



**HAL**  
open science

# The Energy differentiated field theorem: a powerful tool to quantify wireless capacity limits

Philippe Jacquet

► **To cite this version:**

Philippe Jacquet. The Energy differentiated field theorem: a powerful tool to quantify wireless capacity limits. Doctoral. Hammamet (Tunisie), Tunisia. 2024. hal-04596366

**HAL Id: hal-04596366**

**<https://hal.science/hal-04596366>**

Submitted on 31 May 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# The Energy differentiated field theorem: a powerful tool to quantify wireless capacity limits

Philippe Jacquet

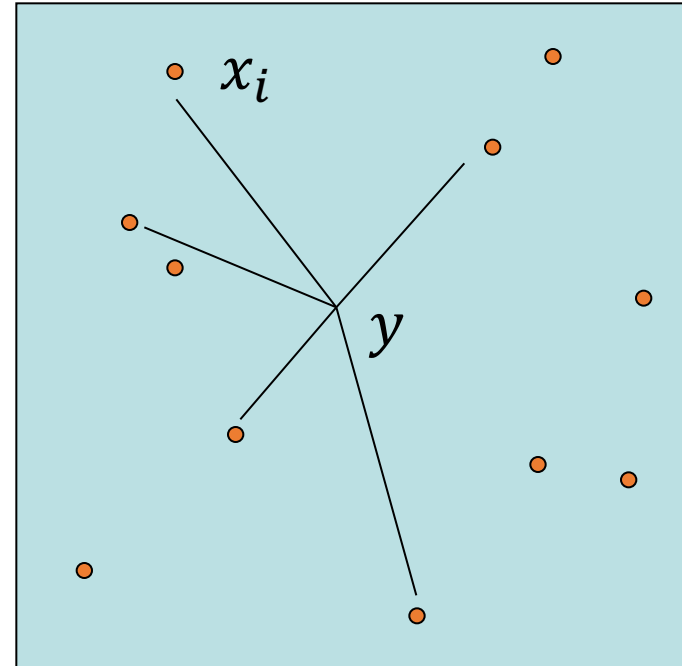
IEEE Fellow

Inria

Workshop IoT & ET Spring School 2024  
Hamamet

# Multiple Input (MI) Shannon capacity model

- emitters  $x_1, \dots, x_i, \dots$  transmit independent flows toward an access point  $y$ .
- Each flow is a gaussian noise for the other flows

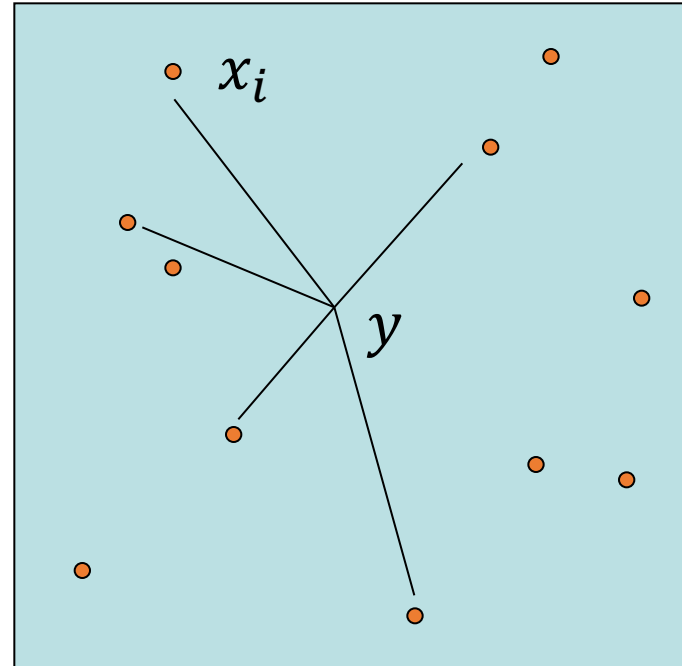


# When a complex equation ...

- Let  $s(x_i)$  be the signal received from transmitter  $x_i$  on the base station  $y$ .
- Let  $B(y)$  the internal noise of the receiver  $y$ .
- The average Shannon capacity of the access point (in bit per Hertz)

$$C(\lambda) = E \left[ \sum_i \log_2 \left( 1 + \frac{s(x_i)}{B(y) + \sum_{j \neq i} s(x_j)} \right) \right]$$

- Average taken on all position points, fading etc...,  $\lambda$  is the transmitter density



# ... results in a simple outcome

- When the signal attenuation is  $\alpha$  and transmitters position are uniform on the infinite plan and  $B(z) \stackrel{\text{def}}{=} 0$  we find:

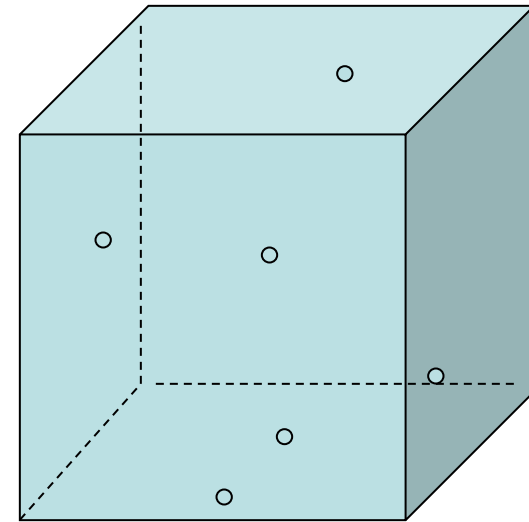
$$C(\lambda) = \frac{\alpha}{2 \log 2} \text{ bits per Herz}$$

- When the transmitters are distributed in an infinite cube

$$C(\lambda) = \frac{\alpha}{3 \log 2}$$

- On an infinite line

$$C(\lambda) = \frac{\alpha}{\log 2}$$



# The deep insights around a simple formula

Physics: the attenuation factor of the  
Electromagnetic wave propagation

$$C = \frac{\alpha}{D \log 2}$$

Geometry: the euclidian  
dimension of the embedding space

Information theory: the intrassic value  
of one bit of entropy

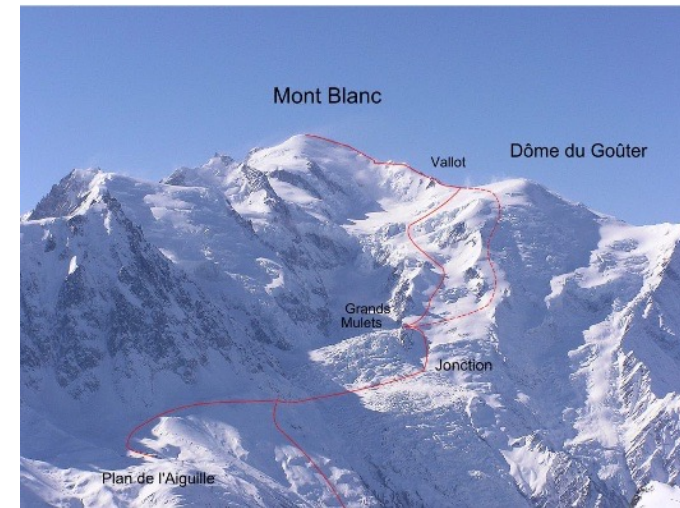
# Two proofs, two climbing ways, same mountain

- The proof a la Italian path



- The proof relies on many steps and requires skills in complex analysis and Laplace transform
- Vertigo assured

- The proof a la French path



- The proof is quiet and smooth thanks to the **Energy differentiated field theorem.**

The hard way short visit, just to contemplate the headache.

- First step: compute the Laplace-Stieljes transform of the received signal  $\mathcal{S}(\lambda) = \sum_i s(z_i): w(\lambda, \theta) = E[e^{-\theta \mathcal{S}(\lambda)}]$ .
- Second step: extract the probability  $p(r, K, \lambda)$  that a transmitter is received at distance  $r$ , with minimal SNR  $K$ .
- Third step: connect to the capacity

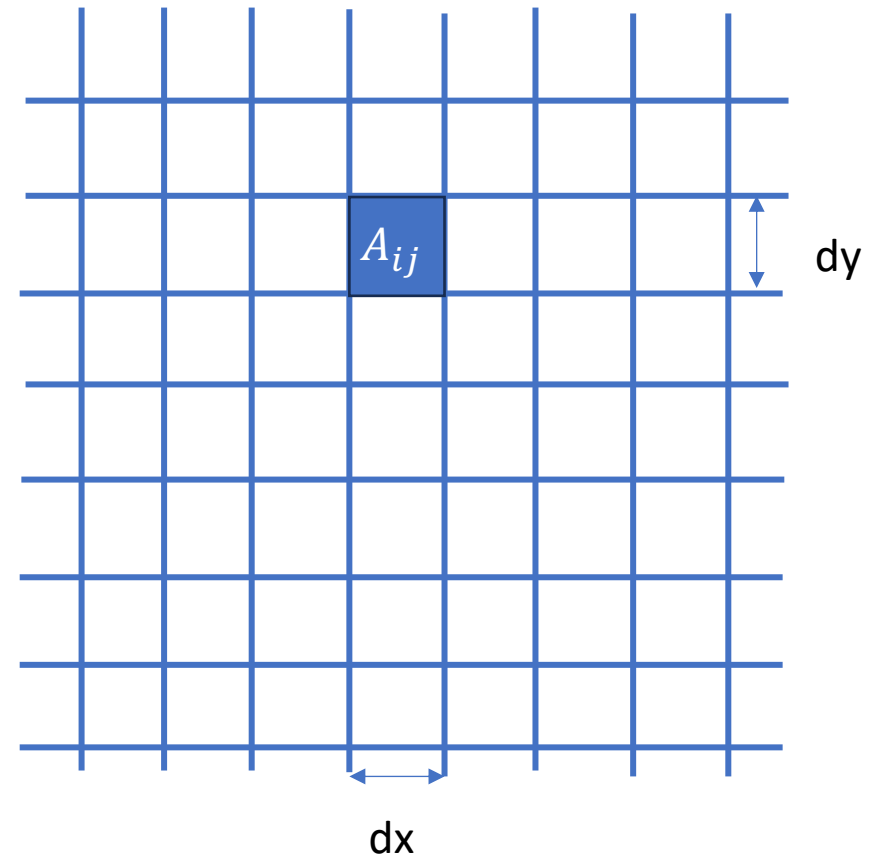
$C(\lambda)$

$$= -2\pi \iint \log_2(1 + K) \frac{\partial}{\partial K} p(r, K, \lambda) r dr dK = \frac{2\pi}{\log 2} \iint p(r, K, \lambda) \frac{r}{1 + K} dr dK$$



# First step overview

- We split the plan in small squares  $A_{ij}$  of size  $dx \times dy$ ,
- Each contribute independently to  $\mathcal{S}(\lambda) = \sum_{ij} S(A_{ij})$ .
- $w(\lambda, \theta) = \prod_{ij} E[e^{-\theta S(A_{ij})}]$
- When  $dx$  and  $dy$  is small
  - $E[e^{-\theta S(A_{ij})}] = 1 - \lambda dx dy + \lambda dx dy e^{-\theta s(x)} + O(dx^2 dy^2)$



# First step overview

- By  $dx, dy \rightarrow 0$  then

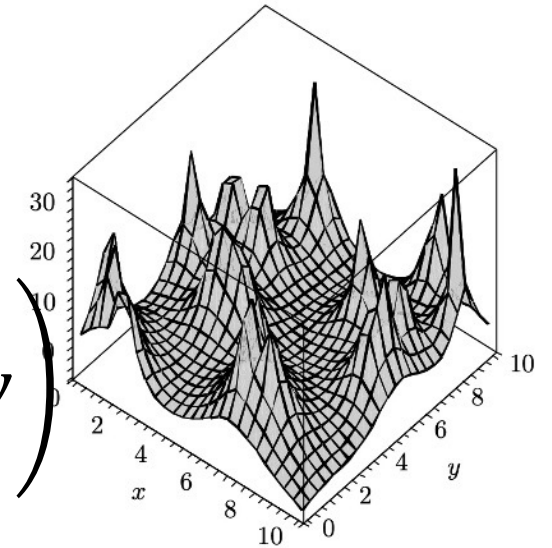
$$w(\lambda, \theta) = \exp\left(-\iint (1 - e^{-\theta s(x)}) dx dy\right)$$

- When  $s(x) = \frac{1}{\|x-y\|^\alpha}$ , then

$$w(\lambda, \theta) = \exp\left(-\lambda\pi\Gamma\left(1 - \frac{2}{\alpha}\right)\theta^{2/\alpha}\right)$$

- When  $s(x) = \frac{F(x)}{\|x-y\|^\alpha}$ , with an independent fading  $F$  then

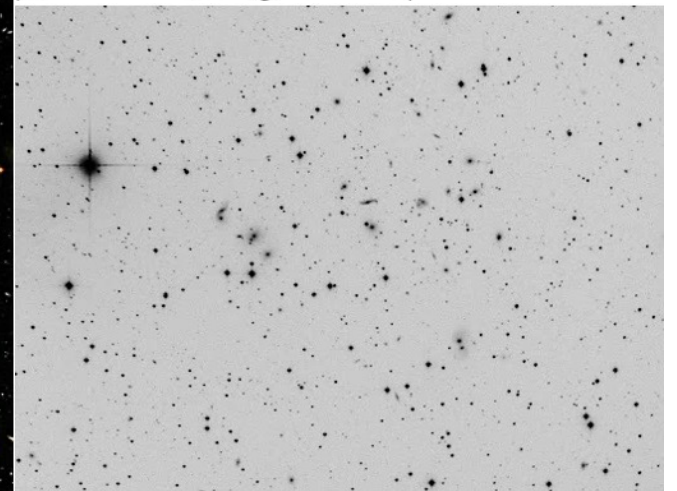
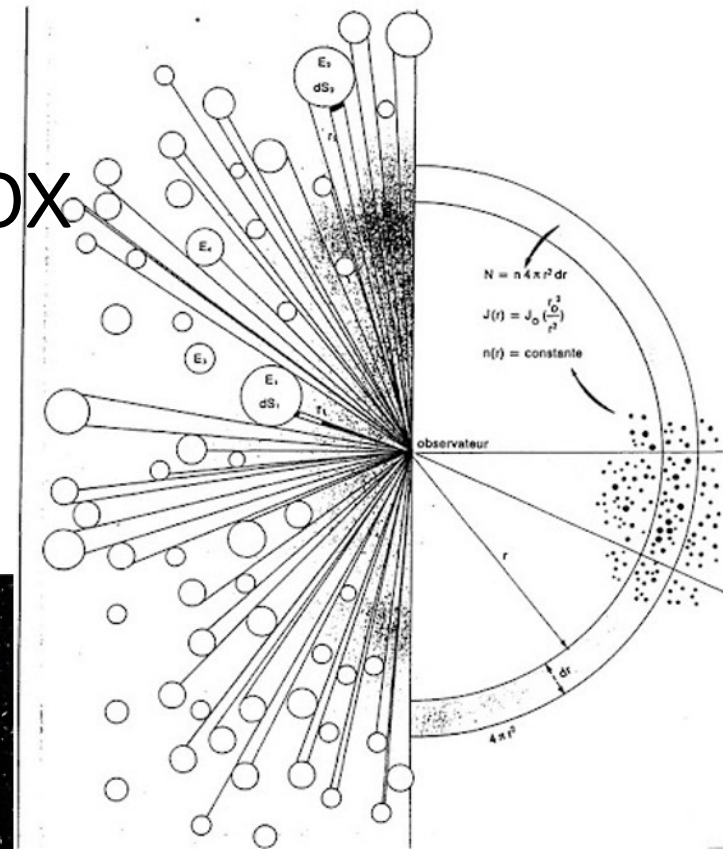
$$w(\lambda, \theta) = \exp\left(-\lambda\pi\Gamma\left(1 - \frac{2}{\alpha}\right)E[F^{\frac{2}{\alpha}}]\theta^{2/\alpha}\right)$$



# Side remark: the dark night paradox

- When  $\alpha < D$ :  $S(\lambda) \equiv \infty$ 
  - Olber's paradox of the dark night ( $D=3$ ,  $\alpha = 2$  in vacuum)

- when  $\alpha > D$ :  $S(\lambda) < \infty$ 
  - But  $E[S(\lambda)] = \infty$



# Second step overview

- When  $s(x) = \frac{1}{\|x-y\|^\alpha}$ , then

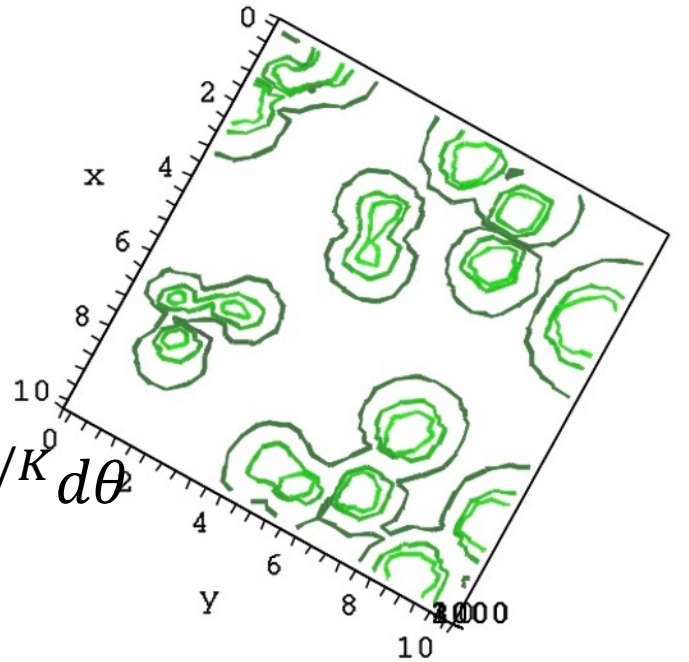
$$p(r, K, \lambda) = \frac{1}{2i\pi} \int_{-c-i\infty}^{-c+i\infty} \frac{1-w(\theta)}{\theta} e^{r^{-\alpha}/K} d\theta$$

- When  $s(x) = \frac{F(x)}{\|x-y\|^\alpha}$ , then

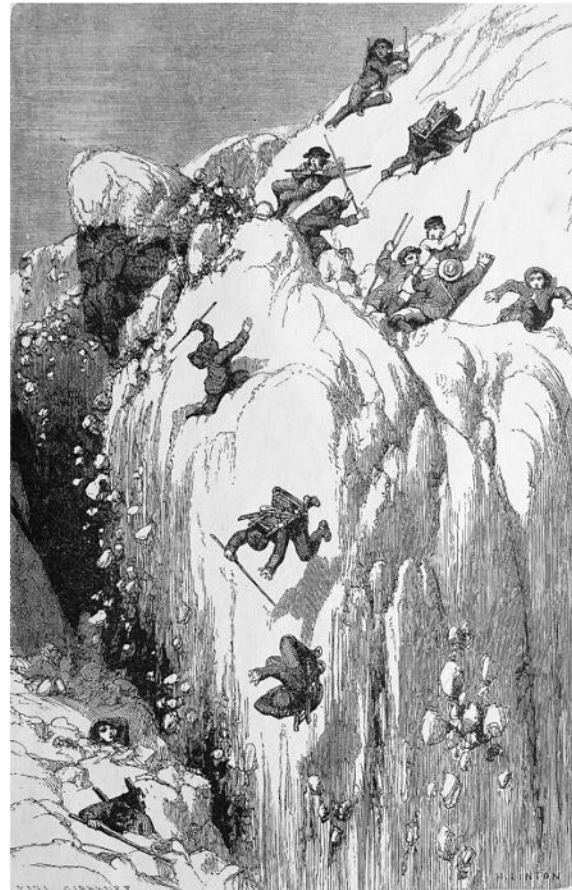
$$p(r, K, \lambda) = \frac{1}{2i\pi} \int_{-c-i\infty}^{-c+i\infty} \frac{1-w(\theta)}{\theta} E[e^{Fr^{-\alpha}/K}] d\theta$$

- Special case non selective Rayleigh fading

$$p(r, K, \lambda) = \exp\left(-\lambda\pi \frac{\frac{2\pi}{\alpha}}{\sin \frac{2\pi}{\alpha}} K^{\frac{2}{\alpha}} r^2\right)$$



Third step (forget it)



# The easy path: via an alternative formula

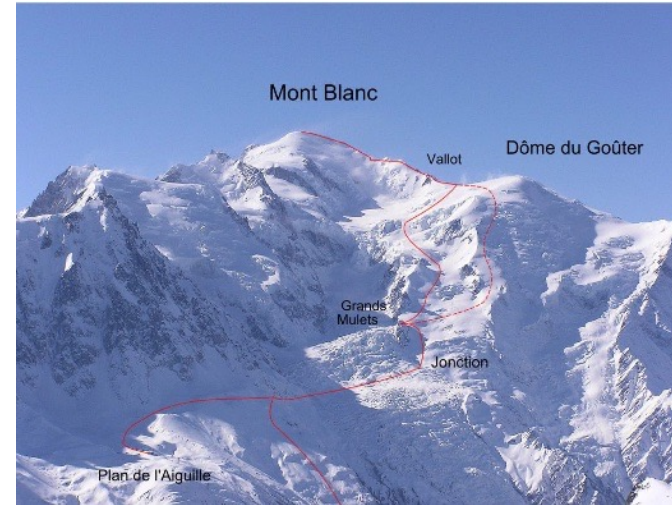
$$C(\lambda) = \iint \lambda E \left[ \log_2 \left( 1 + \frac{s(x)}{S(\lambda)} \right) \right] dx dy$$

- There is a resemblance between

$$C(\lambda) = \iint \lambda (E[\log_2(S(\lambda) + s(x))] - E[\log_2 S(\lambda)]) dx dy$$

- And

$$\frac{\partial}{\partial \lambda} E[\log_2 S(\lambda)] = \iint (E[\log_2(S(\lambda) + s(x))] - E[\log_2 S(\lambda)]) dx dy$$



# The energy differentiated field theorem

Does not depend on energy unit

$$C(\lambda) = \frac{\partial E [\log_2 S(\lambda)]}{\partial \log \lambda}$$



Does not depend on density unit

- The identity has a general scope, the energy  $S(\lambda)$  includes the thermal noise, etc. The transmitter locations should be i.i.d, eg modulated Poisson.

# MI Shannon capacity revisited

- Noiseless case

- Theorem:

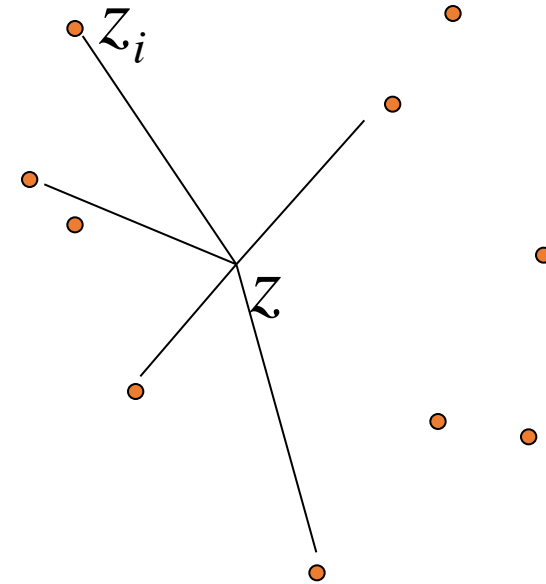
$$C(\lambda) = \frac{\alpha}{2 \log 2}$$

- Proof (the energy differentiated field)

$$C(\lambda) = \lambda \frac{\partial}{\partial \lambda} E(\log_2 S(\lambda))$$

$$E[\log S(x\lambda)] = -\frac{\alpha}{2} \log x + E[\log S(\lambda)]$$

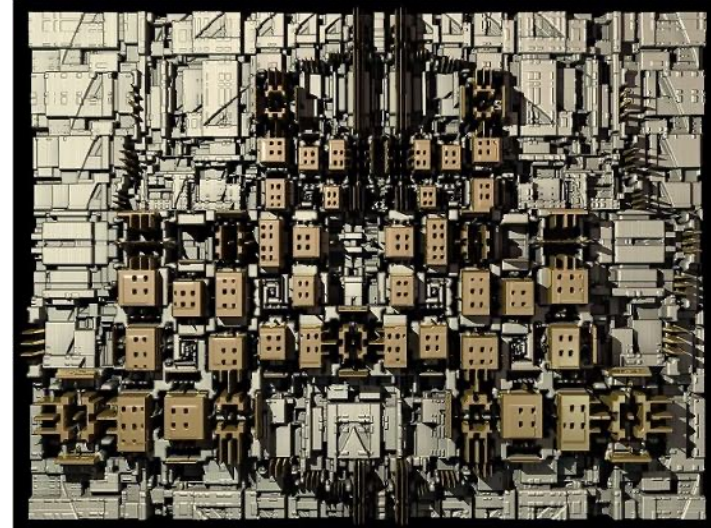
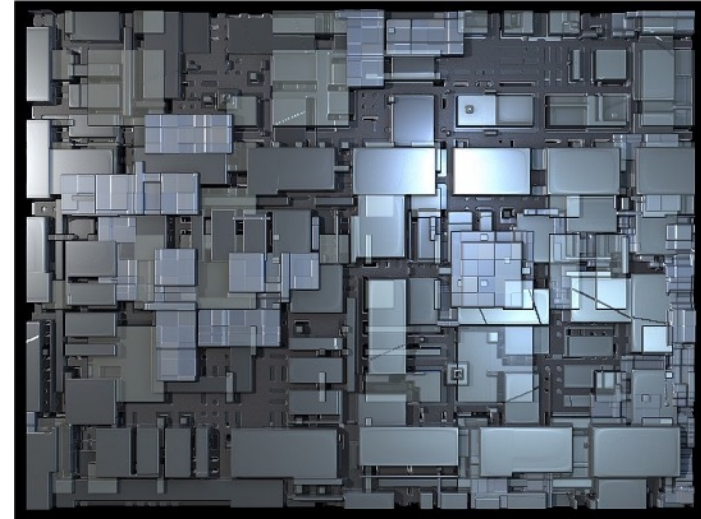
$$E[\log S(\lambda)] = \frac{\alpha}{2} \log \lambda + E[\log S(1)]$$





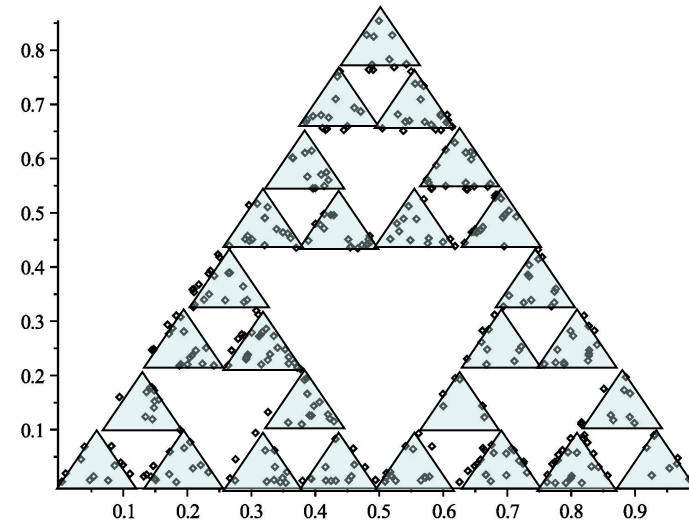
# Uniform Poisson models are not realistic

- The world is fractal
  - Fractal cities



# MI Shannon capacity theorem extended to fractal maps

- Eg Sierpinsky triangle
- Fractal dimension  $d_F = \frac{\log 3}{\log 2} \approx 1.58\dots$
- Extension of MI capacity theorem (access point in the fractal map)

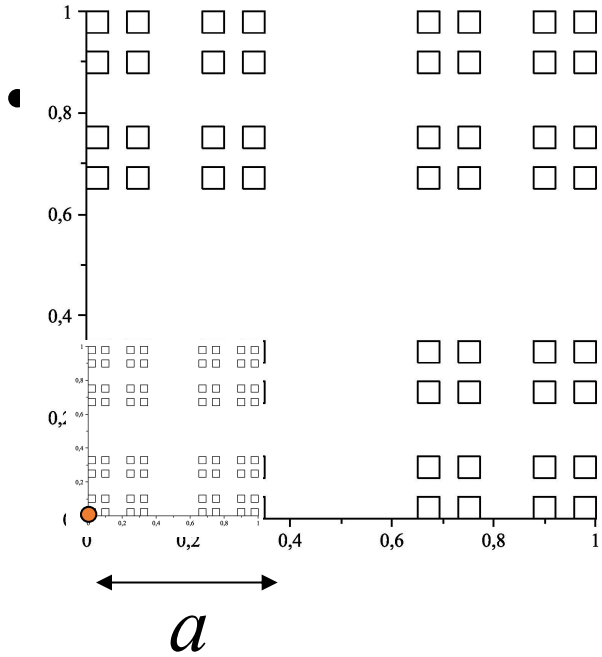


$$C(\lambda) = \frac{\alpha}{d_F \log 2} + P(\log \lambda)$$



Small periodic fluctuations of mean 0

# The Swiss flag fractal map



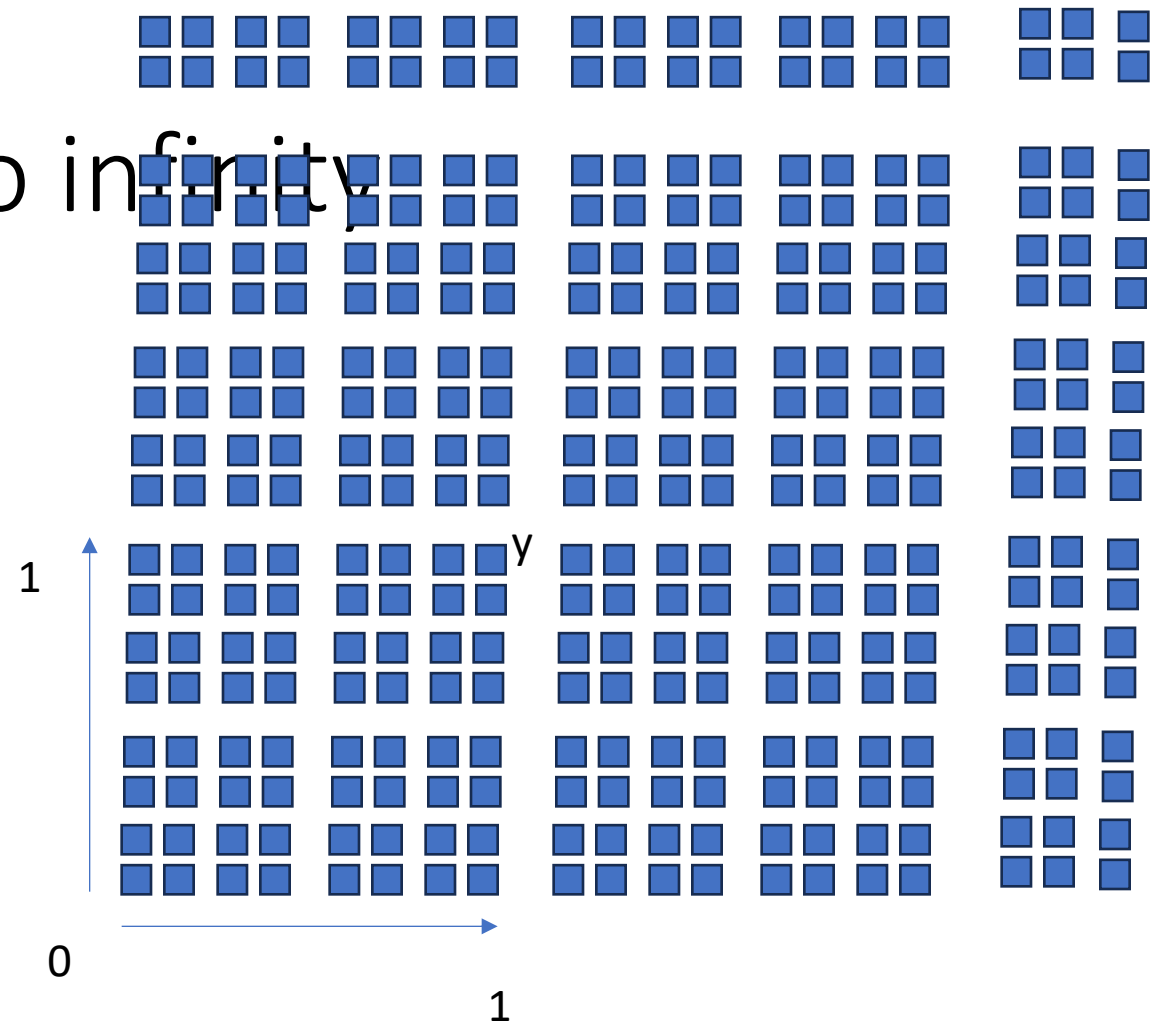
$$d_F = -\frac{\log 4}{\log a}$$



# Extending the swiss flag to infinity

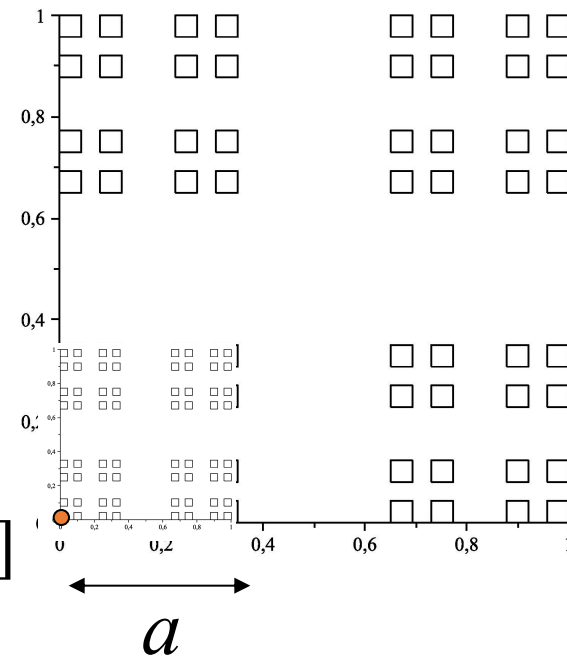
- The full quadran with a density  $\lambda$  in the unit quadran
- The overall density tends to zero.
- The access point position  $y$  must be specified in the MI capacity

$$C(\lambda, y)$$



# MI Shannon Capacity in the swiss flag

- Access point at left corner  $y=0$ 
  - Contraction by factor  $a$
  - Density increases  $\lambda \rightarrow 4\lambda$
  - $S(4\lambda, 0) \rightarrow a^{-\alpha} S(\lambda, 0)$



$$E[\log_2 S(4\lambda, 0)] = -\alpha \log_2 a + E[\log_2 S(\lambda, 0)]$$

$$E[\log_2 S(\lambda, 0)] = -\alpha \frac{\log_2 a}{\log 4} \log \lambda + Q(\log \lambda) \leftarrow \text{periodic}$$

$$C(\lambda, 0) = \lambda \frac{\partial}{\partial \lambda} E[\log_2 S(\lambda, 0)] = -\alpha \frac{\log_2 a}{\log 4} + Q'(\log \lambda) \leftarrow \text{Periodic of Mean zero}$$

# Random Access point in swiss flag

$$E[\log_2 S(4\lambda, ay)] = -\alpha \log_2 a + E[\log_2 S(\lambda, y)]$$

- Averaged over all  $y$ :

$$E[\log_2 S(\lambda)] = -\frac{\alpha}{d_F \log 2} \log \lambda + Q_1(\log \lambda)$$

$$C(\lambda) = \lambda \frac{\partial}{\partial \lambda} E[\log_2 S(\lambda)] = -\frac{\alpha}{d_F \log 2} + Q_1'(\log \lambda)$$

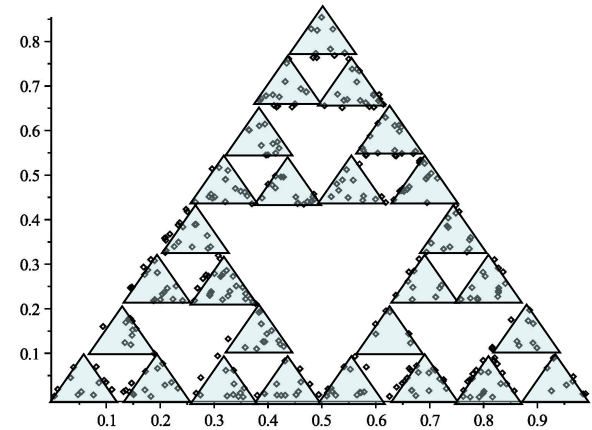
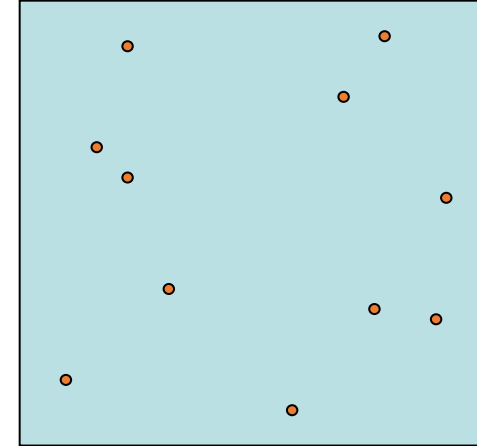
# Gain in capacity due to fractal dimension

- In the uniform plan: with  $\alpha = 4$

$$C(\lambda) = 2.885390082$$

- In the Sierpinsky triangle

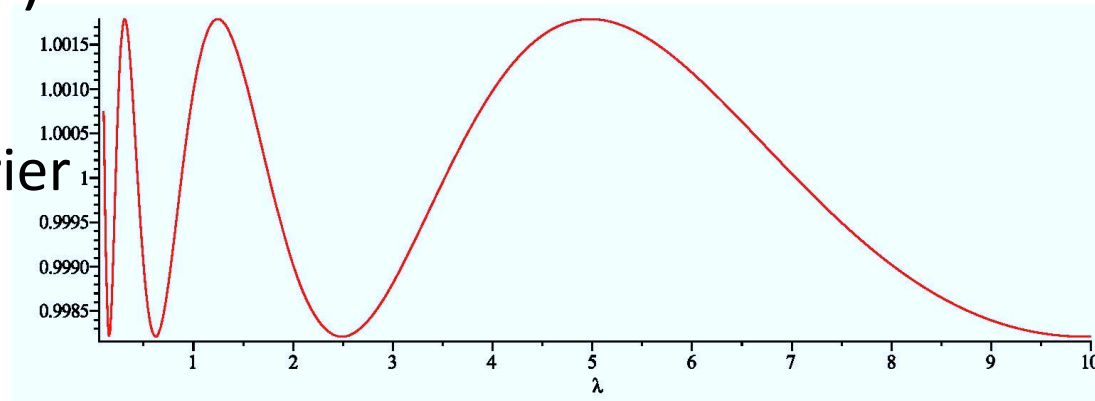
$$C(\lambda) = 3.640956906$$



# Periodic oscillations

- Small indeed (not enough for free lunch?)

- exact analysis via Fourier transform:



- Amplitude of order

$$a = 0.3, \alpha = 4 \quad 10^{-3}$$

P. Jacquet: Capacity of simple MIMO wireless networks in uniform or fractal maps, MASCOTS 2013



# Olber's paradox in a fractal universe

- When  $\alpha < d_F$ :  $S(\lambda) \equiv \infty$
- when  $\alpha > d_F$ :  $S(\lambda) < \infty$ 
  - But  $E[S(\lambda)] = \infty$
  - Fractal dimension of galactic filaments:  $d_F \approx 1.4 < 2$ .

