

Optimized Stokes imaging for highly resolved optical speckle fields, part II: Optimal acquisition & estimation strategies: supplemental document

This document contains supplemental information for the article *Optimized Stokes imaging for highly resolved optical speckle fields, part II: Optimal acquisition & estimation strategies*. It addresses the issue of the generation of a distribution of points across the surface of a sphere (the Poincaré's sphere in the present case) ensuring optimum conditioning number of the matrix defined by the concatenation of the points vector coordinates, and at the same time providing best uniform coverage of the surface of the sphere.

1. SUPPLEMENTAL INFORMATION 1

Generating uniform distributions of points on a sphere is a problem that admits non unique and non exact solutions [1, 2]. To generate the *Spiral* distributions used in this article, we relied on an numerical implementation directly inspired from Ref. [2]. One drawback of such spiral approaches is that they do not guarantee an optimal CN for the generated matrix W . In order to generate a distribution of N points composed of $N_t = N/4$ tetrahedrons that cover as uniformly as possible the Poincaré's sphere, we relied on the following approximate numerical procedure. Using the approach of Ref. [2], we generated a set of Nt points, denoted $\{T_1^A, \dots, T_{N_t}^A\}$ distributed along a spiral trajectory around the polarimetric axis S_3 , and forming the set of vertices A of the N_t tetrahedrons. Then, generating the set of the three remaining vertices $\{T_i^B, T_i^C, T_i^D\}$, $i \in [1, N_t]$ that maximize the uniformity of the total number of points is not as straightforward as it may appear.

Indeed, the natural idea is to apply a similar rotation on the set of Nt vertices A in order to rotate the spiral, these rotation operators being obtained by the rotation matrices allowing the reference point T_1^A of coordinates $[001]^T$ to be rotated on the other vertices of a regular tetrahedron (i.e., here, $T_1^B = [\sqrt{(2/3)} \quad \sqrt{(2/3)} \quad -1/3]^T$, $T_1^C = [0 \quad -\sqrt{(8/3)} \quad -1/3]^T$ and $T_1^D = [-\sqrt{(2/3)} \quad \sqrt{(2/3)} \quad -1/3]^T$). However, this solution does not permit to preserve the spiral structure for the four sets of N_t points, hence leading to a highly non-uniform final distribution of N states across the Poincaré's sphere. This can be clearly understood by analyzing a simple example: upon daily earth rotation, the harbours of Rotterdam (Netherlands) and Marseille (France) which lie at distinct latitudes do not travel the same distance. By generalizing this reasoning, one realizes that two isometric trajectories on the unit sphere are obtained with an axial rotation. The distance between the corresponding points by this isometry varies as a function of the polar angle with respect to the rotation axis, and therefore cannot be linked by a constant edge of a mobile rigid tetrahedron. It is thus impossible to "rotate" a set of points forming a pattern on a sphere (here a spiral) in order to form 3 patterns, isometric with the original one, and forming at the same time the vertices of regular tetrahedrons.

In order to circumvent this difficulty, an approximate solution was therefore adopted. The principle of this solution is to use an initial regular tetrahedron $\{T_1^A, T_1^B, T_1^C, T_1^D\}$ as a starting point, and calculate the rotation matrix R_{A1-A2} that sends point T_1^A to point T_2^A . This isometry is not unique, however, as it can be composed, on the left or right, by a rotation around the axis defined by the line joining the center of the sphere and the point T_1^A or T_2^A respectively. We therefore applied the following procedure: the initial rotation R_{A1-A2} was composed on the right by a rotation of angle θ around the end point T_2^A , which rotation leaves the point T_2^A invariant but modifies the position of points T_2^B , T_2^C , and T_2^D . For each angle θ , we then calculate the angle of "displacement" η^B , η^C and η^D undergone by each point between its original and its final position. Among all the possible angles $\theta \in [-\pi, \pi]$, we selected the value that minimized the following quantity $\|[\eta^B - \eta^C, \eta^C - \eta^D, \eta^D - \eta^B]\|_1$ ($\|\cdot\|_1$ denoting $l1$ -norm) in order to optimize the "uniform" filling of the sphere. This operation was then repeated step by step between the point quadruplets $\{T_i^A, T_i^B, T_i^C, T_i^D\}$ and $\{T_{i+1}^A, T_{i+1}^B, T_{i+1}^C, T_{i+1}^D\}$, until the N_t tetrahedrons have

been constructed. The sets of points $\{T_i^B, T_i^C, T_i^D\}$, $i \in [1, N_i]$ no longer form spiral trajectories, but the final distribution of N points on the sphere tends to be homogeneous, as can be seen in Fig. 2.a of the main article.

REFERENCES

REFERENCES

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