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Economic growth with brown or green capital*

Stefano Bosi[†], Cuong Le Van[‡], Giang Phung[§]

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Abstract

This paper explores a growth model with capital that can be "brown" or "green," impacting other firms' total factor productivity (TFP). Brown capital acts as a negative externality, lowering productivity, while green capital, either human or natural, spills over and generates positive externalities, boosting productivity. We prove the existence of a competitive equilibrium with these externalities and demonstrate the occurrence of two-period cycles through local and global stability analysis. Notably, cycles in the case of negative externalities require strong intertemporal income effects, while positive externalities lead to cycles with significant substitution effects.

Keywords: Romer model, sustainable growth, brown and green capital externalities, competitive equilibrium with externalities, local and global stability.

JEL codes: C62, O44.

1 Introduction

Economic growth has long been considered the engine of human progress, enabling rising living standards, technological advancement, and poverty reduction. However, the current growth paradigm faces a critical challenge: sustainability. Global warming and environmental degradation significantly threaten long-term economic growth and human well-being. Traditional growth models often treat capital accumulation as a homogenous process, neglecting its impact on the environment. Such a myopic perspective has led to a situation where economic prosperity is increasingly threatened by climate change, resource depletion, and environmental degradation. Our paper addresses this critical gap by proposing a new framework for analyzing growth that explicitly considers the

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environmental dimension of capital accumulation. We introduce a one-sector growth model where capital can be categorized as "brown" or "green."

Brown capital represents traditional, polluting forms of capital that generate negative externalities such as environmental damage, reducing other firms' total factor productivity (TFP). Market failures caused by negative externalities promote private investment that ignores climate damage (Rezai, 2011). Existing data paints a concerning picture. From 1850 to 2022, the yearly carbon footprint has multiplied by almost 14 (Jones et al., 2023), the population by more than 6 (Gapminder, 2023 and UN, 2022), and therefore, the individual footprint more than doubled. This rapid growth is primarily driven by fossil fuel dependence, which contributes significantly to greenhouse gas (GHG) emissions (IPCC, 2007). The consequences are severe, potentially leading to a sixth mass extinction event (Barnosky et al., 2011; Ceballos et al., 2016). Economic sectors contribute unevenly to GHG emissions, where factories relying on fossil fuels or industrial processes that release harmful pollutants are the heaviest contributors, accounting for over two-thirds of global emissions. WRI's 2023 report highlights that electricity and heat generation (31%), ground transport (16.51%), manufacturing and construction (12.34%), and agriculture (11.34%), are the biggest polluters. These sectors rely heavily on brown capital, generating negative externalities that reduce the productivity of other firms. The negative impacts of pollution on health, agriculture, and overall economic efficiency are well documented (Xepapadeas, 2005).

Conversely, green capital encompasses knowledge-based and natural capital stocks that create positive externalities, boosting their TFP. Human capital, embodied in education, skills, and innovation, fosters positive externalities through knowledge spillovers from research and development (Romer, 1986), investments in renewable energy, or sustainable resource management practices. Similarly, natural capital, comprising forests, clean water, and biodiversity, provides essential ecosystem services that enhance overall productivity (UNEP, 2021). Green capital investments can lead to cleaner production processes, resource efficiency, and reduced reliance on fossil fuels.

Several theoretical studies have explored the interplay between environmental considerations and economic growth. Gradus and Smulders (1993) investigate how environmental concerns influence technology selection when pollution arises as a side-product of physical capital accumulation. Acemoglu et al. (2012) focus on the dirty sector's negative impact on environmental quality. Boucekkinne et al. (2013) introduce pollution dynamics into a two-sector AK model and calculate the optimal timing for switching from a dirtier technology to a cleaner one. Similarly, Acemoglu et al. (2016) analyze the transition to cleaner technology. However, these studies often focus on models with distinct brown and green capital stocks competing in production, along with the question of optimal switching between them. A key limitation of the existing literature is its reliance on numerical solutions rather than analytical ones. For instance, Rezai (2011) integrates greenhouse gas externalities into the DICE-07 model, showing optimal climate policy can boost welfare: the internalization of climate damage, lowering the private return to capital, favors mitigation and consumption.

Similarly, Rezai et al. (2012) explore how global warming's negative externality can be mitigated, using simulations to show potential Pareto gains from investments in mitigation strategies. Hoel et al. (2019) perform simulations using the RICE-10 model (Nordhaus 2010), suggesting that well-designed international climate agreements, optimizing investments and emissions to distribute welfare across generations, can achieve Pareto improvements (no one loses compared to inaction).

In contrast, our paper adopts a different approach. We present a single-sector market economy model where the capital can be "brown" or "green" depending on its externalities. Negative externalities represent pollution (brown capital), while positive externalities reflect knowledge spillovers, human capital accumulation, or natural capital benefits (green capital). In this respect, our model is more general, allowing us to encompass both negative and positive externalities within a single model. Moreover, we prioritize analytical solutions over solely numerical approaches, offering a clearer understanding of the model's dynamics. This paper deliberately excludes the demographic transition. While population growth undoubtedly impacts sustainability, its inclusion necessitates a more complex and hybrid modeling approach beyond our single-sector framework. Furthermore, according to Liu and Turnovsky (2005), production externalities have a widespread impact, causing persistent distortions in the long-term economic equilibrium regardless of the elasticity of labor supply. Our simplified model allows for a clear examination of the core interplay between capital and externalities, paving the way for future research that integrates additional complexities.

Building upon the literature's gaps, our paper tackles two crucial aspects: (1) the proof of the existence of competitive equilibrium with both positive and negative externalities, and (2) the analytical characterization of local and global dynamics in such scenarios.

(1) The existence of a competitive equilibrium with positive externalities has received limited attention in the literature. We consider works most relevant to ours. Romer (1986) presents a continuous-time infinite-horizon model where private firms use knowledge as a productive input, but the total knowledge stock in the economy positively impacts all firms' outputs. Additionally, the knowledge growth rate depends on the investment rate and can be positive in the long run. Exploring positive externalities and their impact on growth, Le Van et al. (2002) provide a detailed but simple proof of equilibrium existence in a discrete-time version of Romer's model. Similarly, Uzawa (1965) and Lucas (1988) study models where externalities stem from human capital accumulation alongside physical capital. While d'Albis and Le Van (2006) focus solely on effective labor as input, they establish sufficient and necessary conditions for a unique competitive equilibrium with human capital externalities.

Key differences distinguish our work from these studies. In Romer (1986) and Le Van et al. (2002), the positive externality arises from the existing knowledge stock, which depends on the investment rate. Greiner and Semmler (2002) also argue that investment generates positive externalities, promoting knowledge capital accumulation through education, explaining sustained per capita

growth as an endogenous phenomenon driven by constant or increasing returns on capital, supported by empirical evidence. Similarly, for Uzawa (1965) and Lucas (1988), the externality is linked to human capital accumulation based on worker training time. Klenow and Rodriguez-Clare (2005) incorporate international knowledge spillover as externalities in their growth model, suggesting diffusion drives most income above subsistence. From an empirical perspective, Costantini and Monni (2008) show that investment in human capital (education and health) outweighs resource dependence in achieving and sustaining higher future consumption levels. These studies integrate positive externalities on the TFP from knowledge capital into growth models. We extend this framework by incorporating both negative and positive externalities, arising from brown (pollution) and green capital (knowledge spillovers) respectively, impacting the total factor productivity (TFP). Unlike Romer’s model, which exhibits endogenous growth, we focus on bounded externalities within a framework that captures the diminishing returns to capital accumulation observed in the real world. This allows for a more nuanced and realistic representation of the complex relationship between capital accumulation, environmental impact, and economic activity. Furthermore, we analyze the non-linear dynamics of capital (physical or human) through the lens of bifurcations, a topic not addressed in the literature mentioned above.

(2) The influence of externalities on growth and stability properties has been widely studied in the literature. Romer (1986) laid the foundation for the endogenous growth literature with the assumption of positive externalities. Further contributions exploring endogenous cycles and self-fulfilling prophecies under such conditions include seminal works by Benhabib and Farmer (1994) and Boldrin and Rustichini (1994). Withagen (1995) highlights how stock externalities can lead to unbalanced growth patterns compared to traditional models and emphasizes the role of abatement technology in influencing long-term growth rates, without providing a general model. Antoci (2005) analyzes growth dynamics in an economy with leisure, environmental goods, and private goods, finding that negative externalities on agents from production can drive growth through exploiting positive externalities (knowledge accumulation), but coordination failures can lead to undesirable growth paths (either a “poverty trap” or a “private consumption trap”). Regarding negative externalities (pollution), Xepapadeas (2005) provides a comprehensive survey in the *Handbook of Environmental Economics*. Yörük (2007) displays a potential bias in growth analysis in an empirical study of 28 OECD countries, where ignoring negative externalities (CO2 emissions) can lead to an overestimation of convergence for low-income countries. This focus on short-term output growth may overlook environmental and social costs, hindering sustainable development. In an alternative perspective, Bartolini (2003) proposes that economic growth reflects a progressive substitution of free (environmental) resources with costly alternatives (private goods), prompting increased work hours to offset negative externalities, and suggests exploring the implications for environmental economics, particularly the unexplained high work effort but lack of happiness despite income growth, in future research. However, a crucial gap exists in this literature: the proof

of equilibrium existence with negative externalities. While stability properties have been analyzed, the challenging issue of equilibrium existence proofs is often missing. We also highlight the limitations of existing overlapping generations (OG) models in environmental economics. Existence proofs for positive externalities remain elusive in OG models (Howarth, 1998; Dao and Davila, 2014; Karp and Rezai, 2014). Our paper addresses this gap by rigorously demonstrating the existence of a competitive equilibrium in a model with both positive and negative externalities, laying the groundwork for further analysis of sustainable growth dynamics. We demonstrate the emergence of two-period cycles in a simple Ramsey-style setting. Building upon the existing literature on the stability of growth models with externalities (e.g., Benhabib and Farmer, 1994; Boldrin and Rustichini, 1994), we show how the interplay between externalities and income-substitution effects can lead to the emergence of two-period cycles. These cycles occur under significant intertemporal income effects for negative externalities and substantial substitution effects for positive externalities. In our general equilibrium stability analysis, this result holds for both hyperbolic and locally isoelastic externality functions, highlighting the robustness of our findings.

By establishing a robust foundation for further investigation in analyzing growth models with brown and green capital, this model contributes to stimulating further research that can guide policymakers toward strategies that promote sustainable economic development while mitigating environmental damage. To fully explore the model's potential, future research should further investigate the empirical significance of negative externalities and their magnitude, which is crucial for calibrating the model to real-world data. Additionally, justifying the assumption of bounded externalities requires further theoretical development.

The rest of the paper is organized as follows. Section 2 establishes the model, focusing on a basic one-sector market economy and how brown or green capital influences growth. Sections 3 and 4 tackle the crucial challenge of proving equilibrium existence under these conditions. Section 5 explores general results concerning equilibrium stability. To gain further insights, Section 6 assumes hyperbolic externalities and analyzes the stability properties, particularly the possibility of two-period cycles. Section 7 examines locally isoelastic externalities and demonstrates the robustness of two-period cycles within the model. Finally, Section 8 concludes the paper, summarizing the key contributions and outlining potential avenues for future research. All the proof is gathered in the Appendix.

2 Brown or green capital

We consider the Cobb-Douglas production function of firm j with negative or positive aggregate externalities: $F(K_{jt}, L_{jt}) \equiv A(\bar{K}_t) K_{jt}^\alpha L_{jt}^{1-\alpha}$.

Definition 1 Capital \bar{K}_t is said to be: (1) brown when the externality is negative (pollution): $A'(\bar{K}_t) < 0$; (2) green, when the externality is positive (knowledge): $A'(\bar{K}_t) > 0$.

The production per capita is defined as usual

$$f(k_t) \equiv \frac{F(K_t, L_t)}{L_t} = A(\bar{K}_t) k_t^\alpha \quad (1)$$

where $k_t \equiv K_t/L_t$ is the capital intensity.

In order to prove the existence of a competitive equilibrium, we introduce a plausible assumption of bounded external effects. We observe that, for now, we do not require any monotonicity hypothesis. Importantly, Definition 1 is also useful to study the stability issue (section 5).

Assumption 1 The Total Factor Productivity (TFP) is bounded: $0 < a_0 \leq A(\bar{K}_t) \leq a_1 < \infty$ for any $\bar{K}_t > 0$.

Consider a sequence of externalities: $(\bar{K}_t)_{t=1}^\infty$, and, given this sequence, the maximization problem $P(K_0, (\bar{K}_t)_{t=1}^\infty)$ under a sequence of resource constraints:

$$\begin{aligned} \max \sum_{t=0}^{\infty} \beta^t u(c_t) \\ c_t + k_{t+1} \leq A(\bar{K}_t) k_t^\alpha \end{aligned} \quad (2)$$

for any $t \geq 0$. These resource constraints are the reduced forms of the consumer's budget constraints:

$$c_t + k_{t+1} - (1 - \delta) k_t \leq r_t k_t + w_t \quad (3)$$

where, for the sake of simplicity, labor supply is supposed to be inelastic and equal to one, capital depreciation to be full ($\delta = 1$) and capital and labor prices (r_t, w_t) to be determined by a standard profit maximization (firm by firm):

$$\max [A(\bar{K}_t) K_{jt}^\alpha L_{jt}^{1-\alpha} - r_t K_{jt} - w_t L_{jt}] \quad (4)$$

that is by its first-order conditions: $r_t = \alpha A(\bar{K}_t) k_t^{\alpha-1}$ and $w_t = (1 - \alpha) A(\bar{K}_t) k_t^\alpha$.

We observe that the capital intensity $k_t = K_{jt}/L_{jt}$ turns out to be the same for any firm j and coincides with the individual wealth in the consumer's constraint (3). We notice also that \bar{K}_t is not a choice variable. Each producer takes it as given.

Maximizing the intertemporal utility under the sequence of budget constraints: $c_t + k_{t+1} \leq r_t k_t + w_t$, is equivalent to maximizing under the sequence of resources constraints: $c_t + k_{t+1} \leq A(\bar{K}_t) k_t^\alpha$.

Assumption 2 u is increasing, strictly concave and continuously differentiable with $u(0) \geq -\infty$ and $u'(0) = +\infty$.

For instance, this assumption is satisfied by the popular class of isoelastic utility functions:

$$u(c) = \frac{c^{1-1/\sigma}}{1-1/\sigma} \text{ if } \sigma \neq 1 \quad (5)$$

$$u(c) = \ln c \text{ if } \sigma = 1 \quad (6)$$

where $\sigma > 0$ denotes the constant elasticity of intertemporal substitution. We observe that, in the case of dominant income effects ($\sigma < 1$) or balanced income and substitution effects ($\sigma = 1$), $u(0^+) \equiv \lim_{c \rightarrow 0^+} u(c) = -\infty$, that is the utility function is unbounded from below.

3 Existence of an optimal capital path in a growth model with capital externalities

Market clearing requires $C_t + K_{t+1} = A(\bar{K}_t) K_t^\alpha L_t^{1-\alpha}$ where capital letters denote the aggregate variables. We assume that all agents work and total population is normalized to one: $L_t = 1$ for any t . Hence, we recover our resource constraint with equality: $c_t + k_{t+1} L_{t+1}/L_t = c_t + k_{t+1} = A(\bar{K}_t) k_t^\alpha$. Since $L_t = 1$, let $\bar{k}_t \equiv \bar{K}_t/L_t = \bar{K}_t$ for any t in the following and, thus, $(\bar{k}_t)_{t=1}^\infty = (\bar{K}_t)_{t=1}^\infty$ with $k_0 = K_0$.

We reformulate the initial program as a Problem $P(k_0, (\bar{k}_t)_{t=1}^\infty)$ with $k_0 > 0$ and $\bar{k}_t \geq 0$ for any $t > 0$: $\max \sum_{t=0}^\infty \beta^t u(c_t)$ subject to $c_t + k_{t+1} \leq A(\bar{k}_t) k_t^\alpha$, $c_t \geq 0$ and $k_{t+1} \geq 0$ for any $t \geq 0$ with $0 < \alpha < 1$.

Definition 2 $(k_t)_{t=0}^\infty$ is a feasible sequence if $0 \leq k_{t+1} \leq A(\bar{k}_t) k_t^\alpha$ for any $t \geq 0$.

Since $0 < \alpha < 1$, Assumption 1 implies

$$k_t \leq m \equiv \max\{k_0, \kappa_1\} \quad (7)$$

for any $t \geq 0$ where $\kappa_1 \equiv a_1^{1/(1-\alpha)}$.

When $u(0^+) = -\infty$, the function $(k_t)_{t=0}^\infty \mapsto \sum_{t=0}^\infty \beta^t u(A(\bar{k}_t) k_t^\alpha - k_{t+1})$ is upper semi-continuous for the product topology.¹ When $u(0) > -\infty$, this function becomes continuous.²

Lemma 3 For any initial condition $k_0 > 0$ and any sequence $(\bar{k}_t)_{t=1}^\infty$, there exists $\zeta > 0$ such that: $0 < \zeta < A(k_0) k_0^\alpha$ and $0 < \zeta < A(\bar{k}_t) \zeta^\alpha$ for any $t > 0$. In particular, $a_0 \zeta^\alpha - \zeta > 0$. The stationary sequence (ζ, ζ, \dots) is feasible for Problem $P(k_0, (\bar{k}_t)_{t=1}^\infty)$.

¹See Lemma 5.2.3 at page 103 in Le Van and Dana (2002).

²Apply step 3 at pages 17-18 in the proof of Proposition 2.2.1 in Le Van and Dana (2002) to the new function $v(x) \equiv u(x) - u(0)$.

Lemma 4 For any $\varepsilon > 0$, there exists T_ε such that $\lim_{\varepsilon \rightarrow 0} T_\varepsilon = \infty$ and, for any $k_0 > 0$, there is a feasible stationary sequence (ζ, ζ, \dots) with:

- (0) $0 < \zeta < A(k_0) k_0^\alpha$,
- (1) $0 < \zeta < A(\bar{k}_t) \zeta^\alpha$ for any $1 \leq t \leq T_\varepsilon$,
- (2) $0 < \zeta < a_0 \zeta^\alpha - \zeta \leq A(\bar{k}_t) \zeta^\alpha - \zeta$ for any $t > T_\varepsilon$.

Definition 5 $Q(k_0)$ is a set of bounded sequences:

$$Q(k_0) \equiv \{(k_t)_{t=1}^\infty : 0 \leq k_t \leq m \text{ for any } t > 0\}$$

We observe that the set of feasible sequences is a subset of $Q(k_0)$.

Lemma 6 The set $Q(k_0)$ is compact for the product topology and convex.

Let $(k_t^*)_{t=1}^\infty$ denote the optimal sequence of Problem $P(k_0, (\bar{k}_t)_{t=1}^\infty)$, solution to $\max \sum_{t=0}^\infty \beta^t u(A(\bar{k}_t) k_t^\alpha - k_{t+1})$ subject to $0 \leq k_{t+1} \leq A(\bar{k}_t) k_t^\alpha$ for any $t \geq 0$, given $k_0 > 0$.

Proposition 7 Problem $P(k_0, (\bar{k}_t)_{t=1}^\infty)$ has a unique optimal solution

$$(k_{t+1}^* ((\bar{k}_t)_{t=1}^\infty), c_t^* ((\bar{k}_t)_{t=1}^\infty))_{t=0}^\infty$$

given the initial condition k_0 .

It is straightforward to prove that $(k_t^*)_{t=1}^\infty \in Q(k_0)$.

Definition 8 The mapping $\varphi : Q(k_0) \rightarrow Q(k_0)$ is the optimal sequence of capital stocks: $(\bar{k}_t)_{t=1}^\infty \in Q(k_0) \mapsto \varphi((\bar{k}_t)_{t=1}^\infty) = (k_t^*)_{t=1}^\infty \in Q(k_0)$.

Recalling that $0 \leq k_{t+1}^* \leq A(\bar{k}_t) k_t^{*\alpha}$ for any $t \geq 0$, we observe that

$$\sum_{t=0}^\infty \beta^t u(A(\bar{k}_t) k_t^{*\alpha} - k_{t+1}^*) \geq \sum_{t=0}^\infty \beta^t u(A(\bar{k}_t) \zeta^\alpha - \zeta) \geq \sum_{t=0}^\infty \beta^t u(a_0 \zeta^\alpha - \zeta) = \omega > -\infty$$

Continuity of φ is key.

Lemma 9 φ is a continuous mapping for the product topology.

This lemma deserves a few remarks.

If $(k_t)_{t=1}^\infty$ is feasible, then the expression $\sum_{t=0}^\infty \beta^t u(A(\bar{k}_t) k_t^\alpha - k_{t+1})$ exists, that is it is real-valued or equal to $-\infty$. To show this point, let $T \equiv \{t : u(A(\bar{k}_t) k_t^\alpha - k_{t+1}) \geq 0\}$. Since $0 \leq k_{t+1} \leq A(\bar{k}_t) k_t^\alpha \leq a_1 k_t^\alpha$, we have $0 \leq k_{t+1} \leq a_1 m^\alpha$ for any $t \geq 0$. Hence, $0 \leq \sum_{t \in T} \beta^t u(A(\bar{k}_t) k_t^\alpha - k_{t+1}) \leq \sum_{t \in T} \beta^t u(a_1 m^\alpha)$ and $\sum_{t \in T} \beta^t u(A(\bar{k}_t) k_t^\alpha - k_{t+1})$ is a non-negative real value. In this case,

$$\sum_{t=0}^\infty \beta^t u(A(\bar{k}_t) k_t^\alpha - k_{t+1}) = \sum_{t \in T} \beta^t u(A(\bar{k}_t) k_t^\alpha - k_{t+1}) + \sum_{t \notin T} \beta^t u(A(\bar{k}_t) k_t^\alpha - k_{t+1})$$

and it exists since $\sum_{t \neq T}^{\infty} \beta^t u(A(\bar{k}_t) k_t^\alpha - k_{t+1}) = -\infty$ or to a negative real value, as sum of negative terms.

When the production function is $A(k_t) k_t^\alpha$ with $A(k_t) \leq a_1$ for any k , Assumption H3 at page 102 in Le Van and Dana (2002) is satisfied. Then, the function defined by $(k_t)_{t=1}^{\infty}$ feasible $\mapsto \sum_{t=0}^{\infty} \beta^t u(A(\bar{k}_t) k_t^\alpha - k_{t+1})$ is upper semi-continuous for the product topology.³

Moreover, since the stationary sequence (ζ, ζ, \dots) is feasible with $(\tilde{k}_t)_{t=1}^{\infty}$ and $(k_t^*)_{t=1}^{\infty}$ is optimal, we have

$$\sum_{t=0}^{\infty} \beta^t u\left(A(\tilde{k}_t) k_t^{*\alpha} - k_{t+1}^*\right) \geq \sum_{t=0}^{\infty} \beta^t u(a_0 \zeta^\alpha - \zeta) > -\infty$$

Finally, by applying a standard fixed-point theorem (Brouwer), we obtain the existence of a competitive equilibrium with externalities.

Theorem 10 *Since φ maps $Q(k_0)$ into $Q(k_0)$, there exists a fixed point $(k_t^*)_{t=0}^{\infty}$.*

In the next section, we will prove that, actually, this fixed point is the competitive equilibrium with externalities $(k_t^*)_{t=0}^{\infty}$ we are looking for.

4 Existence of a competitive equilibrium with externalities

Let $(\tilde{k}_t)_{t=0}^{\infty}$ denote the fixed points of φ and $\tilde{c}_t = A(\tilde{k}_t) \tilde{k}_t^\alpha - \tilde{k}_{t+1}$ be the associated consumption.

We want to show that there exists a sequence of prices $(\tilde{p}_t)_{t=0}^{\infty} \geq 0$ verifying $\sum_{t=0}^{\infty} \tilde{p}_t < \infty$ such that $(q^*, (\tilde{p}_t, \tilde{c}_t, \tilde{k}_{t+1})_{t=0}^{\infty})$ with initial condition k_0 , is a competitive equilibrium, that is the following three points hold.

(1) π^* is the maximal profit of firm:

$$\pi^* \equiv \max_{(k_t)_{t=1}^{\infty}} \left(\sum_{t=0}^{\infty} \tilde{p}_t \left[A(\tilde{k}_t) k_t^\alpha - k_{t+1} \right] - q^* k_0 \right)$$

(2) $(\tilde{c}_t)_{t=0}^{\infty}$ solves: $\max \sum_{t=0}^{\infty} \beta^t u(c_t)$ subject to $\sum_{t=0}^{\infty} \tilde{p}_t c_t \leq \pi^* + q^* k_0$.

(3) Markets clear: $\tilde{c}_0 + \tilde{k}_1 = A(k_0) k_0^\alpha$ and $\tilde{c}_t + \tilde{k}_{t+1} = A(\tilde{k}_t) \tilde{k}_t^\alpha$ for any $t > 0$.

Observing that $\tilde{k}_t \leq m$ for any t , where m is given by (7), we define $\mu \equiv m^{1-\alpha} / (\alpha a_0)$.

Theorem 11 *Suppose $\mu < 1$. Given the initial condition k_0 , $(\tilde{p}_t, \tilde{c}_t, \tilde{k}_{t+1})_{t=0}^{\infty}$, with $\tilde{p}_t = \beta^t u'(\tilde{c}_t)$ for any t , is a competitive equilibrium with externalities.*

³See again Lemma 5.2.3 at page 103 in Le Van and Dana (2002).

5 Dynamics around the steady state

Theorem 11 with, now, $k_t = \bar{k}_t$ for any $t > 0$ (as in Romer, 1986), paves the way to compute the equilibrium system. Market clearing and Euler condition (equation (46) in the proof of Theorem 11) form a two-dimensional dynamic system:

$$c_t + k_{t+1} = A(k_t) k_t^\alpha \quad (8)$$

$$\frac{u'(c_t)}{u'(c_{t+1})} = \alpha\beta A(k_{t+1}) k_{t+1}^{\alpha-1} \quad (9)$$

The capital intensity of steady state is given by

$$\frac{A(k) k^\alpha}{k} = \frac{1}{\alpha\beta} \quad (10)$$

and the consumption level by

$$\frac{c}{k} = \frac{A(k) k^\alpha}{k} - 1 = \frac{1 - \alpha\beta}{\alpha\beta} \quad (11)$$

We introduce the elasticity of externalities: $\varepsilon(k) \equiv kA'(k)/A(k)$, and the elasticity of intertemporal substitution: $\sigma(c) \equiv -u'(c)/[cu''(c)] > 0$. According to Definition 1, capital is brown if $\varepsilon(k) < 0$, green if $\varepsilon(k) > 0$.

Lemma 12 *The linearization of system (8)-(9) around the steady state is given by*

$$\begin{bmatrix} \frac{dk_{t+1}}{k} \\ \frac{dc_{t+1}}{c} \end{bmatrix} = J \begin{bmatrix} \frac{dk_t}{k} \\ \frac{dc_t}{c} \end{bmatrix} \quad (12)$$

where J is the Jacobian matrix with trace and determinant

$$T = 1 + \frac{\alpha + \varepsilon(k)}{\alpha\beta} + \sigma(c) [1 - \alpha - \varepsilon(k)] \left(\frac{1}{\alpha\beta} - 1 \right) \quad (13)$$

$$D = \frac{\alpha + \varepsilon(k)}{\alpha\beta} \quad (14)$$

Focus now on the two-dimensional space spanned by the eigenvectors associated to the eigenvalues λ_1 and λ_2 . We can study the stability properties considering $T \equiv \lambda_1 + \lambda_2$ and $D \equiv \lambda_1\lambda_2$ instead of the eigenvalues λ_1 and λ_2 . Using a geometrical method introduced by Samuelson (1942) and developed by Grandmont et al. (1998), we can represent these properties in the (T, D) -plane

instead of the Argand plan (Figure 1).

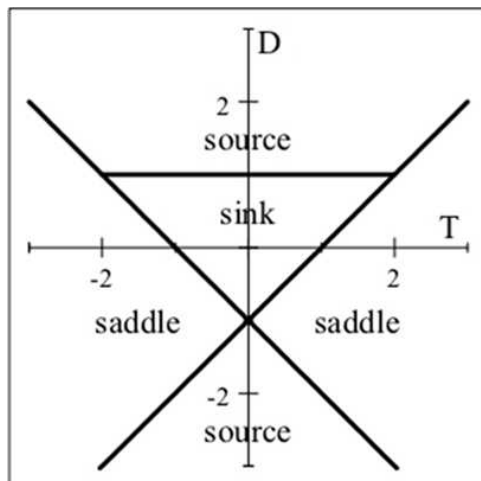


Fig. 1 Stability triangle

We evaluate the characteristic polynomial $P(\lambda) \equiv \lambda^2 - T\lambda + D$ at -1 and 1 : along the line $D = T - 1$, one eigenvalue is equal to 1 because $P(1) = 1 - T + D = 0$; along the line $D = -T - 1$, one eigenvalue is equal to -1 because $P(-1) = 1 + T + D = 0$. Along the segment $D = 1$ with $|T| < 2$, the eigenvalues are nonreal and conjugate with unit modulus. Focus on points outside these lines and the segment. Inside the triangle, the steady state is a sink ($D < 1$ and $|T| < 1 + D$). It is a saddle point if (T, D) lies on the left cone of the lines $D = T - 1$ and $D = -T - 1$, or on the right cone of these lines ($|1 + D| < |T|$). It is a source otherwise. A local bifurcation arises when one eigenvalue crosses the unit circle in the Argand plan or, equivalently, the pair (T, D) goes through one of the lines $D = T - 1$ or $D = -T - 1$, or the segment $D = 1$ with $|T| \leq 2$. When a bifurcation parameter changes, (T, D) moves in the (T, D) -plane. Generically, a generic saddle-node occurs when (T, D) crosses $D = T - 1$, a flip when (T, D) goes through the line $D = -T - 1$, whereas a Neimark-Sacker when (T, D) crosses the segment $D = 1$ with $|T| \leq 2$.

The geometrical method introduced by Grandmont et al. (1998) allows for a complete characterization of local bifurcations if the locus obtained in the (T, D) -plane when a bifurcation parameter varies, is linear.

In the following, for simplicity, we consider isoelastic preferences.

Assumption 3 *The elasticity of intertemporal substitution is constant (expressions (5) and (6)).*

According to Assumption 3, in expressions (13) and (14), we have $\sigma(c) = \sigma$.

6 Hyperbolic externalities

Focus on the hyperbolic case:

$$A(\bar{K}_t) \equiv a + \frac{\eta}{1 + \bar{K}_t} \quad (15)$$

with $a > 0$.

6.1 Existence of a competitive equilibrium

Proposition 13 *If $\eta > -a$ and $\eta \neq 0$ (externalities matter), a competitive equilibrium exists.*

In this case, the Euler equation holds:

$$\frac{u'(c_t)}{u'(c_{t+1})} = \alpha\beta A(k_{t+1}) k_{t+1}^{\alpha-1} \quad (16)$$

and the steady state is given by the Modified Golden Rule:

$$\frac{1}{\alpha\beta} = A(k) k^{\alpha-1} \quad (17)$$

In the following, we consider separately the cases of negative and positive externalities.

6.2 Dynamics with negative externalities

6.2.1 Steady state

The externality parameter η is key. Since $A(k)$ depends on η , the steady state k depends on η as well: $k = k^*(\eta)$. The following proposition shows that, in the case of negative externalities, the steady state is an increasing function of η .

Lemma 14 *If $\eta > 0$ (negative externalities), the steady state $k^*(\eta)$, solution to*

$$\frac{k^{1-\alpha}}{\alpha\beta} = a + \frac{\eta}{1+k} \quad (18)$$

is a well-defined strictly increasing C^1 function of η from $k^(0) = (a\alpha\beta)^{1/(1-\alpha)}$ to $k^*(\infty) = \infty$.*

6.2.2 Local dynamics

The intuition suggests that, in the case of negative externalities, cycles take place because a rise in production today entails a rise in pollution which lowers the productivity tomorrow. The case $\eta > 0$ corresponds precisely to a negative impact $A'(k_t) < 0$ and to a negative elasticity of externalities $\varepsilon(k) < 0$.

As we will see, negative externalities are not sufficient to generate persistent fluctuations even if they generate damped oscillations. However, strong externalities and large intertemporal income effects (that is low elasticity of intertemporal substitution) promote the occurrence of persistent cycles.

Lemma 15 *If $\eta > 0$ (negative externalities), at the steady state, the elasticity of hyperbolic externalities (15) is a C^1 function $\tilde{\varepsilon}(\eta) \equiv \varepsilon(k^*(\eta))$, strictly decreasing from $\tilde{\varepsilon}(0) = 0$ to $\tilde{\varepsilon}(+\infty) = -1$.*

Reconsider the trace and the determinant with $\eta \geq 0$:

$$\begin{aligned} T(\eta) &\equiv 1 + \frac{\alpha + \tilde{\varepsilon}(\eta)}{\alpha\beta} + \sigma[1 - \alpha - \tilde{\varepsilon}(\eta)] \frac{1 - \alpha\beta}{\alpha\beta} \\ D(\eta) &\equiv \frac{\alpha + \tilde{\varepsilon}(\eta)}{\alpha\beta} \end{aligned}$$

where $\tilde{\varepsilon}(\eta) \equiv \varepsilon(k^*(\eta))$. We know that $\tilde{\varepsilon}(0) = 0$, $\tilde{\varepsilon}(\infty) = -1$ and $\tilde{\varepsilon}'(\eta) < 0$. Thus, moving η from 0 to ∞ is equivalent to moving $\tilde{\varepsilon}$ from 0 to -1 . Since $\tilde{\varepsilon}$ is continuously strictly decreasing, any bifurcation value $\tilde{\varepsilon}_B$ for $\tilde{\varepsilon}$ determines unambiguously the corresponding bifurcation value η_B for η : $\eta_B = \tilde{\varepsilon}^{-1}(\tilde{\varepsilon}_B)$.

Hence, we can study directly how the pair (\tilde{T}, \tilde{D}) with

$$\begin{aligned} \tilde{T}(\tilde{\varepsilon}) &\equiv 1 + \frac{\alpha + \tilde{\varepsilon}}{\alpha\beta} + \sigma(1 - \alpha - \tilde{\varepsilon}) \frac{1 - \alpha\beta}{\alpha\beta} \\ \tilde{D}(\tilde{\varepsilon}) &\equiv \frac{\alpha + \tilde{\varepsilon}}{\alpha\beta} \end{aligned}$$

moves when $\tilde{\varepsilon}$ decreases from 0 to -1 .

We need the following assumption to have positive bifurcation values.

Assumption 4 $\alpha < 1/(1 + \beta)$.

We introduce the following critical elasticities of intertemporal substitution

$$\sigma_0 \equiv \frac{2}{2 - \alpha} \frac{1 - \alpha(1 + \beta)}{1 - \alpha\beta} \quad (19)$$

$$\sigma_1 \equiv \frac{1}{1 - \alpha\beta} \quad (20)$$

$$\sigma_2 \equiv \frac{2}{1 - \alpha\beta} \quad (21)$$

Under Assumption 4, $\sigma_0 > 0$. It is easy to see that these critical elasticities are ranked:

$$0 < \sigma_0 < 1 < \sigma_1 < \sigma_2$$

In addition, we introduce the critical elasticity of externalities:

$$\tilde{\varepsilon}_F \equiv -\alpha - \frac{2\alpha\beta + \sigma(1 - \alpha\beta)}{2 - \sigma(1 - \alpha\beta)} \quad (22)$$

Lemma 16 *Under Assumption 4, cycles of period two generically arise through a flip bifurcation at $\tilde{\varepsilon} = \tilde{\varepsilon}_F$ provided that $\tilde{\varepsilon}_F \in (-1, 0)$ or, equivalently, $\sigma < \sigma_0$.*

Proposition 17 *Let Assumption 4 hold.*

If $\sigma < \sigma_0$, the steady state is a source if $\tilde{\varepsilon} \in (-1, \tilde{\varepsilon}_F)$ and a saddle point if $\tilde{\varepsilon} \in (\tilde{\varepsilon}_F, 0)$. The economic system generically undergoes flip and period-doubling bifurcations at $\tilde{\varepsilon} = \tilde{\varepsilon}_F \in (-1, 0)$ (expression (22)) and, in particular, two-period cycles arise. In terms of η , the flip bifurcation point is unambiguously given by $\eta_F = \tilde{\varepsilon}^{-1}(\tilde{\varepsilon}_F)$.

If $\sigma > \sigma_0$, the steady state is always a saddle point.

Remark 18 *Since $\sigma_0 < 1$, the steady state is always a saddle point when $\sigma > 1$ (positive utility).*

Oscillations are possible even when $\sigma > \sigma_0$ (saddle-point stability). Indeed, as shown in the proof of Proposition 17, the determinant becomes negative for low values of $\tilde{\varepsilon}$ (sufficiently close to -1). In this case, the eigenvalues are real because of the saddle configuration, but one becomes negative, while the other remains positive. This means that damped oscillations are possible along the stable branch.

Proposition 19 *The critical value η_F is given by*

$$\eta_F = (1 + k_F) \left(\frac{k_F^{1-\alpha}}{\alpha\beta} - a \right) \quad (23)$$

where k_F is solution to

$$\tilde{\varepsilon}_F = \frac{k}{1+k} \left(\frac{a\alpha\beta}{k^{1-\alpha}} - 1 \right) \quad (24)$$

It is interesting to notice that technology and preferences plays a role on the critical level of externalities. In particular, deriving expression (22), we have the following corollary.

Corollary 20 *The higher the elasticity of intertemporal substitution (dominant substitution effects), the larger the externalities needed to generate cycles:*

$$\frac{\partial \tilde{\varepsilon}_F}{\partial \sigma} = -\frac{2(1-\alpha\beta)(1+\alpha\beta)}{[2-\sigma(1-\alpha\beta)]^2} < 0$$

In other terms, cycles are more likely when agents smooth consumption over time with more difficulty.

6.2.3 Simulations

Let $\alpha = 1/3$, $\beta = 0.96$, $\sigma = 1/4$. In this case, Assumption 4 is satisfied ($\alpha = 1/3 < 1/(1+\beta) = 0.5102$). Using (19), we verify that $\sigma = 1/4 < \sigma_0 = 0.61176$. According to Proposition 17, there is room for a flip bifurcation. Therefore,

we can compute the critical value: $\tilde{\varepsilon}_F = -0.77596 \in (-1, 0)$. Generically, two-period cycles exist when $\tilde{\varepsilon}$ lies in a neighborhood of $\tilde{\varepsilon}_F$. The image of the interval $[-1, 0] \ni \tilde{\varepsilon}$ by the function (\tilde{T}, \tilde{D}) is the blue segment S in Figure 2 connecting the endpoints $E_{-1} \equiv (\tilde{T}(-1), \tilde{D}(-1)) = (-0.19792, -2.0833)$ and $E_0 = (\tilde{T}(0), \tilde{D}(0)) = (2.3958, 1.0417)$.

As you can see, the blue segment S crosses the red line of flip bifurcations $D = -T - 1$ precisely when $\tilde{\varepsilon} = \tilde{\varepsilon}_F = -0.77596$.

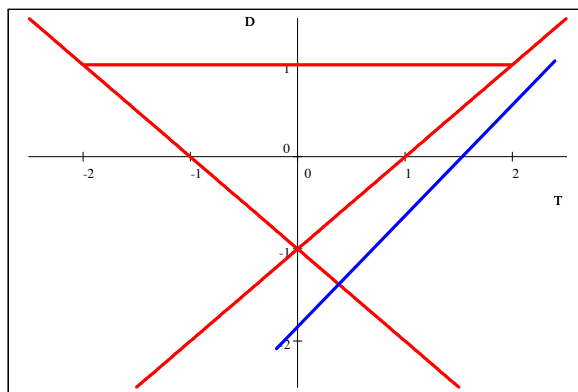


Fig. 2 Flip bifurcation

Applying Proposition 19, we obtain also the critical steady state and the critical value of the fundamental externality parameter: $k_F = 6.0688$ and $\eta_F = 66.427$.

When η is close to its critical value, a persistent two-period cycle arises around the steady state. When η steps aside, the cycle disappears and, instead, we observe damped oscillations and convergence to the steady state.

To illustrate these situations, we consider two values of the externality parameter:

$$\eta_1 = 30 < \eta_2 = 66.4 < \eta_F = 66.427$$

We observe that $\eta_2 = 66.4$ is sufficiently close to the bifurcation value $\eta_F = 66.427$ to generate a cycle.

(1) Let $\eta = \eta_1 = 30$. The steady state is given by (18): $k^* = 3.6659$. The dynamic system

$$\frac{c_{t+1}}{c_t} = \left[\alpha\beta \left(a + \frac{\eta}{1+k_{t+1}} \right) k_{t+1}^{\alpha-1} \right]^\sigma \quad (25)$$

$$c_t + k_{t+1} = \left(a + \frac{\eta}{1+k_t} \right) k_t^\alpha \quad (26)$$

can be rewritten as a second-order difference equation:

$$k_{t+2} = \left(a + \frac{\eta}{1 + k_{t+1}}\right) k_{t+1}^\alpha - \left[\left(a + \frac{\eta}{1 + k_t}\right) k_t^\alpha - k_{t+1}\right] \left[\alpha\beta \left(a + \frac{\eta}{1 + k_{t+1}}\right) k_{t+1}^{\alpha-1}\right]^\sigma \quad (27)$$

Consider a starting point $k_0 = 5$. Using the transition function (27) jointly with this initial and final conditions $k_0 = 5$ and $k_\infty = k^* = 3.6659$, we represent the converging discrete dynamics and its damped oscillations in Figure 3.

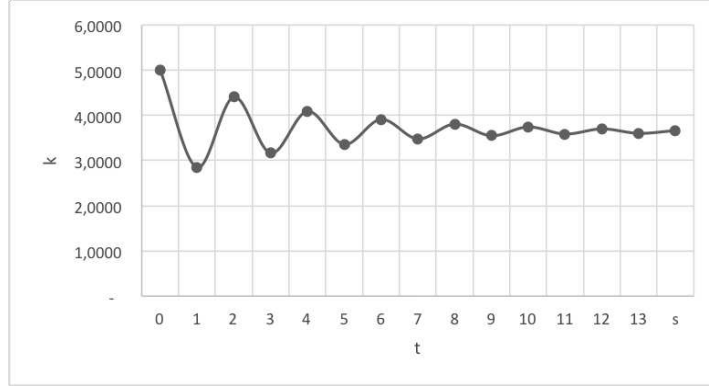


Fig. 3 Damped fluctuations for $\sigma = 1/4$

(2) Let $\eta = \eta_2 = 66.4$. The steady state is now given by $k^* = 6.067237454153$, according to (18). Dynamic system (25)-(26) can be rewritten as

$$k_{t+1} = \varphi(k_t, c_t) \quad (28)$$

$$c_{t+1} = \psi(k_t, c_t) \quad (29)$$

where

$$\varphi(k_t, c_t) \equiv \left(a + \frac{\eta}{1 + k_t}\right) k_t^\alpha - c_t$$

$$\psi(k_t, c_t) \equiv c_t \left(\alpha\beta \left[a + \frac{\eta}{1 + \left(a + \frac{\eta}{1 + k_t}\right) k_t^\alpha - c_t} \right] \left[\left(a + \frac{\eta}{1 + k_t}\right) k_t^\alpha - c_t \right]^{\alpha-1} \right)^\sigma$$

Focus on the second iterate, that is on

$$k_{t+2} = \varphi(k_{t+1}, c_{t+1}) = \varphi(\varphi(k_t, c_t), \psi(k_t, c_t))$$

$$c_{t+2} = \psi(k_{t+1}, c_{t+1}) = \psi(\varphi(k_t, c_t), \psi(k_t, c_t))$$

A cycle of period two is solution to the system

$$\tilde{k} \equiv k_t = k_{t+2}$$

$$\tilde{c} \equiv c_t = c_{t+2}$$

with $\tilde{k} \neq \varphi(\tilde{k}, \tilde{c})$ and $\tilde{c} \neq \psi(\tilde{k}, \tilde{c})$. More explicitly, we have to solve the system

$$\begin{aligned}\tilde{k} &= \left[a + \frac{\eta}{1 + \varphi(\tilde{k}, \tilde{c})} \right] \varphi(\tilde{k}, \tilde{c})^\alpha - \psi(\tilde{k}, \tilde{c}) \\ \tilde{c} &= \psi(\tilde{k}, \tilde{c}) \left[\alpha\beta \left(a + \frac{\eta}{1 + \tilde{k}} \right) \tilde{k}^{\alpha-1} \right]^\sigma\end{aligned}$$

or, equivalently, the reduced system

$$\begin{aligned}\left[\alpha\beta \left(a + \frac{\eta}{1 + \tilde{\varphi}} \right) \tilde{\varphi}^{\alpha-1} \right]^\sigma &= \frac{\left(a + \frac{\eta}{1 + \tilde{\varphi}} \right) \tilde{\varphi}^\alpha - \tilde{k}}{\left(a + \frac{\eta}{1 + \tilde{k}} \right) \tilde{k}^\alpha - \tilde{\varphi}} \\ (\alpha\beta)^2 \left(a + \frac{\eta}{1 + \tilde{k}} \right) \left(a + \frac{\eta}{1 + \tilde{\varphi}} \right) &= \left(\tilde{\varphi} \tilde{k} \right)^{1-\alpha}\end{aligned}$$

with respect to $(\tilde{k}, \tilde{\varphi})$ where $\tilde{\varphi} \equiv \varphi(\tilde{k}, \tilde{c})$.

This system has three solutions: the lower and the higher points of the cycle, k_L^* and k_H^* , and, clearly, the steady state between the two: $k^* \in (k_L^*, k_H^*)$.

Solving the system with $\eta = 66.4$ in a neighborhood of the flip bifurcation point, we get

$$k_L^* = 6.00012865853 < k^* = 6.067237454153 < k_H^* = 6.13502771443$$

Remark 21 *In our model, we obtain the critical flip bifurcation value by linearizing the non-linear system (25)-(26). This approximation entails a loss of information. In particular, the Jacobian matrix does not allow us to know whether the cycle is stable or not. In order to know the subcriticality (instability) or the supercriticality (stability) of the cycle, we would need to know the terms of higher order of the Taylor expansion, through complicated computations. Instead, we observe that, when $\eta = 0$, we recover the Ramsey model and the equilibrium is saddle-point stable. In other terms, the steady state is attractive along the one-dimensional stable branch. This branch is the center manifold where the bifurcation takes place. Increasing η to η_F generates a cycle around the steady state and along the center manifold when, according to our simulations, η enters a left neighborhood of η_F ($\eta = 66.4 < \eta_F$). We conjecture that, by continuity with respect to the Ramsey case, the attractive steady state in the left neighborhood of η_F is surrounded by a repulsive cycle, that is the flip bifurcation is subcritical. Additional simulations based on the second-order difference equation (27) seems to prove that this is indeed the case.*

Remark 22 *We observe that, generically, cycles of period 2^n with $n > 2$ (through period-doubling bifurcations) and, eventually, chaos are also possible in our model. To compute cycles of higher order, the interest reader has to find the stationary solutions to the two-dimensional iterate of order 2^n of system (28)-(29) and discard the stationary solutions of the two-dimensional iterate of order 2^{n-1} .*

6.3 Dynamics with positive externalities

Focus on the case $-a < \eta < 0$ (positive externalities). In this case, there is room for multiple steady state and, surprisingly, persistent cycles are still possible, but now, contrarily to the case of negative externalities, they arise under dominant intertemporal substitution effects (high elasticity σ).

6.3.1 Steady states

Proposition 23 *If $-a < \eta < 0$ (positive externalities), a steady state k^* , solution to (18), always exists.*

(1) *If*

$$a \leq \tilde{a} \equiv \frac{1}{\alpha\beta} \left(\frac{2-\alpha}{\alpha} \right)^\alpha \quad (30)$$

the steady state is unique.

(2) *If $a > \tilde{a}$, then there exist two distinct values η_2 and $\eta_1 > \max\{-a, \eta_2\}$ such that:*

(2.1) *if $\eta \in (\eta_2, \eta_1)$, there are three steady states,*

(2.2) *if $\eta \in \{\eta_2, \eta_1\}$, there are two steady states (non-generic case),*

(2.3) *if $\eta \notin [\eta_2, \eta_1]$, the steady state is unique.*

6.3.2 Local dynamics

Let $\sigma(c) = \sigma > 0$, a constant. We introduce the following critical values: $\varepsilon_P = 1 - \alpha$ and

$$\sigma_F \equiv \frac{2}{1-\alpha\beta} \frac{\varepsilon(k) + \alpha(1+\beta)}{\varepsilon(k) - (1-\alpha)} \quad (31)$$

We observe that the RHS of (31) does not depend on σ .

Proposition 24 (local dynamics) *When $0 < \varepsilon(k) < \varepsilon_P$, the steady state is a saddle point.*

When $\varepsilon_P < \varepsilon(k) < \varepsilon_F$, the steady state is a source.

When $\varepsilon_F < \varepsilon(k)$, the steady state is again a saddle point.

The following proposition shows the possibility of pitchfork and flip bifurcations.

Proposition 25 (pitchfork bifurcation) *If*

$$-a < \eta \leq -a \frac{1-\alpha}{1-\alpha + \left(\frac{\alpha}{2}\right)^2} \quad (32)$$

a pitchfork bifurcation (from one to three steady states) generically arises at $\beta = \beta_-$ and $\beta = \beta_+$, where

$$\beta_- \equiv \frac{\kappa_-^{1-\alpha}}{\alpha \left(a + \frac{\eta}{1+\kappa_-} \right)} \quad \text{and} \quad \beta_+ \equiv \frac{\kappa_+^{1-\alpha}}{\alpha \left(a + \frac{\eta}{1+\kappa_+} \right)}$$

with $0 < \kappa_- < \kappa_+$, provided that $0 < \beta_{\pm} < 1$. The expressions for κ_- and κ_+ are given in the proof.

Proposition 26 (cycles of period two) *If $\varepsilon(k) > 1 - \alpha$, a flip bifurcation generically arises at $\sigma = \sigma_F > 0$.*

It is easy to show that limit cycles are ruled out.

Proposition 27 (no limit cycles) *There is no room for Neimark-Sacker bifurcations.*

The Neimark-Sacker bifurcation is the discrete-time equivalent of the Hopf bifurcation in continuous time.

6.3.3 Simulations

Let $\alpha = 1/3$ and $\beta = 0.96$. In this case, Assumption 4 is satisfied ($\alpha = 1/3 < 1/(1 + \beta) = 0.5102$). We focus on the case $\sigma > \sigma_2 = 2.9412$. Let $\sigma = 6$. Equations (13) and (14) hold, that is

$$D = \frac{T}{1 - \sigma(1 - \alpha\beta)} - \frac{1}{\alpha\beta} \frac{\alpha\beta + \sigma(1 - \alpha\beta)}{1 - \sigma(1 - \alpha\beta)} \quad (33)$$

with $D \equiv (\alpha + \varepsilon) / (\alpha\beta) \geq D_0 = 1/\beta$ and $T \leq T_0 = 10.542$, where T_0 is solution to (33) with $D = 1/\beta$.

Under restriction $T \leq T_0$, (33) becomes the blue half-line in Figure 4.

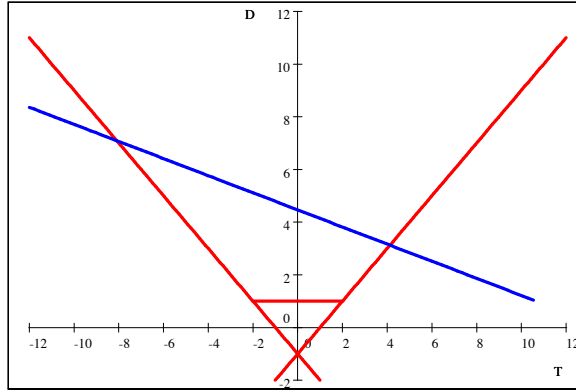


Fig. 4 Pitchfork bifurcation

The pitchfork bifurcation generically arises at $\varepsilon = \varepsilon_P = 1 - \alpha = 2/3$ (intersection of the blue half-line line with the red line $D = T - 1$). We observe that the flip bifurcation value (31) with $\sigma = 6$ is $\varepsilon_F = 1.9359$ solution to

$$\sigma = \frac{2}{1 - \alpha\beta} \frac{\varepsilon + \alpha(1 + \beta)}{\varepsilon - (1 - \alpha)}$$

Definition 28 *The set of admissible points A is the subset of the half-line defined by (33) and $T \leq T_0$, corresponding to a (real) positive steady state k .*

Lemma 29 *There exists $\tilde{a} > 0$, such that, for any $a > \tilde{a}$, the intersection between the set of admissible points A and the flip bifurcation line $D = -T - 1$ is nonempty. This intersection corresponds to the critical value $\varepsilon = \varepsilon_F$.*

In our case,

$$P_F = (1 - \alpha)(1 + \varepsilon_F)^{-\frac{\alpha}{1-\alpha}} \alpha^{\frac{1+\alpha}{1-\alpha}} \beta^{\frac{1}{1-\alpha}} = 0.040663$$

and \tilde{a}_F is solution to $\zeta(a) = \varepsilon_F - P_F a^{1/(1-\alpha)} = 0$, that is $\tilde{a}_F = 13.136 > 0$. Thus, for any $a > \tilde{a}_F$ there exists a steady state $k(a) > 0$ with $\varepsilon_F = 1.9359$ as bifurcation value.

In other terms, for our calibration, the set of admissible points (T, D) which is a subset of the blue half-line crosses the flip bifurcation line $D = -T - 1$.

7 Isoelastic externalities

We consider isoelastic externalities

$$A(k) \equiv Ak^\varepsilon \tag{34}$$

with finite constant elasticity $\varepsilon \in \mathbb{R}$. Externalities are negative when $\varepsilon < 0$ and positive when $\varepsilon > 0$. When $\varepsilon = 0$, we recover the standard Ramsey model. For example, consider $A = 1$ and $\varepsilon = -1, 1/2, 2$: clearly, externalities in Figure 5 are unbounded and don't satisfy Assumption 1. We consider bounded externalities, but locally isoelastic in the next subsection.

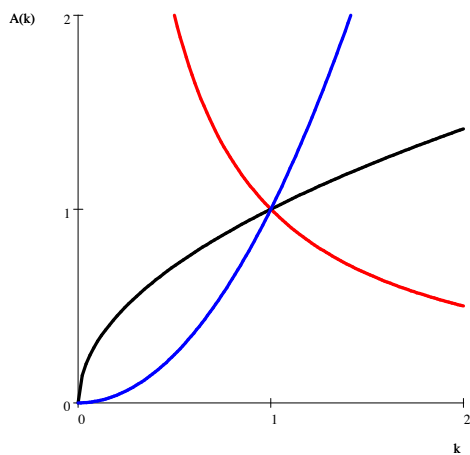


Fig. 5 Isoelastic externalities

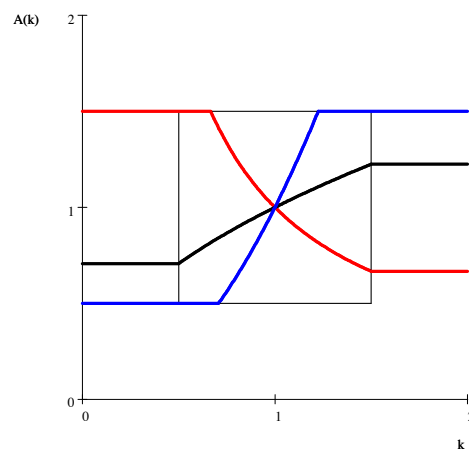


Fig. 6 Bounded externalities

7.1 Existence of a competitive equilibrium

In the local analysis, as long as the entire equilibrium trajectory lies in the neighborhood of an attractor (steady state or cycle), we require the externality to be only locally isoelastic. Thus, we can introduce boundaries for $A(k)$ outside the neighborhood we are interested in, for instance, by considering a square sufficiently large to include the steady state k . For instance, in Figure 5, we can introduce simple boundaries and obtain Figure 6.

In the following, we consider bounded externalities locally isoelastic around the steady state.

Lemma 30 Fix $\mu \in (0, 1 - \alpha)$ and bounds a_0 and a_1 such that $0 < a_0 < b_0 \leq b_1 < a_1$ where

$$\begin{aligned} b_0 &\equiv \min \left\{ A(\alpha\beta A)^{-1-\frac{1-\alpha}{\mu}}, A(\alpha\beta A)^{-1+\frac{1-\alpha}{\mu}} \right\} \\ b_1 &\equiv \max \left\{ A(\alpha\beta A)^{-1-\frac{1-\alpha}{\mu}}, A(\alpha\beta A)^{-1+\frac{1-\alpha}{\mu}} \right\} \end{aligned}$$

Consider elasticity values $\varepsilon \notin (1 - \alpha - \mu, 1 - \alpha + \mu)$.

(1) In the case of negative externalities ($\varepsilon < 0$), define:

$$\begin{aligned} A(k) &\equiv b_1 \text{ if } k < \left(\frac{b_1}{A}\right)^{\frac{1}{\varepsilon}} \\ A(k) &\equiv Ak^\varepsilon \text{ if } \left(\frac{b_1}{A}\right)^{\frac{1}{\varepsilon}} \leq k \leq \left(\frac{b_0}{A}\right)^{\frac{1}{\varepsilon}} \\ A(k) &\equiv b_0 \text{ if } k > \left(\frac{b_0}{A}\right)^{\frac{1}{\varepsilon}} \end{aligned}$$

(1) In the case of positive externalities ($\varepsilon > 0$), define:

$$\begin{aligned} A(k) &\equiv b_0 \text{ if } k < \left(\frac{b_0}{A}\right)^{\frac{1}{\varepsilon}} \\ A(k) &\equiv Ak^\varepsilon \text{ if } \left(\frac{b_0}{A}\right)^{\frac{1}{\varepsilon}} \leq k \leq \left(\frac{b_1}{A}\right)^{\frac{1}{\varepsilon}} \\ A(k) &\equiv b_1 \text{ if } k > \left(\frac{b_1}{A}\right)^{\frac{1}{\varepsilon}} \end{aligned}$$

In both these cases, for any $\varepsilon \notin (1 - \alpha - \mu, 1 - \alpha + \mu)$, the externalities are isoelastic around the steady state and they satisfy Assumption 1 (boundedness).

In the following, σ is constant and ε locally constant.

7.2 Dynamics with negative externalities

7.2.1 Steady state

According to (10), (11) and (34), the steady state is unique and the corresponding levels of capital and consumption are given by

$$k = (A\alpha\beta)^{\frac{1}{1-\alpha-\varepsilon}} \quad (35)$$

$$c = \frac{1-\alpha\beta}{\alpha\beta}k \quad (36)$$

7.2.2 Local dynamics

We introduce the following critical values:

$$\begin{aligned} \sigma_1 &\equiv \frac{1}{1-\alpha\beta} < \sigma_2 \equiv \frac{2}{1-\alpha\beta} \\ \varepsilon_F &\equiv -\alpha - \frac{2\alpha\beta + \sigma(1-\alpha\beta)}{2-\sigma(1-\alpha\beta)} \end{aligned} \quad (37)$$

Remark 31 *This critical values are the same of the hyperbolic case (expressions (20), (21) and (22)), but, now, the elasticity of externalities ε is constant and may be unbounded ($\varepsilon \in (-\infty, 0)$). In the case of hyperbolic bounded externalities, the elasticity of externalities $\tilde{\varepsilon}$ was endogenous and bounded ($\tilde{\varepsilon} \in (-1, 0)$).*

Proposition 32 (flip bifurcation) *Let Assumptions 3 and 5.*

(1) *If $0 < \sigma < \sigma_2$, then the steady state is a source for $\varepsilon < \varepsilon_F$ and a saddle point for $\varepsilon_F < \varepsilon < 0$. Cycles of period two (and powers of two) generically arise through a flip bifurcation (and period-doubling bifurcations) at $\varepsilon = \varepsilon_F$.*

(2) *If $\sigma_2 < \sigma < \infty$, then the steady state is a saddle point for any ε .*

7.2.3 Simulations

To illustrate Proposition 32, let us fix the fundamental parameters as follows $\alpha = 1/3$, $\beta = 0.96$, $\sigma = 1.1$. In this case, we have $0 < \sigma = 1.1 < \sigma_2 = 2.9412$. Thus, there is room for a flip bifurcation. The critical value is given by $\varepsilon_F = -1.442$. Thus the steady state is a saddle if $-1.442 < \varepsilon < 0$ and a source if $\varepsilon < -1.442$. We do not know whether the cycle arising at $\varepsilon_F = -1.442$ is sub or supercritical. The intersection between the red and the blue lines in Figure

7 corresponds to $\varepsilon_F = -1.442$.

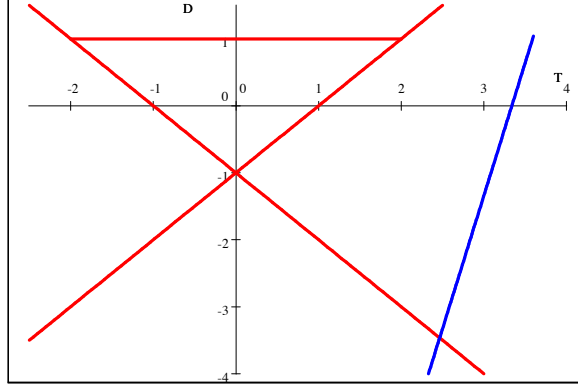


Fig. 7 Flip bifurcation

7.2.4 Global dynamics

In the logarithmic case, we can compute the explicit trajectory.

Proposition 33 *If $u(c_t) \equiv \ln c_t$, the solution of dynamic system (8)-(9) becomes*

$$k_t = (\alpha\beta A)^{\frac{1-(\alpha+\varepsilon)^t}{1-\alpha-\varepsilon}} k_0^{(\alpha+\varepsilon)^t} \quad (38)$$

$$c_t = (1 - \alpha\beta) A k_t^{\alpha+\varepsilon} \quad (39)$$

and, if $|\alpha + \varepsilon| < 1$, the trajectory converges to the steady state: $\lim_{t \rightarrow \infty} k_t = (\alpha\beta A)^{1/(1-\alpha-\varepsilon)}$.

We observe that, when $-1 < \alpha + \varepsilon < 0$ we have a convergence through dumping fluctuations, and persistent fluctuations when $\alpha + \varepsilon = -1$ with

$$k_t = (\alpha\beta A)^{\frac{1-(-1)^t}{2}} k_0^{(-1)^t}$$

Thus, in this case, the two values of the cycle are $k_{2\tau} = k_0$ and $k_{2\tau+1} = \alpha\beta A/k_0$ for any $\tau \geq 0$. Notice also that, in the case of a logarithmic utility, $\sigma = 1$ and the flip bifurcation value (37) becomes $\varepsilon_F = -1 - \alpha$. Precisely, we get $\alpha + \varepsilon_F = -1$.

7.3 Dynamics with positive externalities

7.3.1 Steady state

As in the case of negative externalities, there is a unique non-negative real steady state and the corresponding levels of capital and consumption are given by (35) and (36).

7.3.2 Local dynamics

We introduce the following critical values: $\varepsilon_B \equiv 1 - \alpha$ and the same expression for ε_F as in (37).

Proposition 34 *The steady state is unique.*

(1) *If $0 < \sigma < \sigma_1$, it is a saddle point for $0 < \varepsilon < 1 - \alpha - \mu$ and a source for $\varepsilon > 1 - \alpha + \mu$.*

(2) *If $\sigma_1 < \sigma < \sigma_2$, it is a saddle point for $0 < \varepsilon < 1 - \alpha - \mu$ and a source for $\varepsilon > 1 - \alpha + \mu$.*

(3) *If $\sigma_2 < \sigma < \infty$, it is a saddle point for any $0 < \varepsilon < 1 - \alpha - \mu$, a source for $1 - \alpha + \mu < \varepsilon < \varepsilon_F$ and again saddle point for $\varepsilon_F < \varepsilon$. Cycles of period two (and powers of two) generically arise through a flip bifurcation (and period-doubling bifurcations) at $\varepsilon = \varepsilon_F$.*

Focus on case (3). Beyond $1 - \alpha + \mu$, the steady state become a source. Since the capital intensity k_t is a predetermined variable, if $k_0 \neq k$, then the equilibrium moves away from the steady state, maybe towards a surrounding attractor.

A two-period cycles generically arises in a left neighborhood of ε_F when the flip bifurcation is supercritical (stable cycle) or in the right neighborhood of ε_F when the flip bifurcation is subcritical (unstable cycle). Because of the linearization, local analysis remains uninformative about the stability of the cycle. In order to capture its sub- or super-critical nature a higher-order approximation (Taylor polynomial) of the original system is needed.

Remark 35 *We observe that, now, a flip bifurcation takes place under dominant substitution effects. In other terms, when the externalities are negative, the occurrence of cycles requires dominant income effects; conversely, when the externalities are positive, it requires dominant substitution effects.*

7.3.3 Simulations

To illustrate Proposition 34, let us fix the fundamental parameters as follows $\alpha = 1/3$, $\beta = 0.96$, $\sigma = 6$. We simulate the case $\sigma > \sigma_2 = 2.9412$ with $\varepsilon \notin (1 - \alpha - \mu, 1 - \alpha + \mu)$ (the "hole") and, for instance, $\mu = 1/24$.

The blue half-line is still given by (33) with $D \geq D_0 = 1/\beta$, that is $T \leq T_0 = 10.542$. It crosses the red line $D = T - 1$ at $\varepsilon_B \equiv 1 - \alpha = 2/3$ and the "hole" around is given by (T_+, T_-) where T_- and T_+ are obtained replacing $\varepsilon_- \equiv \varepsilon_B - \mu$ and $\varepsilon_+ \equiv \varepsilon_B + \mu$ in the trace expression (13).

$$\begin{aligned} T_+ &= 1 + \frac{1 + \mu}{\alpha\beta} - \sigma\mu \frac{1 - \alpha\beta}{\alpha\beta} = 3.7240 \\ T_- &= 1 + \frac{1 - \mu}{\alpha\beta} + \sigma\mu \frac{1 - \alpha\beta}{\alpha\beta} = 4.526 \end{aligned}$$

The flip bifurcation value is given by (37): $\varepsilon_F = 1.9359$, determining, according to (13), the intersection $(T_F, D_F) = (-8.0914, 7.0914)$ between the blue

half-line and the flip bifurcation red line $D = -T - 1$ in Figure 8.

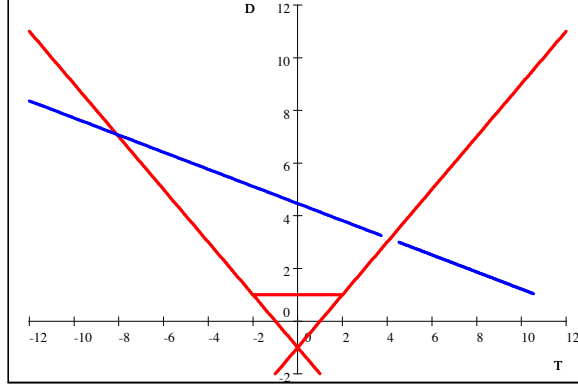


Fig. 8 Flip bifurcation

8 Conclusion

Under the assumption of bounded externalities, we have proven the existence of a competitive equilibrium.

In the case of hyperbolic negative externalities, we have shown the occurrence of flip bifurcations and period-doubling bifurcations, and the emergence of cycles of period two when, under largely dominant income effects, the externalities are sufficiently strong.

Conversely, in the case of positive externalities, the occurrence of cycles require sufficiently large substitution effects.

We have recovered similar results in the case of locally isoelastic externalities.

9 Appendix

Proof of Lemma 3

Let κ_0 satisfy $0 < \kappa_0 = a_0 \kappa_0^\alpha$, that is $\kappa_0 \equiv a_0^{1/(1-\alpha)}$.

(1) If $0 < k_0 \leq \kappa_0$, take whatever $\zeta \in (0, k_0)$. Then, $0 < \zeta < a_0 \zeta^\alpha < A(k_0) k_0^\alpha$ and, clearly, $0 < \zeta < a_0 \zeta^\alpha \leq A(\bar{k}_t) \zeta^\alpha$ for any $t > 0$.

(2) If $k_0 > \kappa_0$, take also whatever $\zeta \in (0, \kappa_0)$. This gives $0 < \zeta < a_0 \zeta^\alpha < a_0 \kappa_0^\alpha < a_0 k_0^\alpha < A(k_0) \zeta^\alpha$ and $0 < \zeta < a_0 \zeta^\alpha \leq A(\bar{k}_t) \zeta^\alpha$ for any $t \geq 0$.

In both cases, we obtain $a_0 \zeta^\alpha - \zeta > 0$. ■

Proof of Lemma 4

For any $\varepsilon > 0$, fix T_ε such that $\beta^{T_\varepsilon} < \varepsilon$. Then, $\varepsilon \rightarrow 0$ implies $T_\varepsilon \rightarrow \infty$. For instance, let $T_\varepsilon \equiv 1 + \lceil \ln \varepsilon / \ln \beta \rceil > \ln \varepsilon / \ln \beta$, where $\lceil x \rceil$ denotes the unique integer $z \in \mathbb{Z}$ such that $z \leq x < 1 + z$.

According to Lemma 9, we know that there exists a sequence $(\tilde{\zeta}, \tilde{\zeta}, \dots)$ such that $a_0 \tilde{\zeta}^\alpha - \tilde{\zeta} > 0$. However, whatever sequence (ζ, ζ, \dots) with $\zeta \in (0, \tilde{\zeta})$ also

satisfies Lemma 9. We can choose $\zeta > 0$ such that $a_0\zeta^\alpha - 2\zeta > 0$ to obtain $a_0\zeta^\alpha - \zeta > \zeta > 0$.

(0) If $t = 0$, clearly, $0 < \zeta < A(k_0)k_0^\alpha$.

(1) If $1 \leq t \leq T_\varepsilon$, the feasible stationary sequence (ζ, ζ, \dots) satisfies $0 < \zeta < A(\bar{k}_t)\zeta^\alpha$.

(2) If $t > T_\varepsilon$, then

$$0 < \zeta < a_0\zeta^\alpha - \zeta \leq A(\bar{k}_t)\zeta^\alpha - \zeta \quad (40)$$

■

Proof of Lemma 6

By definition of compactness and convexity, the proof is immediate. ■

Proof of Proposition 7

The strict concavity of the production function (1) and the utility function u , the compactness and convexity of $Q(k_0)$ imply the existence, the uniqueness and the continuity of the solution $(k_{t+1}^*, (\bar{k}_t)_{t=1}^\infty), c_t^*((\bar{k}_t)_{t=1}^\infty)_{t=0}^\infty$. ■

Proof of Lemma 9

Consider a converging sequence of sequences $((\bar{k}_t^n)_{t=1}^\infty)_{n=0}^\infty \in Q(k_0)$ with $\lim_{n \rightarrow \infty} (\bar{k}_t^n)_{t=1}^\infty = (\tilde{k}_t)_{t=1}^\infty$. Lemma 6 ensures that $(\tilde{k}_t)_{t=1}^\infty \in Q(k_0)$. Since $Q(k_0)$ is compact and $(k_t^{n*})_{t=1}^\infty \in Q(k_0)$ for any $n \geq 0$, where $(k_t^{n*})_{t=1}^\infty$ is solution to Problem $P(k_0, (\bar{k}_t^n)_{t=1}^\infty)$, we have also that $(k_t^{n*})_{t=1}^\infty \rightarrow (k_t^*)_{t=1}^\infty$. Moreover, $0 \leq k_{t+1}^* \leq A(\tilde{k}_t)k_t^{*\alpha}$ for any $t \geq 0$.

Let the sequence $(x_t)_{t=1}^\infty$ satisfy $0 \leq x_1 \leq A(k_0)k_0^\alpha$ and $0 \leq x_{t+1} \leq A(\tilde{k}_t)x_t^\alpha$ for any $t \geq 0$. $(x_t)_{t=1}^\infty$ is a feasible path associated with $(\tilde{k}_t)_{t=1}^\infty$ and $(x_t)_{t=1}^\infty \in Q(k_0)$. To prove that φ is continuous, it is sufficient to prove that $\sum_{t=0}^\infty \beta^t u(A(\tilde{k}_t)k_t^{*\alpha} - k_{t+1}^*) \geq \sum_{t=0}^\infty \beta^t u(A(\tilde{k}_t)x_t^\alpha - x_{t+1})$.

Let $\varepsilon > 0$ and T_ε be associated with ε as in Lemma 4. Consider $\lambda \in (0, 1)$. Then, $0 < \lambda\zeta + (1 - \lambda)x_1 < A(k_0)k_0^\alpha$ and, for any $t > 0$,

$$0 < \lambda\zeta + (1 - \lambda)x_{t+1} < \lambda A(\tilde{k}_t)\zeta^\alpha + (1 - \lambda)A(\tilde{k}_t)x_t^\alpha < A(\tilde{k}_t)[\lambda\zeta + (1 - \lambda)x_t]^\alpha \quad (41)$$

since $f(z) = z^\alpha$ is strictly concave.

Define $z_t \equiv \lambda\zeta + (1 - \lambda)x_t$ for any $t \leq T_\varepsilon$ and $z_t \equiv \zeta$ for any $T > T_\varepsilon$. According to (41), we have $0 < z_1 < A(k_0)k_0^\alpha$ and $0 < z_{t+1} < A(\tilde{k}_t)z_t^\alpha$ for $1 \leq t \leq T_\varepsilon$, and, according to (40), $0 < \zeta < A(\tilde{k}_t)\zeta^\alpha$ for any $t > T_\varepsilon$.

By continuity, there exists N_ε such that, for any $n \geq N_\varepsilon$, we have $0 < z_{t+1} < A(\bar{k}_t^n)z_t^\alpha$ for any $t \leq T_\varepsilon$. Since, according to Lemma 3, $0 < \zeta < a_0\zeta^\alpha$, we get $0 < \zeta < A(\bar{k}_t^n)\zeta^\alpha$ for any $n \geq N_\varepsilon$. Therefore, the sequence $(z_t)_{t=1}^\infty$ is feasible

with $(\bar{k}_t^n)_{t=1}^\infty$. In this case,

$$\begin{aligned} & \sum_{t=0}^{T_\varepsilon} \beta^t u(A(\bar{k}_t^n) z_t^\alpha - z_{t+1}) + \sum_{t=T_\varepsilon+1}^\infty \beta^t u(A(\bar{k}_t^n) \zeta^\alpha - \zeta) \\ & \leq \sum_{t=0}^\infty \beta^t u(A(\bar{k}_t^n) k_t^{n*\alpha} - k_{t+1}^{n*}) \end{aligned} \quad (42)$$

where $(k_t^{n*})_{t=1}^\infty$ is the optimal path associated with $(\bar{k}_t^n)_{t=1}^\infty$.

We observe that $0 < a_0 \zeta^\alpha - \zeta \leq A(\bar{k}_t^n) \zeta^\alpha - \zeta \leq a_1 \zeta^\alpha - \zeta$. Hence,

$$\sum_{t=T_\varepsilon+1}^\infty \beta^t u(a_0 \zeta^\alpha - \zeta) \leq \sum_{t=T_\varepsilon+1}^\infty \beta^t u(A(\bar{k}_t^n) \zeta^\alpha - \zeta) \leq \sum_{t=T_\varepsilon+1}^\infty \beta^t u(a_1 \zeta^\alpha - \zeta)$$

and, according to Lemma 4,

$$\begin{aligned} & \sum_{t=0}^\infty \beta^t u(A(\bar{k}_t^n) k_t^{n*\alpha} - k_{t+1}^{n*}) \\ & = \sum_{t=0}^{T_\varepsilon} \beta^t u(A(\bar{k}_t^n) k_t^{n*\alpha} - k_{t+1}^{n*}) + \sum_{t=T_\varepsilon+1}^\infty \beta^t u(A(\bar{k}_t^n) k_t^{n*\alpha} - k_{t+1}^{n*}) \\ & \leq \sum_{t=0}^{T_\varepsilon} \beta^t u(A(\bar{k}_t^n) k_t^{n*\alpha} - k_{t+1}^{n*}) + \sum_{t=T_\varepsilon+1}^\infty \beta^t u(a_1 m^\alpha) \\ & < \sum_{t=0}^{T_\varepsilon} \beta^t u(A(\bar{k}_t^n) k_t^{n*\alpha} - k_{t+1}^{n*}) + \varepsilon \frac{u(a_1 m^\alpha)}{1 - \beta} \end{aligned}$$

since $\beta^{T_\varepsilon} < \varepsilon$.

In (42), let $\lambda \rightarrow 0$. Then,

$$\sum_{t=0}^{T_\varepsilon} \beta^t u(A(\bar{k}_t^n) z_t^\alpha - z_{t+1}) \rightarrow \sum_{t=0}^{T_\varepsilon} \beta^t u(A(\bar{k}_t^n) x_t^\alpha - x_{t+1})$$

and (42) becomes

$$\begin{aligned} & \sum_{t=0}^{T_\varepsilon} \beta^t u(A(\bar{k}_t^n) x_t^\alpha - x_{t+1}) + \sum_{t=T_\varepsilon+1}^\infty \beta^t u(A(\bar{k}_t^n) \zeta^\alpha - \zeta) \\ & \leq \sum_{t=0}^\infty \beta^t u(A(\bar{k}_t^n) k_t^{*\alpha} - k_{t+1}^*) \\ & = \sum_{t=0}^{T_\varepsilon} \beta^t u(A(\bar{k}_t^n) k_t^{*\alpha} - k_{t+1}^*) + \sum_{t=T_\varepsilon+1}^\infty \beta^t u(A(\bar{k}_t^n) k_t^{*\alpha} - k_{t+1}^*) \\ & \leq \sum_{t=0}^{T_\varepsilon} \beta^t u(A(\bar{k}_t^n) k_t^{*\alpha} - k_{t+1}^*) + \varepsilon \frac{u(a_1 m^\alpha)}{1 - \beta} \end{aligned} \quad (43)$$

We have

$$\sum_{t=T_\varepsilon+1}^{\infty} \beta^t u(a_0 \zeta^\alpha - \zeta) \leq \sum_{t=T_\varepsilon+1}^{\infty} \beta^t u(A(\bar{k}_t^n) \zeta^\alpha - \zeta) \leq \sum_{t=T_\varepsilon+1}^{\infty} \beta^t u(a_1 \zeta^\alpha - \zeta)$$

or

$$u(a_0 \zeta^\alpha - \zeta) \frac{\beta^{T_\varepsilon+1}}{1-\beta} \leq \sum_{t=T_\varepsilon+1}^{\infty} \beta^t u(A(\bar{k}_t^n) \zeta^\alpha - \zeta) \leq u(a_1 \zeta^\alpha - \zeta) \frac{\beta^{T_\varepsilon+1}}{1-\beta} \quad (44)$$

Let $n \rightarrow \infty$. (43) becomes

$$\begin{aligned} & \sum_{t=0}^{T_\varepsilon} \beta^t u(A(\tilde{k}_t) x_t^\alpha - x_{t+1}) + \sum_{t=T_\varepsilon+1}^{\infty} \beta^t u(A(\tilde{k}_t) \zeta^\alpha - \zeta) \\ & \leq \sum_{t=0}^{T_\varepsilon} \beta^t u(A(\tilde{k}_t) k_t^{*\alpha} - k_{t+1}^*) + \varepsilon \frac{u(a_1 m^\alpha)}{1-\beta} \end{aligned} \quad (45)$$

We know that

$$\lim_{T \rightarrow \infty} \sum_{t=0}^T \beta^t u(A(\bar{k}_t) k_t^\alpha - k_{t+1}) = \sum_{t=0}^{\infty} \beta^t u(A(\bar{k}_t) k_t^\alpha - k_{t+1})$$

for any sequence $(\bar{k}_t)_{t=1}^{\infty}$ and any feasible sequence $(k_t)_{t=1}^{\infty}$ associated with $(\bar{k}_t)_{t=1}^{\infty}$.

Take the limit of (45):

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \sum_{t=0}^{T_\varepsilon} \beta^t u(A(\tilde{k}_t) x_t^\alpha - x_{t+1}) + \lim_{\varepsilon \rightarrow 0} \sum_{t=T_\varepsilon+1}^{\infty} \beta^t u(A(\tilde{k}_t) \zeta^\alpha - \zeta) \\ & \leq \lim_{\varepsilon \rightarrow 0} \sum_{t=0}^{T_\varepsilon} \beta^t u(A(\tilde{k}_t) k_t^{*\alpha} - k_{t+1}^*) \\ & \sum_{t=0}^{\infty} \beta^t u(A(\tilde{k}_t) x_t^\alpha - x_{t+1}) + 0 \leq \sum_{t=0}^{\infty} \beta^t u(A(\tilde{k}_t) k_t^{*\alpha} - k_{t+1}^*) \end{aligned}$$

since $\lim_{\varepsilon \rightarrow 0} T_\varepsilon = \infty$, and

$$u(a_0 \zeta^\alpha - \zeta) \frac{\beta^{T_\varepsilon+1}}{1-\beta} \leq \sum_{t=T_\varepsilon+1}^{\infty} \beta^t u(A(\tilde{k}_t) \zeta^\alpha - \zeta) \leq u(a_1 \zeta^\alpha - \zeta) \frac{\beta^{T_\varepsilon+1}}{1-\beta}$$

imply $\lim_{\varepsilon \rightarrow 0} \sum_{t=T_\varepsilon+1}^{\infty} \beta^t u(A(\tilde{k}_t) \zeta^\alpha - \zeta) = 0$.

Finally, let us observe that, since

$$\lim_{T_\varepsilon \rightarrow \infty} \sum_{t=0}^{T_\varepsilon} \beta^t u(A(\tilde{k}_t) x_t^\alpha - x_{t+1}) = \sum_{t=0}^{\infty} \beta^t u(A(\tilde{k}_t) x_t^\alpha - x_{t+1})$$

the path $(k_t^*)_{t=1}^\infty$ is optimal with $(\bar{k}_t)_{t=1}^\infty$ as externality sequence. ■

Proof of Theorem 11

The proof is articulated in three steps.

(1) The Euler equation holds:

$$\beta^t u'(\tilde{c}_t) = \beta^{t+1} u'(\tilde{c}_{t+1}) A(\tilde{k}_{t+1}) \alpha \tilde{k}_{t+1}^{\alpha-1} \quad (46)$$

for any t .

Define $\tilde{p}_t = \beta^t u'(\tilde{c}_t)$ for any t . Then, according to Assumption 1,

$$\frac{\tilde{p}_{t+1}}{\tilde{p}_t} = \frac{1}{A(\tilde{k}_{t+1}) \alpha \tilde{k}_{t+1}^{\alpha-1}} \leq \frac{m^{1-\alpha}}{\alpha a_0} \equiv \mu < 1$$

Hence, $\tilde{p}_t \leq \mu^t \tilde{p}_0$ for any t and $\sum_{t=0}^\infty \tilde{p}_t \leq \tilde{p}_0 / (1 - \mu) < \infty$.

(2) The sequence $(\tilde{c}_t)_{t=0}^\infty$ maximizes the consumer's intertemporal utility function.

Since $\tilde{c}_t + \tilde{k}_{t+1} = A(\tilde{k}_t) \tilde{k}_t^\alpha$ for any $t > 0$, we have $\tilde{p}_t \tilde{c}_t = \tilde{p}_t [A(\tilde{k}_t) \tilde{k}_t^\alpha - \tilde{k}_{t+1}]$ for any t . Summing over t , we find $\sum_{t=0}^\infty \tilde{p}_t \tilde{c}_t = \sum_{t=0}^\infty \tilde{p}_t [A(\tilde{k}_t) \tilde{k}_t^\alpha - \tilde{k}_{t+1}]$. Let $(c_t)_{t=0}^\infty$ satisfy $\sum_{t=0}^\infty \tilde{p}_t c_t \leq \pi^* + q^* k_0$. Since $\pi^* + q^* k_0 = \sum_{t=0}^\infty \tilde{p}_t [A(\tilde{k}_t) \tilde{k}_t^\alpha - \tilde{k}_{t+1}]$, we have $\sum_{t=0}^\infty \tilde{p}_t \tilde{c}_t \geq \sum_{t=0}^\infty \tilde{p}_t c_t$.

Since u is concave, $\sum_{t=0}^\infty \beta^t u'(\tilde{c}_t) - \sum_{t=0}^\infty \beta^t u'(c_t) \geq \sum_{t=0}^\infty \beta^t u'(\tilde{c}_t) (\tilde{c}_t - c_t) = \sum_{t=0}^\infty \tilde{p}_t (\tilde{c}_t - c_t) \geq 0$. Therefore, $\sum_{t=0}^\infty \beta^t u'(\tilde{c}_t) \geq \sum_{t=0}^\infty \beta^t u'(c_t)$.

(3) It remains to prove that the sequence $(\tilde{k}_t)_{t=1}^\infty$ maximizes the profit of the firm.

Let $\Delta_T \equiv \sum_{t=0}^T \tilde{p}_t [A(\tilde{k}_t) \tilde{k}_t^\alpha - \tilde{k}_{t+1}] - \sum_{t=0}^T \tilde{p}_t [A(\tilde{k}_t) k_t^\alpha - k_{t+1}]$, where the sequence $(k_t)_{t=1}^\infty$ satisfies $0 \leq k_{t+1} \leq A(\tilde{k}_t) k_t^\alpha$ for any t . By concavity, $\tilde{k}_t^\alpha - k_t^\alpha \geq \alpha \tilde{k}_t^{\alpha-1} (\tilde{k}_t - k_t)$ and

$$\begin{aligned} \Delta_T &\equiv \sum_{t=0}^T \tilde{p}_t A(\tilde{k}_t) (\tilde{k}_t^\alpha - k_t^\alpha) - \sum_{t=0}^T \tilde{p}_t (\tilde{k}_{t+1} - k_{t+1}) \\ &\geq \sum_{t=0}^T \tilde{p}_t A(\tilde{k}_t) \alpha \tilde{k}_t^{\alpha-1} (\tilde{k}_t - k_t) - \sum_{t=0}^T \tilde{p}_t (\tilde{k}_{t+1} - k_{t+1}) \\ &= \sum_{t=1}^T \tilde{p}_t A(\tilde{k}_t) \alpha \tilde{k}_t^{\alpha-1} (\tilde{k}_t - k_t) - \sum_{t=0}^T \tilde{p}_t (\tilde{k}_{t+1} - k_{t+1}) \\ &= \sum_{t=0}^{T-1} \tilde{p}_{t+1} A(\tilde{k}_{t+1}) \alpha \tilde{k}_{t+1}^{\alpha-1} (\tilde{k}_{t+1} - k_{t+1}) - \sum_{t=0}^T \tilde{p}_t (\tilde{k}_{t+1} - k_{t+1}) \\ &= \sum_{t=0}^{T-1} \tilde{p}_t (\tilde{k}_{t+1} - k_{t+1}) - \sum_{t=0}^T \tilde{p}_t (\tilde{k}_{t+1} - k_{t+1}) \\ &= \tilde{p}_T k_{T+1} - \tilde{p}_T \tilde{k}_{T+1} \geq -\tilde{p}_T \tilde{k}_{T+1} \end{aligned}$$

since $\tilde{k}_0 = k_0$ and $\tilde{p}_{t+1}A(\tilde{k}_{t+1})\alpha\tilde{k}_{t+1}^{\alpha-1} = \tilde{p}_t$ (Euler equation).

Since $\sum_{t=0}^T \tilde{p}_t < \infty$, we have $\lim_{t \rightarrow \infty} \tilde{p}_t \rightarrow 0$. Moreover, since $\tilde{k}_t \leq m$ for any t , in the limit, we obtain $\lim_{T \rightarrow \infty} \Delta_T \geq \lim_{T \rightarrow \infty} (-\tilde{p}_T \tilde{k}_{T+1}) = 0$. This ends the proof of Theorem 11. ■

Proof of Lemma 12

We linearize system (8)-(9) around the steady state. Using (10) and (11), we obtain

$$\begin{aligned} \alpha\beta \frac{dk_{t+1}}{k} &= [\alpha + \varepsilon(k)] \frac{dk_t}{k} - (1 - \alpha\beta) \frac{dc_t}{c} \\ \sigma(c) [1 - \alpha - \varepsilon(k)] \frac{dk_{t+1}}{k} + \frac{dc_{t+1}}{c} &= \frac{dc_t}{c} \end{aligned}$$

that is (12), where

$$J = \begin{bmatrix} \frac{\alpha + \varepsilon(k)}{\alpha\beta} & 1 - \frac{1}{\alpha\beta} \\ \sigma(c) [\alpha + \varepsilon(k) - 1] \frac{\alpha + \varepsilon}{\alpha\beta} & 1 + \sigma(c) [\alpha + \varepsilon(k) - 1] \left(1 - \frac{1}{\alpha\beta}\right) \end{bmatrix}$$

■

Proof of Proposition 13

Case 1: $-a < \eta < 0$. We have $A'(\bar{K}_t) = -\eta / (1 + \bar{K}_t)^2 > 0$, that is A is strictly increasing. We observe that $A(0) = a + \eta$ and $A(\infty) = a$. Fixing $a_0 \equiv a + \eta$ and $a_1 \equiv a$, we obtain $0 < a_0 \leq A(\bar{K}_t) \leq a_1 < \infty$ and Assumption 1 is satisfied (bounded TFP).

Case 2: $\eta > 0$. We have $A'(\bar{k}_t) = -\eta / (1 + \bar{K}_t)^2 < 0$, that is A is strictly decreasing. We observe that $A(0) = a + \eta$ and $A(\infty) = a$. Fixing $a_0 \equiv a$ and $a_1 \equiv a + \eta$, we obtain $0 < a_0 \leq A(\bar{K}_t) \leq a_1 < \infty$ and Assumption 1 is also satisfied (bounded TFP).

The program becomes: $\max \sum_{t=0}^{\infty} \beta^t u(c_t)$ subject to $c_t + k_{t+1} \leq A(\bar{K}_t) k_t^\alpha$. An optimal path always exists (product topology) since $k_{t+1} \leq a_1 k_t^\alpha$ for any t . Moreover, it is interior because of the Inada condition $u'(0) = \infty$. Thus, the Euler equation (16) holds and the steady state is given by (17). ■

Proof of Lemma 14

According to (15) and (17), the steady state is solution to

$$\frac{k^{1-\alpha}}{\alpha\beta} = A(k) = a + \frac{\eta}{1+k} \quad (47)$$

If $\eta > 0$, the RHS of (18) is strictly decreasing in k from $a + \eta > a$ to $a > 0$. The LHS is strictly increasing in k from 0 to ∞ . This implies that the solution is unique, say $k^*(\eta)$. It is easy to see that $k^*(0) = (a\alpha\beta)^{1/(1-\alpha)}$ and $k^*(\infty) = \infty$. Moreover, the upward-sloping curve on the LHS of (18) does not depend on η while the downward-sloping curve on the RHS moves up when η increases. Then, the intersection $k^*(\eta)$ moves on the right when η increases, that is $k^{*'}(\eta) > 0$. ■

Proof of Lemma 15

The elasticity of hyperbolic externalities (15) at the steady state is a function of η .

$$\tilde{\varepsilon}(\eta) \equiv \varepsilon(k^*(\eta)) = \frac{k^*(\eta) A'(k^*(\eta))}{A(k^*(\eta))} = -\frac{k^*(\eta)}{1+k^*(\eta)} \frac{\frac{\eta}{1+k^*(\eta)}}{a + \frac{\eta}{1+k^*(\eta)}} \quad (48)$$

This function is negative and continuous. In addition, $\tilde{\varepsilon}(0) = 0$ and $\tilde{\varepsilon}(\eta) < 0$ for any $0 < \eta < \infty$. Moreover, since $k^*(\infty) = \infty$, according to (18):

$$\lim_{\eta \rightarrow \infty} \frac{\eta}{1+k^*(\eta)} = \lim_{\eta \rightarrow \infty} \left(\frac{[k^*(\eta)]^{1-\alpha}}{\alpha\beta} - a \right) = \infty$$

and, thus, according to (48),

$$\lim_{\eta \rightarrow \infty} \tilde{\varepsilon}(\eta) = -\lim_{\eta \rightarrow \infty} \frac{k^*(\eta)}{1+k^*(\eta)} \lim_{\eta \rightarrow \infty} \frac{\eta/[1+k^*(\eta)]}{a + \eta/[1+k^*(\eta)]} = -1$$

We observe that $k^*(\eta)/[1+k^*(\eta)]$ is increasing in η because, according to Lemma 14, $k^{*\prime}(\eta) > 0$. In addition, $\eta/[1+k^*(\eta)] = [k^*(\eta)]^{1-\alpha}/(\alpha\beta) - a$ is also increasing in η because the RHS is increasing. Then, expression

$$\frac{\eta/[1+k^*(\eta)]}{a + \eta/[1+k^*(\eta)]}$$

is increasing in η and, finally, the expression in the RHS of (48) is decreasing in η , that is $\tilde{\varepsilon}'(\eta) < 0$. ■

Proof of Lemma 16

A flip bifurcation generically occurs at $\tilde{\varepsilon}$ solution to $\tilde{D}(\tilde{\varepsilon}) = -\tilde{T}(\tilde{\varepsilon}) - 1$, that is at $\tilde{\varepsilon} = \tilde{\varepsilon}_F$.

Since $\tilde{\varepsilon} \in (-1, 0)$, we require $\tilde{\varepsilon}_F \in (-1, 0)$. $\tilde{\varepsilon}_F < 0$ is equivalent to $\sigma < \sigma_2$. $\tilde{\varepsilon}_F > -1$ is equivalent to $\sigma < \sigma_0$ or $\sigma > \sigma_2$.

Thus, $\tilde{\varepsilon}_F \in (-1, 0)$ is equivalent to $\sigma < \sigma_0$ and $\sigma < \sigma_2$, that is to $\sigma < \sigma_0$, provided that $\sigma_0 > 0$, that is Assumption 4 holds. ■

Proof of Proposition 17

The locus $S \equiv \left\{ \left(\tilde{T}(\tilde{\varepsilon}), \tilde{D}(\tilde{\varepsilon}) \right) : \tilde{\varepsilon} \in [-1, 0] \right\}$ is linear. Let

$$E_{-1} \equiv \left(\tilde{T}(-1), \tilde{D}(-1) \right) \text{ and } E_0 \equiv \left(\tilde{T}(0), \tilde{D}(0) \right)$$

be its endpoints, that is $S = \{\lambda E_{-1} + (1-\lambda) E_0 : \lambda \in [0, 1]\}$.

Clearly, S is a segment. Indeed, $\tilde{T}(\tilde{\varepsilon})$ is a continuous function and the image of the unit interval $[-1, 0]$ is the compact interval

$$\left[\tilde{T}(-1), \tilde{T}(0) \right] = \left[1 - \frac{1-\alpha}{\alpha\beta} + \sigma(2-\alpha) \frac{1-\alpha\beta}{\alpha\beta}, 1 + \frac{1}{\beta} + \sigma(1-\alpha) \frac{1-\alpha\beta}{\alpha\beta} \right]$$

Similarly, $\tilde{D}(\tilde{\varepsilon})$ is a continuous function and the image of the unit interval $[-1, 0]$ is the compact interval $\left[\tilde{D}(-1), \tilde{D}(0) \right] = \left[-(1-\alpha)/(\alpha\beta), 1/\beta \right]$.

Focus now on Figure 1. We observe that

$$\tilde{D}(0) = \tilde{T}(0) - 1 - \sigma(1 - \alpha)(1 - \alpha\beta) / (\alpha\beta) < \tilde{T}(0) - 1$$

and $\tilde{D}(0) = 1/\beta > 1$. Moreover,

$$\tilde{D}(-1) = \tilde{T}(-1) - 1 - \sigma(2 - \alpha)(1 - \alpha\beta) / (\alpha\beta) < \tilde{T}(-1) - 1$$

and $\tilde{D}(-1) = -(1 - \alpha) / (\alpha\beta) < 0$. The half space below the line $D < T - 1$ is the union of two cones: $C_0 \equiv \{(T, D) : D < T - 1, D < -T - 1\}$ and $C_1 \equiv \{(T, D) : D < T - 1, D > -T - 1\}$, and the half-line between ($D = -T - 1$ and $D < T - 1$).

C_0 is a cone of sources, while C_1 is a cone of saddles (see Figure 1).

The segment S is compact and its endpoints lie below the line $D = T - 1$ (indeed, $\tilde{D}(-1) < \tilde{T}(-1) - 1$ and $\tilde{D}(0) < \tilde{T}(0) - 1$), then all the points of the segment S lie below the line $D = T - 1$. In other words, any point of S belongs to C_0 (source) or C_1 (saddle).

We know that $E_0 \in C_1$. This is coherent with the Ramsey model. Indeed, when $\eta = 0$ (no externalities), that is $\tilde{\varepsilon} = 0$, our model reduces to the basic Cass-Koopmans version of the Ramsey model.

We will prove that, according to Lemma 16, that $E_{-1} \in C_0$ if $\sigma < \sigma_0$ and $E_{-1} \in C_1$ if $\sigma > \sigma_0$.

Thus, under Assumption 4, the segment crosses the line $D = -T - 1$ where the flip bifurcation occurs if and only if $E_{-1} \in C_0$, that is $\sigma < \sigma_0$. In this case, S crosses the flip bifurcation at $\tilde{\varepsilon} = \tilde{\varepsilon}_F \in (-1, 0)$ and the steady state is a source if $\tilde{\varepsilon} \in (-1, \tilde{\varepsilon}_F)$ and it is a saddle if $\tilde{\varepsilon} \in (\tilde{\varepsilon}_F, 0)$. If $\sigma > \sigma_0$, the segment S lies entirely in the cone C_1 and the steady state is always a saddle point. In particular, this is true when $\sigma > 1$ because $1 > \sigma_0$, that is when the utility is positive.

Now, let us show that $E_{-1} \in C_0$ if and only if $\sigma < \sigma_0$. We observe that $\tilde{D}'(\tilde{\varepsilon}) = 1/(\alpha\beta) > 0$ and $\tilde{T}'(\tilde{\varepsilon}) = 1/(\alpha\beta) - \sigma(1 - \alpha\beta) / (\alpha\beta) > 0$ if and only if $\sigma < \sigma_1$.

The slope $s(\sigma)$ of S is given by $D'(T)$, where $D(T) = \tilde{D}(\tilde{T}^{-1}(T))$, and $s(\sigma) = \tilde{D}'(\tilde{\varepsilon}) / \tilde{T}'(\tilde{\varepsilon}) = 1/[1 - \sigma(1 - \alpha\beta)] > 0$ if and only if $\sigma < \sigma_1$.

We observe that $s'(\sigma) > 0$, thus, the segment S rotates in a counterclockwise sense. More precisely, we have:

$$s(0) = 1 < s(\sigma_0) = (2 - \alpha) / [\alpha(1 + 2\beta)] < s(1) = 1/(\alpha\beta) < s(\sigma_1^-) = +\infty$$

and $s(\sigma_1^+) = -\infty < s(\sigma_2) = -1 < s(+\infty) = 0^-$ where the inequality $1 < s(\sigma_0)$ holds under Assumption 4. There is a discontinuity at $\sigma = \sigma_1$.

We consider all the possible cases for σ under the Assumption 4.

(1) $0 < \sigma < \sigma_0$. Then, $\tilde{D}'(\tilde{\varepsilon}) > 0$, $\tilde{T}'(\tilde{\varepsilon}) > 0$ and $1 < s < (2 - \alpha) / [\alpha(1 + 2\beta)]$. According to Lemma 16, there is room for a flip bifurcation occurring at $\tilde{\varepsilon} = \tilde{\varepsilon}_F$. When $-1 < \tilde{\varepsilon} < \tilde{\varepsilon}_F$, the steady state is a source; when $\tilde{\varepsilon}_F < \tilde{\varepsilon} < 0$, it is a saddle. However, we don't know whether this bifurcation is sub- or super-critical.

(2) $\sigma_0 < \sigma < 1$. Then, $\tilde{D}'(\tilde{\varepsilon}) > 0$ and $\tilde{T}'(\tilde{\varepsilon}) > 0$ with $(2 - \alpha) / [\alpha(1 + 2\beta)] < s < 1 / (\alpha\beta)$.

(3) $1 < \sigma < \sigma_1$. Then, $\tilde{D}'(\tilde{\varepsilon}) > 0$ and $\tilde{T}'(\tilde{\varepsilon}) > 0$ with $1 / (\alpha\beta) < s < +\infty$.

(4) $\sigma_1 < \sigma < \sigma_2$. Then, $\tilde{D}'(\tilde{\varepsilon}) > 0$ and $\tilde{T}'(\tilde{\varepsilon}) < 0$ with $-\infty < s < -1$.

(5) $\sigma_2 < \sigma < +\infty$. Then, $\tilde{D}'(\tilde{\varepsilon}) > 0$ and $\tilde{T}'(\tilde{\varepsilon}) < 0$ with $-1 < s < 0$.

In cases (2) to (5), since $\sigma > \sigma_0$, according to Lemma 16, the endpoint E_{-1} belongs to C_1 and, thus, C_1 includes the entire segment S . The equilibrium is always saddle-path stable but, since $\tilde{D}(-1) = -(1 - \alpha) / (\alpha\beta) < 0$, for low values of $\tilde{\varepsilon}$ (sufficiently close to -1), the determinant becomes negative (the eigenvalues are real because of the saddle, but one becomes negative, while the other remains positive). This means that damped oscillations are possible along the stable branch. ■

Proof of Proposition 19

Let $k_F \equiv k^*(\eta_F)$ be the critical steady state. Consider equations (47) and (48). The critical pair (η_F, k_F) is solution to system

$$\frac{k^{1-\alpha}}{\alpha\beta} = a + \frac{\eta}{1+k} \quad (49)$$

$$\tilde{\varepsilon}_F = -\frac{k}{1+k} \frac{\frac{\eta}{1+k}}{a + \frac{\eta}{1+k}} \quad (50)$$

Equation (49) gives

$$\eta = (1+k) \left(\frac{k^{1-\alpha}}{\alpha\beta} - a \right) \quad (51)$$

Replacing (51) in (50), we obtain (24). Replacing in turn the solution k_F to (24) in (51), we obtain (23). ■

Proof of Proposition 23

A steady state k^* is solution to (17), that is to $\varphi(k) \equiv (1+k)(k^{1-\alpha} - a\alpha\beta) = \eta\alpha\beta$. We have $\varphi(0) = -a\alpha\beta < \eta\alpha\beta < 0$ and $\varphi(\infty) = \infty$. Thus, a solution always exists. Moreover, $\varphi'(k) = [1 + (1 - \alpha)(1 + k)/k]k^{1-\alpha} - a\alpha\beta$ and

$$\varphi''(k) = \frac{1-\alpha}{k^\alpha} \left(2 - \alpha \frac{1+k}{k} \right)$$

We observe that $\varphi''(k) \geq 0$ if and only if $k \geq \alpha / (2 - \alpha) \equiv \tilde{k}$ where \tilde{k} is the unique inflexion point of a concave-convex function with $\varphi'(0^+) = +\infty$ and $\varphi'(+\infty) = +\infty$. Moreover, $\varphi'(\tilde{k}) = (2/\alpha - 1)^\alpha - a\alpha\beta$ and $\varphi'(\tilde{k}) \geq 0$ if and only if $a \leq \tilde{a}$, where \tilde{a} is solution to $\varphi'(\tilde{k}) = 0$, that is (30).

If $a \leq \tilde{a}$, solution unique (φ is strictly increasing for $a < \tilde{a}$ and for $a < \tilde{a}$ and $k \neq \tilde{k}$).

If $a > \tilde{a}$, there is a local maximum at k_1 and a local minimum at k_2 with $k_1 < \tilde{k} < k_2$ and $\varphi(k_1) > \varphi(k_2)$.

Let $\eta_1 \equiv \varphi(k_1) / (\alpha\beta) > \eta_2 \equiv \varphi(k_2) / (\alpha\beta)$ and observe that $\eta_1 > -a$, that is $\eta_1 > \max\{-a, \eta_2\}$. Then, if $\eta \in (\eta_2, \eta_1)$, there are three steady states. If

$\eta \in \{\eta_2, \eta_1\}$, there are two steady states. If $\eta \notin [\eta_2, \eta_1]$, there is only one steady state. ■

Proof of Proposition 24

According to (13)-(14):

$$T = 1 + \frac{\alpha + \varepsilon(k)}{\alpha\beta} + \sigma [(1 - \alpha) - \varepsilon(k)] \left(\frac{1}{\alpha\beta} - 1 \right) \quad (52)$$

$$D = \frac{\alpha + \varepsilon(k)}{\alpha\beta} \quad (53)$$

From (48), the elasticity of hyperbolic externalities (15) at the steady state is a function of η .

$$\tilde{\varepsilon}(\eta) \equiv \varepsilon(k^*(\eta)) = \frac{k^*(\eta) A'(k^*(\eta))}{A(k^*(\eta))} = -\frac{k^*(\eta)}{1 + k^*(\eta)} \frac{\frac{\eta}{1+k^*(\eta)}}{a + \frac{\eta}{1+k^*(\eta)}} \quad (54)$$

which is always positive. We observe that, always, $D = [\alpha + \varepsilon(k)] / (\alpha\beta) > 1$. Moreover, when $0 < \varepsilon(k) < \varepsilon_P$, we have $T > 1 + [\alpha + \varepsilon(k)] / (\alpha\beta) = 1 + D$ that is $D < T - 1$.

The subcone $1 < D < T - 1$ is included in a cone of saddle points. Thus, the steady state is a saddle point for $0 < \varepsilon(k) < \varepsilon_P$. ■

Proof of Proposition 25

The occurrence of a pitchfork bifurcation requires $D = T - 1$, that is $\varepsilon(k) \equiv 1 - \alpha$. Using (54), we have

$$-\frac{k}{1+k} \frac{\frac{\eta}{1+k}}{a + \frac{\eta}{1+k}} = 1 - \alpha$$

Solving for k , we get two roots:

$$\begin{aligned} \kappa_- &= \frac{-\eta(2-\alpha) - \sqrt{\eta^2(2-\alpha)^2 + 4a\eta(1-\alpha)}}{2a(1-\alpha)} - 1 > 0 \\ \kappa_+ &= \frac{-\eta(2-\alpha) + \sqrt{\eta^2(2-\alpha)^2 + 4a\eta(1-\alpha)}}{2a(1-\alpha)} - 1 > \kappa_- \end{aligned}$$

These solutions are real if and only if $\eta^2(2-\alpha)^2 + 4a\eta(1-\alpha) \geq 0$, that is (32) holds. We observe that $\kappa_+ > \kappa_-$ and

$$\eta \leq -a \frac{1-\alpha}{1-\alpha + \left(\frac{\alpha}{2}\right)^2} \Rightarrow \eta < -a \frac{1-\alpha}{1-\alpha + \frac{\alpha}{2}} \Rightarrow \kappa_- > 0$$

According to (18), the steady state satisfies

$$\beta = \frac{k^{1-\alpha}}{\alpha \left(a + \frac{\eta}{1+k} \right)}$$

Replacing κ_- and κ_+ , we obtain the pitchfork bifurcation values β_- and β_+ .

■

Proof of Proposition 26

A flip bifurcation generically arises when a real eigenvalue crosses -1 , that is when $D = -T - 1$. Using (52) and (53), and solving for σ equation $D = -T - 1$, we obtain the critical bifurcation value (31). ■

Proof of Proposition 27

Simply notice that, always, $D = [\alpha + \varepsilon(k)] / (\alpha\beta) > 1$. So the product D of the two nonreal and conjugated eigenvalues (square of the modulus) never equals one. ■

Proof of Lemma 29

Replacing $\eta / (1 + k) = k^{1-\alpha} / (\alpha\beta) - a$ in

$$\varepsilon = -\frac{k}{1+k} \frac{\frac{\eta}{1+k}}{a + \frac{\eta}{1+k}}$$

we obtain the steady state equation

$$\varepsilon = -\frac{k}{1+k} \left(1 - a \frac{\alpha\beta}{k^{1-\alpha}} \right) \quad (55)$$

Can we find $a > 0$ such that the equality (55) with $\alpha, \beta \in (0, 1)$ and $\varepsilon > 0$ has a solution $k > 0$?

(55) can be written as

$$\varphi(k) \equiv \varepsilon + (1 + \varepsilon)k - a\alpha\beta k^\alpha = 0 \quad (56)$$

We observe that $\varphi'(k) = 1 + \varepsilon - a\alpha^2\beta k^{\alpha-1}$ and $\varphi''(k) = a\alpha^2\beta(1 - \alpha)k^{\alpha-2} > 0$. Moreover, $\varphi'(0) = -\infty$ and $\varphi'(\infty) = 1 + \varepsilon$.

$$\bar{k} \equiv \left(\frac{a\alpha^2\beta}{1 + \varepsilon} \right)^{\frac{1}{1-\alpha}} \quad (57)$$

is solution to $\varphi'(k) = 0$. Because of the strict convexity of φ , $\varphi(\bar{k}) < \varphi(0) = \varepsilon < \varphi(\infty) = \infty$.

Hence, there exists $a > 0$ such that (55) has a solution k if and only if $\varphi(\bar{k}) < 0$.

Replacing (57) in (56), we obtain $\varphi(\bar{k}) \equiv \varepsilon - Pa^{1/(1-\alpha)} \equiv \zeta(a)$, where

$$P \equiv (1 - \alpha)(1 + \varepsilon)^{-\frac{\alpha}{1-\alpha}} \alpha^{\frac{1+\alpha}{1-\alpha}} \beta^{\frac{1}{1-\alpha}} > 0$$

Therefore, $\varphi(\bar{k}) < 0$ if and only if $\zeta(a) < 0$. We observe that $\zeta(0) = \varepsilon$, $\zeta(\infty) = -\infty$ and

$$\zeta'(a) = -\frac{P}{1 - \alpha} a^{\frac{\alpha}{1-\alpha}} < 0$$

Therefore, there exists a unique solution $\tilde{a} > 0$ to $\zeta(a) = 0$ and $\zeta(a) < 0$ (entailing a strictly positive steady state k) if and only if $a > \tilde{a}$. ■

Proof of Lemma 30

According to (34), externalities are bounded if $a_0 < Ak^\varepsilon < a_1$. From (10) and (34), we obtain the steady state $k = (\alpha\beta A)^{1/(1-\alpha-\varepsilon)}$. Thus, we require $a_0 < A(\alpha\beta A)^{\varepsilon/(1-\alpha-\varepsilon)} < a_1$.

Fix $0 < \mu < 1 - \alpha$ and consider the following functions

$$\varphi(\varepsilon) \equiv \frac{\varepsilon}{1 - \alpha - \varepsilon} \text{ for any } \varepsilon \notin (1 - \alpha - \mu, 1 - \alpha + \mu)$$

and $\psi(\varphi) \equiv A(\alpha\beta A)^\varphi$. φ is strictly increasing and ψ is strictly monotonic if $\alpha\beta A \neq 1$.

$$\begin{aligned} \min_{\varepsilon \notin (1-\alpha-\mu, 1-\alpha+\mu)} \varphi(\varepsilon) &= -1 - \frac{1-\alpha}{\mu} < -2 \\ \max_{\varepsilon \notin (1-\alpha-\mu, 1-\alpha+\mu)} \varphi(\varepsilon) &= -1 + \frac{1-\alpha}{\mu} > 0 \end{aligned}$$

φ can take all the values in the set $[-1 - (1 - \alpha) / \mu, -1 + (1 - \alpha) / \mu] \setminus \{1\}$.
Let

$$\begin{aligned} b_0 &\equiv \min \left\{ A(\alpha\beta A)^{-1 - \frac{1-\alpha}{\mu}}, A(\alpha\beta A)^{-1 + \frac{1-\alpha}{\mu}} \right\} \\ b_1 &\equiv \max \left\{ A(\alpha\beta A)^{-1 - \frac{1-\alpha}{\mu}}, A(\alpha\beta A)^{-1 + \frac{1-\alpha}{\mu}} \right\} \end{aligned}$$

If a_0 and a_1 are chosen such that $0 < a_0 < b_0 \leq b_1 < a_1$, then $a_0 < Ak^\varepsilon < a_1$.

■

Proof of Proposition 32

Reconsider (13)-(14) with $\varepsilon(k) = \varepsilon$ and $\sigma(c) = \sigma$. Define

$$T(\varepsilon) \equiv 1 + \frac{\alpha + \varepsilon}{\alpha\beta} + \sigma(1 - \alpha - \varepsilon) \left(\frac{1}{\alpha\beta} - 1 \right) \quad (58)$$

$$D(\varepsilon) \equiv \frac{\alpha + \varepsilon}{\alpha\beta} \quad (59)$$

The set $\{T(\varepsilon), D(\varepsilon)\}_{-\infty \leq \varepsilon \leq 0}$ is a half-line.

We can compute the half-line $\tilde{D}(T)$ in the (T, D) -plane. Replacing $\varepsilon = \alpha\beta D - \alpha$ in (13), we find

$$\tilde{D}(T) = \frac{T}{1 - \sigma(1 - \alpha\beta)} - \frac{1}{\alpha\beta} \frac{\alpha\beta + \sigma(1 - \alpha\beta)}{1 - \sigma(1 - \alpha\beta)}$$

The origin of this half-line is given by

$$\begin{aligned} T(0) &\equiv 1 + \frac{1}{\beta} + \sigma(1 - \alpha) \left(\frac{1}{\alpha\beta} - 1 \right) \geq 1 + \frac{1}{\beta} > 2 \\ D(0) &\equiv \frac{1}{\beta} > 1 \end{aligned}$$

and the slope by

$$s = \tilde{D}'(T) \equiv \frac{D'(\varepsilon)}{T'(\varepsilon)} \equiv \frac{1}{1 - \sigma(1 - \alpha\beta)} \quad (60)$$

where $D'(\varepsilon) = 1/(\alpha\beta) > 0$ and $T'(\varepsilon) = [1 - \sigma(1 - \alpha\beta)]/(\alpha\beta)$.

We observe that: $0 < \sigma < \sigma_1$ if and only if $1 < s < \infty$; $\sigma_1 < \sigma < \sigma_2$ if and only if $-\infty < s < -1$; $\sigma_2 < \sigma < \infty$ if and only if $-1 < s < 0$. The flip bifurcation value for ε is given by the intersection between the half-line $\{T(\varepsilon), D(\varepsilon)\}_{-\infty \leq \varepsilon \leq 0}$ and the line $D = -T - 1$ in the (T, D) -plane, that is by ε_F . Notice that, always $D'(\varepsilon) > 0$. That is, when ε goes from 0 to $-\infty$, the determinant decreases. The following holds.

(1) If $0 < \sigma < \sigma_1$, then $1 < s < \infty$. Then saddle point for $\varepsilon_F < \varepsilon < 0$ and source for $\varepsilon < \varepsilon_F$. Flip bifurcation at $\varepsilon = \varepsilon_F$.

(2) If $\sigma_1 < \sigma < \sigma_2$, then $-\infty < s < -1$. Then saddle point for $\varepsilon_F < \varepsilon < 0$ and source for $\varepsilon < \varepsilon_F$. Flip bifurcation at $\varepsilon = \varepsilon_F$.

(3) If $\sigma_2 < \sigma < \infty$, then $-1 < s < 0$. Then saddle point for any ε .

Summing up, we have Proposition 32. ■

Proof of Proposition 33

System (8)-(9) becomes: $c_t + k_{t+1} = Ak_t^{\alpha+\varepsilon}$ and $c_{t+1}/c_t = \alpha\beta Ak_{t+1}^{\alpha+\varepsilon-1}$, with solution $k_{t+1} = \alpha\beta Ak_t^{\alpha+\varepsilon}$. Indeed, $c_t = Ak_t^{\alpha+\varepsilon} - k_{t+1} = (1 - \alpha\beta) Ak_t^{\alpha+\varepsilon}$ and

$$\frac{c_{t+1}}{c_t} = \frac{(1 - \alpha\beta) Ak_{t+1}^{\alpha+\varepsilon}}{(1 - \alpha\beta) Ak_t^{\alpha+\varepsilon}} = \frac{Ak_{t+1}^{\alpha+\varepsilon}}{Ak_t^{\alpha+\varepsilon}} = \frac{Ak_{t+1}^{\alpha+\varepsilon}}{k_{t+1}/(\alpha\beta)} = \alpha\beta Ak_{t+1}^{\alpha+\varepsilon-1}$$

The explicit trajectory (38)-(39) is computed by induction and the steady state is given by

$$\lim_{t \rightarrow \infty} k_t = \lim_{t \rightarrow \infty} \left[(\alpha\beta A)^{\frac{1-(\alpha+\varepsilon)t}{1-\alpha-\varepsilon}} k_0^{(\alpha+\varepsilon)t} \right] = (\alpha\beta A)^{\frac{1}{1-\alpha-\varepsilon}} k_0^0 = (\alpha\beta A)^{\frac{1}{1-\alpha-\varepsilon}}$$

provided that $|\alpha + \varepsilon| < 1$. ■

Proof of Proposition 34

As in the case of negative externalities, reconsider (13)-(14) with constant elasticities of externality and intertemporal substitution, that is expressions (58) and (59). The set $\{T(\varepsilon), D(\varepsilon)\}_{0 \leq \varepsilon < \infty}$ is an half-line.

We can compute the half-line $\tilde{D}(T)$ in the (T, D) -plane. Replacing $\varepsilon = \alpha\beta D - \alpha$ in (13), we find

$$\tilde{D}(T) = \frac{T}{1 - \sigma(1 - \alpha\beta)} - \frac{1}{\alpha\beta} \frac{\alpha\beta + \sigma(1 - \alpha\beta)}{1 - \sigma(1 - \alpha\beta)}$$

The origin of this half-line is given by

$$T(0) \equiv 1 + \frac{1}{\beta} + \sigma(1 - \alpha) \left(\frac{1}{\alpha\beta} - 1 \right) \geq 1 + \frac{1}{\beta} > 2$$

and $D(0) \equiv 1/\beta > 1$. We notice that $D(0) > 1$ and $D(0) < T(0) - 1$ for any $\sigma > 0$, that is, when ε is close to 0, the steady state is a saddle point as in the basic Ramsey model (no externalities).

The slope is still given by (60). As above, we observe that: $0 < \sigma < \sigma_1$ if and only if $1 < s < \infty$; $\sigma_1 < \sigma < \sigma_2$ if and only if $-\infty < s < -1$; $\sigma_2 < \sigma < \infty$ if and only if $-1 < s < 0$. Notice that, always $D'(\varepsilon) > 0$. That is, when ε goes from 0 to ∞ , the determinant increases. The following holds.

(1) If $0 < \sigma < \sigma_1$, then $1 < s < \infty$. Then saddle point for $0 < \varepsilon < 1 - \alpha - \mu$ and source for $\varepsilon > 1 - \alpha + \mu$.

(2) If $\sigma_1 < \sigma < \sigma_2$, then $-\infty < s < -1$. Then saddle point for $0 < \varepsilon < 1 - \alpha - \mu$ and source for $\varepsilon > 1 - \alpha + \mu$.

(3) If $\sigma_2 < \sigma < \infty$, then $-1 < s < 0$. Then saddle point for any $0 < \varepsilon < 1 - \alpha - \mu$, source for $1 - \alpha + \mu < \varepsilon < \varepsilon_F$ and saddle for $\varepsilon_F < \varepsilon$.

$\varepsilon = 1 - \alpha$ corresponds to the solution to equation $D(\varepsilon) = T(\varepsilon) - 1$. Equation (35) is equivalent to

$$k^{\varepsilon+\alpha-1} = (A\alpha\beta)^{-1} \quad (61)$$

A degenerate bifurcation takes place at $\varepsilon = 1 - \alpha$. Indeed, in this case, equation (61) become $1 = k^0 = 1/(A\alpha\beta)$ with, generically, no solution.

The flip bifurcation value for ε is given by the intersection between the half-line $\{T(\varepsilon), D(\varepsilon)\}_{0 \leq \varepsilon \leq \infty}$ and the line $D = -T - 1$ in the (T, D) -plane. The critical value ε_F (expression (37)) corresponds to the solution to equation $D(\varepsilon) = -T(\varepsilon) - 1$. Summing up, we obtain Proposition 34. ■

10 References

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