r \rightarrow +\infty} k(r) = +\infty$,

$$k(r)^{\frac{1}{\alpha+1}} \leq cr^{N-2}$$

We would then have

$$(N-1)(1+\alpha) + (1-\gamma)\frac{\alpha+2}{\alpha+1} + 1 \leq (N-2)(1+\alpha)$$

which would imply $\alpha + 2 - \gamma \leq 0$, a contradiction. We have obtained that u takes the value 0 somewhere.

2.3 The stability of the principal eigenvalue and related eigenfunctions

The results of the present section give, as a corollary, the proof of Theorem 1.2.

Let us start by proving the stability with respect to the ϵ -regularization of the singular potential. We recall that $r_\epsilon = (r^2 + \epsilon^2)^{\frac{1}{2}}$ and $\bar{\lambda}_\gamma^\epsilon = \bar{\lambda}(F, \frac{1}{r_\epsilon}, B(0, 1))$.

Theorem 2.14. *One has*

$$\bar{\lambda}'_\gamma = \lim_{\epsilon \rightarrow 0} \bar{\lambda}_\gamma^\epsilon.$$

Furthermore, if $\{u_\epsilon\}$ is the sequence of the eigenfunctions associated with the eigenvalue $\bar{\lambda}_\gamma^\epsilon$ and satisfies $u_\epsilon(0) = 1$, then, one can extract from $\{u_\epsilon\}$ a subsequence uniformly converging on $\overline{B(0, 1)}$ to the eigenfunction associated with $\bar{\lambda}'_\gamma$ which takes the value 1 at zero.

Proof. Let $\{u_\epsilon\}$ be the sequence as in the statement. Then, each u_ϵ is a $\mathcal{C}_{1,\alpha}$ positive radial function satisfying in particular

$$|\nabla u_\epsilon|^\alpha F(D^2 u_\epsilon) + \bar{\lambda}_\gamma^\epsilon \frac{u_\epsilon^{1+\alpha}}{r^\gamma} \geq 0 \quad \text{in } B(0, 1) \setminus \{0\},$$

so that, by Theorem 2.9, one has $\bar{\lambda}_\gamma^\epsilon \geq \bar{\lambda}'_\gamma$. Moreover, the sequence $\{\bar{\lambda}_\gamma^\epsilon\}$ is monotone increasing with respect to ϵ . Thus, we deduce

$$\mu := \lim_{\epsilon \rightarrow 0} \bar{\lambda}_\gamma^\epsilon \geq \bar{\lambda}'_\gamma.$$

On the other hand, by the monotonicity properties of radially symmetric solutions of elliptic equations, we know that $u'_\epsilon(r) \leq 0$ for $r \in [0, 1]$. Since

$$\mathcal{M}^+(D^2 u_\epsilon) \geq F(D^2 u_\epsilon)$$

we deduce that, independently of the sign of $u''_\epsilon(r)$, one has

$$|u'_\epsilon|^\alpha u''_\epsilon + (N_+ - 1)|u'_\epsilon|^\alpha \frac{u'_\epsilon}{r} \geq -\frac{\bar{\lambda}_\gamma^\epsilon}{\lambda} u_\epsilon^{1+\alpha} r_\epsilon^{-\gamma}.$$

This implies

$$(|u'_\epsilon|^\alpha u'_\epsilon r^{(N_+-1)(1+\alpha)})' \geq -(1+\alpha) \frac{\bar{\lambda}_\gamma^\epsilon}{\lambda} u_\epsilon^{1+\alpha} \frac{r^{(N_+-1)(1+\alpha)}}{r_\epsilon^\gamma} \geq -(1+\alpha) \frac{\bar{\lambda}_\gamma^\epsilon}{\lambda} r^{(N_+-1)(1+\alpha)-\gamma},$$

and therefore, by integrating, using $u'_\epsilon(0) = 0$,

$$0 \geq u'_\epsilon(r) \geq - \left(\frac{\bar{\lambda}_\gamma^\epsilon(\alpha + 1)}{\lambda((N_+ - 1)(1 + \alpha) + 1 - \gamma)} \right)^{\frac{1}{1+\alpha}} r^{\frac{1-\gamma}{1+\alpha}}.$$

Hence, on $\overline{B(0, 1)}$, the functions u_ϵ are uniformly Lipschitz continuous if $\gamma \leq 1$, and uniformly $\frac{2+\alpha-\gamma}{1+\alpha}$ -Hölder continuous if $\gamma > 1$. In both cases, up to a subsequence, $\{u_\epsilon\}$ is uniformly converging to a continuous radial function $u \in C(\overline{B(0, 1)})$ which satisfies $u(0) = 1$ and

$$|\nabla u|^\alpha F(D^2 u) + \mu \frac{u^{1+\alpha}}{r^\gamma} = 0.$$

Hence, u is $\mathcal{C}_{1,\alpha}$ and, by the standard strong maximum principle, u is strictly positive in $B(0, 1)$. This yields, by definition, $\mu \leq \bar{\lambda}'_\gamma$. Hence, $\mu = \bar{\lambda}'_\gamma$ and the conclusion follows from Proposition 2.13. Note that since $u' < 0$ for $r \neq 0$, u is $\mathcal{C}^2(B(0, 1) \setminus \{0\})$. □

As a consequence of the previous theorem, we finally obtain the following

Corollary 2.15. *One has*

$$\bar{\lambda}_\gamma = \lim_{\delta \rightarrow 0} \bar{\lambda}_\gamma(B(0, 1) \setminus \overline{B(0, \delta)}) = \bar{\lambda}'_\gamma.$$

Proof. We observe that the function $\delta \mapsto \bar{\lambda}_\gamma(B(0, 1) \setminus B(0, \delta))$ is monotone increasing. Moreover, by their own definition, we have that

$$\bar{\lambda}'_\gamma \leq \bar{\lambda}_\gamma \leq \bar{\lambda}_\gamma(B(0, 1) \setminus B(0, \delta)) \quad \text{for all } \delta \geq 0.$$

On the other hand, by Theorem 2.14, for any $\eta > 0$ there exists $\epsilon_0 > 0$ such that

$$\bar{\lambda}_\gamma^{\epsilon_0} \leq \bar{\lambda}'_\gamma + \frac{\eta}{2}.$$

Furthermore, by using the continuity of the principal eigenvalue with respect to the domain for equations with regular coefficients, there exists $\delta_0 > 0$ such that

$$\bar{\lambda}_\gamma^{\epsilon_0}(B(0, 1) \setminus B(0, \delta_0)) \leq \bar{\lambda}_\gamma^{\epsilon_0} + \frac{\eta}{2} \leq \bar{\lambda}'_\gamma + \eta.$$

Now, since $\epsilon \mapsto \bar{\lambda}_\gamma^\epsilon(B(0, 1) \setminus B(0, \delta_0))$ decreases when ϵ decreases to zero, one gets

$$\bar{\lambda}_\gamma(B(0, 1) \setminus B(0, \delta_0)) \leq \bar{\lambda}_\gamma^{\epsilon_0}(B(0, 1) \setminus B(0, \delta_0)) \leq \bar{\lambda}'_\gamma + \eta,$$

which gives the conclusion. □

3 The case $\gamma = \alpha + 2$

In this section we suppose that the operator F is \mathcal{M}^+ . We will sometimes denote $\bar{\lambda}_\gamma$ or $\bar{\lambda}_\gamma(|\nabla \cdot|^\alpha \mathcal{M}^+)$ the eigenvalue. Let us introduce the space of functions

$$\mathcal{V} = \{u \in C^2([0, 1]) : u'(0) = 0, \text{ supp}(u) \text{ compact in } [0, 1]\},$$

endowed with the norm

$$\|u\| = \left(\int_0^1 |u'|^{\alpha+2} r^{(N_+-1)(1+\alpha)} dr \right)^{1/\alpha+2},$$

and let us denote by $\mathcal{H}_0^{1+\alpha}$ the closure of \mathcal{V} for this norm. We define

$$\bar{\lambda}_{var,\gamma} = \inf_{v \in \mathcal{H}_0^1, \int_0^1 |v|^{\alpha+2} r^{(N_+-1)(1+\alpha)-\gamma} = 1} \int_0^1 |v'|^{\alpha+2} r^{(N_+-1)(1+\alpha)} dr$$

Theorem 3.1. *One has*

$$\bar{\lambda}_{var,2+\alpha} = \left(\frac{(N_+-2)(\alpha+1)}{\alpha+2} \right)^{\alpha+2} := \tau^{\alpha+2}.$$

Proof. We begin to prove that $\bar{\lambda}_{var,2+\alpha} \geq \left(\frac{(N_+-2)(\alpha+1)}{\alpha+2} \right)^{\alpha+2}$ or equivalently that for any $v \in \mathcal{H}_0^{1+\alpha}$

$$\tau^{\alpha+2} \int_0^1 v^{2+\alpha} r^{(N_+-1)(1+\alpha)} dr \leq \int_0^1 |v'|^{2+\alpha} r^{(N_+-1)(1+\alpha)} dr.$$

For that aim we consider

$$u(r) = r^{-\tau}$$

where $\tau = \frac{(N_+-2)(\alpha+1)}{2+\alpha}$. Then u satisfies for any $r > 0$ in $B(0, 1)$

$$\frac{d}{dr} (|u'|^\alpha u' r^{(N_+-1)(1+\alpha)}) = -\tau^{\alpha+2} u^{1+\alpha} r^{(N_+-1)(1+\alpha)-2-\alpha}.$$

Let us multiply this equation by $\frac{|v|^{2+\alpha}}{u^{1+\alpha}}$, for $v \in \mathcal{V}$. Using the definition of u one observes that $\lim \frac{|u'|^\alpha u'}{u^{1+\alpha}} r^{(N_+-1)(1+\alpha)} = 0$ since $N_+ > 2$. Integrating by parts one has

$$\begin{aligned} (\alpha+2) \int_0^1 |u'|^\alpha u' |v|^{\alpha} v u^{-1-\alpha} v' r^{(N_+-1)(1+\alpha)} dr &= (\alpha+1) \int_0^1 |u'|^{\alpha+2} |v|^{\alpha+2} u^{-2-\alpha} r^{(N_+-1)(1+\alpha)} dr \\ &= \tau^{\alpha+2} \int_0^1 |v|^{2+\alpha} r^{(N_+-1)(1+\alpha)-2-\alpha} dr. \end{aligned}$$

We denote $X = |u'|^\alpha u' |v|^\alpha v u^{-1-\alpha}$, $Y = v'$, $p = \alpha+2$ and $p' = \frac{\alpha+2}{\alpha+1}$. Using the convexity inequality

$$XY \leq \frac{1}{p} |Y|^p + \frac{1}{p'} |X|^{p'}.$$

We obtain

$$\tau^{\alpha+2}(\alpha+2)^{-1} \int_0^1 |v|^{2+\alpha} r^{(N_+-1)(1+\alpha)-2-\alpha} \leq \frac{1}{\alpha+2} \int_0^1 |v'|^{\alpha+2} r^{(N_+-1)(1+\alpha)} dr$$

Passing to the infimum one gets $\bar{\lambda}_{var,2+\alpha} \geq \tau^{\alpha+2}$.

We prove the reverse inequality. Let for $\epsilon > 0$, $\epsilon < \inf(\frac{\tau}{2}, \frac{\tau \log 2}{4})$ $u(r) = r^{-\tau+\epsilon}(-\log r)$. We want to prove that for some constant c

$$\int_0^1 |u'|^{2+\alpha} r^{(N_+-1)(1+\alpha)} dr \leq \tau^{2+\alpha} \int_0^1 |u|^{2+\alpha} r^{(N_+-1)(1+\alpha)-2} (1+c\epsilon).$$

We define

$$I_\epsilon = \int_0^1 (-\log r)^{2+\alpha} r^{-1+\epsilon(2+\alpha)} dr.$$

Note that

$$I_\epsilon \geq \int_0^{\frac{1}{2}} (\log 2)^{2+\alpha} r^{-1+(2+\alpha)\epsilon} dr = \frac{C}{\epsilon}. \quad (3.1)$$

We also introduce

$$J_\epsilon = \int_0^{e^{\frac{-1}{\tau-\epsilon}}} ((\tau-\epsilon)(-\log r) + 1)^{2+\alpha} r^{-1+\epsilon(2+\alpha)} dr,$$

$$\text{and } K_\epsilon = \int_{e^{\frac{-1}{\tau-\epsilon}}}^1 ((\tau-\epsilon)(-\log r) + 1)^{2+\alpha} r^{-1+\epsilon(2+\alpha)} dr.$$

Note for the sequel that

$$\int |u'|^{2+\alpha} r^{N_+-1(1+\alpha)} = \tau^{1+\alpha} (J_\epsilon + K_\epsilon)$$

We use the inequality for $0 \leq u \leq 1$

$$(1+u)^{2+\alpha} \leq 1 + (2+\alpha)2^{1+\alpha}u$$

to get when $r < e^{\frac{-1}{\tau-\epsilon}}$

$$((\tau-\epsilon)(-\log r) + 1)^{2+\alpha} \leq (\tau-\epsilon)(-\log r)^{2+\alpha} + 2^{1+\alpha}(2+\alpha)((\tau-\epsilon)(-\log r))^{1+\alpha}$$

and next we use for any $b > 0$, $b < 1$

$$\int_0^b (-\log r)^{1+\alpha} r^{-1+\epsilon(2+\alpha)} = c_1 + c_2 \epsilon \int_0^b (-\log r)^{2+\alpha} r^{-1+\epsilon(2+\alpha)}$$

From this one derives

$$J_\epsilon \leq \tau^{2+\alpha} I + (2+\alpha)2^{1+\alpha} \tau^{1+\alpha} \int_0^{e^{\frac{-1}{\tau-\epsilon}}} (-\log r)^{1+\alpha} r^{-1+\epsilon(2+\alpha)} \leq \tau^{2+\alpha} I + c\epsilon I_\epsilon \quad (3.2)$$

On the other hand one has

$$K_\epsilon \leq \int_{e^{\frac{-1}{\tau-\epsilon}}}^1 ((\tau-\epsilon)(-\log r) + 1)^{2+\alpha} r^{-1} dr = \frac{1}{\tau-\epsilon} \left[\frac{((\tau-\epsilon)(-\log r) + 1)^{3+\alpha}}{-(3+\alpha)} \right]_{e^{\frac{-1}{\tau-\epsilon}}}^1 \leq c(\tau, \alpha) \epsilon I_\epsilon$$

We have obtained

$$\int_0^1 |u'|^{2+\alpha} r^{(N_+-1)(1+\alpha)} dr = J_\epsilon + K_\epsilon \leq \tau^{2+\alpha} I_\epsilon (1 + c(\tau, \alpha) \epsilon)$$

which is the desired result since ϵ is arbitrarily small. \square

In order to establish the relationship between $\bar{\lambda}_{\gamma, var}$ and $\bar{\lambda}_\gamma$ we need to investigate on the monotonicity and convexity properties of the functions u realizing the infimum in the definition of $\bar{\lambda}_{\gamma, var}$.

Proposition 3.2. *Let $\gamma > 1$. Let v_γ be in $\mathcal{H}_o^{1+\alpha}(B(0, 1))$, > 0 which realizes the minimum defining $\bar{\lambda}_{var, \gamma}$. Then v_γ is bounded, $v'_\gamma < 0$ and $(|v'_\gamma|^\alpha v'_\gamma)' > 0$.*

Proof. Since v_γ is a minimum, it satisfies for any $u \in \mathcal{H}_o^{1+\alpha}$

$$\int_0^1 |v'_\gamma|^\alpha v'_\gamma u' r^{(N_+-1)(1+\alpha)} dr = \bar{\lambda}_{var, \gamma} \int_0^1 |v_\gamma|^\alpha v_\gamma u r^{(N_+-1)(1+\alpha)-\gamma} dr \quad (3.3)$$

and then in the distribution sense

$$(|v'_\gamma|^\alpha v'_\gamma)' + (1 + \alpha) \frac{N_+ - 1}{r} |v'_\gamma|^\alpha v'_\gamma = -\bar{\lambda}_{var, \gamma} v_\gamma^{1+\alpha} r^{-\gamma}. \quad (3.4)$$

In particular by the regularity properties of solutions of equations in the distribution sense, one gets $v_\gamma \in \mathcal{C}_{1, \alpha}$. Let us multiply the Euler equation (3.4) above by $u r^{(N_+-1)(1+\alpha)}$ where $u \in \mathcal{H}_o^{1+\alpha}$. Integrating by parts on $]\epsilon, r[$ we obtain

$$\begin{aligned} \int_\epsilon^r |v'_\gamma|^\alpha (s) v'_\gamma (s) u' (s) s^{(N_+-1)(1+\alpha)} ds + |v'_\gamma|^\alpha v'_\gamma (\epsilon) u (\epsilon) \epsilon^{(N_+-1)(1+\alpha)} \\ = \bar{\lambda}_{var, \gamma} \int_\epsilon^r v_\gamma^{1+\alpha} (s) u (s) s^{(N_+-1)(1+\alpha)-\gamma} ds. \end{aligned}$$

Letting ϵ go to zero, since $u(0)$ can be chosen $\neq 0$, we obtain from (3.3) that

$$|v'_\gamma|^\alpha v'_\gamma (\epsilon) \epsilon^{(N_+-1)(1+\alpha)} \rightarrow 0. \quad (3.5)$$

We now prove that v_γ is bounded. Since $v_\gamma > 0$, by the equation above integrated between ϵ and r , $|v'_\gamma|^\alpha v'_\gamma (r) r^{(N_+-1)(1+\alpha)} \leq 0$, hence v_γ is decreasing. From (3.5) one also gets that there exists some constant c so that $v'_\gamma \geq -c r^{1-N_+}$ and this implies that

$$v_\gamma \leq c r^{2-N_+}$$

which in turn implies

$$|v'_\gamma|^\alpha v'_\gamma (r) r^{(N_+-1)(1+\alpha)} \geq -c \int_0^r v_\gamma^{1+\alpha} r^{(N_+-1)(1+\alpha)-\gamma} \geq -c r^{2+\alpha-\gamma}$$

or equivalently for some constant c_1

$$v'_\gamma \geq -c_1 r^{\frac{2+\alpha-\gamma}{\alpha+1}+1-N_+}.$$

Integrating one gets for some constant d_1

$$v_\gamma(r) \leq d_1 r^{\frac{2+\alpha-\gamma}{\alpha+1}+2-N_+}$$

Iterating the above inequalities, we obtain that for all integers $j \geq 0$ such that $2 - N_+ + j \frac{2+\alpha-\gamma}{1+\alpha} < 0$, there exist positive constants c_j and d_j satisfying

$$v'_\gamma(r) \geq -c_j r^{1-N_++j(\frac{2+\alpha-\gamma}{1+\alpha})} \quad (3.6)$$

$$v_\gamma(r) \leq d_j r^{2-N_++j(\frac{2+\alpha-\gamma}{1+\alpha})} \quad (3.7)$$

Indeed, by (3.5) inequality (3.6) is true for $j = 0$, and implies (3.7) for $j = 0$ by integrating. We now suppose that (3.7) is true for j , then

$$\begin{aligned} |v'_\gamma|^\alpha v'_\gamma r^{(N_+-1)(1+\alpha)} &= -\bar{\lambda}_{var,\gamma} \int_0^r v_\gamma^{1+\alpha} r^{(N_+-1)(1+\alpha)-\gamma} \\ &\geq -c \int_0^r d_j^{1+\alpha} r^{(2-N_+)(1+\alpha)+j(2+\alpha-\gamma)} r^{(N_+-1)(1+\alpha)-\gamma} \\ &\geq -c r^{(j+1)(2+\alpha-\gamma)} \end{aligned}$$

which implies

$$v'_\gamma \geq -c r^{1-N_++(j+1)\frac{2+\alpha-\gamma}{1+\alpha}}.$$

If $1 - N_+ + (j + 1) \frac{2+\alpha-\gamma}{1+\alpha} < 0$, integrating this inequality we obtain (3.7) for $j + 1$.

Now, if there exists $j \in \mathbb{N}$ such that $2 - N_+ + j \frac{2+\alpha-\gamma}{1+\alpha} = 0$, i.e. if $\frac{(N_+-2)(1+\alpha)}{2+\alpha-\gamma} \in \mathbb{N}$, then, by integrating the estimates obtained at the $(j - 1)$ -th step, we obtain

$$v_\gamma(r) \geq -c_j r^{-1}, \quad v_\gamma(r) \leq d_j (-\ln r).$$

Integrating once more, we finally deduce

$$v'_\gamma(r) \geq -c_{j+1} (-\ln r) r^{1-\gamma} \implies v_\gamma(r) \leq d_{j+1} = \frac{c_{j+1}}{(2-\gamma)^2}.$$

On the other hand, if $\frac{(N_+-2)(1+\alpha)}{2+\alpha-\gamma}$ is not integer, by integrating estimates (3.6) for $j = \left\lfloor \frac{(N_+-2)(1+\alpha)}{2+\alpha-\gamma} \right\rfloor$, we obtain

$$v'_\gamma(r) \geq -c_{j+1} r^{1-N_++(j+1)(\frac{2+\alpha-\gamma}{1+\alpha})} \implies v_\gamma(r) \leq d_{j+1} = \frac{c_{j+1}}{2 - N_+ + (j + 1)(\frac{2+\alpha-\gamma}{1+\alpha})}.$$

This shows that, in any case, v_γ is bounded. We need to prove that $(|v'_\gamma|^\alpha v'_\gamma)' \geq 0$. For that aim we introduce $y = (1 + \alpha)(N_+ - 1)|v'_\gamma|^\alpha v'_\gamma + \bar{\lambda}_{var,\gamma} v_\gamma^{1+\alpha} r^{1-\gamma}$. It is sufficient to prove that $y \leq 0$. One has

$$y' = (N_+-1)(1+\alpha)(|v'_\gamma|^\alpha v'_\gamma)' + \bar{\lambda}_{var,\gamma}(1+\alpha)v_\gamma^\alpha v'_\gamma r^{1-\gamma} + \bar{\lambda}_{var,\gamma}(1-\gamma)r^{-\gamma}v_\gamma^{1+\alpha} = (N_+-1)(1+\alpha)\frac{-y}{r} + (\leq 0)$$

hence

$$(yr^{(N_+-1)(1+\alpha)})' \leq 0$$

and then since $(N_+ - 1)(1 + \alpha) + 1 - \gamma > 0$, $\lim_{r \rightarrow 0} v_\gamma(r)^{1+\alpha} r^{1-\gamma} r^{(N_+-1)(1+\alpha)}(r) = 0$ and since we proved above that $\lim_{r \rightarrow 0} |v'_\gamma|^\alpha v'_\gamma r^{N_+-1(1+\alpha)} = 0$, we get $y \leq 0$, hence $(|v'_\gamma|^\alpha v'_\gamma)' \geq 0$. \square

Corollary 3.3. *Let $\gamma \in]1, 2 + \alpha[$. Then*

$$\bar{\lambda}_\gamma(|\nabla \cdot |^\alpha \mathcal{M}^+) = \frac{\Lambda}{\alpha + 1} \bar{\lambda}_{\gamma, var}.$$

Proof. By proposition 3.2, v_γ is bounded, satisfies $\lim_{r \rightarrow 0} v'_\gamma r^{N_+-1} = 0$, $v'_\gamma \leq 0$ and $(|v'_\gamma|^\alpha v'_\gamma)' \geq 0$, so v_γ satisfies

$$|\nabla v_\gamma|^\alpha \mathcal{M}^+(D^2 v_\gamma) = -\frac{\Lambda \bar{\lambda}_{var, \gamma} v_\gamma^{1+\alpha}}{\alpha + 1 r^\gamma}$$

In particular by the definition of $\bar{\lambda}_\gamma$, $\Lambda(\alpha + 1)^{-1} \bar{\lambda}_{var, \gamma} \leq \bar{\lambda}_\gamma$. Furthermore, analyzing the boundary condition, we get, by regularity, that $v_\gamma(1) = 0$ in the classical sense, and then, if one had $(\alpha + 1)^{-1} \Lambda \bar{\lambda}_{var, \gamma} < \bar{\lambda}_\gamma$ one would get by the maximum principle, since $v_\gamma = 0$ on $\partial B(0, 1)$, that $v_\gamma \leq 0$, a contradiction. \square

Corollary 3.4. *One has*

$$\lim_{\gamma \rightarrow 2+\alpha} \bar{\lambda}_\gamma(|\nabla \cdot |^\alpha \mathcal{M}^+) = \Lambda(\alpha + 1)^{-1} \left(\frac{(N_+ - 2)(\alpha + 1)}{\alpha + 2} \right)^{2+\alpha}.$$

Proof. By Corollary 3.3, it is sufficient to prove that $\bar{\lambda}_{\gamma, var} \rightarrow \bar{\lambda}_{2+\alpha, var}$ as $\gamma \rightarrow 2 + \alpha$.

We first observe that, by their own definition, $\bar{\lambda}_{2+\alpha, var} \leq \bar{\lambda}_{\gamma, var}$.

On the other hand, for any $\epsilon > 0$ there exists $v \in \mathcal{H}_0^{1+\alpha}$, $v \neq 0$ such that

$$\int_0^1 |v'|^{2+\alpha} r^{(N_+-1)(1+\alpha)} dr \leq (\bar{\lambda}_{2+\alpha, var} + \epsilon) \int_0^1 |v|^{2+\alpha} r^{(N_+-1)(1+\alpha)-2-\alpha} dr.$$

Take γ_0 sufficiently close to $2 + \alpha$ in order that, for $2 + \alpha > \gamma \geq \gamma_0$,

$$\int_0^1 |v|^{2+\alpha} r^{(N_+-1)(1+\alpha)-\gamma} dr \geq (1 - \epsilon) \int_0^1 |v|^{2+\alpha} r^{(N_+-1)(1+\alpha)-2-\alpha} dr.$$

Thus, one has

$$\int_0^1 |v'|^{2+\alpha} r^{(N_+-1)(1+\alpha)} dr \leq (\bar{\lambda}_{2+\alpha, var} + \epsilon)(1 - \epsilon)^{-1} \int_0^1 |v|^{2+\alpha} r^{(N_+-1)(1+\alpha)-\gamma} dr$$

which yields

$$\bar{\lambda}_{\gamma, var} \leq (\bar{\lambda}_{2+\alpha, var} + \epsilon)(1 - \epsilon)^{-1}.$$

\square

Corollary 3.5. *One has*

$$\bar{\lambda}_{2+\alpha}(|\nabla \cdot |^\alpha \mathcal{M}^+) = \frac{\Lambda}{1 + \alpha} \left(\frac{(N_+ - 2)(1 + \alpha)}{2 + \alpha} \right)^{2+\alpha}.$$

Proof. One has for $\gamma > 1$, $\gamma < 2 + \alpha$

$$\bar{\lambda}_{2+\alpha} \leq \bar{\lambda}_\gamma = \Lambda(\alpha + 1)^{-1} \bar{\lambda}_{var,\gamma} \rightarrow (\alpha + 1)^{-1} \Lambda \bar{\lambda}_{var,2+\alpha} = \Lambda(\alpha + 1)^{-1} \tau^{\alpha+2}$$

Furthermore since the function $u(r) = r^{-\tau}$ satisfies

$$|\nabla u|^\alpha \mathcal{M}^+(D^2 u) + \Lambda(\alpha + 1)^{-1} \tau^{\alpha+2} u^{1+\alpha} r^{-2-\alpha} = 0$$

one gets that $\bar{\lambda}_{2+\alpha} \geq \Lambda(\alpha + 1)^{-1} \tau^{\alpha+2}$. □

4 The case $\gamma > \alpha + 2$

We prove Theorem 1.5. Suppose that there exists a radial $\mathcal{C}_{1,\alpha}$ positive supersolution, of the inequation

$$|\nabla u|^\alpha \mathcal{M}^-(D^2 u) + \mu u^{1+\alpha} r^{-\gamma} \leq 0$$

with $\mu > 0$ and $\gamma > \alpha + 2$.

Acting as in the proof of Theorem 1.2 one easily obtains that u' does not change sign in a neighborhood of zero. We prove that the sign is ≤ 0 . Indeed, if we had $u' \geq 0$ one would have

$$\Lambda(|u'|^\alpha u')' + \lambda \frac{(N-1)(1+\alpha)}{r} |u'|^\alpha u' \leq -\mu(1+\alpha) u r^{-\gamma}$$

and then

$$(|u'|^\alpha u' r^{(N+1)(1+\alpha)})' \leq 0$$

hence $(|u'|^\alpha u' r^{(N+1)(1+\alpha)})$ has a limit when r goes to zero, if this limit was > 0 one would get that u becomes large negative near zero. Then $u' \leq 0$. Coming back to the inequation, whatever is the sign of $(|u'|^\alpha u')'$ one has

$$(|u'|^\alpha u')' + \frac{\Lambda(N-1)(1+\alpha)}{\lambda r} |u'|^\alpha u' \leq -\frac{\mu}{\Lambda} (1+\alpha) u r^{-\gamma} \quad (4.1)$$

we then have

$$(|u'|^\alpha u' r^{(N-1)(1+\alpha)})' \leq 0$$

which implies that $|u'|^\alpha u' r^{(N-1)(1+\alpha)}$ has a limit ≤ 0 . If it is < 0 , then there exists some constant $c > 0$ so that

$$u' \leq -c r^{1-N_-}$$

and integrating

$$u(r) \geq c r^{2-N_-}.$$

We prove that for all integer $j \geq 0$ such that $(N_- - 2)(1 + \alpha) + j(2 + \alpha - \gamma) > 0$ and for r sufficiently small, one has, for some $c_j > 0$,

$$u(r) \geq c_j r^{\frac{j(2+\alpha-\gamma)}{1+\alpha}}. \quad (4.2)$$

Indeed, (4.2) holds true for $j = 0$, since $u' < 0$ and u is positive. Let us suppose that (4.2) is true for j and that $N_- - 2 + (j + 1)\frac{2+\alpha-\gamma}{1+\alpha} > 0$. Then, by (4.2),

$$\begin{aligned} -|u'|^\alpha u'(r)r^{(N_- - 1)(1+\alpha)} &\geq \frac{\mu}{\Lambda} c_j \int_0^r s^{(N_- - 1)(1+\alpha) + j(2+\alpha-\gamma) - \gamma} ds \\ &= \frac{\mu}{\Lambda} \frac{c_j}{(N_- - 1)(1+\alpha) + j(2+\alpha-\gamma) + 1 - \gamma} r^{(N_- - 1)(1+\alpha) + j(2+\alpha-\gamma) + 1 - \gamma}, \end{aligned}$$

which implies

$$-u' \geq cr^j \frac{2+\alpha-\gamma}{1+\alpha} + \frac{1-\gamma}{1+\alpha} \quad (4.3)$$

which, by integration, yields (4.2) for $j + 1$.

Now, let us assume that $\frac{(N_- - 2)(1+\alpha)}{\gamma - 2 - \alpha}$ is not integer. Then, using estimate (4.2) with $j = \left\lceil \frac{(N_- - 2)(1+\alpha)}{\gamma - 2 - \alpha} \right\rceil$ jointly with (4.1), we deduce for $r_0 > r > 0$

$$\begin{aligned} -|u'(r_0)|^\alpha u'(r_0)r_0^{(N_- - 1)(1+\alpha)} + |u'(r)|^\alpha u'(r)r^{(N_- - 1)(1+\alpha)} \\ \geq \frac{\mu c_j}{\Lambda ((N_- - 2)(1+\alpha) + (j + 1)(2+\alpha-\gamma))} \left[s^{(N_- - 2)(1+\alpha) + (j + 1)(2+\alpha-\gamma)} \right]_r^{r_0}. \end{aligned}$$

Since $(N_- - 2)(1+\alpha) + (j + 1)(2+\alpha-\gamma) < 0$, this yields the contradiction

$$\lim_{r \rightarrow 0} |u'(r)|^\alpha u'(r)r^{(N_- - 1)(1+\alpha)} = +\infty.$$

On the other hand, if $\frac{(N_- - 2)(1+\alpha)}{\gamma - 2 - \alpha} = j + 1$ is integer, then $(N_- - 2)(1+\alpha) + j(2+\alpha-\gamma) = \gamma - 2 - \alpha > 0$, and from (4.2) it follows that

$$u(r) \geq c_j r^{\frac{\gamma+\alpha}{1+\alpha} - N_-},$$

hence

$$u^{1+\alpha}(r)r^{(N_- - 1)(1+\alpha) - \gamma} \geq c_j r^{-1}.$$

From (4.1) we then deduce, for $0 < r < r_0$,

$$-|u'(r_0)|^\alpha u'(r_0)r_0^{(N_- - 1)(1+\alpha)} + |u'(r)|^\alpha u'(r)r^{(N_- - 1)(1+\alpha)} \geq \frac{\mu c_j}{\Lambda} (\ln r_0 - \ln r)$$

and we reach, also in this case, the contradiction

$$\lim_{r \rightarrow 0} u'(r)r^{N_- - 1} = +\infty.$$

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