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# A differential variational inequality in the study of contact problems with wear

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## Abstract

We start with a mathematical model which describes the sliding contact of a viscoelastic body with a moving foundation. The contact is frictional and the wear of the contact surfaces is taken into account. We prove that this model leads to a differential variational inequality in which the unknowns are the displacement field and the wear function. Then, inspired by this model, we consider a general differential variational inequality in reflexive Banach spaces, governed by four parameters. We prove the unique solvability of the inequality as well as the continuous dependence of its solution with respect to the parameters. The proofs are based on arguments of monotonicity, compactness, convex analysis and lower semicontinuity. Then, we apply these abstract results to the mathematical model of contact for which we deduce the existence of a unique solution as well as the existence of optimal control for an associate optimal control problem. We also present the corresponding mechanical interpretations.

**AMS Subject Classification :** 74M15, 74M10, 47J22, 49J40, 49J21, 34G25.

**Key words :** viscoelastic material, frictional contact, wear, differential variational inequality, convergence results, optimal control, optimal pair, weak solution.

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# 1 Introduction

The Mathematical Theory of Contact Mechanics deals with the variational analysis of systems of partial differential equations which describes contact phenomena between a deformable body and an obstacle, the so-called foundation. It provides results of existence, uniqueness, control and numerical approximation of the weak solution of the corresponding models which, usually, are expressed in terms of variational or hemivariational inequalities. References in the field include the books [3, 5, 6, 10, 12, 22, 23, 24] and, more recently, [29, 30].

Contact processes with elastic, viscoelastic or viscoplastic bodies abound in industry and in everyday life. They are accompanied by a number of phenomena among which the main one is the friction. Nevertheless, more is involved in contact than just friction. Indeed, during a contact process, elastic or plastic deformations of the surface asperities may happen. Also, some or all of the following may take place: squeezing of oil or other fluids, breaking of the asperities' tips and production of debris, motion of the debris, formation or welding of junctions, creeping, fracture, etc. Moreover, frictional contact is associated with heat generation, material damage, wear and adhesion of contacting surfaces. [References on the evolution of damage in solids](#) are [13, 20], for instance. There, the “flow rule” for the damage parameter is rate-independent. [A comprehensive reference to the theory of rate-independent systems with applications to inelastic processes such as plasticity, damage, phase transformations, or adhesive-type contacts both at small strains and at finite strains,](#) is [19]. There, a rigorous mathematical treatment of the corresponding problems is provided, by using the concept of energetic solution, among others.

Due to its crucial role, there exists an considerable interest in study of contact models with wear, both in the [engineering](#) and mathematical literature. References in the field are [1, 4, 7, 8, 11, 14, 26, 27, 31, 32, 33], among others. Wear in sliding systems is often very slow but it is persisting, continuous and cumulative. There may be increase in the conformity of the surfaces and their smoothness, or increase of the surface roughness, fogging of the surface, generation of scratches and grooves, initiation of cracks and generation of debris which may change the contact characteristics. This motivates the study of contact with wear, in the engineering literature. On the other hand, taking into account the wear of the contact surfaces leads to a weak formulation which is in the form of a so-called differential variational inequality, i.e., a system which couples a variational or hemivariational inequality for the displacement field with a differential equation for the wear function. Differential variational inequalities have been introduced by Aubin and Cellina in [2] and intensively studied in [2, 9, 15, 17, 18, 21, 25, 28], for instance. In particular, the results in [17] concern the existence of a unique solution for a class of differential variational inequality. They have completed in [28] with the study of an associated optimal control problem and an application in the study of a contact model with viscoelastic materials. [Moreover, results on history-dependent differential variational-hemivariational inequalities with](#)

applications to contact problems with wear can be found in [16].

Our aim in this paper is two folds. The first one is to prove the analysis and control of a new model of frictional contact with wear for viscoelastic materials. Taking into account the wear of the contact surface leads to a weak formulation of the model which is in a form of new type of differential variational inequality, governed by a history-dependent operator, for which the abstract results in [17, 28] do not work. Therefore, motivated by this contact model, our second aim is to provide a study this kind of differential variational inequality, in an abstract framework. Extending the results in [17, 28] to a new class of differential variational inequalities represents the first trait of novelty of our work. The second novelty is that, in contrast with various results in the literature, in the current paper we consider a new model of contact with viscoelastic materials and, in addition, besides its unique weak solvability, we present results on its optimal control, which could be of interest in real world applications.

The rest of the manuscript is organized as follows. In Section 2 we present our mathematical model of contact, list the assumption on the data and derive a variational formulation, in a form of differential variational inequality in which the unknowns are the wear function and the displacement field. Moreover, we formulate an optimal control problem associated to this mathematical model. Motivated by this contact model, in Section 3 we introduce an abstract differential variational inequality and prove its unique solvability. In Section 4 we study the dependence of the solution of this inequality with respect to the data and parameters and provide a convergence result. Finally, in Section 5 we turn back to the contact model introduced in Section 2 and illustrate the applicability of these abstract results in the study of this model.

## 2 The contact model

We consider a viscoelastic body which occupies a bounded domain  $\Omega \subset \mathbb{R}^d$  ( $d = 2, 3$ ) with a Lipschitz continuous boundary  $\Gamma$ , divided into three measurable parts  $\Gamma_1$ ,  $\Gamma_2$  and  $\Gamma_3$  such that  $meas(\Gamma_1) > 0$  and, in addition,  $\Gamma_3$  is plane. The body is subject to the action of body forces of time-dependent density  $\mathbf{f}_0$ . It is fixed on  $\Gamma_1$  and surfaces tractions act on  $\Gamma_2$ . On  $\Gamma_3$ , the body is in frictional contact with a moving obstacle, assumed to be made of a hard material covered by a layer of soft material of thickness  $g$ . We denote by  $\mathbf{v}^* \neq \mathbf{0}$  the velocity of the foundation which is supposed to be a constant vector in the plane of  $\Gamma_3$ . The physical setting is depicted in Figure 1.

The friction implies the wear of the foundation that we model with a surface variable, the wear function. Its evolution is governed by a simplified version of Archard's law, see [27]. Moreover, we assume that the soft material is deformable and, therefore, its penetration is allowed. Based on these ingredients, we model the contact with a normal compliance condition with unilateral constraint, which takes into account the wear of the foundation. We associate this condition to a sliding version of Coulomb's law of dry friction, adopt the framework of the small strain theory and assume that

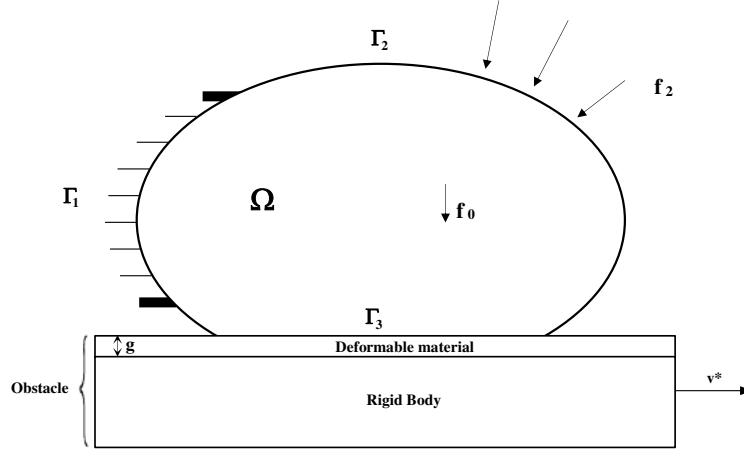


Figure 1: The viscoelastic body in contact with a moving foundation.

the contact process is quasistatic and it is studied in the interval of time  $[0, T]$  with  $T > 0$ . In addition, we denote by  $\mathbb{S}^d$  the space of second order symmetric **tensors** on  $\mathbb{R}^d$  and use the notation “ $\cdot$ ”,  $\|\cdot\|$ ,  $\mathbf{0}$  for the inner product, the Euclidian norm and the zero element on the spaces  $\mathbb{R}^d$  and  $\mathbb{S}^d$ , respectively. We denote by  $\boldsymbol{\nu}$  the outward unit vector to  $\Gamma$  and use the indices  $\nu$  and  $\tau$  to represent the normal and tangential components of vectors and tensors, respectively. Finally, a dot above will represent the derivative with respect to the time and, for simplicity, we sometimes do not indicate explicitly the dependence of various functions on the spatial variable  $\boldsymbol{x} \in \Omega \cup \Gamma$ .

With these preliminaries, the classical formulation of the contact problem under consideration is the following.

**Problem  $\mathcal{P}$ .** Find a stress field  $\boldsymbol{\sigma} : \Omega \times [0, T] \rightarrow \mathbb{S}^d$ , a displacement field  $\boldsymbol{u} : \Omega \times [0, T] \rightarrow \mathbb{R}^d$  and a wear function  $w : \Gamma_3 \times [0, T] \rightarrow \mathbb{R}_+$  such that

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \alpha \int_0^t e^{-(t-s)} \boldsymbol{\varepsilon}(\boldsymbol{u}(s)) ds \quad \text{in } \Omega, \quad (2.1)$$

$$\text{Div } \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \mathbf{0} \quad \text{in } \Omega, \quad (2.2)$$

$$\boldsymbol{u}(t) = \mathbf{0} \quad \text{on } \Gamma_1, \quad (2.3)$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \theta(t)\boldsymbol{f}_2 \quad \text{on } \Gamma_2, \quad (2.4)$$

$$\left. \begin{aligned} u_\nu(t) \leq g, \quad \sigma_\nu(t) + p(u_\nu(t) - w(t)) \leq 0, \\ (u_\nu(t) - g) \left( \sigma_\nu(t) + p(u_\nu(t) - w(t)) \right) = 0 \end{aligned} \right\} \quad \text{on } \Gamma_3, \quad (2.5)$$

$$-\boldsymbol{\sigma}_\tau(t) = \mu p(u_\nu(t) - w(t)) \boldsymbol{n}^* \quad \text{on } \Gamma_3, \quad (2.6)$$

$$\dot{w}(t) = k(t) \|\boldsymbol{v}^*\| p(u_\nu(t) - w(t)) \quad \text{on } \Gamma_3, \quad (2.7)$$

for all  $t \in [0, T]$  and, in addition,

$$w(0) = 0 \quad \text{on } \Gamma_3. \quad (2.8)$$

We now provide a brief explanation for the equations and conditions in Problem  $\mathcal{P}$  and we refer to [31] for more details. First, equation (2.1) represents the viscoelastic constitutive law of the material in which  $\mathcal{A}$  denotes the elasticity operator and  $\alpha$  represents a relaxation coefficient. Equation (2.2) is the equilibrium equation in which  $\text{Div}$  represents the divergence operator for tensor-valued functions. Conditions (2.3) and (2.4) are the displacement and traction boundary conditions, respectively, in which  $\theta(t)\mathbf{f}_2$  denotes the density of surface tractions, assumed to be time-dependent. The function  $\theta$  is assumed to be such that  $\theta(0) = 1$  and, therefore,  $\mathbf{f}_2$  represents the initial density of surface tractions.

Next, condition (2.5) represents the contact condition in which  $p$  is a positive function which will be described below. Moreover, condition (2.6) represents a sliding version of the classical Coulomb's law of dry friction in which  $\mu$  is the friction coefficient and  $\mathbf{n}^*$  is the unitary vector given by

$$\mathbf{n}^* = -\frac{\mathbf{v}^*}{\|\mathbf{v}^*\|}. \quad (2.9)$$

This condition was derived in [31] assuming that the velocity of the foundation is large enough in comparison with the tangential velocity on the body's surface  $\Gamma_3$ . Finally, the differential equation (2.7) **represents a version of Archard's law** in which  $k$  represents the wear coefficient, assumed to be time-dependent. This assumption makes sense if, for instance, we assume that  $k$  depends on the temperature field which plays the role of a parameter. **Note that Archard's law can be derived by thermodynamic principles, as explained in [32].** Finally, (2.8) represents the initial condition for the wear function, which shows that at the initial moment the foundation is new.

Next, we adopt standard notation for the Lebesgue and Sobolev spaces associated to  $\Omega$  and  $\Gamma$ . In particular, the inner products on the Hilbert spaces  $L^2(\Omega; \mathbb{R}^d)$  and  $L^2(\Gamma; \mathbb{R}^d)$  are given by

$$(\mathbf{u}, \mathbf{v})_{L^2(\Omega; \mathbb{R}^d)} = \int_{\Omega} \mathbf{u} \cdot \mathbf{v} \, dx, \quad (\mathbf{u}, \mathbf{v})_{L^2(\Gamma; \mathbb{R}^d)} = \int_{\Gamma} \mathbf{u} \cdot \mathbf{v} \, da$$

with the associated norms  $\|\cdot\|_{L^2(\Omega; \mathbb{R}^d)}$  and  $\|\cdot\|_{L^2(\Gamma; \mathbb{R}^d)}$ , respectively. Moreover, for the displacement and the stress field we consider the spaces

$$V = \{ \mathbf{v} \in H^1(\Omega; \mathbb{R}^d) : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_1 \}, \quad (2.10)$$

$$Q = \{ \boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega; \mathbb{R}) : \tau_{ij} = \tau_{ji} \}. \quad (2.11)$$

These are real Hilbert spaces endowed with the inner products

$$(\mathbf{u}, \mathbf{v})_V = \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \, dx, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_{\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{\tau} \, dx,$$

and the associated norms  $\|\cdot\|_V$  and  $\|\cdot\|_Q$ , respectively. Here  $\boldsymbol{\varepsilon}$  represents the [strain operator](#) given by

$$\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \forall \mathbf{u} \in H^1(\Omega; \mathbb{R}^d),$$

where an index that follows a comma denotes the partial derivative with respect to the corresponding component of  $\mathbf{x}$ , i.e.,  $u_{i,j} = \frac{\partial u_i}{\partial x_j}$ . Recall that the completeness of the space  $(V, \|\cdot\|_V)$  follows from the assumption  $meas(\Gamma_1) > 0$ , which allows the use of Korn's inequality. We still use  $\mathbf{u}$  for the trace of  $\mathbf{u} \in V$  on the boundary  $\Gamma$ . Moreover, recall that, for any given  $\mathbf{u} \in V$ , the normal and the tangential components of  $\mathbf{u}$  on  $\Gamma$  are defined by  $u_\nu = \mathbf{u} \cdot \boldsymbol{\nu}$ ,  $\mathbf{u}_\tau = \mathbf{u} - u_\nu \boldsymbol{\nu}$ , respectively. In addition, the [trace theorem](#) guarantees that there exists a positive constant  $c_{tr}$  which depends on  $\Omega$ ,  $\Gamma$  and such that

$$\|\mathbf{u}\|_{L^2(\Gamma; \mathbb{R}^d)} \leq c_{tr} \|\mathbf{u}\|_V \quad \forall \mathbf{u} \in V. \quad (2.12)$$

Finally, for any Banach space  $X$  we shall use the notation  $C([0, T]; X)$  and  $C^1([0, T]; X)$  for the space of continuous and continuously differentiable functions defined on  $[0, T]$  with values in  $X$ , respectively. Recall that  $C([0, T]; X)$  is a Banach space endowed with the norm of the uniform convergence, i.e.,

$$\|x\|_{C([0, T]; X)} = \max_{t \in [0, T]} \|x(t)\|_X. \quad (2.13)$$

In the study of Problem  $\mathcal{P}$  we consider the following assumptions on the data.

$$\left\{ \begin{array}{l} \text{(a) } \mathcal{A} : \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d. \\ \text{(b) There exists } L_{\mathcal{A}} > 0 \text{ such that} \\ \quad \|\mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_2)\| \leq L_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\ \quad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(c) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ \quad (\mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\ \quad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(d) The mapping } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega \\ \quad \text{for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(e) } \mathcal{A}(\mathbf{x}, \mathbf{0}) = \mathbf{0} \text{ a.e. } \mathbf{x} \in \Omega. \end{array} \right. \quad (2.14)$$

$$\alpha \in L^\infty(\Omega) \quad (2.15)$$

$$\mathbf{f}_0 \in C([0, T]; L^2(\Omega; \mathbb{R}^d)). \quad (2.16)$$

$$\theta \in C([0, T]; \mathbb{R}), \quad \theta(0) = 1. \quad (2.17)$$

$$\mathbf{f}_2 \in L^2(\Gamma_2; \mathbb{R}^d). \quad (2.18)$$

$$\mu \in L^\infty(\Gamma_3), \quad \mu(\mathbf{x}) \geq 0 \quad \text{a.e. } \mathbf{x} \in \Gamma_3. \quad (2.19)$$

$$k \in C([0, T]; L^\infty(\Gamma_3)), \quad k(\mathbf{x}, t) \geq 0 \quad \forall t \in [0, T], \text{ a.e. } \mathbf{x} \in \Gamma_3. \quad (2.20)$$

$$g > 0. \quad (2.21)$$

$$\left\{ \begin{array}{l} \text{(a) } p : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}_+. \\ \text{(b) There exists } L_p > 0 \text{ such that} \\ \quad |p(\mathbf{x}, \theta_1) - p(\mathbf{x}, \theta_2)| \leq L_p(|\theta_1 - \theta_2|) \\ \quad \text{for all } \theta_1, \theta_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ \text{(c) } (p(\mathbf{x}, \theta_1) - p(\mathbf{x}, \theta_2))(\theta_1 - \theta_2) \geq 0 \\ \quad \text{for all } \theta_1, \theta_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ \text{(d) The mapping } p(\cdot, \theta) \text{ is measurable on } \Gamma_3 \\ \quad \text{for all } \theta \in \mathbb{R}. \\ \text{(e) } p(\mathbf{x}, \theta) = 0 \quad \text{for all } \theta \leq 0, \text{ a.e. } \mathbf{x} \in \Gamma_3. \end{array} \right. \quad (2.22)$$

Consider the set  $K$  given by

$$K = \{\mathbf{u} \in V : u_\nu \leq g \text{ a.e. on } \Gamma_3\}. \quad (2.23)$$

Using standard arguments we obtain that, if  $\boldsymbol{\sigma}$ ,  $\mathbf{u}$ ,  $w$  represent a smooth solution to Problem  $\mathcal{P}$ , then

$$\begin{aligned} \mathbf{u}(t) \in K, \quad & (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q + \int_{\Gamma_3} p(u_\nu(t) - w(t))(v_\nu - u_\nu(t)) da \\ & + \int_{\Gamma_3} \mu p(u_\nu(t) - w(t)) \mathbf{n}^* \cdot (\mathbf{v}_\tau - \mathbf{u}_\tau(t)) da \\ & \geq \int_{\Omega} \mathbf{f}_0(t) \cdot (\mathbf{v} - \mathbf{u}) dx + \int_{\Gamma_2} \theta(t) \mathbf{f}_2 \cdot (\mathbf{v} - \mathbf{u}) da \quad \forall \mathbf{v} \in K, t \in [0, T]. \end{aligned} \quad (2.24)$$

We now substitute the constitutive law (2.1) in (2.24) and gather the resulting inequality with (2.7) and (2.8) to obtain the following variational formulation of Problem  $\mathcal{P}$ .

**Problem  $\mathcal{P}_V$ .** Find a displacement field  $\mathbf{u} : [0, T] \rightarrow K$  and a wear function  $w : [0, T] \rightarrow L^2(\Gamma_3)$  such that

$$\begin{aligned} & (\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}(t)), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q + (\alpha \int_0^t e^{-(t-s)} \boldsymbol{\varepsilon}(\mathbf{u}(s)) ds, \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}(t)))_Q \\ & + \int_{\Gamma_3} p(u_\nu(t) - w(t))(v_\nu - u_\nu(t)) da + \int_{\Gamma_3} \mu p(u_\nu(t) - w(t)) \mathbf{n}^* \cdot (\mathbf{v}_\tau - \mathbf{u}_\tau(t)) da \\ & \geq \int_{\Omega} \mathbf{f}_0(t) \cdot (\mathbf{v} - \mathbf{u}) dx + \int_{\Gamma_2} \theta(t) \mathbf{f}_2 \cdot (\mathbf{v} - \mathbf{u}) da \quad \forall \mathbf{v} \in K, t \in [0, T], \end{aligned} \quad (2.25)$$

$$\dot{w}(t) = k(t) \|\mathbf{v}^*\| p(u_\nu(t) - w(t)) \quad \forall t \in [0, T], \quad (2.26)$$

$$w(0) = w_0. \quad (2.27)$$



Our first aim in the study of Problem  $\mathcal{P}_V$  is to prove its unique solvability. Our second aim is to minimize the wear of the deformable layer at the end of the contact process. More precisely, we would like to find a minimizer for the cost functional

$$\mathcal{L}(\mathbf{f}_2) = \int_{\Gamma_3} w^2(T) da \quad (2.28)$$

where  $\mathbf{f}_2$  represents the initial density of surface tractions, on which the solution  $(\mathbf{u}, w)$  depends. The answer of these two questions will be provided in Section 5. It is based on abstract existence, uniqueness and continuous dependence results which we present in Sections 3 and 4 of this manuscript.

We end this section with an additional comments on our contact model. Assume that (2.1)–(2.8) has a classical solution. Then, since  $k$  and  $p$  are positive functions, it follows from (2.7) that  $\dot{w}(t) \geq 0$  for all  $t$ , i.e. the wear is **nondecreasing**, in each point of the contact surface. Moreover, if at a moment  $t_0$  we have  $w(t_0) = g$ , then, using equation (2.7), the constraint  $u_\nu(t) \leq g$  and the property (2.22) of the function  $p$ , it can be easily proved that  $w(t_0) = g$  for all  $t \geq t_0$ . This behavior shows that the wear of the foundation is limited by the constraint  $w(t) \leq g$ , which means that rigid layer of the foundation does not wear.

### 3 An abstract differential variational inequality

Note that Problem  $\mathcal{P}_V$  represents a system which couples a history-dependent variational inequality for the displacement field, (2.25), with a differential equation, for the wear function, (2.26)–(2.27). Therefore, using the terminology in the Introduction, it represents a differential variational inequality. Inspired by this contact model, in this section we consider an abstract differential variational inequality for which we prove an existence and uniqueness result.

The functional framework that we adopt here and in the next section is the following. First,  $X$  is a Banach spaces,  $V$  is a reflexive Banach space and  $Z$  is a Hilbert space endowed with the inner product  $(\cdot, \cdot)_Z$ . The norm on these spaces will be denoted by  $\|\cdot\|_X$ ,  $\|\cdot\|_V$  and  $\|\cdot\|_Z$ , respectively. The strong topological dual space of  $V$  is denoted by  $V^*$  and the duality pairing of  $V$  and  $V^*$  is denoted by  $\langle \cdot, \cdot \rangle$ . The symbols " $\rightharpoonup$ " and " $\rightarrow$ " will represent the weak and strong convergence in various normed spaces to be specified. All the limits, upper and lower limits are considered as  $n \rightarrow \infty$ , even if we do not mention it explicitly. Finally,  $T > 0$  and the dot above represent the derivative with respect to the time.

Let  $F : [0, T] \times X \times V \rightarrow X$ ,  $S : C([0, T]; V) \rightarrow C([0, T]; V^*)$ ,  $A : X \times V \rightarrow V^*$ ,  $j : X \times V \times V \rightarrow \mathbb{R}$ ,  $\pi : V \rightarrow Z$ ,  $f : [0, T] \rightarrow Z$  and  $K \subset V$ . Then, the abstract problem we consider in this section is stated as follows.

**Problem  $\mathcal{P}_a$ .** Find  $x \in C^1([0, T]; X)$  and  $u \in C([0, T]; K)$  such that

$$\dot{x}(t) = F(t, x(t), u(t)) \quad \forall t \in [0, T], \quad (3.1)$$

$$x(0) = x_0, \quad (3.2)$$

$$\begin{aligned} \langle A(x(t), u(t)) + Su(t), v - u(t) \rangle + j(x(t), u(t), v) - j(x(t), u(t), u(t)) \\ \geq (f(t), \pi v - \pi u(t))_Z \quad \forall v \in K, t \in [0, T]. \end{aligned} \quad (3.3)$$

Note that here and below for any  $u \in C([0, T]; V)$  we use the symbol  $Su(t)$  to represent the value of the function  $S(u)$  at the point  $t \in [0, T]$ . Moreover, inclusion  $u \in C([0, T]; K)$  means that  $u \in C([0, T]; V)$  and  $u(t) \in K$  for all  $t \in [0, T]$ .

In the study of Problem  $\mathcal{P}_a$  we consider the following assumptions on the data.

$$\left\{ \begin{array}{l} F : [0, T] \times X \times V \rightarrow X \text{ is such that:} \\ \text{(a) } F(\cdot, x, u) \text{ is continuous on } [0, T] \text{ for all } x \in X, u \in V. \\ \text{(b) There exists } L_F > 0 \text{ such that} \\ \quad \|F(t, x_1, u_1) - F(t, x_2, u_2)\|_X \leq L_F (\|x_1 - x_2\|_X + \|u_1 - u_2\|_V) \\ \quad \text{for all } x_1, x_2 \in X, u_1, u_2 \in V, t \in [0, T]. \end{array} \right. \quad (3.4)$$

$$x_0 \in X. \quad (3.5)$$

$$K \text{ is a nonempty closed convex subset of } V. \quad (3.6)$$

$$\left\{ \begin{array}{l} A : X \times V \rightarrow V^* \text{ is such that:} \\ \text{(a) There exists } L_A^X > 0 \text{ such that} \\ \quad \|A(x_1, u) - A(x_2, u)\|_{V^*} \leq L_A^X \|x_1 - x_2\|_X \\ \quad \text{for all } x_1, x_2 \in X, u \in V. \\ \text{(b) There exists } L_A^V > 0 \text{ such that} \\ \quad \|A(x, u_1) - A(x, u_2)\|_{V^*} \leq L_A^V \|u_1 - u_2\|_V \\ \quad \text{for all } x \in X, u_1, u_2 \in V. \\ \text{(c) There exists } m_A > 0 \text{ such that} \\ \quad \langle A(x, u_1) - A(x, u_2), u_1 - u_2 \rangle \geq m_A \|u_1 - u_2\|_V^2 \\ \quad \text{for all } x \in X, u_1, u_2 \in V. \end{array} \right. \quad (3.7)$$

$$\left\{ \begin{array}{l} S : C([0, T]; V) \rightarrow C([0, T]; V^*) \text{ and there exists } l_S, L_S > 0 \text{ such that} \\ \|Su_1(t) - Su_2(t)\|_{V^*} \leq l_S \|u_1(t) - u_2(t)\|_V + L_S \int_0^t \|u_1(s) - u_2(s)\|_V ds \\ \text{for all } u_1, u_2 \in C([0, T]; V), t \in [0, T]. \end{array} \right. \quad (3.8)$$

$$\left\{ \begin{array}{l} j : X \times V \times V \rightarrow \mathbb{R} \text{ is such that:} \\ \text{(a) For all } x \in X, \text{ and } u \in V, j(x, u, \cdot) \text{ is convex} \\ \text{and lower semicontinuous (l.s.c) on } V. \\ \text{(b) There exists } \alpha_j > 0 \text{ and } \beta_j > 0 \text{ such that} \\ j(x_1, u_1, v_2) - j(x_1, u_1, v_1) + j(x_2, u_2, v_1) - j(x_2, u_2, v_2) \\ \leq \alpha_j \|x_1 - x_2\|_X \|v_1 - v_2\|_V + \beta_j \|u_1 - u_2\|_V \|v_1 - v_2\|_V, \\ \text{for all } x_1, x_2 \in X, u_1, u_2 \in V, v_1, v_2 \in V. \end{array} \right. \quad (3.9)$$

$$m_A - \beta_j > l_S. \quad (3.10)$$

$$f \in C([0, T]; Z). \quad (3.11)$$

$$\left\{ \begin{array}{l} \pi : V \rightarrow Z \text{ is a linear continuous operator, i.e.,} \\ \text{there exists } c_0 > 0 \text{ such that } \|\pi v\|_Z \leq c_0 \|v\|_V \quad \forall v \in V. \end{array} \right. \quad (3.12)$$

Our first result in the section is the following.

**Theorem 3.1.** *Assume (3.4)–(3.12). Then, Problem  $\mathcal{P}_a$  has a unique solution  $(x, u) \in C^1([0, T]; X) \times C([0, T]; K)$ .*

Note that Problem  $\mathcal{P}_a$  was studied in [28] in the particular case when  $S$  vanishes. Therefore, in order to avoid repetition, we skip the details and resume the proof of Theorem 3.1 in three steps, as follows.

*Proof. Step i).* Using Theorem 2.1 in [28], it follows that for any  $\eta \in C([0, T]; V^*)$ , there exists a unique couple of functions  $x_\eta \in C^1([0, T]; X)$ ,  $u_\eta \in C([0, T]; K)$  such that

$$\dot{x}_\eta(t) = F(t, x_\eta(t), u_\eta(t)) \quad \forall t \in [0, T], \quad (3.13)$$

$$x_\eta(0) = x_0, \quad (3.14)$$

$$\begin{aligned} \langle A(x_\eta(t), u_\eta(t)), v - u_\eta(t) \rangle + j(x_\eta(t), u_\eta(t), v) - j(x_\eta(t), u_\eta(t), u_\eta(t)) \\ \geq \langle \eta(t), v - u_\eta(t) \rangle \quad \forall v \in K, t \in [0, T]. \end{aligned} \quad (3.15)$$

Moreover, if  $(x_i, u_i)$  represents the solution of problem (3.13)–(3.15) with  $\eta = \eta_i \in C([0, T]; V^*)$ ,  $i = 1, 2$ , then by standard arguments we find that

$$\|x_1(t) - x_2(t)\|_X \leq L_F \left( \int_0^t \|x_1(s) - x_2(s)\|_X ds + \int_0^t \|u_1(s) - u_2(s)\|_V ds \right),$$

$$\|u_1(t) - u_2(t)\|_V \leq \frac{L_A^X + \alpha_j}{m_A - \beta_j} \|x_1(t) - x_2(t)\|_X + \frac{1}{m_A - \beta_j} \|\eta_1(t) - \eta_2(t)\|_{V^*}$$

for all  $t \in [0, T]$ . We now add these inequalities, use a Gronwall argument and, after some algebra we find that

$$\|u_1(t) - u_2(t)\|_V \leq \frac{1}{m_A - \beta_j} \|\eta_1(t) - \eta_2(t)\|_{V^*} + C \int_0^t \|\eta_1(s) - \eta_2(s)\|_{V^*} ds \quad (3.16)$$

for any  $t \in [0, T]$ , with a constant  $C > 0$  which does not depend on  $t$ .

*Step ii).* We now define the operator  $\Lambda : C([0, T]; V^*) \rightarrow C([0, T]; V^*)$  by equality

$$\langle \Lambda \eta(t), v \rangle = (f(t), \pi v)_Z - \langle Su_\eta(t), v \rangle \quad \forall \eta \in C([0, T]; V), \quad t \in [0, T]. \quad (3.17)$$

We use (3.16) and (3.17) and the properties of the operators  $S$  and  $\pi$  to see that there exists  $C' > 0$  such that

$$\|\Lambda \eta_1(t) - \Lambda \eta_2(t)\|_{V^*} \leq \frac{l_S}{m_A - \beta_j} \|\eta_1(t) - \eta_2(t)\|_{V^*} + C' \int_0^t \|\eta_1(s) - \eta_2(s)\|_{V^*} ds$$

for all  $\eta_1, \eta_2 \in C([0, T]; V)$ ,  $t \in [0, T]$ . This inequality combined with the smallness assumption (3.10) allows us to use Theorem 25 in [30, p. 41] in order to deduce that  $\Lambda$  has a unique fixed point  $\eta^* \in C([0, T]; V)$ .

*Step iii).* We use (3.13)–(3.15), the definition (3.17) of the operator  $\Lambda$  and equality  $\Lambda \eta^* = \eta^*$  to see that

$$\begin{aligned} \dot{x}_{\eta^*}(t) &= F(t, x_{\eta^*}(t), u_{\eta^*}(t)) \quad \forall t \in [0, T], \\ x_{\eta^*}(0) &= x_0, \\ \langle A(x_{\eta^*}(t), u_{\eta^*}(t)) + Su_{\eta^*}(t), v - u_{\eta^*}(t) \rangle + j(x_{\eta^*}(t), u_{\eta^*}(t), v) \\ &\quad - j(x_{\eta^*}(t), u_{\eta^*}(t), u_{\eta^*}(t)) \geq (f(t), \pi v - \pi u_{\eta^*}(t))_Z \quad \forall v \in K, \quad t \in [0, T]. \end{aligned}$$

We conclude from here that  $(x_{\eta^*}, u_{\eta^*})$  is the solution of Problem  $\mathcal{P}_a$ . This proves the existence part of Theorem 3.1. The uniqueness part follows from the uniqueness of the fixed point of the operator  $\Lambda$ , guaranteed by Theorem 25 in [30].  $\square$

## 4 A convergence result

The solution  $(x, u)$  to Problem  $\mathcal{P}_a$  depends on the data  $F, x_0, A, S, K, j$  and  $f$ . Its behaviour with respect to perturbations of all these data can be studied, by using arguments similar to those used in our previous paper [28]. Nevertheless, since our main interest is to provide tools in the study of the contact problem  $\mathcal{P}$ , we restrict ourselves to study the behaviour of the solution with respect to perturbations of  $F, S, K$  and  $f$ . To this end for each  $n \in \mathbb{N}$  we consider a function  $F_n$ , a set  $K_n$ , an operator  $S_n$  and a function  $f_n$  that satisfy the assumptions (3.4), (3.6), (3.8), (3.11) respectively, with constants  $L_{F_n}, l_{S_n}, L_{S_n}$ . To avoid any confusion, we use notation  $(3.4)_n, (3.6)_n, (3.8)_n, (3.10)_n, (3.11)_n$  when we refer to these assumptions for the perturbation data.

The sequences  $\{L_{F_n}\}, \{l_{S_n}\}$  and  $\{L_{S_n}\}$  are assumed to be bounded and, therefore, without the loss of generality we assume that

$$L_{F_n} \leq L_F, \quad l_{S_n} \leq l_S, \quad L_{S_n} \leq L_S \quad (4.1)$$

for all  $n \in \mathbb{N}$ . Then, for each  $n \in \mathbb{N}$  we consider the following problem.

**Problem  $\mathcal{P}_a^n$ .** Find  $x_n \in C^1([0, T]; X)$  and  $u_n \in C([0, T]; K_n)$  such that

$$\dot{x}_n(t) = F_n(t, x_n(t), u_n(t)) \quad \forall t \in [0, T], \quad (4.2)$$

$$x_n(0) = x_0, \quad (4.3)$$

$$\begin{aligned} \langle A(x_n(t), u_n(t)) + S_n u_n(t), v - u_n(t) \rangle + j(x_n(t), u_n(t), v) \\ - j(x_n(t), u_n(t), u_n(t)) \geq (f_n(t), \pi v - \pi u_n(t))_Z \quad \forall v \in K_n, t \in [0, T], \end{aligned} \quad (4.4)$$

The unique solvability of Problem  $\mathcal{P}_a^n$  is a direct consequence of Theorem 3.1. To study the behavior of the solution  $(x_n, u_n)$  as  $n \rightarrow \infty$ , we consider the following additional assumptions.

$$\left\{ \begin{array}{l} \text{There exists } \{\Gamma_n\} \subset \mathbb{R}_+ \text{ such that :} \\ \text{(a) } \|F_n(t, x, u) - F(t, x, u)\|_X \leq \Gamma_n (\|x\|_X + \|u\|_V + 1) \\ \quad \forall t \in [0, T], x \in X, u \in V, n \in \mathbb{N}. \\ \text{(b) } \lim \Gamma_n = 0. \end{array} \right. \quad (4.5)$$

$$\left\{ \begin{array}{l} \text{The sequence } \{K_n\} \text{ converges to } K \text{ in the sense of Mosco, i.e.,} \\ \text{(a) for any } v \in K \text{ there exists a sequence } \{v_n\} \text{ such that} \\ \quad v_n \in K_n \quad \forall n \in \mathbb{N} \text{ and } v_n \rightarrow v \text{ in } V. \\ \text{(b) for any sequence } \{v_n\} \subset X \text{ such that} \\ \quad v_n \in K_n \quad \forall n \in \mathbb{N} \text{ and } v_n \rightharpoonup v \text{ in } V \text{ we have } v \in K. \end{array} \right. \quad (4.6)$$

$$\left\{ \begin{array}{l} \text{There exists } \{\Delta_n\} \subset \mathbb{R}_+ \text{ such that :} \\ \text{(a) } \|S_n u - S u\|_{C([0, T]; V^*)} \leq \Delta_n (\|u\|_{C([0, T]; V)} + 1) \\ \quad \forall u \in C([0, T]; V), n \in \mathbb{N}. \\ \text{(b) } \lim \Delta_n = 0. \end{array} \right. \quad (4.7)$$

$$\left\{ \begin{array}{l} \text{(a) There exists } \gamma_j \geq 0 \text{ such that} \\ \quad j(x, u, v_1) - j(x, u, v_2) \leq \gamma_j (\|x\|_X + \|u\|_V + 1) \|v_1 - v_2\|_V \\ \quad \forall x \in X, u \in V, v_1, v_2 \in V. \\ \text{(b) For any } x \in X \text{ and any sequences } \{u_n\} \subset V, \{v_n\} \subset V \\ \quad \text{such that } u_n \rightharpoonup u \text{ in } V \text{ and } v_n \rightarrow v \text{ in } V, \text{ we have} \\ \quad \limsup_{n \rightarrow \infty} [j(x, u_n, v_n) - j(x, u_n, u_n)] \leq j(x, u, v) - j(x, u, u). \end{array} \right. \quad (4.8)$$

$$f_n(t) \rightharpoonup f(t) \text{ in } Z \text{ for all } t \in [0, T]. \quad (4.9)$$

$$\left\{ \begin{array}{l} \text{For any sequence } \{v_n\} \subset V \text{ such that} \\ v_n \rightharpoonup v \text{ in } V, \text{ we have } \pi v_n \rightarrow \pi v \text{ in } Z. \end{array} \right. \quad (4.10)$$

Our main result in this section. is the following.

**Theorem 4.1.** *Assume (3.4)–(3.12), (3.4)<sub>n</sub>, (3.6)<sub>n</sub>, (3.8)<sub>n</sub>, (3.11)<sub>n</sub> for each  $n \in \mathbb{N}$ , (4.1) and (4.5)–(4.10). Then, the sequence  $\{(x_n, u_n)\}$  of solutions for Problem  $\mathcal{P}_a^n$  converges to the solution  $(x, u)$  of Problem  $\mathcal{P}_a$  at any time moment, i.e.,*

$$x_n(t) \rightarrow x(t) \quad \text{in } X \quad \text{and} \quad u_n(t) \rightarrow u(t) \quad \text{in } V,$$

for each  $t \in [0, T]$ .

The proof of Theorem 4.1 will be divided into several steps that we present in what follows. To this end, everywhere below we keep the assumptions of Theorem 4.1, even if we do not mention it explicitly. We start with the following result of Minty type for time-dependent quasivariational inequalities.

**Lemma 4.2.** *Let  $(x, u) \in C^1([0, T]; X) \times C([0, T]; V)$ . Then, there exists a unique function  $\tilde{u} \in C([0, T]; K)$  such that*

$$\begin{aligned} & \langle A(x(t), \tilde{u}(t)) + Su(t), v - \tilde{u}(t) \rangle + j(x(t), \tilde{u}(t), v) - j(x(t), \tilde{u}(t), \tilde{u}(t)) \\ & \geq (f(t), \pi v - \pi \tilde{u}(t))_Z \quad \forall v \in K, t \in [0, T]. \end{aligned} \quad (4.11)$$

Moreover,  $\tilde{u} \in C([0, T]; K)$  is the solution of inequality (4.11) if and only if

$$\begin{aligned} & \langle A(x(t), v) + Su(t), v - \tilde{u}(t) \rangle + j(x(t), \tilde{u}(t), v) - j(x(t), \tilde{u}(t), \tilde{u}(t)) \\ & \geq (f(t), \pi v - \pi \tilde{u}(t))_Z \quad \forall v \in K, t \in [0, T]. \end{aligned} \quad (4.12)$$

*Proof.* The first part of the lemma is the direct consequence of Lemma 2.1 in [28]. Moreover, it follows from the monotonicity of the operator  $A$  that for any  $v \in V$  and  $t \in [0, T]$  we have

$$\langle A(x(t), v), v - \tilde{u}(t) \rangle \geq \langle A(x(t), \tilde{u}(t)), v - \tilde{u}(t) \rangle,$$

which implies that any solution of problem (4.11) is also the solution of problem (4.12).

Assume now that  $\tilde{u} \in C([0, T]; K)$  is solution of problem (4.12). Let  $t \in [0, T]$ ,  $w \in K$  and  $s \in (0, 1]$  and denote  $v = \tilde{u}(t) + s(w - \tilde{u}(t))$ . Then  $v \in K$  and (4.12) yields

$$\begin{aligned} & \langle A(x(t), \tilde{u}(t) + s(w - \tilde{u}(t))) + Su(t), s(w - \tilde{u}(t)) \rangle + j(x(t), \tilde{u}(t), \tilde{u}(t) + s(w - \tilde{u}(t))) \\ & - j(x(t), \tilde{u}(t), \tilde{u}(t)) \geq s(f(t), \pi w - \pi \tilde{u}(t))_Z. \end{aligned} \quad (4.13)$$

We use assumption (3.9)(a) to see that

$$j(x(t), \tilde{u}(t), \tilde{u}(t) + s(w - \tilde{u}(t))) \leq sj(x(t), \tilde{u}(t), w) + (1 - s)j(x(t), \tilde{u}(t), \tilde{u}(t))$$

and, therefore, (4.13) implies that

$$\begin{aligned} & s\langle A(x(t), \tilde{u}(t) + s(w - \tilde{u}(t))) + Su(t), w - \tilde{u}(t) \rangle + sj(x(t), \tilde{u}(t), w) \\ & - sj(x(t), \tilde{u}(t), \tilde{u}(t)) \geq s(f(t), \pi w - \pi \tilde{u}(t))_Z. \end{aligned}$$

We now divide this inequality with  $s > 0$ , then we pass to the limit as  $s \rightarrow 0$  and use the assumption (3.7)(b) to see that inequality (4.11) holds, which completes the proof of the lemma.  $\square$

**Remark 1.** Let  $(x, u) \in C^1([0, T]; X) \times C([0, T]; K)$  be the solution of Problem  $\mathcal{P}_a$  and let  $\tilde{u} \in C([0, T]; K)$ . Then, Theorem 3.1 and Lemma 4.2 imply that  $u = \tilde{u}$  if and only if  $\tilde{u}$  satisfies the inequality (4.12).

Next, for each  $n \in \mathbb{N}$ , we use the solution  $(x, u) \in C^1([0, T]; X) \times C([0, T]; K)$  of Problem  $\mathcal{P}_a$ , to construct the following auxiliary problem.

**Problem  $\tilde{\mathcal{P}}_n$ .** Find  $\tilde{u}_n \in C([0, T]; K_n)$  such that

$$\begin{aligned} & \langle A(x(t), \tilde{u}_n(t)) + S_n u(t), v - \tilde{u}_n(t) \rangle + j(x(t), \tilde{u}_n(t), v) - j(x(t), \tilde{u}_n(t), \tilde{u}_n(t)) \\ & \geq (f_n(t), \pi v - \pi \tilde{u}_n(t))_Z \quad \forall v \in K_n, t \in [0, T]. \end{aligned} \quad (4.14)$$

The next step of the proof is the following.

**Lemma 4.3.** For each  $n \in \mathbb{N}$ , Problem  $\tilde{\mathcal{P}}_n$  has a unique solution  $\tilde{u}_n \in C([0, T]; V)$ . Moreover, there exists a constant  $C > 0$  such that

$$\|\tilde{u}_n(t)\|_V \leq C \quad \forall n \in \mathbb{N}, t \in [0, T]. \quad (4.15)$$

*Proof.* The unique solvability of Problem  $\tilde{\mathcal{P}}_n$  is the direct consequence of Lemma 4.2. To prove the second part of the lemma we fix  $n \in \mathbb{N}$ ,  $t \in [0, T]$  and  $u_0 \in K$ . We conclude from (4.6) that there exists a sequence  $\{u_n\}$  such that

$$u_n \in K_n \quad \forall n \in \mathbb{N} \quad \text{and} \quad u_n \rightarrow u_0 \quad \text{in } V.$$

Taking  $v = u_n \in K_n$  in (4.14), one has

$$\begin{aligned} & \langle A(x(t), \tilde{u}_n(t)), \tilde{u}_n(t) - u_n \rangle \leq \langle S_n u(t), u_n - \tilde{u}_n(t) \rangle + j(x(t), \tilde{u}_n(t), u_n) \\ & - j(x(t), \tilde{u}_n(t), \tilde{u}_n(t)) + (f_n(t), \pi \tilde{u}_n(t) - \pi u_n)_Z \end{aligned}$$

and, therefore,

$$\begin{aligned} & \langle A(x(t), \tilde{u}_n(t)) - A(x(t), u_n), \tilde{u}_n(t) - u_n \rangle \leq \langle A(x(t), u_n) + S_n u(t), u_n - \tilde{u}_n(t) \rangle \\ & + j(x(t), \tilde{u}_n(t), u_n) - j(x(t), \tilde{u}_n(t), \tilde{u}_n(t)) + (f_n(t), \pi \tilde{u}_n(t) - \pi u_n)_Z. \end{aligned}$$

Using now assumption (3.7)(c) on  $A$  we have

$$\begin{aligned} m_A \|\tilde{u}_n(t) - u_n\|_V^2 &\leq \langle A(x(t), u_n) + S_n u(t), u_n - \tilde{u}_n(t) \rangle \\ &\quad + j(x(t), \tilde{u}_n(t), u_n) - j(x(t), \tilde{u}_n(t), \tilde{u}_n(t)) + (f_n(t), \pi \tilde{u}_n(t) - \pi u_n)_Z. \end{aligned} \quad (4.16)$$

On the other hand, since  $u_n \rightarrow u$  in  $V$ , we can find  $D > 0$  such that

$$\|u_n\|_V \leq D, \quad \|x(t)\|_X \leq D, \quad \|u(t)\|_V \leq D. \quad (4.17)$$

Here and below  $D, D', D'', D'''$  and  $D_0$  are positive constants which do not depend on  $t$  and  $n$ . Then, using assumptions (3.7) and (4.7) we deduce that there exists a constant  $D'$  such that

$$\|A(x(t), u_n) + S_n u(t)\|_{V^*} \leq D'. \quad (4.18)$$

Moreover, conditions (3.9)(b) and (4.8)(a) imply that

$$\begin{aligned} &j(x(t), \tilde{u}_n(t), u_n) - j(x(t), \tilde{u}_n(t), \tilde{u}_n(t)) \\ &\leq j(x(t), u_n, u_n) - j(x(t), u_n, \tilde{u}_n(t)) + \beta_j \|u_n - \tilde{u}_n(t)\|_V^2 \\ &\leq \gamma_j (\|x(t)\|_X + \|u_n\|_V + 1) \|u_n - \tilde{u}_n(t)\|_V + \beta_j \|u_n - \tilde{u}_n(t)\|_V^2. \end{aligned}$$

and, using (4.17) we find that that there exists  $D'' > 0$  such that

$$\begin{aligned} &j(x(t), \tilde{u}_n(t), u_n) - j(x(t), \tilde{u}_n(t), \tilde{u}_n(t)) \\ &\leq D'' \|u_n - \tilde{u}_n(t)\|_V + \beta_j \|u_n - \tilde{u}_n(t)\|_V^2. \end{aligned} \quad (4.19)$$

Finally assumptions (4.9) and (3.12) guarantee that there exists  $D''' > 0$  such that

$$(f_n(t), \pi \tilde{u}_n(t) - \pi u_n)_Z \leq D''' \|u_n - \tilde{u}_n(t)\|_V. \quad (4.20)$$

We now gather the bounds (4.16), (4.18)–(4.20) so see that

$$(m_A - \beta_j) \|\tilde{u}_n(t) - u_n\|_V^2 \leq (D' + D'' + D''') \|\tilde{u}_n(t) - u_n\|_V.$$

This inequality combined with the smallness condition  $m_A - \beta_j > l_s \geq 0$ , implied by (3.10), shows that there exists a constant  $D_0$  such that  $\|\tilde{u}_n(t) - u_n\|_V \leq D_0$ . We now use the bound (4.17) to conclude the proof.  $\square$

The next step is the following.

**Lemma 4.4.** *The sequence  $\tilde{u}_n$  of solutions for Problem  $\tilde{\mathcal{P}}_n$  converges weakly to  $u$  at any time moment, i.e.,*

$$\tilde{u}_n(t) \rightharpoonup u(t) \quad \text{in } V, \quad \forall t \in [0, T].$$



*Proof.* Let  $t \in [0, T]$ . The reflexivity of  $V$  and Lemma 4.3 guarantee that there exists a subsequence of  $\{\tilde{u}_n(t)\}$  still denoted by  $\{\tilde{u}_n(t)\}$ , and an element  $\tilde{u}(t) \in V$  such that  $\tilde{u}_n(t) \rightharpoonup \tilde{u}(t)$  in  $V$ . Moreover, since  $\tilde{u}_n(t) \in K_n$  for each  $n \in \mathbb{N}$ , assumption (4.6)(b) implies that  $\tilde{u}(t) \in K$ . We only need to prove that  $\tilde{u}(t) = u(t)$  and, from Remark 1 it is enough to prove that the following inequality holds:

$$\begin{aligned} & \langle A(x(t), v) + Su(t), \tilde{u}(t) - v \rangle \\ & \leq j(x(t), \tilde{u}(t), v) - j(x(t), \tilde{u}(t), \tilde{u}(t)) + (f(t), \pi\tilde{u}(t) - \pi v)_Z \quad \forall v \in K. \end{aligned} \quad (4.21)$$

Let  $v \in K$  and let  $\{v_n\} \subset V$  such  $v_n \in K_n$  for each  $n \in \mathbb{N}$  and  $v_n \rightarrow v$  in  $V$ . Let  $n \in \mathbb{N}$ . We use (4.14) to see that

$$\begin{aligned} \langle A(x(t), \tilde{u}_n(t)) + S_n u(t), \tilde{u}_n(t) - v_n \rangle & \leq j(x(t), \tilde{u}_n(t), v_n) - j(x(t), \tilde{u}_n(t), \tilde{u}_n(t)) \\ & \quad + (f_n(t), \pi\tilde{u}_n(t) - \pi v_n)_Z \end{aligned}$$

and, using the inequality

$$A(x(t), v_n), \tilde{u}_n(t) - v_n \rangle \leq A(x(t), \tilde{u}_n(t)), \tilde{u}_n(t) - v_n \rangle,$$

guaranteed by assumption (3.7)(c), we find that

$$\begin{aligned} \langle A(x(t), v_n) + S_n u(t), \tilde{u}_n(t) - v_n \rangle & \leq j(x(t), \tilde{u}_n(t), v_n) - j(x(t), \tilde{u}_n(t), \tilde{u}_n(t)) \\ & \quad + (f_n(t), \pi v_n - \pi\tilde{u}_n(t))_Z. \end{aligned} \quad (4.22)$$

Then, writing

$$Su(t) = Su(t) - S_n u(t) + S_n u(t)$$

and using (4.22) we deduce that

$$\begin{aligned} & \langle A(x(t), v_n) + Su(t), \tilde{u}_n(t) - v_n \rangle \\ & = \langle S_n u(t) - Su(t), v_n - \tilde{u}_n(t) \rangle + \langle A(x(t), v_n) + S_n u(t), \tilde{u}_n(t) - v_n \rangle \\ & \leq \langle S_n u(t) - Su(t), v_n - \tilde{u}_n(t) \rangle + j(x(t), \tilde{u}_n(t), v_n) - j(x(t), \tilde{u}_n(t), \tilde{u}_n(t)) \\ & \quad + (f_n(t), \pi\tilde{u}_n(t) - \pi v_n)_Z. \end{aligned} \quad (4.23)$$

We now estimate each term of the right hand side above.

First, we use (4.7) to find that

$$\langle S_n u(t) - Su(t), v_n - \tilde{u}_n(t) \rangle \leq \Delta_n (\|u\|_{C([0, T]; V)} + 1) \|v_n - \tilde{u}_n(t)\|_V. \quad (4.24)$$

Since the sequence  $\{v_n\}$  and  $\{\tilde{u}_n(t)\}$  are bounded in  $V$ , inequality (4.24) and (4.7)(b) imply that

$$\limsup \langle S_n u(t) - Su(t), v_n - \tilde{u}_n(t) \rangle \leq 0. \quad (4.25)$$

On the other hand, since  $v_n \rightarrow v$  and  $\tilde{u}_n(t) \rightharpoonup \tilde{u}(t)$  in  $V$ , assumption (4.8) (b) shows that

$$\begin{aligned} & \limsup [j(x(t), \tilde{u}_n(t), v_n) - j(x(t), \tilde{u}_n(t), \tilde{u}_n(t))] \\ & \leq j(x(t), \tilde{u}(t), v) - j(x(t), \tilde{u}(t), \tilde{u}(t)). \end{aligned} \quad (4.26)$$

Finally, note that

$$\begin{aligned}
& (f_n(t), \pi \tilde{u}_n(t) - \pi v_n)_Z \\
&= (f_n(t), \pi \tilde{u}_n(t) - \pi \tilde{u}(t))_Z + (f_n(t), \pi \tilde{u}(t) - \pi v)_Z + (f_n(t), \pi v - \pi v_n)_Z \\
&\leq \|f_n(t)\|_Z \|\pi \tilde{u}_n(t) - \pi \tilde{u}(t)\|_Z + (f_n(t), \pi \tilde{u}(t) - \pi v)_Z + \|f_n(t)\|_Z \|\pi v - \pi v_n\|_Z.
\end{aligned}$$

Thus, using the convergences  $v_n \rightarrow v$  and  $\tilde{u}_n(t) \rightarrow \tilde{u}(t)$ , both in  $V$ , and assumptions (4.9), (4.10), we deduce that

$$\limsup (f_n(t), \pi \tilde{u}_n(t) - \pi v_n)_Z \leq (f(t), \pi \tilde{u}(t) - \pi v)_Z. \quad (4.27)$$

We now pass to the upper limit in (4.23) and use inequalities (4.25)–(4.27) to obtain that

$$\begin{aligned}
& \limsup \langle A(x(t), v_n) + Su(t), \tilde{u}_n(t) - v_n \rangle \\
&\leq j(x(t), \tilde{u}(t), v) - j(x(t), \tilde{u}(t), \tilde{u}(t)) + (f(t), \pi \tilde{u}(t) - \pi v)_Z. \quad (4.28)
\end{aligned}$$

On the other hand, from the Lipschitz continuity (3.7)(b) of the operator  $A$  and the convergences  $v_n \rightarrow v$ ,  $\tilde{u}_n(t) \rightarrow \tilde{u}(t)$  in  $V$ , it follows that

$$\begin{aligned}
& \limsup \langle A(x(t), v_n) + Su(t), \tilde{u}_n(t) - v_n \rangle \\
&= \langle A(x(t), v) + Su(t), \tilde{u}(t) - v \rangle. \quad (4.29)
\end{aligned}$$

We now use (4.28) and (4.29) to find that (4.21) holds, which concludes the proof.  $\square$

We now prove the following strong convergence result.

**Lemma 4.5.** *The sequence  $\{\tilde{u}_n\}$  of solutions for Problem  $\tilde{\mathcal{P}}_n$  converges strongly to  $u$  at any time moment, i.e.,*

$$\tilde{u}_n(t) \rightarrow u(t) \quad \text{in } V \quad \forall t \in [0, T].$$

*Proof.* Let  $t \in [0, T]$  and let  $\{v_n\} \subset V$  be a sequence such that  $v_n \in K_n$  for all  $n \in \mathbb{N}$  and  $v_n \rightarrow u(t)$  in  $V$ . Let  $n \in \mathbb{N}$ . We use (4.14) to see that

$$\begin{aligned}
& \langle A(x(t), \tilde{u}_n(t)), \tilde{u}_n(t) - v_n \rangle \\
&\leq \langle S_n u(t) - Su(t), v_n - \tilde{u}_n(t) \rangle + \langle Su(t), v_n - \tilde{u}_n(t) \rangle + j(x(t), \tilde{u}_n(t), v_n) \\
&\quad - j(x(t), \tilde{u}_n(t), \tilde{u}_n(t)) + (f_n(t), \pi \tilde{u}_n(t) - \pi v_n)_Z
\end{aligned}$$

and, therefore,

$$\begin{aligned}
& \langle A(x(t), \tilde{u}_n(t)), \tilde{u}_n(t) - u(t) \rangle \\
&\leq \langle A(x(t), \tilde{u}_n(t)), v_n - u(t) \rangle + \langle S_n u(t) - Su(t), v_n - \tilde{u}_n(t) \rangle + \langle Su(t), v_n - \tilde{u}_n(t) \rangle \\
&\quad + j(x(t), \tilde{u}_n(t), v_n) - j(x(t), \tilde{u}_n(t), \tilde{u}_n(t)) + (f_n(t), \pi \tilde{u}_n(t) - \pi v_n)_Z. \quad (4.30)
\end{aligned}$$

for all  $v_n \in K_n$ . This inequality and arguments similar to those used to obtain (4.25)–(4.27) yield

$$\limsup \langle A(x(t), \tilde{u}_n(t)), \tilde{u}_n(t) - u(t) \rangle \leq 0$$

and, therefore,

$$\begin{aligned} & \limsup \langle A(x(t), \tilde{u}_n(t)) - A(x(t), u(t)), \tilde{u}_n(t) - u(t) \rangle \\ & \leq \limsup \langle A(x(t), u(t)), u(t) - \tilde{u}_n(t) \rangle = 0. \end{aligned} \quad (4.31)$$

It follows now from condition (3.7)(c) and inequality (4.31) that

$$\limsup \|\tilde{u}_n(t) - u(t)\|_V \leq 0,$$

which concludes the proof.  $\square$

We are now in a position to provide the proof of Theorem 4.1.

*Proof.* Let  $t \in [0, T]$ . We test in (4.4) and (4.14) with  $v = \tilde{u}_n(t)$  and  $v = u_n(t)$ , respectively, and then we add the resulting inequalities to deduce that

$$\begin{aligned} & \langle A(x_n(t), u_n(t)) - A(x(t), \tilde{u}_n(t)), u_n(t) - \tilde{u}_n(t) \rangle \\ & \leq \langle S_n u(t) - S_n u_n(t), u_n(t) - \tilde{u}_n(t) \rangle + j(x_n(t), u_n(t), \tilde{u}_n(t)) \\ & \quad - j(x_n(t), u_n(t), u_n(t)) + j(x(t), \tilde{u}_n(t), u_n(t)) - j(x(t), \tilde{u}_n(t), \tilde{u}_n(t)). \end{aligned}$$

It follows from here that

$$\begin{aligned} & \langle A(x(t), u_n(t)) - A(x(t), \tilde{u}_n(t)), u_n(t) - \tilde{u}_n(t) \rangle \\ & \leq \langle A(x(t), u_n(t)) - A(x_n(t), u_n(t)), u_n(t) - \tilde{u}_n(t) \rangle \\ & \quad + \langle S_n u(t) - S_n u_n(t), u_n(t) - \tilde{u}_n(t) \rangle + j(x_n(t), u_n(t), \tilde{u}_n(t)) \\ & \quad - j(x_n(t), u_n(t), u_n(t)) + j(x(t), \tilde{u}_n(t), u_n(t)) - j(x(t), \tilde{u}_n(t), \tilde{u}_n(t)). \end{aligned}$$

Therefore, using conditions (3.7), (3.8)<sub>n</sub>, (3.9) and (4.1) we find that

$$\begin{aligned} & m_A \|u_n(t) - \tilde{u}_n(t)\|_V^2 \\ & \leq (\|A(x(t), u_n(t)) - A(x_n(t), u_n(t))\|_{V^*} + \|S_n u(t) - S_n u_n(t)\|_{V^*}) \|u_n(t) - \tilde{u}_n(t)\|_V \\ & \quad + \alpha_j \|x_n(t) - x(t)\|_X \|\tilde{u}_n(t) - u_n(t)\|_V + \beta_j \|u_n(t) - \tilde{u}_n(t)\|_V^2 \\ & \leq \left( L_A^X \|x_n(t) - x(t)\|_X + l_S \|u(t) - u_n(t)\|_V \right. \\ & \quad \left. + L_S \int_0^t \|u(s) - u_n(s)\|_V ds \right) \|u_n(t) - \tilde{u}_n(t)\|_V \\ & \quad + \alpha_j \|x_n(t) - x(t)\|_X \|\tilde{u}_n(t) - u_n(t)\|_V + \beta_j \|\tilde{u}_n(t) - u_n(t)\|_V^2, \end{aligned}$$

which implies that

$$\begin{aligned} (m_A - \beta_j) \|u_n(t) - \tilde{u}_n(t)\|_V &\leq (L_A^X + \alpha_j) \|x_n(t) - x(t)\|_X \\ &+ l_S \|u_n(t) - u(t)\|_V + L_S \int_0^t \|u(s) - u_n(s)\|_V ds. \end{aligned} \quad (4.32)$$

We now write

$$\|u_n(t) - u(t)\|_V \leq \|u_n(t) - \tilde{u}_n(t)\|_V + \|\tilde{u}_n(t) - u(t)\|_V,$$

multiply this inequality by  $m_A - \beta_j > 0$  and use (4.32) together with the smallness assumption (3.10) to deduce that

$$\begin{aligned} \|u_n(t) - u(t)\|_V &\leq C_1 (\|x_n(t) - x(t)\|_X + \|\tilde{u}_n(t) - u(t)\|_V) \\ &+ C_2 \int_0^t \|u_n(s) - u(s)\|_V ds. \end{aligned} \quad (4.33)$$

Here and below in this paper  $C_i$ ,  $i = 1, \dots, 8$  represent positive constants which may depend on  $x$ ,  $u$  and  $T$ , but do not depend on  $n$  and  $t$ .

On the other hand, (3.1)–(3.2) and (4.2)–(4.3) imply that

$$x(t) = x_0 + \int_0^t F(s, x(s), u(s)) ds, \quad x_n(t) = x_0 + \int_0^t F_n(s, x_n(s), u_n(s)) ds$$

and so

$$\|x_n(t) - x(t)\|_X \leq \int_0^t \|F_n(s, x_n(s), u_n(s)) - F(s, x(s), u(s))\|_X ds. \quad (4.34)$$

We now use the hypotheses (3.4)<sub>n</sub>(b), (4.5) and (4.1) to see that

$$\begin{aligned} &\|F_n(s, x_n(s), u_n(s)) - F(s, x(s), u(s))\|_X \\ &\leq \|F_n(s, x_n(s), u_n(s)) - F_n(s, x(s), u(s))\|_X \\ &\quad + \|F_n(s, x(s), u(s)) - F(s, x(s), u(s))\|_X \\ &\leq L_F (\|x_n(s) - x(s)\|_X + \|u_n(s) - u(s)\|_V) \\ &\quad + \Gamma_n (\|x(s)\|_X + \|u(s)\|_X + 1), \end{aligned} \quad (4.35)$$

for any  $s \in [0, T]$ . Then, inequalities (4.34) and (4.35) show that

$$\|x_n(t) - x(t)\|_X \leq C_3 \Gamma_n + L_F \int_0^t h_n(s) ds, \quad (4.36)$$

where  $h_n : [0, T] \rightarrow \mathbb{R}$  is the function defined by

$$h_n(s) = \|x_n(s) - x(s)\|_X + \|u_n(s) - u(s)\|_X \quad \forall s \in [0, T]. \quad (4.37)$$

We now combine inequalities (4.33) and (4.36) to see that

$$\|u_n(t) - u(t)\|_V \leq C_4(\Gamma_n + \|\tilde{u}_n(t) - u(t)\|_V) + C_5 \int_0^t h_n(s) ds. \quad (4.38)$$

Finally, we add inequalities (4.36) and (4.38) to deduce that

$$h_n(t) \leq C_6(\Gamma_n + \|\tilde{u}_n(t) - u(t)\|_V) + C_7 \int_0^t h_n(s) ds. \quad (4.39)$$

and, after using the Gronwall argument yields

$$h_n(t) \leq C_8 \left( \Gamma_n + \|\tilde{u}_n(t) - u_n(t)\|_V + \int_0^t \|\tilde{u}_n(s) - u(s)\|_V ds \right). \quad (4.40)$$

We now use (4.40), assumption (4.5)(b), Lemmas 4.5 and 4.3 and the Lebesgue dominated convergence theorem to deduce that

$$h_n(t) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \quad (4.41)$$

We now combine (4.41) and (4.37) to conclude the proof.  $\square$

## 5 Analysis and control of the contact model

In this section we apply the abstract results in Sections 3 and 4 in the study of Problem  $\mathcal{P}_V$ . To this end, everywhere below we use the space  $V$  and the set  $K$  defined by (2.10) and (2.23), respectively. The unique solvability of this problem is given by the following existence and uniqueness result.

**Theorem 5.1.** *Assume (2.14)–(2.22) and, in addition, assume that*

$$m_A > L_p c_{tr} \|\mu\|_{L^\infty(\Gamma_3)}. \quad (5.1)$$

*Then, Problem  $\mathcal{P}_V$  has a unique solution  $(w, \mathbf{u}) \in C^1([0, T]; L^2(\Gamma_3)) \times C([0, T]; K)$ . Moreover, the solution depends continuously on the data  $\alpha$ ,  $\mathbf{f}_2$ ,  $k$  and  $g$ , that is, if for each  $n \in \mathbb{N}$ ,  $(w_n, \mathbf{u}_n)$  represent the solution of Problem  $\mathcal{P}_V$  with the data  $\alpha_n$ ,  $\mathbf{f}_{2n}$ ,  $g_n$  and  $k_n$  which satisfy (2.15), (2.18), (2.20) and (2.21) and*

$$\alpha_n \rightarrow \alpha \quad \text{in } L^\infty(\Omega), \quad (5.2)$$

$$\mathbf{f}_{2n} \rightarrow \tilde{\mathbf{f}}_2 \quad \text{in } L^2(\Gamma_2, \mathbb{R}^d), \quad (5.3)$$

$$k_n \rightarrow k \quad \text{in } C([0, T]; L^\infty(\Gamma_3)), \quad (5.4)$$

$$g_n \rightarrow g \quad \text{in } \mathbb{R} \quad (5.5)$$

*then, for each  $t \in [0, T]$ , following convergences **hold**:*

$$w_n(t) \rightarrow w(t) \quad \text{in } L^2(\Gamma_3), \quad \mathbf{u}_n(t) \rightarrow \mathbf{u}(t) \quad \text{in } V. \quad (5.6)$$

*Proof.* We use Theorems 3.1 and 4.1 with  $X = L^2(\Gamma_3)$  and  $Z = L^2(\Omega; \mathbb{R}^d) \times L^2(\Gamma_2; \mathbb{R}^d)$ . To this end we consider the operators  $A : L^2(\Gamma_3) \times V \rightarrow V^*$ ,  $S : C([0, T]; V) \rightarrow C([0, T]; V^*)$ ,  $\pi : V \rightarrow Z$ , and the functions  $F : [0, T] \times X \times V \rightarrow X$ ,  $j : X \times V \times V \rightarrow \mathbb{R}$ ,  $\mathbf{f} : [0, T] \rightarrow Z$  defined by the equalities

$$\langle A(w, \mathbf{u}), \mathbf{v} \rangle = (\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_Q + \int_{\Gamma_3} p(u_\nu - w)v_\nu da \quad (5.7)$$

$$\forall \mathbf{u}, \mathbf{v} \in V, w \in X, \quad (5.8)$$

$$\langle S\mathbf{u}(t), \mathbf{v} \rangle = (\alpha \int_0^t e^{t-s} \boldsymbol{\varepsilon}(\mathbf{u}(s)) ds, \boldsymbol{\varepsilon}(\mathbf{v}))_Q \quad (5.9)$$

$$\forall \mathbf{u} \in C([0, T]; V), \mathbf{v} \in V, t \in [0, T], \quad (5.10)$$

$$F(t, w, \mathbf{u}) = k(t) \|\mathbf{v}^*\| p(u_\nu - w) \quad \forall w \in X, \mathbf{u} \in V, t \in [0, T], \quad (5.11)$$

$$j(w, \mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} \mu p(u_\nu - w) \mathbf{n}^* \cdot \mathbf{v}_\tau da \quad \forall w \in X, \mathbf{u}, \mathbf{v} \in V, \quad (5.12)$$

$$\pi \mathbf{v} = (\mathbf{v}, \mathbf{v}|_{\Gamma_2}) \quad \forall \mathbf{v} \in V, \quad (5.13)$$

$$\mathbf{f}(t) = (\mathbf{f}_0(t), \theta(t) \mathbf{f}_2) \quad \forall t \in [0, T]. \quad (5.14)$$

Then, it is easy to see that Problem  $\mathcal{P}_V$  is equivalent to finding a pair  $(w, \mathbf{u}) \in C^1(I; X) \times C(I; K)$  such that

$$\dot{w}(t) = F(t, w(t), \mathbf{u}(t)) \quad \forall t \in I, \quad (5.15)$$

$$w(0) = 0, \quad (5.16)$$

$$\begin{aligned} \langle A(w(t), \mathbf{u}(t)) + S\mathbf{u}(t), \mathbf{v} - \mathbf{u}(t) \rangle + j(w(t), \mathbf{u}(t), \mathbf{v}) - j(w(t), \mathbf{u}(t), \mathbf{u}(t)) \\ \geq (\mathbf{f}(t), \pi \mathbf{v} - \pi \mathbf{u}(t))_Z \quad \forall \mathbf{v} \in K, t \in [0, T]. \end{aligned} \quad (5.17)$$

Using assumptions (2.14)–(2.22), we obtain that conditions (3.4)–(3.12) hold with  $L_F = L_p \|k\|_{C([0, T]; L^\infty(\Gamma_3))} \|\mathbf{v}^*\| (c_{tr} + 1)$ ,  $L_A^V = L_{\mathcal{A}} + L_p c_{tr}^2$ ,  $L_A^X = L_p c_{tr}$ ,  $m = m_{\mathcal{A}}$ ,  $l_S = 0$ ,  $L_S = \|\alpha\|_{L^\infty(\Omega)} e^T$ ,  $\alpha_j = L_p c_{tr}^2 \|\mu\|_{L^\infty(\Gamma_3)}$ ,  $\beta_j = L_p c_{tr} \|\mu\|_{L^\infty(\Gamma_3)}$ ,  $c_0 = 1 + c_{tr}$ . For instance, the regularity (3.11) can be derived from (2.16) and (2.18) and the smallness condition (3.10) follows from (5.1). Then, it follows from Theorem 3.1 that there exists a unique  $(w, \mathbf{u}) \in C^1([0, T]; X) \times C([0, T]; K)$  such that (5.15)–(5.17) hold. This proves the first part of the Theorem.

For the second part we assume that  $\alpha_n$ ,  $\mathbf{f}_{2n}$ ,  $g_n$  and  $k_n$  have the regularities (2.15), (2.18), (2.20) and (2.21), respectively, and, moreover, (5.2), (5.3), (5.4) and (5.5) hold. Then, it follows that **there exists a unique pair  $(w_n, \mathbf{u}_n) \in C^1([0, T]; X) \times C([0, T]; K_n)$**

such that

$$\dot{w}_n(t) = F_n(t, w_n(t), \mathbf{u}_n(t)) \quad \forall t \in [0, T], \quad (5.18)$$

$$w_n(0) = 0, \quad (5.19)$$

$$\begin{aligned} & \langle A(w_n(t), \mathbf{u}_n(t)) + S_n \mathbf{u}_n(t), \mathbf{v} - \mathbf{u}_n(t) \rangle + j(w_n(t), \mathbf{u}_n(t), \mathbf{v}) \\ & - j(w_n(t), \mathbf{u}_n(t), \mathbf{u}_n(t)) \geq (\mathbf{f}(t), \pi \mathbf{v} - \pi \mathbf{u}_n(t))_Z \quad \forall \mathbf{v} \in K_n, t \in [0, T], \end{aligned} \quad (5.20)$$

where the set  $K_n$ , the operator  $S_n : C([0, T]; V) \rightarrow C([0, T]; V^*)$  and the functions  $F_n : [0, T] \times X \times V \rightarrow X$ ,  $\mathbf{f}_n : [0, T] \rightarrow Z$  are defined by the equalities

$$K_n = \{ \mathbf{v} \in V \mid \mathbf{v} \leq g_n \text{ on } \Gamma_3 \}, \quad (5.21)$$

$$\begin{aligned} \langle S_n \mathbf{u}(t), \mathbf{v} \rangle &= (\alpha_n \int_0^t (e^{t-s} \boldsymbol{\varepsilon}(\mathbf{u}(s))) ds, \boldsymbol{\varepsilon}(\mathbf{v}))_Q \\ &\forall \mathbf{u} \in C([0, T]; V), \mathbf{v} \in V, t \in [0, T], \end{aligned} \quad (5.22)$$

$$F_n(t, w, \mathbf{u}) = k_n(t) \| \mathbf{v}^* \| p(u_\nu - w) \quad \forall w \in X, \mathbf{u} \in V, t \in [0, T], \quad (5.23)$$

$$\mathbf{f}_n(t) = (\mathbf{f}_0(t), \theta(t) \mathbf{f}_{2n}) \quad \forall t \in [0, T]. \quad (5.24)$$

Then, using the properties of the function  $p$  and the compactness of the trace operator, it is routine to check that conditions (4.5)–(4.10) hold. For instance, the validity of condition (4.6) follows from equality  $K_n = \frac{g_n}{g} K$  and the smallness condition (3.10) follows from (5.1) since, in our case,  $l_S = 0$ . To conclude, the convergence result (5.6) is a direct consequence of Theorem 4.1.  $\square$

Besides the mathematical interest in the convergence result in Theorem 4.1 it is important from mechanical point of view. Indeed, it shows that at each time moment the solution of the viscoelastic frictional contact with wear depends continuously on the relaxation coefficient  $\alpha$ , the initial density of surface tractions  $\mathbf{f}_2$ , the wear coefficient  $k$  and the thickness of the soft layer  $g$ . In particular if  $\alpha$  vanishes the convergence (5.6) shows that the solution of the elastic contact problem with wear can be approached by the solution of the viscoelastic problem with wear; if  $k$  vanishes it follows from (5.6) that the solution of the viscoelastic contact problem without wear can be approached by the solution of a viscoelastic problem with wear; finally, if  $g$  vanishes, (5.6) shows that the solution of the viscoelastic contact problem with a rigid obstacle can be approached by the solution of the viscoelastic problem with wear and a rigid-deformable obstacle.

In addition, Theorem 4.1 allows us to consider various optimal control **problems** associated to the contact model (2.1)–(2.8). Here we restrict ourselves to consider an optimal control problem associated to the cost functional (2.28) which, recall, represents a measure of the wear of the deformable layer at the end of the contact process. Our approach is as follows. First, note that, under assumptions of Theorem 4.1, the solution of Problem  $\mathcal{P}_V$  depends on  $\mathbf{f}_2$  and, therefore, we denote it in what

follows by  $(\mathbf{u}_{\mathbf{f}_2}, w_{\mathbf{f}_2})$ . Moreover, we recall the regularity  $w_{\mathbf{f}_2} \in C(0, T; L^2(\Gamma_3))$ . Let  $M_2 > 0$  and let

$$W_2 = \{ \mathbf{f}_2 \in L^2(\Gamma_2; \mathbb{R}^d) : \|\mathbf{f}_2\|_{L^2(\Gamma_2; \mathbb{R}^d)} \leq M_2 \}. \quad (5.25)$$

Then, the optimal control **problem** we consider is the following.

**Problem Q.** Find  $\mathbf{f}_2^* \in W_2$  such that

$$\int_{\Gamma_3} w_{\mathbf{f}_2^*}^2(T) da \leq \int_{\Gamma_3} w_{\mathbf{f}_2}^2(T) da \quad \forall \mathbf{f}_2 \in W_2. \quad (5.26)$$

Our main result in the study of this problem is the following.

**Theorem 5.2.** Assume (2.14)–(2.17), (2.19)–(2.22) and (5.1). Then, the optimal control problem **Q** has at least one solution  $\mathbf{f}_2^*$ .

*Proof.* Let  $\mathcal{L} : W_2 \rightarrow \mathbb{R}$  be the cost function given by

$$\mathcal{L}(\mathbf{f}_2) = \int_{\Gamma_3} w_{\mathbf{f}_2}^2(T) da \quad \forall \mathbf{f}_2 \in W_2. \quad (5.27)$$

Using Theorem 5.1 it is easy to see that  $\mathcal{L}$  is a weakly continuous function and, therefore, it is weakly lower semicontinuous. On the other hand, the set  $W_2$  is a bounded nonempty weakly closed subset of the reflexive Banach space  $L^2(\Gamma_2; \mathbb{R}^d)$ . The existence of at least one solution for Problem **Q** is now a direct consequence of the Weierstrass theorem.  $\square$

The mechanical interpretation of Problem **Q** is the following: given a contact process described by the boundary value problem (2.1)–(2.8) and a constant  $M_2 > 0$ , we are looking for an initial density of surface tractions  $\mathbf{f}_2 \in L^2(\Gamma_2; \mathbb{R}^d)$  with  $\|\mathbf{f}_2\|_{L^2(\Gamma_2; \mathbb{R}^d)} \leq M_2$  such that the corresponding wear of the foundation at the end of the contact process is as small as possible. Such kind of problems could have relevant applications in various industrial settings and, in particular, in the automotive industry.

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## References

- [1] K.T. Andrews, A. Klarbring, M. Shillor and S. Wright, A dynamic contact problem with friction and wear, *Int. J. Egnng. Sci.* **35** (1997), 1291–1309.
- [2] J. P. Aubin , A. Cellina, *Differential inclusions*, Springer-Verlag, NewYork, 1984.
- [3] A. Capatina, *Variational Inequalities and Frictional Contact Problems*, Advances in Mechanics and Mathematics **31**, Springer, Heidelberg, 2014.
- [4] A. Chudzikiewicz and A. Myśliński, Thermoelastic wheel-rail contact problem with elastic graded-materials, *Wear* **271** (2011), 417–425.
- [5] G. Duvaut and J.-L. Lions, *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin, 1976.
- [6] C. Eck, J. Jarušek and M. Krbeč, *Unilateral Contact Problems: Variational Methods and Existence Theorems*, Pure and Applied Mathematics **270**, Chapman/CRC Press, New York, 2005.
- [7] R.J. Gu, K.L. Kuttler and M. Shillor, Frictional wear of a thermoelastic beam, *J. Math. Anal. Appl.* **242** (2000), 212–236.
- [8] R.J. Gu and M. Shillor, Thermal and wear analysis of an elastic beam in sliding contact, *Int. J. Solids Structures* **38** (2001), 2323–2333.
- [9] J. Gwinner, On a new class of differential variational inequalities and a stability result, *Mathematical Programming* **139** (2013), 205–221.
- [10] W. Han and M. Sofonea, *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity*, Studies in Advanced Mathematics **30**, American Mathematical Society, Providence, RI–International Press, Somerville, MA, 2002.
- [11] P. Kalita, P. Szafraniec, M. Shillor, A frictional contact problem with wear diffusion, *Zeitschrift fur angewandte Mathematik und Physik*, **70** (2019), 70–96.
- [12] N. Kikuchi and J.T. Oden, *Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods*, SIAM, Philadelphia, 1988.
- [13] D. Knees, R. Rossi, C. Zanini, A quasilinear differential inclusion for viscous and rate-independent damage systems in non-smooth domains, *Nonlinear Analysis: Real World Applications* **24** (2015), 126–162.
- [14] K. L.Kuttler and M. Shillor, Dynamic contact with normal compliance wear and discontinuous friction coefficient, *SIAM J. Math. Anal.* **34** (2002), 1–27.
- [15] X. Li, N. Huang, A class of impulsive differential variational inequalities in finite dimensional space, *Journal of the Franklin Institute* **353** (2016), 3151–3175.

- [16] Z.H. Liu, V.T. Nguyen, J.C. Yao, S.D. Zeng, History-dependent differential variational-hemivariational inequalities with applications to contact mechanics, *Evol. Equ. Control Theory* **9** (2020), 1073–1087.
- [17] Z.H. Liu, M. Sofonea, Differential quasivariational inequalities in contact mechanics, *Math. Mech. Solids* **24** (2018), 845–861.
- [18] Z. H. Liu, S. D. Zeng, Differential variational inequalities in infinite Banach spaces, *Acta Mathematica Scientia* **37** (2017), 26–32.
- [19] A. Mielke, T. Roubíček, Rate-independent systems. Theory and Application, Applied Mathematical Sciences **193**, Springer, 2015.
- [20] A. Mielke, T. Roubíček, J. Zeman, Complete damage in elastic and viscoelastic media and its energetics, *Computer Methods in Applied Mechanics and Engineering*, **199** (2010), 1242–1253.
- [21] S. Migórski, Z. Liu, S. Zeng. A class of history-dependent differential variational inequalities with application to contact problems, *Optimization*, **69** (2020), 743–775.
- [22] Z. Naniewicz and P. D. Panagiotopoulos, *Mathematical Theory of Hemivariational Inequalities and Applications*, Marcel Dekker, Inc., New York, Basel, Hong Kong, 1995.
- [23] P.D. Panagiotopoulos, *Inequality Problems in Mechanics and Applications*, Birkhäuser, Boston, 1985.
- [24] P. D. Panagiotopoulos, *Hemivariational Inequalities, Applications in Mechanics and Engineering*, Springer-Verlag, Berlin, 1993.
- [25] J. S. Pang, D. E. Stewart, Differential variational inequalities. Mathematical programming, **113** (2008), 345–424.
- [26] J. Rojek, J.J. Telega and S. Stupkiewicz, Contact problems with friction, adhesion and wear in orthopaedic biomechanics. II: Numerical implementation and application to implanted knee joints, *J. Theor. Appl. Mech.* **39** (2001), 679–706.
- [27] M. Shillor, M. Sofonea and J.J. Telega, *Models and Analysis of Quasistatic Contact*, Lecture Notes in Physics **655**, Springer, Berlin, 2004.
- [28] M. Sofonea, J. Bollati, D.A. Tarzia, Optimal control of differential quasivariational inequalities with applications in contact mechanics, *Journal of Mathematical Analysis and Applications*, <https://doi.org/10.1016/j.jmaa.2020.124567>, to appear.

- [29] M. Sofonea and A. Matei, *Mathematical Models in Contact Mechanics*, London Mathematical Society Lecture Note Series **398**, Cambridge University Press, 2012.
- [30] M. Sofonea and S. Migórski, *Variational-Hemivariational Inequalities with Applications*, Pure and Applied Mathematics, Chapman & Hall/CRC Press, Boca Raton-London, 2018.
- [31] M. Sofonea, F. Pătrulescu, Y. Souleiman, Analysis of a contact problem with wear and unilateral constraint, *Applicable Analysis*, **95** (2016), 2590–2607.
- [32] N. Strömberg, L. Johansson and A. Klarbring, Derivation and analysis of a generalized standard model for contact, friction and wear, *Int. J. Solids Structures* **33** (1996), 1817–1836.
- [33] A. Zmitrowicz, Variational descriptions of wearing out solids and wear particles in contact mechanics, *Journal of Theoretical and Applied Mechanics* **39** (2001), 791–808.