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Optimized Stokes imaging for highly resolved optical speckle fields, part II: Optimal acquisition & estimation strategies

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Abstract: In this second article of a three-paper series focusing on Stokes polarimetry of optical speckle fields resolved at the individual speckle grain scale, a theoretical study based on numerical simulations is presented in order to establish the optimum sensing, estimation and processing strategies that guarantee the best precision, accuracy and robustness for Stokes polarimetry in this specific context. In particular, it is demonstrated that the so-called State Of Polarization Analysis by Full Projection on the Poincaré space (SOPAFP) approach can be optimized in order to ensure best estimation performance. These numerical simulations also make it possible to establish that the SOPAFP approach provides better results in terms of robustness to residual experimental imperfections of the setup when compared to classical Stokes polarimetry approaches.

1. Introduction

This article is the second of a three-paper series titled “Optimized Stokes imaging for highly resolved optical speckle fields”. In the first article of the series [1], we have extensively described an optimized experimental Stokes polarimetric imaging system resolved at the speckle grain scale, and shown some preliminary results validating the experimental setup. In this second article, we will rely on numerical simulations to investigate and establish the optimal sensing and estimation strategies to perform Stokes imaging in the context of highly resolved speckle pattern.

For that purpose, we will compare the performances of various estimation schemes for (full-Stokes) polarimetry:

- either with approaches commonly used in standard polarimetric imaging involving a limited number (≥ 4) of intensity measurements along different polarization analyzing directions;
- or with the so-called SOPAFP approach (State Of Polarization Analysis by Full Projection on the Poincaré space) proposed initially in [2], and which, as its name suggests, probes numerous (several 10’s to several 100’s) polarimetric states located at the surface of the Poincaré sphere.

These various approaches and the optimal choice for these probe states are discussed in Section 3, after Section 2 recalls the mathematical formalism required. In Section 4, based on numerical simulations of realistic experiments, we evidence (for the first time to our knowledge) the fundamental interest of the SOPAFP approach by showing how it allows to guarantee the highest estimation accuracy and precision independently of the signal-to-noise (SNR) level or the degree of polarization (DOP) of light, while superseding classical approaches in terms of robustness or unbiasedness to certain experimental imperfections (such as calibration defects, misalignments,...). These comparisons are based on numerical simulations to evaluate various experimental influences encountered when studying the speckle field in a highly resolved manner. Indeed, the influence of SNR, orientation error and polarimetric analyzer calibration error on the quality of estimation of the state of polarization is evaluated, as is the ability of each approach to estimate partially depolarized states.

45 **2. Mathematical formalism and notations**

46 Throughout this paper, we will rely on the so-called Mueller/Stokes formalism to describe the
 47 polarimetric light-matter (linear) interaction by matrix calculation. The polarimetric state \mathbf{S}^{out}
 48 resulting from the interaction of an incident polarization state \mathbf{S}^{in} with a given sample can be
 49 calculated as follows [3] as $\mathbf{S}^{out} = M \mathbf{S}^{in}$, where M denotes a real 4×4 Mueller matrix which
 50 characterizes the polarimetric response of the sample. In the above equation, \mathbf{S}^{in} (respectively
 51 \mathbf{S}^{out}) denotes the Stokes vector (4-component real, column vector) of the input (resp. output
 52 light). This Stokes formalism is widely used in polarimetric imaging due to its ability to
 53 describe any polarimetric state of a light source, whether the light is fully polarized or not,
 54 and to the accessibility of its parameters through experimental intensity measurements. It
 55 is therefore perfectly suited for polarimetric imaging of highly resolved speckle fields and is
 56 classically defined [4] as $\mathbf{S} = [S_0, S_1, S_2, S_3]^T$, where \cdot^T denotes matrix transposition and with
 57 $S_0 = E_{0x}^2 + E_{0y}^2$, $S_1 = E_{0x}^2 - E_{0y}^2$, $S_2 = E_{0x}^2 E_{0y}^2 \cos(\phi)$ and $S_3 = E_{0x}^2 E_{0y}^2 \sin(\phi)$, E_{0x} (respectively
 58 E_{0y}) denoting the horizontal (respectively vertical) component of the transverse electric field
 59 of the light. S_0 represents the total intensity, S_1 is the portion of light linearly polarized along
 60 the vertical or horizontal axis, S_2 is the portion linearly polarized along the $\pm 45^\circ$ axes and S_3
 61 represents the portion circularly polarized right or left [4]. The degree of polarization can be
 62 determined from the Stokes vector as follows:

$$DOP = \frac{\sqrt{S_1^2 + S_2^2 + S_3^2}}{S_0} = \sqrt{s_1^2 + s_2^2 + s_3^2} = \|\mathbf{s}\|, \quad (1)$$

63 where the $s_i = S_i/S_0$, $i=1, \dots, 3$ are the components of the reduced 3-dimensional Stokes vector \mathbf{s}
 64 and with $\|\cdot\|$ denoting the vector quadratic norm.

65 Experimental estimation of the full Stokes vector therefore requires a minimum of four
 66 intensity measurements I_k resulting from the projection of the incident state \mathbf{S}^{in} with the
 67 probed polarimetric state $\mathbf{S}_{PSA,k}$ where k represents the index of the probed state of the
 68 polarimetric analyzer. The intensity measured for each acquisition of index k results from a
 69 scalar product of the input Stokes vector with the analysis Stokes vector, as $I_k = \mathbf{S}_{PSA,k}^T \cdot \mathbf{S}^{in}$.
 70 More generally, the set of measured intensities \mathbf{I} can be written as a k -dimensional column
 71 vector: $\mathbf{I} = [I_1, \dots, I_k, \dots, I_N]^T = W \mathbf{S}^{in}$ where W represents the *analysis* matrix composed of
 72 N probe states $\mathbf{S}_{PSA,k}$ ($k \in [1, N]$). The matrix can be defined through the following equation:
 73 $W^T = [\mathbf{S}_{PSA,1} \dots \mathbf{S}_{PSA,N}]$.

74 The concept of estimating the polarization state using a Mueller matrix description of the
 75 polarization state analyzer is not new and has provided a rich (both theoretical and experimental)
 76 literature even recently with non standard approaches [5–7]. Thus, to estimate the Stokes vector
 77 for a single polarization state, there exists a variety of available W sensing matrices. The selection
 78 of this analysis matrix depends mainly on the experimental conditions, such as the accessibility
 79 of the probe states and the number of states to be probed as a function of the available time to
 80 perform the measurement. Some probe states are more easily accessible with conventional optical
 81 elements, but the use of a variable delay phase plates such as liquid-crystal variable retarders
 82 (LCVRs) now makes it possible to access to any probe state. Indeed, as extensively described
 83 in the Part I of this article series [1], using a pair of LCVRs with appropriate orientations,
 84 followed by a polarizer, it is possible to access any probe state across the Poincaré's sphere,
 85 by addressing the LCVR1 (respectively LCVR2) with voltage level V_1 (resp. V_2), inducing an
 86 optical phase delay ϕ_1 (resp. ϕ_2) which is deduced from V_1 from a calibration step performed
 87 at fixed temperature and given wavelength. It is important to notice that, throughout this article
 88 series, experiments and simulations were performed at a fixed monochromatic wavelength of
 89 $\lambda = 532$ nm. Non-monochromatic illuminations could also be envisaged, but would require
 90 accounting for the dispersion of the optical elements and the optical phase delay of LCVRs. The

91 optimal choice of the W analysis matrix is discussed in more details in next section.

92 3. Analysis matrices W and associated estimation methods

93 This section describes the classical polarization analysis states (or equivalently, the analysis
94 matrices W) commonly used in polarimetric imaging and the associated estimation approaches.
95 The most widely used sets of polarization analysis states will be referred here as *Stokes-4*
96 (abbreviated *St-4*) and *Stokes-6* (abbr. *St-6*), and are represented respectively on the top left and
97 top right corners of Fig. 1. The bottom left corner of Fig. 1 shows the classical *Tetrahedron* (abbr.
98 *Tet*) analysis matrix for which optimality results for the estimation performance under various
99 noise models hold [8–14]. The last set of polarization analysis states, represented in the bottom
100 right part of Fig. 1, and which will be referred to as *Spiral* (abbr. *Sp*), is much less widespread
101 in the polarimetric imaging community. However, it has shown great experimental interest for
102 polarimetric imaging of highly resolved speckle fields [15], and is at the core of the so-called
103 SOPAFP approach described above.

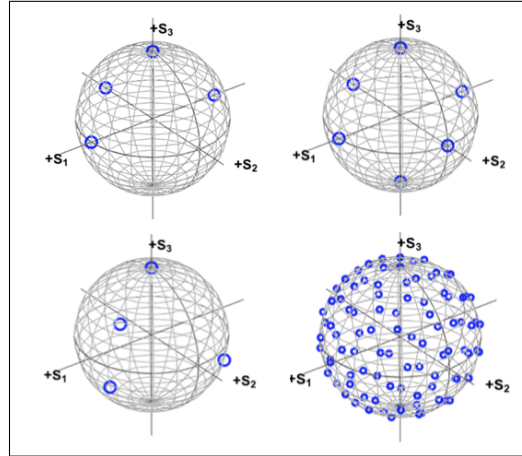


Fig. 1. Poincaré's sphere representation of various sets of polarization analysis states used in Stokes polarimetric imaging: (top left) *Stokes-4*, (top right) *Stokes-6*, (bottom left) *Tetrahedron* and (bottom right) *Spiral*.

104 All of these approaches are based on the measurement of a set of intensity values, related to
105 the incident polarimetric state by a system of linear equations recalled above. The performance
106 of the various sets of analysis states (i.e., of matrices W) in solving such system is studied in
107 the following sections, in terms of estimation accuracy and precision, as well as robustness to
108 experimental imperfections. For this purpose, we need to introduce the condition number (CN) of
109 an analysis matrix W , defined as $CN(W) = \|W\| \cdot \|W^{-1}\|$ [16], where $\|\cdot\|$ denotes the Euclidian
110 matrix norm. In this case, this CN is equal to the ratio between the largest and the smallest
111 singular value of the matrix W . In dimension 4, for the mathematical space of the Stokes vectors,
112 it is well known that its optimal value is $\sqrt{3}$ (≈ 1.7321) [11].

113 3.1. Stokes-4

114 The *Stokes-4* acquisition analysis matrix enables all these parameters to be evaluated in a
115 minimum of four measurements. Typically, the four intensities measured are I_H , I_V , I_{+45} and
116 I_L , corresponding to analyzing the light through standard directions of polarizations. These
117 intensities are easily accessible with conventional optical elements (polarizer, quarter-wave phase
118 plate, etc.), and correspond to the linear probe states (horizontal, vertical and $+45^\circ$) and the

119 circular probe state (circular left in our case). The resulting conditioning number is not optimal
 120 and is equal to 3.226. With this configuration, the estimates of the incident Stokes vector are
 121 generally obtained directly through:

$$\widehat{\mathbf{S}}_{St-4} = \begin{cases} S_0 = I_H + I_V \\ S_1 = I_H - I_V \\ S_2 = 2 \cdot I_{+45} - S_0 \\ S_3 = 2 \cdot I_L - S_0 \end{cases} \quad (2)$$

122 where $\widehat{\mathbf{S}}_{St-4}$ represents the estimated Stokes vector.

123 3.2. Stokes-6

124 The *Stokes-6* analysis matrix is used to determine the Stokes vector from the following six intensity
 125 measurements: $I_H, I_V, I_{+45}, I_{-45}, I_G$ and I_D . These intensities correspond to the six probe states:
 126 horizontal, vertical, $\pm 45^\circ$ and circular left and right. They are easily accessible experimentally,
 127 as described above. However, since this analysis matrix forms a regular octahedron within the
 128 Poincaré sphere (platonic solid), it allows an optimal CN equal to $\sqrt{3}$ to be obtained. The incident
 129 Stokes vector is also estimated by a direct computation of:

$$\widehat{\mathbf{S}}_{St-6} = \begin{cases} S_0 = I_H + I_V \\ S_1 = I_H - I_V \\ S_2 = I_{+45} - I_{-45} \\ S_3 = I_L - I_R \end{cases} \quad (3)$$

130 3.3. Tetrahedron

131 With technological advances, and in particular owing to the democratization of variable delay
 132 LCVRs, it has become possible to choose the polarization analysis states at will across the
 133 Poincaré's sphere. As a result, it is well-established that the optimum CN is obtained with
 134 minimum of 4 measurements when these states are located on the summits of a regular
 135 tetrahedron [8–14]. The incident Stokes vector is in that case estimated by a direct inversion of
 136 the analysis matrix W (full-rank) as follows:

$$\widehat{\mathbf{S}} = W^{-1}\mathbf{I} \quad (4)$$

137 3.4. SOPAFP approaches

138 An alternative introduced as SOPAFP (State Of Polarization Analysis by Full Projection on
 139 the Poincaré space) was implemented for the first time by J. Dupont *et al.* in the context of
 140 Stokes imaging of highly resolved speckle grains [2, 15]. It consists in probing an important
 141 number N of polarimetric states uniformly distributed across the surface of the Poincaré's sphere.
 142 This approach makes it possible to acquire finer polarimetric results than previous approaches,
 143 particularly in low-intensity areas, at the expense of a longer acquisition time. In this seminal
 144 work, the distribution of the analysis states formed a spiral across the Poincaré's sphere, and we
 145 shall refer to this set as "*Spiral*".

146 3.4.1. Spiral

147 The set of states used in [15] were chosen so as to form a spiral winding around the S_3 axis (see
 148 bottom right of Fig. 1). In this work, the Stokes vector was estimated at each pixel of the CCD

149 sensor by using a non-linear regression [2, 15] applied to an intensity curve obtained at each pixel
 150 of the sensor. Indeed, when the various sensing states are applied successively, the measured
 151 intensity at each pixel describes a curve as a function of the polarization state probed, and the
 152 shape of the curve unequivocally correspond to a given input state. An example of such curve
 153 is shown in Fig. 6 of the first article of this series [1], for a spiral composed of $N = 96$ states,
 154 providing a CN of $1.742 > \sqrt{3}$, but close to the optimal value.

155 This approach, coupled to a non-linear regression estimation technique, was applied to the
 156 experimental bench in our laboratory and has been tested in this work on simulation results in
 157 order to compare the performance with other estimation approaches. In our implementation, a
 158 non-linear regression was applied to each intensity curve to estimate the parameters p , χ and ψ
 159 corresponding respectively to the DOP, the polarization ellipticity and the polarization azimuth
 160 of the input light polarization, at a given pixel. These estimated parameters were finally used to
 161 derive the incident reduced Stokes vector, described in spherical coordinates as $\mathbf{s}_{in}^T = [s_1 \ s_2 \ s_3]^T$,
 162 with $s_1 = p \cos(2\psi) \cos(2\chi)$, $s_2 = p \sin(2\psi) \cos(2\chi)$ and $s_3 = p \sin(2\chi)$.

163 We can note that this approach is similar to that used by Dupont, *et al.* in [2, 15], except that
 164 they relied on a different parameterization of the input Stokes vector. In our case, the intensity
 165 measured by the CCD sensor can be rewritten as a function of the three parameters p , χ and ψ
 166 of the incident Stokes vector and of the optical delays $\phi_{1,k}$ and $\phi_{2,k}$ introduced by the LCVR1 and
 167 LCVR2 plates for each state $k = 1, \dots, N$ to be analyzed as:

$$I_k = \frac{I_0}{2} \left[1 - p \cdot \cos(2\chi) \cos(2\psi) \cos(\phi_{2,k}) \right. \\
 - p \cdot \sin(2\chi) \cos(\phi_{1,k}) \sin(\phi_{2,k}) \\
 \left. + p \cdot \cos(2\chi) \sin(2\psi) \sin(\phi_{1,k}) \sin(\phi_{2,k}) \right], \quad (5)$$

168 with I_0 the total intensity of the input state. As described in Section 4, this equation has been
 169 used in the present work to simulate the intensity measurements (direct model).

170 On the other hand, to evaluate the Stokes vector at each pixel from the measured intensity
 171 vector, we used an optimization algorithm (implemented on GNU Octave using the function
 172 `fminsearch()` [17]), to determine the parameters p , χ and ψ , given that the values of $\phi_{1,k}$ and
 173 $\phi_{2,k}$ are in principle perfectly known and correspond to the optical phase delays introduced by
 174 the LCVRs if the system is properly calibrated. The criterion to minimize simply consisted in a
 175 root-mean-square (RMS) error, defined as :

$$\hat{\epsilon}(p, \chi, \psi) = \sum_{k=1}^N |I_{simul,k}(p, \chi, \psi, \phi_{1,k}, \phi_{2,k}) - I_k|^2 \quad (6)$$

176 This optimization algorithm, which is based on the Nelder-Mead's [18] method, requires an
 177 approximate knowledge of the values of the parameters to be determined (initialization values)
 178 to prevent the optimisation of the algorithm from converging to a local minimum. To satisfy
 179 this requirement, an intensity look-up-table (LUT) has been generated for the phase shifts $\phi_{1,k}$
 180 and $\phi_{2,k}$ by varying each parameter p , χ and ψ from 0 to 1 in steps of 0.1. In this way, 1331
 181 intensity curves were simulated in the LUT, allowing the initialization value of the parameters
 182 vector in the optimization algorithm to be determined by picking up the set of parameters that
 183 minimizes the RMS residual error described by (6) between the measured intensity curve and
 184 the 1331 tabulated curves. Once seeded with such initialization, the minimization algorithm
 185 was shown to converge easily towards an estimated value. An example of a result obtained is
 186 shown in Fig. 6 of the first paper of this series [1], where we simulated the intensity curve that
 187 would be obtained by illuminating a metal plate with a light source with left circular incident
 188 polarization, and analyzing the backscattered light through $N = 96$ *spiral* analysis states. The

189 curve I_{th} corresponds to the theoretical polarization input state, whereas I_{opt} stands for the
 190 intensity variation obtained with the parameters estimated by numerical optimization.

191 However, in an imaging context, this optimization must be carried out at each location (pixel)
 192 of the image, which requires significant post-processing time for images of the order of 700×700
 193 pixels. For example, with a 4×4 pixel binning factor, the post-processing time lasts about
 194 $5.3 \times 10^3 s$, i.e. almost 90 minutes on an ordinary computer. To minimize this processing time,
 195 we proposed to rely on a classical direct inversion scheme, involving the pseudo-inverse of W ,
 196 since in this case W is not an invertible square matrix anymore, i.e.,

$$\widehat{\mathbf{S}}_{Sp} = (W^T W)^{-1} W^T \mathbf{I}. \quad (7)$$

197 As shown below, such inversion improves the processing time by several orders of magnitude.

198 3.4.2. Spiral of tetrahedrons analyzing states

199 As stated in the previous subsection, for $N = 96$ states probed, the *Spiral* analysis matrix is close
 200 to the optimum CN, but it has the disadvantage that this CN depends on the number of states
 201 probed (see Fig. 2.b). By optimizing the location of the polarization probe states, it is possible to
 202 ensure that the CN of the selected matrix W is strictly equal to $\sqrt{3}$. For that purpose, we propose
 203 to use a set of analysis states, referred to as *Spiral of tetrahedrons* (abbr. *SpTet*), which consists
 204 of a set of N_T regular tetrahedrons inscribed in the Poincaré's sphere, providing $N = 4 N_T$ probe
 205 states (i.e., a $4 \times N$ analysis matrix). As a result it is possible to build a set of states with optimum
 206 CN of any size N , provided N is a multiple of 4. An example of such set of states is provided
 207 in Fig. 2.a: $N = 96$ states have been generated from $N_T = 24$ tetrahedrons using the following
 208 method (detailed in Supplemental Information 1): starting with a first reference tetrahedron,
 209 regular rotations are applied so that one of the vertices of the tetrahedrons follows the trajectory
 210 of a spiral wounded along the S_3 axis (blue markers in Fig. 2.a) according to the previous *Spiral*
 211 approach. However, this rotation does not modify all four vertices in the same way, producing a
 212 less uniform distribution on the surface of the Poincaré's sphere, and the procedure detailed in
 213 Supplemental Information 1 is used to warrant the maximal uniformity of the N probe states
 214 across the sphere.

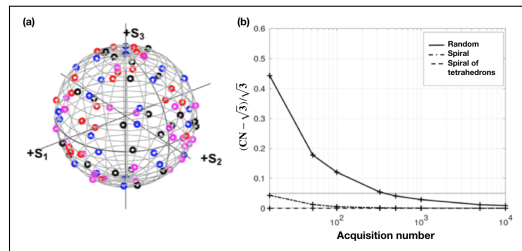


Fig. 2. (a) Distribution of analysis states for the *Spiral of tetrahedrons* analysis matrix, where each colour represents the state resulting from the rotation of one of the vertices of the reference tetrahedron. (b) Evolution of the CN of three analysis matrices (*Spiral of tetrahedrons*, *Spiral*, and uniformly distributed random set of points) as a function of the number of states $N \in \{16, 52, 100, 324, 500, 1000, 5000, 10000\}$.

215 In Fig. 2.b, we represented the relative excess in CN, with respect to the minimum optimal
 216 value of $\sqrt{3}$, for three different sets of probe states with increasing number of points $N \in$
 217 $\{16, 52, 100, 324, 500, 1000, 5000, 10000\}$: the *Spiral of tetrahedrons* set, the *Spiral* set, and a
 218 uniformly distributed random set of states. This plot confirms that, by construction, the *Spiral of*
 219 *tetrahedrons* set maintains optimal CN of the matrix W independently of the number of probe
 220 states. The conditioning of the "Spiral" matrix is very close to the optimal value when the number

221 of probe states is above 100. Finally, the CN obtained with the random set is very dependent on
 222 the number of states and increases very rapidly when the number of probe states decreases.

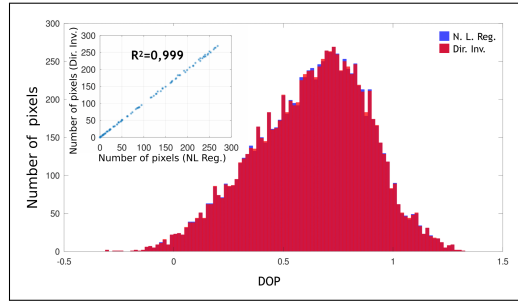


Fig. 3. Example of histogram of the DOP estimated by non-linear regression (blue) or direct inversion (red) on an experimental speckle image of 10^4 pixels. Inset: The number of pixels in each class obtained by the direct inversion method is plotted against those obtained by non-linear regression.

223 To conclude this section, we provide a comparison between the estimation results obtained on
 224 experimental data by the non-linear regression approach (as used in anterior implementations
 225 of the SOPAFP method [2, 15]), and by the direct inversion procedure of Eq. (7) in order to
 226 validate that this latter approach is able to provide the same results but with a huge acceleration
 227 in processing time. In Fig. 3, we plotted the histogram of the DOP values estimated on 10^4 pixels
 228 of a speckle image obtained by illuminating a metal plate with a vertically polarized incident
 229 homogeneous light beam. The experimental setup corresponds to the one described in the first
 230 part of this article series [1]. This graph shows that the two estimation methods are very similar,
 231 which is confirmed by the coefficient of determination R^2 of 0.999. Across the whole field
 232 of view of 700×700 pixels accessible in this experiment, using a binning factor of 2×2 , the
 233 non-linear regression estimation requires $> 2 \times 10^4$ s, whereas only 14.4 s are required for direct
 234 inversion, i.e., an speed-up factor of more than 1500 times. In the following of this article series,
 235 direct inversion estimation will then be used when the SOPAFP approach is implemented, and in
 236 particular in the simulation results presented in the next section.

237 4. Comparative performances of analysis matrices for Stokes imaging

238 In this section, we describe the methodology and performance metrics used to generate exhaustive
 239 simulations of the estimation performances (in terms of accuracy and precision) of the various
 240 sensing strategies detailed above. We then comparatively analyze the efficiency of these various
 241 matrices W while varying the simulated experimental conditions in terms of signal-to-noise ratio
 242 (SNR), DOP and influence of an imperfect calibration/alignment of the experimental setup.

243 4.1. Simulation methodology

244 The five choices of analysis matrices W presented above will be compared in this section in
 245 terms of Stokes vector estimation performance. To compare these various sensing approaches
 246 on a fair basis, we will consider a set of $N = 96$ simulated experimental measurements for each
 247 matrix W . For all matrices W requiring fewer analysis states, we will consider a repetition of
 248 independent measurements in order to obtain the same number of $N = 96$ intensity measurements,
 249 as summarized in Table 1.

250 Now, in order to characterize the "average" estimation performance of each analysis matrix W
 251 over all the input polarimetric states \mathbf{S}_{in} that could be encountered, we further simulated $M = 10^3$
 252 randomly selected input Stokes vectors. These randomly simulated states were in practice

Table 1. Comparison between the number of analysis and repetition states for each analysis matrix.

Matrix W	Stokes 4	Stokes 6	Tetrahedron	Spiral	Spiral of tetrahedrons
# distinct states	4	6	4	96	96
# repetitions	24	16	24	1	1
Total measures	$N = 96$				

253 produced so as to obtain Stokes vectors corresponding to a purely polarized light ($DOP = 1$),
 254 by drawing $M = 10^3$ independent uniformly distributed values between -1 and 1 of the three
 255 reduced Stokes parameters s_1 , s_2 and s_3 . After normalization of the vector \mathbf{s}_{in} , and multiplication
 256 by a constant value S_0 , we obtain the set of simulated Stokes vectors as: $S_{in}^T = S_0 \left[1 \quad \frac{\mathbf{s}_{in}^T}{\|\mathbf{s}_{in}\|} \right]$.

257 Finally, we modeled the influence of the photon noise on the simulated acquisitions. This is
 258 important as, in the context of speckle imaging at the grain scale, the acquired intensity images
 259 are composed of bright grains (corresponding to constructive interferences) and dark areas
 260 (corresponding to destructive interferences). The SNR is therefore higher in the bright areas,
 261 while it is degraded in the regions of lower intensity. The SNR therefore varies spatially in the
 262 intensity image, and we must verify the robustness of the estimation procedure w.r.t. noise. For
 263 that purpose, all measurements were repeated over a number of $R = 10^3$ independent realizations
 264 of Poisson noise affecting the intensity measurement simulated. For each simulated acquisition
 265 with average intensity I , we generated R noisy versions of the measure n_{I_j} , $j \in [1, R]$ with
 266 n_{I_j} identically and independently distributed according to a Poisson probability distribution of
 267 parameter I (i.e., $P_{n_{I_j}}(n_{I_j}) = e^{-I} I^{n_{I_j}} / n_{I_j}!$). Poisson noise being multiplicative, the SNR of the
 268 simulated acquisition is controlled by the value of the total intensity of the input light S_0 .

269 In summary, the simulation results reported below will compare the performance of various
 270 sensing matrices W over a set of $M = 10^3$ distinct input Stokes vectors, each of them requiring
 271 $N = 96$ intensity measurements to be estimated, each estimation being repeated itself over
 272 $R = 10^3$ noise realizations.

273 4.2. Mathematical performance descriptors

274 In the following, we may resort to two definitions of performance metrics for the estimation
 275 problem at stakes, to account respectively on the accuracy and precision of the estimation method.

276 4.2.1. Accuracy

277 For given input Stokes vector, the accuracy of the estimation is evaluated through the computation
 278 of an average relative systematic bias \hat{b} , defined by

$$\hat{b} = \frac{\|\mathbf{S}_{sim} - \langle \hat{\mathbf{S}} \rangle\|}{\|\mathbf{S}_{sim}\|}, \quad (8)$$

279 where the known simulated Stokes vector \mathbf{S}_{sim} is compared to the mean estimated Stokes vector
 280 $\langle \hat{\mathbf{S}} \rangle$ resulting from the $R = 10^3$ noise realizations for each input Stokes vector.

281 A further averaging over the $M = 10^3$ distinct input Stokes vectors finally allows the average
 282 bias $\langle \hat{b} \rangle$ to be computed. This quantity will be used in the following as a global figure of merit of
 283 the estimation accuracy for the different matrices W .

284 4.2.2. Precision

285 Similarly, we proposed to evaluate the relative precision of the Stokes vector estimate by computing
 286 the norm of the vector of standard deviations of the Stokes vector components, evaluated over the
 287 $R = 10^3$ noise realizations, and normalized by the norm of the average Stokes vector estimate:

$$\hat{\sigma} = \frac{\left\| \sqrt{\langle \hat{\mathbf{S}}^2 \rangle} - \langle \hat{\mathbf{S}} \rangle \right\|}{\|\langle \hat{\mathbf{S}} \rangle\|}. \quad (9)$$

288 A further averaging over the $M = 10^3$ distinct input Stokes vectors finally allows the average
 289 relative precision $\langle \hat{\sigma} \rangle$ to be computed. This quantity will be used in the following as a global
 290 figure of merit of the estimation precision for the different matrices W .

291 4.2.3. Graphical representation

292 In the following graphs, these descriptors \hat{b} and $\hat{\sigma}$ will be represented in the form of a Turkey
 293 box indicating the distribution of the data in the form of quartiles (first and third quartiles), with
 294 the median represented by a red line, and the minimum and maximum value of the sample given
 295 by the vertical error bar. In addition, the mean values $\langle \hat{b} \rangle$ or $\langle \hat{\sigma} \rangle$ over the $M = 10^3$ various input
 296 Stokes vectors will also be represented with a pink square, as shown in Fig 4.b for instance.

297 4.3. Simulation results and analysis

298 In this subsection, we focus on the influence of the main experimental errors present on a
 299 polarimetric imaging system, such as the measurement noise, presence of partial depolarization,
 300 and finally a misalignment or a calibration error of the PSA.

301 4.3.1. Influence of the SNR

302 To estimate the influence of the SNR on the estimation of the Stokes vector, we generated similar
 303 simulations for three average intensities corresponding to 10^3 , 10^4 and 10^5 photons for a fixed
 304 exposure time, leading respectively to SNR values of 31.6, 100 and 316.2. The results shown
 305 below, intending to compare the 5 different matrices W for the estimation of the Stokes vector,
 306 have been carried out on the same set of simulated measurements.

307 The simulation results in terms of estimation accuracy are displayed in Fig. 4, for the various
 308 SNRs and matrices W studied. As expected, the relative average bias $\langle \hat{b} \rangle$ decreases as the SNR
 309 increases, as can be clearly seen in Fig. 4.a. On the one hand, we observe that the accuracy
 310 is clearly improved at low SNR (Fig. 4.b) when the "Tetrahedron", "Spiral" and "Spiral of
 311 Tetrahedrons" analysis matrices are considered: the mean bias is lower, but also the maximum
 312 bias observed is clearly diminished by a factor of about 3. This is interesting in the context of our
 313 study of polarimetry of highly resolved speckle grains, as the SNR will be severely degraded in
 314 low-intensity regions.

315 On the one hand, at high SNR (Fig. 4.d) the relative average bias seems almost similar for
 316 the various analysis matrices. However, if we look at the distribution of estimated biases, the
 317 dispersion of the biases appears to be slightly lower for the "Spiral" and "Spiral of Tetrahedrons"
 318 matrices. This lower bias dispersion seems to indicate that an analysis matrix with many
 319 states distributed uniformly over the Poincaré's sphere can provide a systematic bias that is less
 320 dependent on the polarimetric state to be analyzed, than with a matrix W comprising only a few
 321 probed states.

322 We now focus on the estimation precision of the different analysis matrices, which is evaluated
 323 through the computation of the normalized standard deviation of the vector estimates as in Eq. (9).
 324 As shown in Fig. 5 a., the precision logically increases with the SNR value. We also observe that

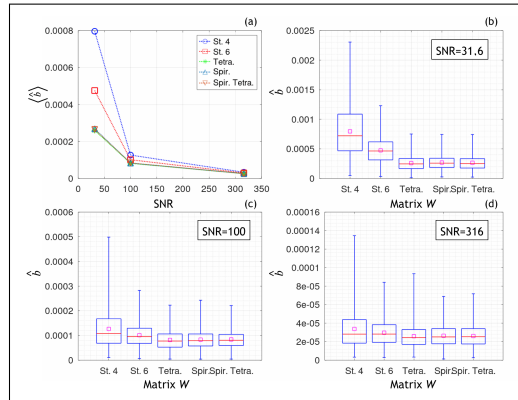


Fig. 4. (a) Average estimation bias over 1000 random polarimetric input states as a function of the SNR and for different analysis matrices W . (b-d) Distribution of the relative estimation bias over the $M = 10^3$ different polarization states, as a function of different analysis matrices for three distinct SNRs.

325 for a given SNR, the mean relative precision $\langle \hat{\sigma} \rangle$ remains of the same order of magnitude for
 326 the various matrices W . More interesting to analyze is the dispersion of $\hat{\sigma}$ over the $M = 10^3$
 327 input states. We can clearly see that the span of values reached by the estimated relative standard
 328 deviation $\hat{\sigma}$ is 11 times (respectively 2.75 times) higher for the *Stokes-4* approach (respectively
 329 for the *Stokes-6* approach) than for the last three matrices W , which seem to be a much better
 330 choice to avoid extreme values of standard deviations when the input Stokes vector is varied.

331 In order to compare the relative performances of these last three approaches, two of them
 332 being by construction optimal as they reach the minimum CN of the matrix W (*Tetrahedron* and
 333 *Spiral of Tetrahedrons*), we plot in Fig. 5.b the distribution of their relative standard deviations
 334 $\hat{\sigma}$, for a SNR of 100. It can be seen that precision obtained for each matrix is very similar (a
 335 feature that occurred to be independent of the SNR, as tested during further simulations which
 336 are not reported here for the sake of conciseness). As discussed later, the non-optimality of the
 337 *Spiral* set of sensing states cannot be detected here, due to the important number of states $N = 96$
 338 considered.

339 As an intermediate conclusion, these first simulation results confirm that the SOPAFP
 340 approaches (using either *Spiral* and *Spiral of tetrahedrons* matrix) can be employed to determine
 341 any Stokes vector with a better accuracy than usual Stokes imaging approaches, in particular at
 342 low SNR. In addition, these matrices have the supplementary advantage of providing near-optimal
 343 estimation precision, with a much lower dispersion of the estimation standard deviation compared
 344 with the more conventional *Stokes-4* and *Stokes-6* measurement methods.

345 4.3.2. Influence of depolarization

346 In this subsection, we analyze the influence of a possibly non unitary DOP of the input Stokes
 347 vector to estimate. This is common in the context of applications of polarization imaging in
 348 the biomedical or industrial field, but this can also be relevant in our context of speckle grains
 349 polarimetry. Indeed, depending on the polarization properties of the illumination light source,
 350 the DOP within a speckle grain can range from 0 to 1 in the general case. Therefore, we analyzed
 351 the robustness of the various sensing matrices W to a partial depolarization of the input Stokes
 352 vector. To this end, we simulated three different experiments, strictly similar to each other, except
 353 for the DOP p of the simulated input Stokes vector, that was varied among $p \in \{1, 0.5, 0.1\}$.
 354 This is easily operated by forming, for each normalized Stokes vector \mathbf{s}_{in} generated, the following

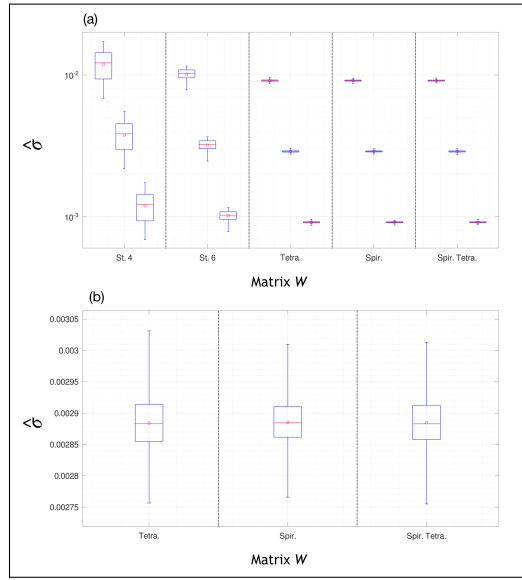


Fig. 5. (a) Influence of the SNR on the relative precision of the Stokes vector estimates for various analysis matrices W . For each analysis matrix, the three candlesticks positioned from left to right correspond respectively to SNR levels of: 31.6, 100 and 316. (b) Dispersion of the precision \hat{b} of the Stokes vector estimates for the *Tetrahedron*, *Spiral* and *Spiral of Tetrahedrons* analysis matrices for a SNR of 100.

355 Stokes vector: $S_{in}^T = s_0 \left[1 \quad p \frac{S_{in}^T}{\|S_{in}\|} \right]$.

356 Photon noise with a SNR of 100 was again applied to the simulated signals and $R = 10^3$
 357 realizations of noise were carried out for each of the $M = 10^3$ random states to be estimated.
 358 Concerning the estimation accuracy, Fig. 6 shows that the average relative bias increases as the
 359 DOP decreases, in a similar way for all the analysis matrices studied. To be more quantitative,
 360 we note that the ratio between the average relative bias obtained for a DOP of 1 and that of 0.1 is
 361 of the order of 10.

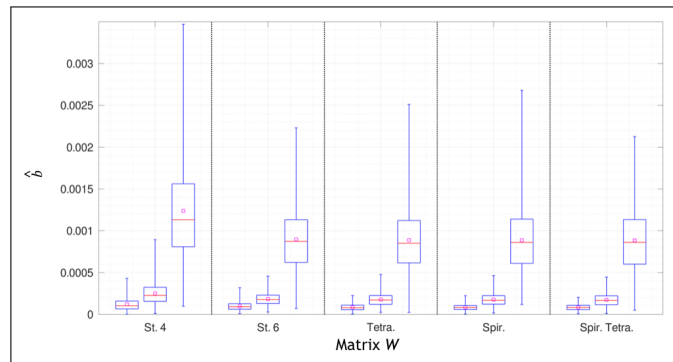


Fig. 6. Influence of the DOP of the input Stokes vector on the estimation quality of each analysis matrix. For the same analysis matrix, the three candlesticks positioned from left to right correspond respectively to a DOP of 1, 0.5 and 0.1. These simulations were carried out with an SNR of 100 and over $M = 10^3$ distinct input Stokes vectors.

362 Furthermore, for a given DOP, it can be observed that the average bias (pink squares) is
 363 approximately comparable, regardless of the analysis matrix chosen, except for the *Stokes-4*
 364 configuration, where the average bias obtained is 1.4 times higher than that of the other matrices.
 365 However, in terms of dispersion of the bias, Fig. 6 tends to show that for a DOP of 0.1 the *Spiral*
 366 *of tetrahedrons* is the best analysis matrix, the dispersion for the other matrices being slightly
 367 higher.

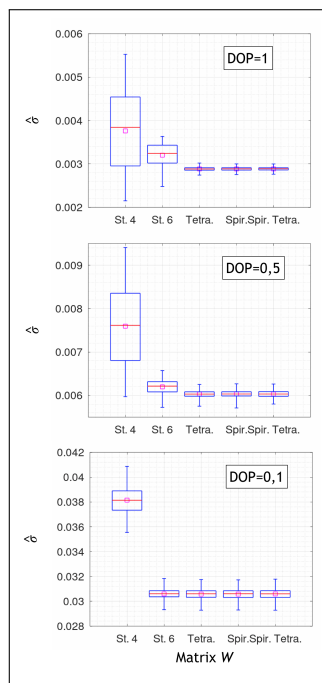


Fig. 7. Influence of depolarization on the estimation precision of the Stokes vector for each analysis matrix W . Three values of DOP are studied: 1, 0.5 and 0.1 respectively.

368 As for the estimation precision, Fig. 7 shows that the relative precision is degraded as the DOP
 369 decreases, by a factor of 10 between that obtained for a DOP of 1 and 0.1, independently of the
 370 analysis matrix. This ratio of precision degradation is similar to that calculated for the estimation
 371 accuracy. It can be seen that the *Tetrahedron*, *Spiral* and *Spiral of Tetrahedrons* analysis matrices
 372 have comparable accuracy (average value and dispersion) and evolve with the DOP in a similar
 373 way. The *Stokes-4* configuration is still the one with the highest precision dispersion, and is
 374 therefore of little interest in our case for fine polarimetric analysis of the speckle pattern. As for
 375 the *Stokes-6* approach, the performance is lower than that of the *Tetrahedron*, *Spiral* and *Spiral of*
 376 *tetrahedrons* matrices for a high DOP, but as the polarization rate decreases, its precision and
 377 precision dispersion become comparable to those of the other three matrices.

378 Once again, the sensing configuration proposed in this article (*Spiral of Tetrahedrons*) allows
 379 one to guarantee the best compromise between estimation accuracy and precision. This is further
 380 evidence in the next subsection, where we analyze the robustness to experimental imperfections
 381 of the setup.

382 4.3.3. Influence of experimental imperfections

383 In this section, we study the influence of a misalignment or of an incorrect calibration of one of
 384 the elements of the PSA used in the Stokes imaging setup developed to study the polarimetry of

385 highly resolved speckle fields. The PSA classically consists of two LCVRs ($LCVR_1$ and $LCVR_2$
 386 respectively), whose fast axes are aligned respectively along the vertical axis and at -45° from the
 387 horizontal axis in a clockwise direction, followed by a linear polarizer along the vertical direction.

388 The Mueller matrix of such PSA reads $M_{PSA} = M_P(\theta_p)M_{LCVR_2}(\theta_2, \phi_2)M_{LCVR_1}(\theta_1, \phi_1)$
 389 where $M_P(\theta_p)$ is the Mueller matrix of an ideal polarizer whose eigenaxis is oriented along an
 390 angle θ_p and is defined as :

$$M_P(\theta_p) = \frac{1}{2} \begin{pmatrix} 1 & c(2\theta_p) & s(2\theta_p) & 0 \\ c(2\theta_p) & c^2(2\theta_p) & c(2\theta_p)s(2\theta_p) & 0 \\ s(2\theta_p) & c(2\theta_p)s(2\theta_p) & s^2(2\theta_p) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (10)$$

391 where we set $c(x) = \cos(x)$ and $s(x) = \sin(x)$. In our case, θ_p is ideally set equal to $\pi/2$. The
 392 Mueller matrix of a variable phase plate is defined as :

$$M_{LCVR}(\theta, \phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c^2(2\theta) + c(\phi)s^2(2\theta) & c(2\theta)s(2\theta) [1 - c(\phi)] & -s(\phi)s(2\theta) \\ 0 & c(2\theta)s(2\theta) [1 - c(\phi)] & s^2(2\theta) + c(\phi)c^2(2\theta) & s(\phi)c(2\theta) \\ 0 & s(\phi)s(2\theta) & -s(\phi)s(2\theta) & c(\phi) \end{pmatrix}, \quad (11)$$

393 where θ and ϕ represent respectively the orientation of the eigenaxes and the voltage-controllable
 394 optical phase delay applied between the two propagation eigenaxes in the liquid crystal. In the
 395 ideal case, $\theta_1 = \pi/2$ and $\theta_2 = -\pi/4$.

396 In this section, to analyze the robustness of the estimation methods to experimental imperfec-
 397 tions, an orientation error ϵ of 1° was introduced successively on one of the PSA elements. As
 398 described below, we also simulated a possible incorrect value of the optical phase delays due to
 399 an imperfect calibration. The analyzer Mueller matrix is thus written as :

$$M_{PSA} = M_P(\theta'_p)M_{LCVR_2}(\theta'_2, \phi'_2)M_{LCVR_1}(\theta'_1, \phi'_1), \quad (12)$$

400 with $\theta'_i = \theta_i + \epsilon_i$, $i \in \{p, 1, 2\}$, and $\phi'_i = \phi_i + \delta\phi_i$, $i \in \{1, 2\}$. For instance, simulating an
 401 orientation error in LCVR1 only will thus consist in fixing $\epsilon_1 = 1^\circ$, and $\epsilon_2 = \epsilon_p = 0^\circ$, $\delta\phi_i = 0$
 402 for $i \in \{1, 2\}$. As in previous simulations, the generated signals are perturbed with photon noise
 403 corresponding to an SNR of 100, and $M = 10^3$ input Stokes vectors are tested to compute an
 404 average accuracy/precision of estimation with different analysis matrices W .

405 **Misalignment error:** We analyze in Fig. 8 the influence of a misalignment of a PSA element
 406 on the accuracy of the estimation of the Stokes vector for the various analysis matrices. The main
 407 conclusion that can be drawn from this plot is that the *Spiral* and *Spiral of Tetrahedrons* analysis
 408 matrices have a similar influence on the mean relative bias and its dispersion, regardless of which
 409 optical element of the LCVR has an orientation error. Contrarily, for the other analysis matrices,
 410 the bias introduced by an orientation error has a much higher dependency on the misoriented
 411 element, particularly if the orientation error relates to one of the variable delay plates. Thus,
 412 by probing numerous states distributed uniformly over the surface of the Poincaré's sphere, an
 413 error in the orientation of the axes of one of the PSA's optical elements will generate a bias that
 414 is comparatively low, and roughly constant irrespectively of which element is disoriented. In
 415 addition, for these two analysis matrices, the distribution of third quartiles of the estimated states
 416 is clearly lower than for other sensing strategies. These simulations thus prove that one strong

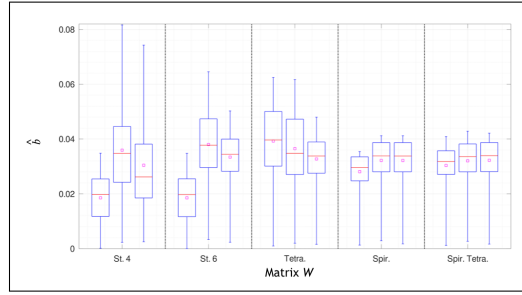


Fig. 8. Influence of a misalignment of a PSA element on the accuracy of the estimation of the Stokes vector for each analysis matrix. For each analysis matrix, the three candlesticks positioned from left to right correspond respectively to an orientation error of 1° on LCVR1, LCVR2 and polarizer.

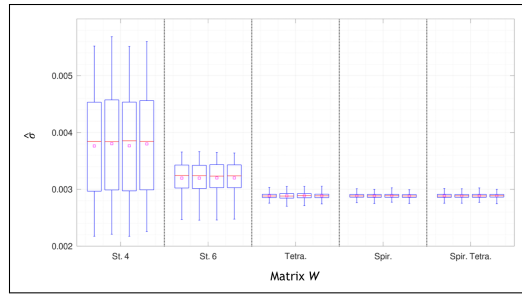


Fig. 9. Influence of a misalignment of a PSA element on the precision of the estimation of the Stokes vector for each analysis matrix. For each analysis matrix, the three candlesticks positioned from left to right correspond respectively to an orientation error of 1° on LCVR1, LCVR2 and polarizer.

417 interest in using the SOPAFP approach and its variants lies in its capacity to better limit the
 418 influence of experimental biases on the accuracy of the estimated results.

419 The precision measurement is analyzed in Fig. 9, which shows that a limited misalignment
 420 of one of the elements of the PSA has no influence on the accuracy of the estimate of the
 421 Stokes vector. This observation can be interpreted by the fact that a (limited) misalignment
 422 of the PSA simply leads to the estimation of an inaccurate (biased) polarization state. In this
 423 way, analysis matrices composed of many states distributed uniformly over the surface of the
 424 Poincaré's sphere have an rather constant bias regardless of the misaligned element (as shown
 425 above) and consequently, a less significant bias dispersion giving them greater robustness to
 426 experimental orientation errors of the optical setup.

427 **LCVR calibration error:** Another source of experimental error can be encountered when
 428 an incorrect calibration of the LCVRs voltage setpoint is used (due to an imprecise calibration,
 429 temperature drifts...). Fig. 10.a shows the calibration curve of the LCVRs used in our experiment
 430 providing the optical phase delay generated when a control voltage V is applied on the cell, at a
 431 stabilized temperature of $24^\circ C$. As mentioned in the first article of this series [1], a polynomial
 432 fit of these calibration curves allowed us to easily simulate the influence of a systematic error δV_i
 433 in the voltage setpoints, by simulating the following Mueller matrix for the PSA: .

$$M_{PSA} = M_P(\theta_p)M_{LCVR2}(\theta_2, \phi'_2)M_{LCVR1}(\theta_1, \phi'_1), \quad (13)$$

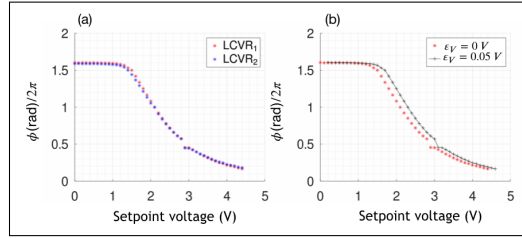


Fig. 10. (a) Calibration curve for the two LCVR plates for 45 different voltages ranging from 0 to 4.5 V, in steps of 0.1 V. (b) LCVR1 calibration curve with and without a set-point voltage error $\epsilon_V = 0.05$.

434 with $\phi'_i = \phi_2(V_i + \delta V_i)$, $i \in \{1, 2\}$. An error of $\delta V_1 = 0.05$ V has been simulated and provides
 435 the erroneous response in optical phase delay shown in Fig. 10.b.

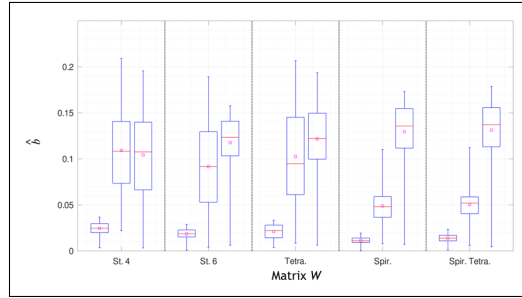


Fig. 11. Influence of a calibration error on one of the PSA variable delay plates on the accuracy of the Stokes vector estimation for each analysis matrix. For each analysis matrix, the three candlesticks positioned from left to right correspond respectively to a perfect calibration, a calibration error on LCVR1 or on LCVR2.

436 Figure 11 shows the influence of such an imperfect voltage calibration on the accuracy of the
 437 Stokes vector estimation. It can first be observed that the average relative bias $\langle \hat{b} \rangle$ for a voltage
 438 error on LCVR2 is higher than that for an error on LCVR1 (except for the *Stokes-4* matrix). In
 439 the case of a setpoint error on LCVR1, the average bias for the *Spiral* and *Spiral of Tetrahedrons*
 440 analysis matrices is half that of the other matrices. Furthermore, in this case they have a lower
 441 bias dispersion. However, this analysis is different if the setpoint voltage error affects LCVR2: in
 442 that case the *Spiral* and *Spiral of Tetrahedrons* analysis matrices are affected by a bias comparable
 443 to that of other sensing approaches. This difference in behaviour between the two LCVRs can be
 444 explained by looking at Eq. (5): the optical phase delay induced by LCVR2, ϕ_2 , modifies three
 445 of the four terms of the equation, whereas the delay induced by LCVR1 only affects two of them.

446 On the other hand, an error in the setpoint voltage on one of the variable delay plates has very
 447 little effect on the average precision of each analysis matrix as shown in Fig. 12. As a result, to
 448 conclude this section, it can be again observed that for a voltage calibration error on the LCVR
 449 setpoints, the *Spiral* and *Spiral of Tetrahedrons* analysis matrices exhibit a lower dispersion than
 450 the other analysis matrices in terms of accuracy (relative bias) and precision (relative standard
 451 deviation).

452 5. Discussion and conclusion

453 The numerical simulations presented in this study have permitted to clearly validate the relevance
 454 of SOPAFP sensing approaches in Stokes polarimetry, i.e., performing polarimetric sensing

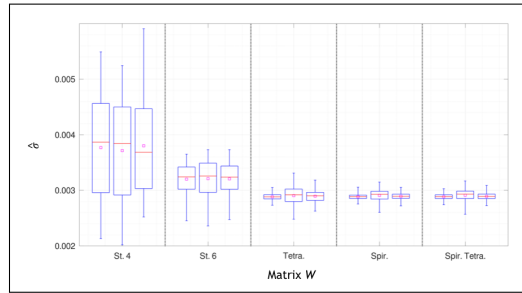


Fig. 12. Influence of a calibration error on one of the PSA variable delay plates on the precision of the determination of the Stokes vector for each analysis matrix. For each analysis matrix, the three candlesticks positioned from left to right correspond respectively to a perfect calibration, a calibration error on LCVR1 and on LCVR2.

455 using many (apparently superfluous) intensity measurements along a strong diversity of analysis
 456 states. Even though such approaches should be limited to particular experimental contexts where
 457 the acquisition time is not a limiting factor, the gain in estimation quality with the SOPAFP
 458 approach had already been noticed in the seminal paper of Dupont *et al.* [15], but without a clear
 459 demonstration of its origin.

460 In this paper, we first demonstrated that the non-linear regression procedure implemented
 461 in [15] could be advantageously replaced by a direct matrix inversion (or pseudo-inversion), with
 462 a huge acceleration of processing times by a factor of more than 1500. We then proposed an
 463 alternative set of polarization analysis states, referred to as *Spiral of Tetrahedrons*, that allows
 464 one to perform SOPAFP with an important number of sensing states covering as uniformly as
 465 possible the Poincaré's sphere, while at the same time ensuring an optimum CN of the analysis
 466 matrix to ensure optimal precision in the estimation by direct inversion.

467 Using extensive simulations of realistic imaging/sensing situations, we demonstrated that
 468 SOPAFP approaches involving the *Spiral* and *Spiral of Tetrahedrons* analysis matrices could
 469 provide optimal estimation accuracy (in terms of relative estimation bias) and precision whatever
 470 the SNR or DOP of the input light. More interestingly, these SOPAFP approaches were shown to
 471 clearly supersede other classical sensing strategies involving only a limited number of analysis
 472 states in the case of slight experimental imperfections of the setup, resulting from improper
 473 alignment or bad calibration of the LCVRs for instance, that can happen in practice, due to
 474 temperature drifts. Indeed, we demonstrated that the SOPAFP approaches provide a low average
 475 relative bias and, more importantly, a very low dispersion of this relative bias as a function of the
 476 input polarization state to estimate, and for distinct imperfections of the PSA.

477 Now, a last topic that can be discussed in this study is the effective interest of using the *Spiral*
 478 *of Tetrahedrons* analysis matrix we proposed instead of any kind of uniformly distributed set of
 479 points across de Poincaré's sphere. As shown in Section 3.4.2, maximal coverage of the sphere
 480 does not guarantee that the CN of the analysis matrix is minimum: therefore using a non-optimal
 481 set of points should lead to a lower estimation precision. Although this was not detected in the
 482 simulations presented above due to the important number of sensing points used ($N = 96$), we
 483 compare in a last simulated experiment the relative performance in terms of precision between
 484 the *Spiral* and *Spiral of Tetrahedrons* analysis matrices, when the number of points N is varied,
 485 and for three values of the DOP of the input Stokes vectors to estimate. The results obtained
 486 and displayed in Fig. 13 confirm the fact that the slight non-optimality of the CN of the *Spiral*
 487 analysis matrix which can be observed in Fig. 2.b) leads to a small degradation of the estimation
 488 precision with respect to the *Spiral of Tetrahedrons* analysis matrix which has an optimum CN,
 489 this degradation being more pronounced for lower DOP of the input Stokes vector. Depending



Fig. 13. Relative precision defined by the ratio of the precision of the *Spiral* and *Spiral of Tetrahedrons* analysis matrices for three levels of degree of polarization (1, 0.5 and 0.1) for W matrices composed of 8 to 324 probe states.

490 on the application at hand, one could choose a different compromise between acquisition time
 491 and estimation precision/robustness by diminishing slightly the number of states composing the
 492 *Spiral of Tetrahedrons*. In the context of this article series, we stuck to a number of sensing
 493 points of $N = 96$, leading in our case to acceptable acquisition times.

494 As a conclusion, the SOPAFP approach and the proposed optimal alternative implementation
 495 using the *Spiral of Tetrahedrons* analysis matrix can prove very efficient in applicative contexts
 496 where very good accuracy/precision of the Stokes vector estimation is required, and in particular
 497 in low SNR contexts and in the presence of depolarized input states. Moreover, we proved in this
 498 study that such approaches are optimal also in terms of robustness to experimental imperfections.
 499 As suggested in Ref. [15], the experimental challenge of performing Stokes polarimetry of highly
 500 resolved speckle patterns at the individual grain scale is a perfect context for applying SOPAFP
 501 approaches. Using the experimental setup detailed in the first article of this series [1], we shall
 502 use the optimizes SOPAFP approach in the last article of the series [19] which details some of
 503 the imaging results obtained experimentally with such optimized setup, as well as new graphical
 504 representations of the polarization information, and an analysis of the distribution of polarization
 505 states across a speckle pattern, and in vicinity of polarization singularities.

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508 **Disclosures.** The authors declare no conflicts of interest.

509 **Data availability.** Data underlying the results presented in this paper are not publicly available at this time
 510 but may be obtained from the authors upon reasonable request.

511 **Supplemental document.** See Supplement 1 for supporting content.

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