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A theory of stratification learning

Eddie Aamari* Clément Berenfeld†

Abstract

Given i.i.d. sample from a stratified mixture of immersed manifolds of different dimensions, we study the minimax estimation of the underlying stratified structure. We provide a constructive algorithm allowing to estimate each mixture component at its optimal dimension-specific rate adaptively. The method is based on an ascending hierarchical code-detection of points belonging to different layers, which also identifies the number of layers and their dimensions, assigns each data point to a layer accurately, and estimates tangent spaces optimally. These results hold regardless of any ambient assumption on the manifolds or on their intersection configurations. They open the way to a broad clustering framework, where each mixture component models a cluster emanating from a specific nonlinear correlation phenomenon.

1 Introduction

1.1 Context

Union of manifold hypothesis With the ubiquitous availability of high-dimensional and unstructured data, the so-called manifold hypothesis is often raised as an explanation for the surprisingly good performances of prediction [16] and generative methods [26]. It posits that high-dimensional data typically reside close to some (unknown) submanifold \( M \) of low intrinsic dimension \( d \), significantly smaller than the ambient dimension \( D \). The support \( M \) models all the possibly non-linear constraints or correlations that data may present, and \( d \) represents its true local degrees of freedom. Statistically speaking, the success of modern machine learning methods is then attributed to estimation rates depending mostly (if not only) on this lower intrinsic dimension \( d \), hence enabling effective learning on reasonably-sized datasets \( X_1, \ldots, X_n \) even when \( D \gg n \). This approach has proven fruitful in a quite wide range of applications, including cancer classification [11], RNA sequencing [47], image analysis [23], location-based services [40] and thermodynamics [51].

The geometric nature of the manifold hypothesis brings an entire field of possible data modelings based on differential geometry [29] and geometric measure theory [31]. Manifold estimation being fundamentally non-parametric, one can see each geometric modeling (or statistical model) as a specific regularity constraint on the problem. The most crude (and studied) one consists in assuming that \( M \) is a connected \( C^2 \)-submanifold without boundary [34]. Most existing manifold estimation techniques assume that \( M \) is a single well-behaved such manifold.

However, recent empirical studies suggest that real data actually exhibit mixed dimensionalities [21]. This work addresses this latter general scenario, where data actually arise from a stratification of manifolds of possibly different dimensions. In probabilistic terms, we consider the estimation and clustering problem of arbitrary mixtures of distributions with individual components having \( C^2 \) submanifolds as a support.

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A manifold approach to generalized clustering  Clustering (or unsupervised learning) is a statistical task that aims at grouping data points into simple groups (called clusters) that are as separated as possible from each other. Cluster separation is most commonly quantified via the distance between clusters, while simplicity shall be measured through some notion of smallness or connectedness. However, achieving separation based on location only may not always be relevant, especially in high dimension. Following [39], the generalized clustering problem consists in decomposing the distribution $P$ of data into "simple" components $(P_k)_{1 \leq k \leq K}$ while maximizing a specified distance function between them.

As mentioned above, local manifold parametrizations model local data correlations arising from numerous non-linear constraints of the generating distribution. Hence, membership in different parametrizations yields a natural geometric criterion for cluster separation. The sample counterpart is a fine non-linear clustering of a point cloud that both includes positional and directional information given by tangent spaces locally [54]. See Figure 1 for an illustration.

Figure 1: A union of curves $M_1$ and a surface $M_2$ in dimension $D = 3$ intersecting arbitrarily (top right). None of them have boundary, and drawings are to be continued outside the picture if necessary. The two completely visible closed curves intersect tangentially, coincide over a non-trivial interval, and are both completely included in the surface. Despite the topology of a pre-image of $M_1 \subset \mathbb{R}^D$ by an immersion being not uniquely defined when $M_1$ self-intersects tangentially, the dimension-wise clustering and reconstruction of the overall support still makes sense (top left). After sampling (bottom right), learning the stratification consists in labeling points dimension-wise, as well as approximating the underlying manifold structures (bottom left).

1.2 Contribution

Informal results

This work presents a first attempt to build an estimation theory of general unions of compact $C^2$-submanifolds of $\mathbb{R}^D$. Its framework steps away from ambient convexity-type assumptions on the target set such as the reach, $\mu$-reach or rolling ball conditions [24, 50]. It shows that
bounded volume and directional curvature are sufficient to obtain fast convergence rates as well as dimension-wise clustering.

Let $X_1, \ldots, X_n \in \mathbb{R}^D$ be drawn i.i.d. from a union $\bigcup_{k=1}^K M_k$ of immersed compact $C^2$-manifolds, with unknown number of mixture components $K$ and unknown dimensions $d_1 < \ldots < d_K$ (see Figure 1). With $\mathcal{X}_n := \{X_1, \ldots, X_n\}$ and scale parameters as input, we propose a constructive algorithm called Slabeling, which outputs an approximation of the stratified structure of $M$. This output has a layer-by-layer structure, each of which corresponds to a fixed dimension. It allows to recover the structure $(M_k)_{1 \leq k \leq K}$ in the following sense.

(Discrete stratification learning) With high probability, the discrete structure of $M = \bigcup_{k=1}^K M_k$ can be identified by Slabeling through $\hat{K} = K$ and $\{\hat{d}_1, \ldots, \hat{d}_K\} = \{d_1, \ldots, d_K\}$. (Theorem 4.2)

Furthermore, data points $\mathcal{X}_{k,n} := M_k \cap \mathcal{X}_n$ from each layer $M_k$ can be clustered by dimension through a constructive decomposition $\mathcal{X}_n = \bigsqcup_{1 \leq k \leq K} \mathcal{X}_{k,n}$ of the data set such that

$$\frac{\#\mathcal{X}_{k,n} \triangle \mathcal{X}_{k,n}}{\max\{\#\mathcal{X}_{k,n}, 1\}} \lesssim \left(\frac{\log n}{n}\right)^{2/d_k},$$

(Theorem 4.4)

with accurate dimension labeling (Theorem 4.3).

(Approximation) On the same event of high probability, all the manifolds $M_k$ can be estimated in Hausdorff distance at the layer-adaptive rate

$$d_H(M_k, \overline{M}_k) \lesssim \left(\frac{\log n}{n}\right)^{2/d_k},$$

(Theorem 4.5)

and tangent spaces at $X_i$’s can be estimated, in the sense that points $x \in M_k$ and $y \in \overline{M}_k$ at distance $\|x - y\| \lesssim (\log n/n)^{2/d_k}$ can be put in correspondence such that

$$\angle(T_x M_k, T_y \overline{M}_k) \lesssim \left(\frac{\log n}{n}\right)^{1/d_k},$$

(Theorem 4.6)

The algorithm does not require any prior information about the number $K$ of mixture components, but only a maximal intrinsic dimension $d_{\max}$ past which points shall be considered coming from a too-high dimension and hence possibly considered clutter noise (see Remark 2.6). Naturally, the above results hold if scale parameters are properly chosen and $d_{\max} \geq d_K$. Otherwise, the results extend to the learning of the thresholded structure $\bigcup_{d_k \leq d_{\max}} M_k$.

Consequences The main high-level consequence of our results is that for manifold estimation, intrinsic geometric regularity constraints (namely curvature and volume bounds) are sufficient to obtain fast estimation rates. That is, no assumption on the reach or any other ambient regularity parameters is required. Possible (self-)intersections and interplay between layers of different dimensions are also harmless for the final rate. This phenomenon also applies to tangent space estimation, but is in sharp contrast to other geometric and topological features such as homology or geodesic distances (see Section 2.3), which become impossible to estimate without reach condition.

Going from single-source to mixtures, manifolds turn out to be much easier to handle than classical density-based (Gaussian) mixture models where source separation requirements are needed to maintain distinguishability [42]. Our algorithm resolves and separates all the layers up to dimension $d$ in time $O(n (\log n)^d)$, that is the same order of magnitude as dealing with the $d$-dimensional layer taken separately.
1.3 Related works

**Single-manifold estimation and reach condition** The minimax theory of support estimation from data drawn on smooth sets is now well understood. In full dimension, all the studied regularity constraints build upon some notion of convexity. This includes standard convexity [30], \(r\)-convexity [7], rolling ball conditions [27] and reach positiveness [31], all yielding rates of order \(O\left((\log n/n)^{2(D)}\right)\) or \(O\left((\log n/n)^{2/(D+1)}\right)\) in Hausdorff distance.

For embedded structures of dimension \(d < D\), regularity constraints always borrow from both ambient convexity-type constraints and differential geometry simultaneously. Usually, the ambient constraint is formalized through a positive lower bound on the reach. Intrinsic differential smoothness takes the form of \(C^k\) parametrizations for \(k \geq 2\) [34, 4, 5, 28] — yielding rate \(O\left((\log n/n)^{k/d}\right)\) — with possibly non-empty boundary for \(k = 2\) [1] — yielding rate \(O\left((\log n/n)^{2/(d+1)}\right)\). These methods rely on local polynomials or local convex hulls depending on the context.

Seen as a regularity parameter, the reach prevents quasi-self intersections to be arbitrarily narrow [3]. It has been used as a key scaling factor in all the works cited above, besides (and sometimes instead of) curvature bounds. Beyond the technical easiness induced by reach constraints [31], the systematic assumption of reach positiveness probably is to be attributed to the fact that some geometric and topological quantities become impossible to estimate under intrinsic constraints only (see Section 2.3). Yet, as proved in the present article, several key geometric quantities remain estimable without convexity-type constraints and even for general unions of immersed manifolds, with no deterioration in the estimation rates. This includes, among others, the support and tangent spaces.

**A concise review of the multi-manifold literature** To our knowledge, the only works coming with provable geometric or statistical guarantees require either linearity, or mixture component to intersect transversally. The others are of heuristic or numerical nature. Theses pieces of work can be grouped as follows.

- **(Multiple PCA)** The cluster recovery of mixture components formed by low-dimensional objects in high dimension has first been carried out in the case of linear subspaces [44, 20]. The main method relies on generalized Principal Component Analysis (PCA) [53], which consists in the minimization of a tweaked least square functional.

- **(Local PCA)** When coping with non-linearity, the most studied methods are based on local PCA [9, 52, 45] and kernelized version of it [25]. The rationale is that properly-localized covariance matrices estimate local tangent directions through their largest singular values and associated singular vectors. Conversely, if the dimension is known to be \(d\), a covariance with at least \(d + 1\) singular values significantly larger than the others indicates the presence of a local \((d + 1)\)-dimensional space of variations, and hence a branching (or intersection) point. Somehow related is the kernel-based clustering method of [39], which builds upon a preliminary featurization step onto a Hilbert space, intended to linearize the geometric setting globally.

- **(Homology-based clustering)** By definition, the geometry of neighborhoods at an intersection point is not that of a single \(d\)-dimensional ball, but that of a union of several \(d\)-dimensional balls. This can be characterized topologically by the \((d-1)\)-dimensional homology of annuli being that of a sphere or not.

This idea has been exploited in dimension \(d = 1\), where the union of closed curves is often called metric graph [6, 15, 43]. The method uses persistent homology, which compares topology at different scales. Although the method manages to handle noise, the curved graph edges are assumed to intersect transversally at vertices. We will show that in fact, no transversality
assumption is required to estimate a union of curves with fast rate, as long as each of them is \(C^2\) smooth.

- **(Dynamic methods)** Instead of finely clustering raw data points via local geometric or topological descriptors, the *mean-shift* algorithm [33] moves them incrementally towards high density regions and then uses a simple \(k\)-mean clustering method. In a mixture of manifolds setting, [54] proposed to perform an optimization scheme that alternates between local structure preservation and cluster distinguishability to account for tangential information and not only positions.

Stepping slightly aside from the manifold-based hypotheses, low-dimensional structures can also be modeled in a relaxed way. Data can be assumed to be generated *near* a submanifold \(M\), i.e. drawn according to a distribution having a very peaky ambient density in a neighborhood of \(M\). In such approximate models, the existence of a likelihood has been leveraged in at least two orthogonal directions.

- **(Density ridges)** The existence of a density allows to define the so-called *density ridges*. They form a stratification of the ambient space through generalizations of density modes, called ridges, and indexed by dimension. Their estimation is now well understood, both theoretically and algorithmically [49, 48]. Although this line of work parallels quite closely the notion of mixture of manifolds, note that density ridges — as name suggests — only make sense if data is generated by a distribution having a density with respect to the ambient Lebesgue measure. In the same vein, let us also mention poissonized non-linear likelihood models [38], allowing for likelihood maximization.

- **(Bayesian approaches)** Putting a mixture prior on the density leads to estimators that are structurally stable to mixtures. As a result, the super-level sets of the posterior density behaves well when distributions are mixed. In the linear case, (nearly-)singular Gaussian mixtures have been studied in [20] on union of (thickened) subspaces. In the non-parametric curved case, local Gaussian mixtures have been put to use in [17].

Let us insist on the fact that all the above cited articles either require transversality at intersection points, or do not come with a statistical analysis. Furthermore, we are not aware of any work that goes beyond the clustering-by-dimensionality problem (or dimension labeling), and that would allow for a geometric reconstruction of the mixture components.

## 2 Geometric and statistical model

### 2.1 Euclidean immersions, curvature and volume

We recall that a Euclidean \(C^2\)-immersion of a compact differentiable manifold \(M_0\) is a twice continuously differentiable map \(\Psi_0 : M_0 \to \mathbb{R}^D\) whose differential is everywhere injective. The image set \(M := \Psi_0(M_0) \subset \mathbb{R}^D\) is called an *immersed manifold*.

The differential structure of \(M_0\) at any point \(x_0 \in M_0\) is canonically pushed forward via the differential of \(\Psi_0\). We write \(T_{x_0}(\Psi_0) M := d\Psi_0(x_0)[T_{x_0}M_0]\), where \(T_{x_0}M_0\) stands for the tangent space of \(M_0\) at \(x_0\). Accordingly, the *tangent cone* at a point \(x \in M = \Psi_0(M_0)\) is the set

\[
T_x(\Psi_0)M := \bigcup_{x_0 \in \Psi_0^{-1}(x)} d\Psi_0(x_0)[T_{x_0}M_0]
\]

of pushed forward tangents of the pre-image \(\Psi_0^{-1}(x_0)\). Now looking at structure of order two, let us recall that the Euclidean metric on \(\mathbb{R}^D\) pulls back canonically onto \(M_0\) via \(\Psi_0\), so that
$M_0$ inherits a Riemannian structure $(M_0, g_0)$ making $\Psi_0 : (M_0, g_0) \to (\mathbb{R}^D, \langle \cdot, \cdot \rangle)$ an isometric immersion in the Riemannian sense. This allows to define the second fundamental form [29, Chapter 6] $\Pi^{(\Psi_0)}_{x_0} : T_{x_0}M_0 \times T_{x_0}M_0 \to T_{x_0}^{(\Psi_0)}M$. We write

$$\kappa^{(\Psi_0)}(M) := \sup_{x_0 \in M_0} \| \Pi^{(\Psi_0)}_{x_0} \|_{\text{op}}$$

for the maximal curvature radius of the immersion of $M_0$, and $\text{vol}(M_0)$ its Riemannian volume.

In what follows, we will infer properties of some (a priori unstructured) geometric set $M \subset \mathbb{R}^D$ under regularity assumptions induced by the existence of some $C^2$ immersion $\Psi_0 : M_0 \to M$. Hence, $M_0$, its Riemannian and differential structures shall be seen as a latent unobserved structure. However, there may not be a unique manifold $M_0$ and immersion $\Psi_0 : M_0 \to \mathbb{R}^D$ with geometric image $M$. Actually, neither the geometry nor the topology of $(M_0, g_0)$ is uniquely defined (see Figure 2).

![Figure 2: The immersed 8-shape curve $M \subset \mathbb{R}^2$ (middle) can originate from either one or two topological circles, shown as $M_0$ and $M_0'$ (left). Furthermore, running through the curve $M$ twice also yields an immersion with the same image, but from a manifold $\text{vol}(M_0'')$ having twice the common volume of $M_0$ and $M_0'$ (right).](image)

If $M \subset \mathbb{R}^D$, we let $\mathbb{I}(M)$ denote the set of couples $(M_0, \Psi_0)$ such that $M_0$ is an abstract $C^2$ manifold, and $\Psi_0 : M_0 \to \mathbb{R}^D$ is a $C^2$ immersion with image $\Psi_0(M_0) = M$. Despite $\mathbb{I}(M)$ possibly containing several very different geometric structures, the following result asserts that their pushforward tangent spaces and maximal curvature actually are all equal.

**Lemma 2.1.** Given $M \subset \mathbb{R}^D$ with $\mathbb{I}(M) \neq \emptyset$ and $x \in M$, then for all immersions $(M_0, \Psi_0)$ and $(M_0', \Psi_0')$ in $\mathbb{I}(M)$, there holds

$$T_x^{(\Psi_0')}M = T_x^{(\Psi_0')}M \quad \text{and} \quad \kappa^{(\Psi_0)}(M) = \kappa^{(\Psi_0')}(M).$$

The proof can be found in Section A.3. Lemma 2.1 allows to define unambiguously the tangent spaces of $M$ at $x \in M$ as

$$T_xM := T_x^{(\Psi_0)}M,$$

and the maximal curvature of $M$ as

$$\kappa(M) := \kappa^{(\Psi_0)}(M),$$
which do not depend on the choice of an immersion of \((M_0, \Psi_0) \in \mathcal{I}(M)\). On the other hand, the volume \(\text{vol}(M_0)\) obviously depends on the immersion (see Figure 2). We define the minimal immersion-realizable volume of \(M \subset \mathbb{R}^D\) as

\[
\mathcal{V}(M) := \inf_{(M_0, \Psi_0) \in \mathcal{I}(M)} \text{vol}(M_0).
\]

By construction, notice in particular that \(\text{vol}(M) \leq \mathcal{V}(M)\), where \(\text{vol}(M)\) stands for the \(d\)-dimensional Hausdorff measure of \(M\) seen as a rectifiable subset of \(\mathbb{R}^D\). We are now in position to define the volume- and curvature-constrained model of \(C^2\) immersed manifolds at a given dimension.

**Definition 2.2** (Geometric model for immersed \(C^2\) manifolds). For \(d \in \{1, \ldots, D - 1\}\), \(\kappa_{\text{max}} > 0\) and \(V_{\text{max}} > 0\), we let \(\mathcal{M}(d)(\kappa_{\text{max}}, V_{\text{max}})\) denote the set of compact immersed \(C^2\) manifolds \(M\) of \(\mathbb{R}^D\) of dimension \(d\) and such that

i) \(\kappa(M) \leq \kappa_{\text{max}}\);

ii) \(\mathcal{V}(M) \leq V_{\text{max}}\).

Constraining \(\text{vol}(M)\) rather that \(\mathcal{V}(M)\) may first seem more natural. In fact, bounding \(\text{vol}(M)\) only would still allow for pathological subsets \(M \subset \mathbb{R}^D\) that exhibit points \(x \in M\) with arbitrarily wild local neighborhood fibers \(\Psi_0^{-1}(B(x, r))\) (see Figure 3). In contrast, a (stronger) bound on \(\mathcal{V}(M)\) conveys enough rigidity to assert that neighborhoods \(M \cap B(x, r)\) all resemble a quantitatively bounded number of Euclidean \(d\)-balls (see Lemma A.3).

\[\begin{figure}
\framebox[.9\textwidth]{Figure 3: An increasing family of immersed curves \((M_\ell)_{\ell \in \mathbb{N}}\), each with \(\ell\) nested \(C^2\) bumps of exponentially decaying height and width. The curves have uniformly bounded lengths \(\text{vol}(M_\ell) \leq \text{vol}(\bigcup_{\ell \in \mathbb{N}} M_\ell) < \infty\). However, the nestedness of the bumps forces immersions \(\Psi_\ell : M_\ell \to M_{\ell'}\) describing \(\ell\) bumps to go around the upper unbumped part \(M^{(u)}\) (dashed) at least \(\ell\) times, so that \(\text{vol}(M_\ell) \geq \ell \text{vol}(M^{(u)})\) \(\ell \to \infty\).}
\end{figure}\]

### 2.2 Manifold mixtures

Let \(M \in \mathcal{M}(d)(\kappa_{\text{max}}, V_{\text{max}})\) have an associated \(C^2\) immersion \((M_0, \Psi_0)\). As a rectifiable subset of \(\mathbb{R}^D\) parametrized by a compact \(C^2\) manifold, \(M\) has finite non-zero \(d\)-dimensional Hausdorff measure \(\mathcal{H}^d(M) = \text{vol}(M)\) [32, Section 3.2]. We let \(\text{vol}_M\) denote the volume measure on \(M\), defined as

\[
\text{vol}_M(\text{d}x) := 1_M \mathcal{H}^d(\text{d}x).
\]

In general, the volume measure \(\text{vol}_M\) on \(M\) and the pushforward \(\Psi_0^\# \text{vol}_{M_0}\) of the Riemannian volume measure on \(M_0\) for some \((M_0, \Psi_0) \in \mathcal{I}(M)\) do not coincide. For example in Figure 2, one can take \(\Psi_0\) to describe the upper circle twice and the other only once, leaving the latter with half the mass relatively to the upper circle. However, these two measures do coincide locally, in the sense that for subsets \(A \subset M_0\) of small enough diameter, \(\text{vol}_{M_0}(A) = \text{vol}_M(\Psi_0(A))\).

In what follows, we will consider distributions over \(M\) that are essentially uniform, in the sense that they have densities with respect to \(\text{vol}_M\) bounded away from 0 and \(\infty\).
Definition 2.3 (Single-manifold model). For \( d \in \{1, \ldots, D - 1\} \), \( 0 < \nu_{\text{max}} < \infty \) and \( 0 < a_{\text{min}} \leq a_{\text{max}} < \infty \), we let \( \mathcal{P}^{(d)}_{\kappa_{\text{max}}\nu_{\text{max}}}(a_{\text{min}}, a_{\text{max}}) \) denote the set of Borel probability distributions over \( \mathbb{R}^D \) such that

i) \( M := \text{support}(P) \) belongs to \( \mathcal{M}(d)(\kappa_{\text{max}}, \nu_{\text{max}}(\kappa_{\text{max}})^{-d}) \);

ii) \( P \) admits a density \( f : M \to \mathbb{R}_+ \) with respect to \( \text{vol}_M \) such that for all \( x \in M \),

\[
a_{\text{min}} \kappa_d^d \leq f(x) \leq a_{\text{max}} \kappa_{\text{max}}^d.
\]

Remark 2.4 (On model parameters). At this stage, let us emphasize two points.

- The above model could as well be parametrized by \( f_{\text{min}} := a_{\text{min}} \kappa_{\text{max}}^d \), \( f_{\text{max}} = a_{\text{max}} \kappa_{\text{max}}^d \) and \( \nu_{\text{max}} = \nu_{\text{max}}(\kappa_{\text{max}})^{-d} \). We choose instead to use a single scale parameter \( \kappa_{\text{max}} \) and non-dimensionalized constants \( a_{\text{min}}, a_{\text{max}}, \nu_{\text{max}} \) so that that they can be shared across several models \( \mathcal{P}^{(d)}_{\kappa_{\text{max}}\nu_{\text{max}}}(a_{\text{min}}, a_{\text{max}}) \) with different values of \( d \).

- Closed manifolds with curvature bounded by \( \kappa_{\text{max}} \) have volume bounded below by that of a sphere of radius \( 1/\kappa_{\text{max}} \) [8]. Hence, one needs to request that \( \nu_{\text{max}} \geq \sigma_d \) to have a non-empty model in dimension \( d \), where \( \sigma_d \) is the volume of the \( d \)-dimensional sphere. In order to make sure that all our models are non-empty for \( d \in \{1, \ldots, D - 1\} \), we will hence assume without loss of generality that

\[
\nu_{\text{max}} \geq \max_{d \geq 1} \sigma_d = \sigma_6 = 16\pi^3/15, \quad (4)
\]

for the rest of the paper.

This model generalizes that of [34, 4, 28], in which the support \( M \) is additionally required to be an \emph{embedded} submanifold with positive reach, instead of an \emph{immersed} one with no reach constraint. In addition, the present article considers general mixtures of such distributions with varying intrinsic dimensions \( d \).

Definition 2.5 (Manifold mixture model). Given \( \alpha_{\text{min}} > 0 \), let \( \tilde{\mathcal{P}}(\alpha_{\text{min}}, \kappa_{\text{max}}, \nu_{\text{max}}, a_{\text{min}}, a_{\text{max}}) \) denote the set of mixture probability distributions \( P \) over \( \mathbb{R}^D \) of the form

\[
P = \sum_{k=1}^{K} \alpha_k P_k,
\]

where \( K \in \{1, \ldots, D - 1\} \) varies, and

i) \( 1 \leq d_1 < d_2 < \ldots < d_K \leq D - 1 \),

ii) \( \min_{1 \leq k \leq K} \alpha_k \geq \alpha_{\text{min}} > 0 \),

iii) \( P_k \in \mathcal{P}_{\kappa_{\text{max}}\nu_{\text{max}}}(a_{\text{min}}, a_{\text{max}}) \) for all \( k \in \{1, \ldots, K\} \).

Model \( \tilde{\mathcal{P}} \) is parametrized by \( \alpha_{\text{min}}, \kappa_{\text{max}}, a_{\text{min}}, a_{\text{max}} \) and \( \nu_{\text{max}} \), all left implicit for sake of brevity.

Model \( \tilde{\mathcal{P}} \) includes mixture structures of different intrinsic dimensions with varying sequences of dimensions \( (d_k)_{k \leq K} \) in a single unified model. If present, a mixture component \( P_k \) is assigned significant weight \( \alpha_k \geq \alpha_{\text{min}} > 0 \). A random variable \( X \sim \sum_{k=1}^{K} \alpha_k P_k \) can be represented in the following way:

- Draw layer index \( Y \in \{1, \ldots, K\} \) according to \( \mathbb{P}(Y = k) = \alpha_k \);

- Conditionally on \( Y \), draw \( X \) according to \( P_Y \), that is \( X \mid \{Y = k\} \sim P_k \).
As any reasonable non-parametric model built upon a $C^2$-type geometric model, the statistical model $\bar{P}$ can be shown to be stable (up to constants) under ambient $C^2$-diffeomorphisms (see [4, Proposition 1]). Additionally, model $\bar{P}$ is structurally stable (up to constants) under mixture. Indeed, the union of stratified manifolds in the model still belongs to the model. At the level of data points, this means that merging several datasets generated from different distributions yields a dataset that can itself be seen a being generated from the model.

**Remark 2.6** (Reinterpreting clutter noise). A random variable $X$ drawn from model $\bar{P}$ originate from a mixture distribution $\sum_{k=1}^K \alpha_k P_k$ over manifolds of different dimensions. It generalizes the clutter noise model studied in [4, 34], where data could be drawn either from (signal) a smooth $d$-dimensional submanifold $M \subset \mathbb{R}^D$ with positive reach, or (clutter) from a uniform distribution over an ambient $D$-dimensional ball $B(0, R)$ such that $B(0, R/2) \supset M$. To see this, one simply has to consider $B(0, R)$ as a $D$-dimensional (flat) submanifold of $\mathbb{R}^D$ and take $\{d_1, d_2\} = \{d, D\}$. Note the slight abuse here since $B(0, R)$ has non-empty boundary, but [4, 34] artificially prevent boundary effects by imposing that $M \supset B(0, R/2)$.

### 2.3 Statistical setting

Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a i.i.d. sequence with $X_i \sim \sum_{k=1}^K \alpha_k P_k \in \bar{P}$ and $X_i \mid Y_i \sim P_{Y_i}$. We observe the unlabeled i.i.d. sample $X_1, \ldots, X_n$ distributed according to the mixture $P = \sum_{k=1}^K \alpha_k P_k$, and we write

$$N_k := \sum_{i=1}^n 1_{Y_i = k}$$

for the (unobserved) number of data points generated by the $d_k$-dimensional measure $P_k$. In what follows, we denote by

$$P_{k,n} := \frac{1}{N_k} \sum_{i=1}^n 1_{Y_i = k} \delta_{X_i}$$

for the (unobserved) empirical distribution of the population distribution $P_k = P_{k,\infty}$ obtained with the true layer labels $Y_i$’s.

**What can and cannot be estimated on immersed manifolds** As mentioned above, reach positiveness [31] is violated when immersions are not embeddings. As it has long been seen as a key regularity parameter and even considered to yield a minimal smoothness notion (see references in [3, 2]), one may wonder whether various geometric quantities of a support $M = \bigcup_{k=1}^K M_k$ can be estimated from sample without reach condition, at least in the large sample limit. We will say that a given distribution and data dependent quantity of interest $\theta_{X_n}(P)$ cannot be estimated uniformly over the model $\bar{P}$ for the loss $\ell(\cdot, \cdot)$ if

$$\liminf_{n \to \infty} \inf_{P \in \bar{P}} \sup_{\hat{\theta}_n} \mathbb{E}_{P^n} \left[ \ell\left( \theta_{X_n}(P), \hat{\theta}(X_n) \right) \right] > 0,$$

where the infimum ranges among all the estimators $\hat{\theta}(X_n)$ based on $n$ sample $X_n := \{X_1, \ldots, X_n\}$. Let us list briefly some important geometric quantities that become impossible to estimate on mixtures of immersions.

- The dimension from which a given point (say $X_1$) cannot be estimated. For instance, a small-dimensional component $M_1$ may be completely included in another higher-dimensional one $M_2$ (see orange points in Figure 5). In this case, points from $M_2$ drawn very close to $M_1$ are statistically indistinguishable from that drawn exactly on $M_1$, and the dimension at
$X_i$ becomes impossible to estimate. Yet, we shall show that the dimensions involved in the mixture can actually be estimated (Theorem 4.2), as well as the individual dimensions of the $X_i$’s far enough from intersections (Theorem 4.3), yielding an overall precise dimension-wise clustering of the points (Theorem 4.4).

- Tangent spaces at $X_i$’s cannot be estimated consistently, because of points near transversal self-intersections, see [3, Theorem 1 and Figure 2]. This fact is also clear given the fact that estimating a tangent space requires to identify the manifold dimension at a given point. Yet, as we shall show that accurate tangent directions of the manifold mixture can be estimated in small neighborhoods (Theorem 4.6).

- Homology also becomes inaccessible even asymptotically without reach condition, because topology is unstable without global constraints [12, Theorem 1]. While the homology of general immersed manifolds can be estimated when self-intersections are transversal [14, 52], this feature becomes not identifiable for general immersions (see Figure 2).

- Geodesic distances are inestimable for similar reasons as homology. Since connectedness (i.e. zeroth order Betti number) cannot be detected without reach condition, one cannot even test for $d_M(X_1, X_2) < \infty$ consistently. Under an additional connectedness assumption, one may also construct branching structures with very different behaviors at arbitrarily small Euclidean scales with bottleneck structures [3].

- The volume behaves the same, as the volume of a single sphere can hardly be distinguished from the union of two spheres with same center, with one of them having an arbitrarily close (but different) radius.

- If non-empty, the boundary $\partial M$ cannot be estimated without reach condition [1], even if its own curvature is forced to remain bounded. To see this, consider a fixed circle $M$ and remove an infinitesimal segment to it. Changing the location of this deletion yields significant displacement of $\partial M$, while its smallness makes it undetectable. Cartesian products with spheres yields curvature-bounded examples in higher dimensions.

We insist on the fact that all the above (counter-)examples yielding inestimability of some functionals can also be built under $C^{\infty}$ constraints instead of $C^2$ curvature bounds.

3 The Slabelling algorithm

3.1 Subspace detection with slabs

The overall strategy of Slabelling is to identify intrinsic neighborhoods in data points, that is to say aligned subsets of data likely to emanate from the same low-dimensional parametrization. Let us develop on how to make this idea precise.

Let $P_k \in \mathcal{P}_{r_{\max},r_{\min}}(a_{\min}, a_{\max})$ be the $d_k$-dimensional mixture component of $P = \sum_{k=1}^{K} \alpha_k P_k \in \mathcal{P}$. In order to learn the stratification of $P$ from sample $\{x_1, \ldots, x_n\} = \mathcal{X} \subset M = \bigcup_{k=1}^{K} M_k$, we will be interested in the empirical mass of specific neighborhoods $S \in \mathcal{S}$ of points in space. To simplify the exposition, a three-step reduction is in order.

- (From sample to distributions) Given a class $\mathcal{S}$ of measurable subsets of $\mathbb{R}^D$, a Vapnik-Chernovenkis argument (see Section B) yields that with high probability, for all $S \in \mathcal{S}$ and all $k \in \{1, \ldots, K\}$ simultaneously,

\[ -\sqrt{P_k(S) \text{VC}(S) \log n/n} \leq P_{k,n}(S) - P_k(S) \leq \sqrt{P_{k,n}(S) \text{VC}(S) \log n/n}. \]
Hence, if relying only on empirical counts over elements of a class $\mathcal{S}$ with bounded Vapnik-Chernovenkis dimension $VC(\mathcal{S})$, one can reason at the population level $P_k = P_{k,\infty}$ up to fluctuations.

- (From distributions to manifolds) By definition of the model, distributions $P_k(dx)$ behave roughly like the volume measure $\text{vol}_{M_k}(dx) = 1_{M_k}\mathcal{H}^{d_k}(dx)$, up to constants depending on $a_{\min}, a_{\max}$ and $\kappa_{\max}$ (Definition 2.5). Distributional considerations hence boil down to volumic ones.

- (From manifolds to linear spaces) Because each $M_k$ is an immersed manifold with curvature bounded by $\kappa_{\max}$, the neighborhood $B(x, h_\ell) \cap M_k$ of each point $x \in M_k$ looks like a finite union $\bigcup_{\ell \in N_0} B_{T_\ell}(y_\ell, h_\ell)$ of $d_k$-dimensional tangent balls (Section A.2). This approximation holds for $h_\ell \lesssim 1/\kappa_{\max}$ up to an error of order $\kappa_{\max}h_\ell^2$ in the direction normal to the tangent spaces nearby $x$ (Section A.1).

To best identify the local structure $M_k \cap B(x, h_\ell) \simeq \bigcup_{\ell \in N_0} B_{T_\ell}(y_\ell, h_\ell)$ up to quadratic terms, we make use of thickened low-dimensional balls, called slabs.

**Definition 3.1** (Slab). Given $x \in \mathbb{R}^d$ and a linear subspace $T \subset \mathbb{R}^d$, we write

$$S_T(x, h_\ell, h_\perp) := x + B_T(0, h_\ell) + B_{T^\perp}(0, h_\perp).$$

For inference, the normal parameter $h_\perp$ is to be chosen of order $\kappa_{\max}h_\ell^2$. Indeed if $h_\perp := a\kappa_{\max}h_\ell^2$, we show in Section A.4 that for all $d$-dimensional linear subspace $T$ (to be detected), and all (candidate) $d'$-dimensional linear subspace $T'$ with $d' \leq d$, then as $h_\ell \to 0$,

$$\mathcal{H}^d\left( B_T(y_\ell, h_\ell) \cap S_T(x, h_\ell, h_\perp) \right) \gtrsim h_\ell^{d'} \iff \begin{cases} d' = d, \\ \ell(T, T') \lesssim h_\ell, \\ \|y - x\| \lesssim h_\ell^2, \end{cases}$$

where $\ell(T, T') := \|\pi_T - \pi_{T'}\|_{op}$ stands for the largest principal angle between $T$ and $T'$. This volume characterization naturally extends to finite unions of subspaces (see Figure 4), in the sense that if $d' \lesssim \min_{\ell \in N_0} \text{dim}(T_\ell)$, then

$$\mathcal{H}^d\left( \bigcup_{\ell \in N_0} B_{T_\ell}(y_\ell, h_\ell) \cap S_{T'}(x, h_\ell, h_\perp) \right) \gtrsim h_\ell^{d'} \iff \exists \ell \in N_0, \begin{cases} d' = \text{dim}(T_\ell), \\ \ell(T_\ell, T') \lesssim h_\ell, \\ \|y_\ell - x\| \lesssim h_\ell^2. \end{cases}$$

(5)

To detect the location and orientation of all the structures of dimensions at most $d_{\max}$, this property suggests to evaluate the intersected volumes of slabs of dimension $d' \in \{1, \ldots, d_{\max}\}$ in increasing order, which leads to the Slab Labeling algorithm.

### 3.2 Nonlinear subspace detection from sample

At the sample level, the heuristic on slabs from the previous section naturally casts to sample-aligned slabs, which are defined as follows.

**Definition 3.2** (Slab of a $(d + 1)$-tuple). Let $d \in \{1, \ldots, D\}$. Given $y = \{y_0, \ldots, y_d\} \subset \mathbb{R}^D$, the slab defined by $y$, tangent width $h_\parallel > 0$ and normal width $h_\perp > 0$ is

$$S(y, h_\parallel, h_\perp) := S_{T(y)}(\bar{y}, h_\parallel, h_\perp),$$

where $\bar{y} = \frac{1}{d+1} \sum_{j=0}^d y_j$ is the barycenter of $y$ and $T(y) := \text{span}(y_1 - y_0, \ldots, y_d - y_0)$ its linear span.
Slab labeling} enumerates dimensions $d$ from 1 to a predefined one $d_{\text{max}}$ in increasing order. Starting from the full dataset $\mathcal{X}^{(1)} := \mathcal{X}$, it constructs decreasing subsets $\mathcal{X}^{(1)} \supset \ldots \supset \mathcal{X}^{(d_{\text{max}}+1)}$. When at current step $d$, points from $\mathcal{X}^{(d)}$ are added to the output $\mathcal{X}^{(d)}$ by $(d + 1)$-tuples $y := (y_0, \ldots, y_d)$. Tuple $y$ is added to $\mathcal{X}^{(d)}$ when

- it has radius $\text{rad}(y) \leq r_d$ (where $\text{rad}(y)$ is the radius of the smallest enclosing ball of $y$), and
- its associated slab $S(y, h_d, \kappa_d h_d^2)$ contains at least $n_d$ points from $\mathcal{X}^{(d)}$.

In practice, $h_d \approx (\log n/n)^{1/d}$ corresponds to the typical sampling scale of a $n$-sample drawn uniformly over a $d$-dimensional ball. Following the intuition from the population level (Section 3.1), this criterion indicates that $\{y_0, \ldots, y_d\}$ are likely to jointly lie on a single $d$-dimensional stratum of $M = \bigcup_{k=1}^{K} M_k$: we say that they have been co-detected. Slab labeling also stores this alignment information by adding $y$ to a list $\mathcal{U}^{(d)}$ of $(d + 1)$-tuples.

At the end of step $d$, Slab labeling also adds to $\mathcal{X}^{(d)}$ the set of points $\mathcal{R}_d \subset \mathcal{X}_d$ that are at distance at most $\delta_d$ (of order $\kappa_d h_d^2$ in practice) from the co-detected convex hulls of $\mathcal{U}^{(d)}$. This extra step is made to ensure that after dimension $d = d_k$ has been parsed, all the points of $M_k$ have been included $\mathcal{X}^{(d)}$. This avoids any contamination in $\mathcal{X}^{(d_k+1)} = \mathcal{X}^{(d_k)} \setminus \mathcal{X}^{(d_k)}$ from such possibly forgotten points when then parsing higher-dimensional layers.

Some of the lists $(\mathcal{U}^{(d)})_{1 \leq d \leq d_{\text{max}}}$ may be empty. The main output of the Slab labeling algorithm is the collection of non-empty tuples $(\mathcal{U}_k)_{1 \leq k \leq \hat{K}}$. From it, the approximate stratified structure of $M$ can be constructed. Namely, the estimated number of layers and intrinsic dimensions are taken to be

$$\hat{K} := \text{Card} \{d \mid \mathcal{U}^{(d)} \neq \emptyset\} \quad \text{and} \quad \{\hat{d}_1, \ldots, \hat{d}_\hat{K}\} := \{d \mid \mathcal{U}_d \neq \emptyset\},$$

and the dimension-wise data points clusters and support estimators are, for all $k \in \{1, \ldots, \hat{K}\}$

$$\hat{\mathcal{X}}_k := \mathcal{X}^{(\hat{d}_k)} \quad \text{and} \quad \hat{\mathcal{M}}_k := \bigcup_{y \in \mathcal{U}_k} \text{conv}(y).$$

**Remark 3.3** (On leftover data points). The points remaining in $\mathcal{X}^{d_{\text{max}}+1} = \mathcal{X} \setminus \bigcup_{1 \leq k \leq \hat{K}} \mathcal{X}_k$ after sieving all the dimensions 1 through $d_{\text{max}}$ are likely to have been generated from a distribution that has intrinsic dimension at least $d_{\text{max}}+1$. As such, they can be considered as originating from too many degrees of freedom, making them hardly analyzable statistically speaking if one has fixed a dimension threshold over which the curse of dimensionality is considered too strong. Hence, they can naturally be labeled as clutter noise, in the same spirit as Remark 2.6.
Algorithm 1: Slab labeling

Data: Point cloud $\mathcal{X} \subset \mathbb{R}^D$
- Maximal dimension $d_{\text{max}}$
- Slab radii $(h_d)_{1 \leq d \leq d_{\text{max}}}$
- Slab aspect ratios $(\kappa_d)_{1 \leq d \leq d_{\text{max}}}$
- Localization radii $(r_d)_{1 \leq d \leq d_{\text{max}}}$
- Pruning radii $(\delta_d)_{1 \leq d \leq d_{\text{max}}}$
- Size cutoffs $(n_d)_{1 \leq d \leq d_{\text{max}}}$

Result: Number of components $\hat{K}$
- Set of dimensions $(\hat{d}_1, \ldots, \hat{d}_{\hat{K}})$
- Labeled datapoints $(\hat{X}_1, \ldots, \hat{X}_{\hat{K}})$
- Collection of tuples $(\hat{U}_1, \ldots, \hat{U}_{\hat{K}})$

$d \leftarrow 0$
$\text{dim list} \leftarrow \emptyset$
$\text{points list} \leftarrow \emptyset$
$\text{tuples list} \leftarrow \emptyset$
$\hat{K} \leftarrow 0$
$\mathcal{X} \leftarrow \mathcal{X}$

while $\mathcal{X} \neq \emptyset$ and $d < d_{\text{max}}$ do
  $d \leftarrow d + 1$
  $\mathcal{X} \leftarrow \emptyset$
  $\mathcal{U} \leftarrow \emptyset$
  for all $(d + 1)$-tuple $y = (y_0, \ldots, y_d)$ of distinct points of $\mathcal{X}$ do
    if $\text{rad}(y) \leq r_d$ and $\text{Card}(\mathcal{X} \cap S(y, h_d, \kappa_d h_d^2)) \geq n_d$ then
      $\mathcal{U} \leftarrow \mathcal{U} \cup \{y\}$
      $\mathcal{X} \leftarrow \mathcal{X} \cup \{y_0, \ldots, y_d\}$
    end
  end
  $\mathcal{X} \leftarrow \mathcal{X} \cup \mathcal{R}$
  for all $z$ in $\mathcal{X} \setminus \hat{X}$ do
    if $\text{dist}(z, \bigcup_{\sigma \in \mathcal{U}} \text{conv} \sigma) \leq \delta_d$ then
      $\mathcal{X} \leftarrow \mathcal{X} \cup \{z\}$
    end
  end
  if $\mathcal{X} \neq \emptyset$ then
    Append $d$ to dim list;
    Append $\mathcal{X}$ to points list;
    Append $\mathcal{U}$ to tuples list;
    $\mathcal{X} \leftarrow \mathcal{X} \setminus \hat{X}$;
    $\hat{K} \leftarrow \hat{K} + 1$;
  end
end
Figure 5: A typical configuration described by Lemma C.2. At step $d = 1$, all the close-enough pairs of sample points are jointly considered and added to $U^{(1)}$ if the slab they define contains enough empirical mass. In the displayed layout, pairs $y$ and $y''$ yield co-detection (blue), while $y'$ does not (red). When co-detected, the convex hull of points $y = \{y_0, \ldots, y_d\}$ contribute to $M_d$ (7). Points from higher-dimensional layers (orange) or pairs from different immersions (green) can actually be co-detected. If so, they provably yield convex hulls that are $O(h_2^2)$-close to $M_1$ (Lemma C.2), so that they do not impair the estimation rates.

4 Optimal stratification learning

In what follows, we will denote by $C, C', C''$, $c, c', c''$ generic positive numerical constants. Their value can differ from one line of an equation to an other. Likewise, we will use $C_d, C'_d, C''_d, c_d, c'_d, c''_d$ for generic constants depending on dimension $d \in \{1, \ldots, D\}$ only.

4.1 Geometric loss functions

In the results to come, we are interested in the exact identification of the integers given by the number of clusters $K$, as well as dimensions $\{d_1, \ldots, d_K\}$. To measure precision on point location and set estimation, the distance function $\text{dist}(\cdot \mid A)$ to a subset $A \subset \mathbb{R}^D$ is defined for all $x \in \mathbb{R}^D$ as

$$\text{dist}(x \mid A) := \inf_{a \in A} \|x - a\|.$$ 

The one-sided Hausdorff distance from $A \subset \mathbb{R}^D$ to $B \subset \mathbb{R}^D$ is

$$d_H(A \mid B) := \sup_{a \in A} \text{dist}(a \mid B),$$

and the Hausdorff distance between $A$ and $B$ is the symmetrized quantity

$$d_H(A, B) := \max \{d_H(A \mid B), d_H(B \mid A)\}.$$ 

When dealing with linear subspaces $T, T' \subset \mathbb{R}^D$, we will be interested in the principal angle

$$\varepsilon(T, T') := \|\pi_T - \pi_{T'}\|_{op},$$

where $\pi_T, \pi_{T'} : \mathbb{R}^D \to \mathbb{R}^D$ stand for the orthogonal projections onto $T$ and $T'$ respectively, and $\|\cdot\|_{op}$ is the (Euclidean) operator norm on endomorphisms of $\mathbb{R}^D$. When comparing two union
of flats index by \(T\) and \(T'\), we let
\[
\varepsilon \left( \bigcup_{T \in \mathcal{T}} T \mid \bigcup_{T' \in \mathcal{T}'} T' \right) := \sup \inf_{T \in \mathcal{T}, T' \in \mathcal{T}'} \varepsilon (T, T').
\]
which coincides with the previous definition when \(T\) and \(T'\) are restricted to single flats.

### 4.2 Parameter settings and algorithmic guarantees

We are now in position to state the main properties of the \textit{Slabeling} algorithm. In all the results below, we shall pick the following collection of parameters to ensure good properties:

\[
\left\{
\begin{array}{l}
d_{\text{max}} = D - 1 \\
h_d = \frac{48d}{\kappa_{\text{max}}} \left( 1 + \frac{1}{d} \right) \left( \Upsilon \gamma \log n \right)^{1/d} \\
\kappa_d = \kappa_{\text{max}} \\
r_d = h_d \\
n_d = \sigma \gamma \log n
\end{array}
\right.
\]  

(8)

where

\(\Upsilon\) is to be taken large enough compared to a constant depending on \(D\) and \(\nu_{\text{max}}\),

\(\sigma\) is to be taken of order \(D\) up to multiplicative constants,

\(\gamma\) is to be taken at least of order \((q \vee D^2 \log D)/\alpha_{\text{min}}\), where

\(q > 0\) is a parameter that will tune the decay of probability of success of the algorithm.

We refer to Lemma C.2 for a more precise statement regarding these constants. This lemma gathers the key geometric properties kept invariant throughout the execution of \textit{Slabeling} on input \(X_n = \{X_1, \ldots, X_n\}\) generated independently and identically distributed from some unknown \(P \in \mathcal{P}\) (Definition 2.5). With probability at least \(1 - O(n^{-a})\), at all steps \(d \in \{1, \ldots, D - 1\}\), the sample counterpart of (5) holds. That is,

- \textit{(Discrete stratification learning)} \(\mathcal{U}^{(d)} \neq \emptyset\) if and only if \(d \in \{d_1, \ldots, d_K\}\).
- \textit{(Approximation)} If \(d = d_k\) for some \(k \in \{1, \ldots, K\}\), then
  - For all \(y = (y_0, \ldots, y_{d_k}) \in \mathcal{U}_{d_k}\) co-detected by \textit{Slabeling} and all \(y \in \text{Conv}(y)\), there exists \(x \in \mathcal{B}(y, O(h_{d_k}^2)) \cap M_k\) such that
    \[
    \|y - x\| \leq h_{d_k} \quad \text{and} \quad \varepsilon \left( \text{span}(y_1 - y_0, \ldots, y_{d_k} - y_0), T_x M_k \right) \leq h_{d_k}.
    \]
  - For all \(x \in M_k\) and all \(d_k\)-dimensional subspace \(T \subset T_x M_k\), there exists \(y = (y_0, \ldots, y_{d_k}) \in \mathcal{U}_{d_k}\) co-detected by \textit{Slabeling}, and \(y \in \mathcal{B}(x, O(h_{d_k}^2)) \cap \text{Conv}(y)\) such that
    \[
    \|x - y\| \leq h_{d_k} \quad \text{and} \quad \varepsilon \left( T, \text{span}(y_1 - y_0, \ldots, y_{d_k} - y_0) \right) \leq h_{d_k}.
    \]

These properties are sufficient to establish the main results of this article.

**Remark 4.1** (On average time complexity). The above choice of parameters happens to turn \textit{Slabeling} into a quasi-linear time algorithm on average, provided that one already has computed local neighborhood information on \(X_n\) [37]. Indeed, Lemma C.2 asserts with high probability, that no point from any \(d'\)-dimensional layer is ever forgotten along the way for all \(d' < d\). Therefore, at step \(d\), each remaining point \(X_i \in X_n\) has at most \(O(n r_d^d) = O((\log n)^d)\) candidate neighbors for its co-detection. As a result, \textit{Slabeling} enumerates at most \(O((\log n)^d)\) slabs containing \(X_i\). This results in an overall time complexity \(O(n (\log n)^{d'})\) on average.
4.3 Main results

Using notation from Section 2.3, we let \( X_n := \{X_1, \ldots, X_n\} \) be drawn from some unknown \( P = \sum_{k=1}^{K} \alpha_k P_k \in \mathcal{P} \) and associated labels \((Y_1, \ldots, Y_n)\), and \( X_{k,n} := \{X_i \mid Y_i = k\} \). The proof of the results below can be found in Section D.

**Stratification identification** Under this sampling scheme and parameter choice as above, the estimated discrete structure

\[
\hat{d}_k := \{d \mid \mathcal{U}^{(d)} \neq \emptyset\} \quad \text{and} \quad \hat{K} := \text{Card} (\{d_k\}),
\]

identifies the underlying one precisely with high probability. In all the subsequent results, \( n \) is taken large enough compared to a constant depending on the parameters of \( \mathcal{P} \) only.

**Theorem 4.2** (Mixture structure identification). With parameters chosen as in (8), for \( n \) large enough, with probability at least \( 1 - 6Kn^{-q} \),

\[
\hat{K} = K \quad \text{and} \quad \hat{d}_k = d_k \quad \text{for all} \quad k \in \{1, \ldots, K\}.
\]

The recovery of the number of clusters \( K \) is, to our knowledge, the first mathematically grounded result of this type (see [54] for empirical evidences on various methods). The dimension(s) estimation part extends the results of [41], which hold only under reach condition and with a single mixture component (\( K = 1 \)).

Beyond dimension(s) identification, **Slabelling** also assigns a dimension to each data point. Recall that for all \( k \in \{1, \ldots, \hat{K}\} \),

\[
\hat{X}_{k,n} := \mathcal{X}^{(d_k)} \quad \text{and} \quad \hat{M}_k := \bigcup_{y \in \hat{d}_k} \text{conv} \{y\}.
\]

For all \( X_i \in X_n \), we let \( \hat{d}(X_i) \) denote the dimension \( d_k \) to which it belongs in the disjoint union \( X_n = \sqcup_{k=1}^K \hat{X}_{k,n} \), i.e. \( \hat{d}(X_i) = \hat{d}_k \) if and only if \( X_i \in \hat{X}_{k,n} \). As mentioned in Section 2.3, the exact individual dimension \( d_{Y_i} = \dim(M_k) \) from which \( X_i \) has been drawn cannot be identified. It may not even be defined uniquely, for instance if \( M_k \subset M_{k+1} \). However, it can be underestimated locally in the following sense.

**Theorem 4.3** (Local dimension estimation). With parameters chosen as in (8), for \( n \) large enough, with probability at least \( 1 - 6Kn^{-q} \),

\[
\text{for all} \quad i \in \{1, \ldots, n\}, \quad \min \left\{ d_k \mid \text{dist}(X_i \mid M_k) \leq \tau_k \left(\frac{\log n}{n}\right) \frac{d_k}{2} \right\} \leq \hat{d}(X_i) \leq d_{Y_i},
\]

where

\[
\tau_k := C_{d_k} \nu_{\max} \frac{a_{\max}}{a_{\min} \kappa_{\max}} \left( \frac{\chi_{\gamma}}{a_{\min}} \right)^{2/d_k}.
\]

In accordance to intuition, the above pointwise statement leaves estimation ambiguity only nearby intersections between layers of different dimensions. In particular, it turns into exact dimension labeling for embedded submanifolds with reach bounded below by \( 1/\kappa_{\max} \). Also note that the localization radius \( O(h_{\hat{d}_k}^2) \) is dimension-dependent: the lower the intrinsic dimension, the most sharply points can be identified as coming from it.

Coming back to global sample identification properties, the following statement asserts that most of the data points are assigned to the correct layer. Recall that on average we have that \( N_k := \#X_{k,n} \) satisfies \( \mathbb{E}_P[N_k] = \alpha_k n \times n \), because \( \alpha_{\min} > 0 \) is considered fixed.
Theorem 4.4 (Dimension-wise clustering). With parameters chosen as in (8), for \( n \) large enough, for all \( k \in \{1, \ldots, K\} \) simultaneously,

\[
\mathbb{E}_P \left[ \frac{\# \left( \widehat{\mathcal{X}}_{k,n} \setminus \mathcal{X}_{k,n} \right)}{N_k \vee 1} \right] \leq \eta_k \left( \frac{\log n}{n} \right)^{a_{\max} - a_{\min} \cdot 2^2} \left( \log \left( \frac{n}{n} \right) \right)^{2^2/\gamma^2} + 6Kn^{-q},
\]

where \( \mathcal{X} \triangle \mathcal{X}' := (\mathcal{X} \setminus \mathcal{X}') \cup (\mathcal{X}' \setminus \mathcal{X}) \) stands for the symmetric difference between \( \mathcal{X}, \mathcal{X}' \subset \mathbb{R}^D \). Here, we used convention \( d_0 = 0 \) and \( d_{K+1} = \infty \).

The above result provides a rare example of a blessing of dimensionality phenomenon. Indeed, varying \((d_{k-1}, d_{k+1})\) while leaving \(d_k\) fixed, exponent \( \beta_k := \frac{d_k - d_{k-1}}{d_{k-1}} \cdot \frac{d_{k-1} - d_k}{d_k} \) ranges

- from \( \beta_k = 2/d_k \) in worst case when \( d_{k+1} = d_k + 1 \), yielding the simpler clustering rates

\[
\mathbb{E}_P \left[ \frac{\# \left( \widehat{\mathcal{X}}_{k,n} \setminus \mathcal{X}_{k,n} \right)}{N_k \vee 1} \right] \leq \eta_k \left( \frac{\log n}{n} \right)^{2/d_k};
\]

- to \( \beta_k = 2(d_k - 1) \) when \((d_{k-1}, d_{k+1}) = (1, \infty)\) (lower layers best case) and \( \beta_k = 2D-1/d_k \) when \((d_{k-1}, d_{k+1}) = (0, D-1)\) (higher layers best case).

This accords with the intuition that layers of more heterogeneous dimensions shall be more easily separated, and interfere less with each other.

Stratification estimation. Now moving to approximation properties at order zero (Hausdorff distance), we show that the support is estimated at dimension-dependent rate by the local convex hulls \( \overline{\mathcal{M}}_k := \bigcup_{y \in d_k} \text{conv}(y) \) of co-detected points.

Theorem 4.5 (Stratified manifold estimation). With parameters chosen as in (8), for \( n \) large enough, with probability at least \( 1 - 6Kn^{-q} \), for all \( k \in \{1, \ldots, K\} \) simultaneously,

\[
d_H(\mathcal{M}_k, \overline{\mathcal{M}}_k) \leq \rho_k \left( \frac{\log n}{n} \right)^{2/d_k},
\]

where \( \rho_k = C_{d_k} \gamma_{\max} \frac{a_{\max}}{a_{\min} \gamma_{\max}} \left( \frac{\gamma_{\max}}{a_{\min}} \right)^{2/d_k} \).

This rate is minimax optimal [41], even over the smaller model composed of embedded sub-manifolds with reach bounded below and no proper mixture \((K = 1)\). Strikingly, the absence of reach condition and the presence of interacting layers of different dimensions do not impact the Hausdorff estimation rate of each layer. As mentioned in Remark 2.6, this result generalizes [34, Theorem 5] and [4, Theorem 2.9], which considered the estimation of a single manifold \( M_d \) given an i.i.d. sampling of it corrupted with outliers generated by Huber’s contamination model. In addition to having weaker assumptions, note that Theorem 4.5 also benefits from an explicit algorithmic description (absent of [34]). It also processes the estimation of the \( d \)-dimensional layer with a single pass on data, as opposed to the intricate iterative algorithm of [4].

At order one, the piecewise linear structure of \( \overline{\mathcal{M}}_k = \bigcup_{y \in d_k} \text{conv}(y) \) also provides consistent estimates of local tangent spaces. For \( x \in \overline{\mathcal{M}}_k \), we let

\[
T_x \overline{\mathcal{M}}_k := \bigcup_{\sigma \in d_k} \text{span} \sigma.
\]
As noted in Section 2.3, the angle criterion for tangent space estimation needs to be weakened for similar reasons to dimension estimation. We consider a natural localized version of it by introducing a spatial tolerance $\Delta > 0$. Then, we write

$$\angle_\Delta(T M_k \mid T \bar{M}_k) := \sup_{x \in M_k} \inf_{y \in \bar{M}_k \cap B(x, \Delta)} \angle(T_x M_k \mid T_y \bar{M}_k),$$

and

$$\angle_\Delta(T \bar{M}_k \mid T M_k) := \sup_{y \in \bar{M}_k} \inf_{x \in M_k \cap B(y, \Delta)} \angle(T_y \bar{M}_k \mid T_x M_k).$$

We can then define the symmetric loss

$$\angle_\Delta(T \bar{M}_k, T M_k) := \angle_\Delta(T \bar{M}_k \mid T M_k) \lor \angle_\Delta(T M_k \mid T \bar{M}_k),$$

which we bound in the following result.

**Theorem 4.6 (Tangent space estimation).** With parameters chosen as in (8), for $n$ large enough and with probability at least $1 - 6Kn^{-q}$, for all $k \in \{1, \ldots, K\}$ simultaneously,

$$\angle_\Delta(T \bar{M}_k, T M_k) \leq \iota_k \left(\frac{\log n}{n}\right)^{-1/d_k},$$

where $\iota_k := C_{d_k} \Upsilon_{\max} \frac{a_{\max}}{a_{\min}} \left(\frac{\Upsilon \gamma}{a_{\min}}\right)^{1/d_k}$,

and $\Delta_k := c_{d_k} \Upsilon_{\max} \frac{a_{\max}}{a_{\min}} \kappa_{\max} \left(\frac{\Upsilon \gamma}{a_{\min}}\right)^{2/d_k} \left(\frac{\log n}{n}\right)^{2/d_k}.$

In words, this result asserts that the $d_k$-dimensional tangent structure of $M = \bigcup_{k=1}^{K} M_k$ can be estimated up to angle $O(h_{d_k})$ in neighborhoods of size $O(h_{d_k}^2)$. Again, this rate is minimax optimal [5] for embedded submanifolds with reach bounded below and no proper mixture ($K = 1$). As for dimension estimation, this result turns into an actual (zero-range) tangent space, when applied to for embedded submanifolds with reach bounded below by $1/\kappa_{\max}$.

## 5 Conclusion and open questions

The present work shows that with high probability, a $n$-sample of a mixture of immersed $C^2$-manifolds with bounded curvature allows to infer their $d$-dimensional structure at scale $O\left((\log n/n)^{2/d}\right)$ in locations and $O\left((\log n/n)^{1/d}\right)$ in directions. We believe that the present formalism of general mixtures of immersed manifolds lies at a sweet spot in terms of statistical modeling. It combines both the desirable properties induced by $C^2$ smoothness allowing the use of standard differential geometry, and the versatility of mixtures and self-intersections induced by general immersions. Natural extensions of the model of Definition 2.5 shall include higher order $C^k$ smoothness for faster rates, and the ability to handle immersions of manifolds with boundary and corners. In the latter framework, stratification shall actually emerge via re-sampling and projecting in the ambient space onto the boundary/corners, along the same lines as [46]. Models with additive noise also constitute relevant generalizations, although the simpler case of a single embedded manifold with positive reach has not yet been completely understood [35, 34, 22].
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A Geometry and volume of immersed manifolds

Below, immersions $\Psi : M \rightarrow \mathbb{R}^D$ are always taken to be isometric in the Riemannian sense (see Section 2.1). Since $M$ is compact and without boundary, the inherited Riemannian metric $(M,g)$ yields a globally defined exponential map at each point $x_0 \in M$, which we denote by $\exp_{x_0}^M : T_{x_0}M \rightarrow M$.

A.1 Geodesic estimates

Lemma A.1 ([52, Lem 2.3 (iii)]). Let $\Psi : M \rightarrow \mathbb{R}^D$ be a $C^2$-immersion of a compact $d$-dimensional manifold $\mathbb{M}$ such that $\sup_{x_0 \in \mathbb{M}} \| \Pi_{x_0}^{(\Psi)} \|_\text{op} \leq \kappa_{\text{max}}$. Then for all $x_0, y_0 \in \mathbb{M}$,

$$\left(1 - \frac{\kappa_{\text{max}}}{2}\right) d_M(x_0, y_0) \leq \| \Psi(x_0) - \Psi(y_0) \|.$$

As a result, for all $r < 2/\kappa_{\text{max}}$, $\Psi$ is an embedding of the closed geodesic ball $B_{\mathbb{M}}(x_0, r)$ onto its image. Hence, if we lift the exponential map $\exp_{x_0}^M : T_{x_0}M \rightarrow M$ onto $T_{x_0}^\mathbb{M}M := d\Psi(x_0)[T_{x_0}M]$ via

$$\exp_{x_0}^M: T_{x_0}^\mathbb{M}M \rightarrow M,$$

we get that $\exp_{x_0}^M$ is an embedding from the tangent ball $B_{T_{x_0}^\mathbb{M}M}(0, r)$ to $\Psi(B_{\mathbb{M}}(x_0, r))$. This embedding fulfills the following estimates.

Lemma A.2 (Geometric approximation bounds). Let $\Psi : M \rightarrow \mathbb{R}^D$ be a $C^2$-immersion of a compact $d$-dimensional manifold $\mathbb{M}$ such that $\sup_{x_0 \in \mathbb{M}} \| \Pi_{x_0}^{(\Psi)} \|_\text{op} \leq \kappa_{\text{max}}$. Fix $x_0 \in \mathbb{M}$ and write $x = \Psi(x_0)$.

1. For all $y \in \Psi(B_{\mathbb{M}}(x_0, 1/(4\kappa_{\text{max}})))$,
   $$\| x - y \| \leq \left\| \left(\exp_{x_0}^M\right)^{-1}(y) \right\| \leq 2 \| x - y \|,$$

2. For all $v \in B_{T_{x_0}^\mathbb{M}M}(0, 1/(4\kappa_{\text{max}}))$,
   $$\| \exp_{x_0}^M(v) - x - v \| \leq \kappa_{\text{max}} \| v \|^2,$$

3. For all $y \in \Psi(B_{\mathbb{M}}(x_0, 1/(4\kappa_{\text{max}})))$,
   $$\| y - x - \left(\exp_{x_0}^M\right)^{-1}(y) \| \leq 2\kappa_{\text{max}} \| x - y \|^2,$$

4. For all measurable set $A \subset B_{T_{x_0}^\mathbb{M}M}(0, 1/(4\kappa_{\text{max}}))$,
   $$2^{-d}\mathcal{H}^d(A) \leq \mathcal{H}^d(\exp_{x_0}^M(A)) \leq 2^d\mathcal{H}^d(A).$$

5. For all $y \in \Psi(B_{\mathbb{M}}(x_0, 1/(4\kappa_{\text{max}})))$,
   $$\| y - x - \pi_{T_{x_0}^\mathbb{M}M}(y - x) \| \leq \frac{\kappa_{\text{max}}}{2} \| y - x \|^2.$$

6. For all $y_0 \in B_{\mathbb{M}}(x_0, 1/(8\kappa_{\text{max}}))$,
   $$\angle(T_{x_0}^\mathbb{M}M, T_{y_0}^\mathbb{M}M) \leq 5\kappa_{\text{max}} \| x - y \|.$$
Proof. Points 1 to 5 are trivial applications of [5, Lem 1]. For point 6, we let \( y = \Psi(y_0) \) and \( v \in B_{T_{y_0}}(4, 1/4\kappa_{\text{max}}) \) and let \( z = \exp_{y_0}^M(v) \), which lies in \( B_M(x_0, 1/(4\kappa_{\text{max}})) \) by triangle inequality. Using point 5 twice, we first get that
\[
\| z - y - \pi_{T_{y_0}}(z - y) \| \leq \frac{\kappa_{\text{max}}}{2} \left\{ \| z - x \|^2 + \| y - x \|^2 \right\}.
\]
Furthermore, thanks to point 2, there holds
\[
\| z - y - v \| \leq \kappa_{\text{max}} \| v \|^2.
\]
Since \( \| z - y \| \leq 2 \| v \| \) from point 1, we obtain
\[
\| v - \pi_{T_{y_0}}(v) \| \leq \frac{\kappa_{\text{max}}}{2} \left\{ \| y - x \|^2 + \| z - x \|^2 \right\} + 2\kappa_{\text{max}} \| v \|^2 \\
\leq \frac{\kappa_{\text{max}}}{2} \left\{ 2\| y - x \|^2 + \| v \|^2 + 2\| x - y \| \| v \| \right\} + 2\kappa_{\text{max}} \| v \|^2.
\]
By taking \( \| v \| = \| y - x \| \leq 1/(8\kappa_{\text{max}}) \), we find
\[
\| v - \pi_{T_{y_0}}(v) \| \leq 5\kappa_{\text{max}} \| y - x \|^2 = 5\kappa_{\text{max}} \| y - x \| \| v \|.
\]
By symmetry between \( x_0 \) and \( y_0 \), we obtain \( \zeta(T_{x_0}^\Psi M, T_{y_0}^\Psi M) \leq 5\kappa_{\text{max}} \| y - x \| \), ending the proof. \( \square \)

## A.2 Fiber estimates

Under our assumptions, intuition suggests that small enough Euclidean neighborhoods of the image \( M = \Psi(M) \) of a \( C^2 \) immersion of a compact manifold \( M \) write as a union of \( L < \infty \) \( d \)-dimensional balls up to diffeomorphisms. The following result provides a quantitative bound on this number \( L \), depending on intrinsic dimension, curvature and volume of \((M, \Psi)\). We write \( \omega_d := \mathcal{H}^d(B_d(0, 1)) \) for the volume of the \( d \)-dimensional unit Euclidean ball.

**Lemma A.3** (Upper bound on local fibers). Let \( M = \Psi(M) \) be the image of a \( C^2 \)-immersion \( \Psi : M \to \mathbb{R}^D \) of a compact \( d \)-dimensional manifold \( M \) such that \( \sup_{x_0 \in M} \| \Pi_0(x_0) \|_{\text{op}} \leq \kappa_{\text{max}} \). Then for all \( x \in \mathbb{R}^D \) and \( r \leq 1/(4\kappa_{\text{max}}) \), there exist \( U_1, \ldots, U_L \subset M \) such that
\[
\Psi^{-1}(M \cap B(x, r)) = \bigcup_{\ell=1}^L U_\ell,
\]
with:

- All the \( U_\ell \)'s are path-connected and have geodesic diameter at most \( 4r \).
- \( L \leq N_0 \), where \( N_0 := \frac{\text{vol}(M)}{\omega_d} (4\kappa_{\text{max}})^d \).

**Proof of Lemma A.3.** We let
\[
\Psi^{-1}(M \cap B(x, r)) = \bigcup_{U \in \mathcal{U}} U,
\]
be a path-connected decomposition of \( \Psi^{-1}(M \cap B(x, r)) \) that is maximal for the inclusion. That is, for all other such decomposition
\[
\Psi^{-1}(M \cap B(x, r)) = \bigcup_{V \in \mathcal{V}} V,
\]
if for all $U \in \mathcal{U}$ there exists a $V \in \mathcal{V}$ such that $U \subset V$, then $\mathcal{U} = \mathcal{V}$. Such a decomposition exists from Zorn’s lemma.

Note that the elements of such a decomposition are pairwise disjoint by minimality and connectedness. We let $U_i, U_2$ be two distinct elements of $\mathcal{U}$. We pick a point $x_i$ in each of the $U_i$. We let $\gamma_0$ be a unit-speed shortest path of $\mathbb{M}$ going from $x_1$ to $x_2$ and we let $\gamma = \Psi \circ \gamma_0$. The path $\gamma$ is also unit-speed, and because $\gamma_0$ is locally a geodesic, we have that $\|\dot{\gamma}\| \leq \kappa_{\max}$ everywhere. We know that $\gamma$ has to leave $B(x, r)$, otherwise we could replace $U_1$ and $U_2$ by the path-connected set $U_1 \cup \Psi^{-1}(1\mathbb{M}) \cup U_2$, which would contradict the maximality of $\mathcal{U}$. We let $\sigma(t) := (\gamma(t) - x, \dot{\gamma}(t))$. At the first time $t_0$ where $\gamma$ leaves $B(x, r)$, we have $\sigma(t) \geq 0$, and at the first time $t_1 \geq t_0$ when $\gamma$ returns to $B(x, r)$, we must have $\sigma(t_1) \leq 0$. However, notice that for $t \geq t_0$,

$$\dot{\sigma}(t) = \|\dot{\gamma}(t)\|^2 + \langle \gamma(t) - x, \dot{\gamma}(t) \rangle$$

$\geq 1 - \kappa_{\max}\|\gamma(t) - x\|$, so that for $\sigma$ to turn negative, one must have $\|\gamma(t) - x\| \geq 1/\kappa_{\max}$ for some $t \in [t_0, t_1]$. This implies that

$$d_{\mathcal{M}}(x_1, x_2) = d_{\mathcal{M}}(x_1, \Psi^{-1}(\gamma(t))) + d_{\mathcal{M}}(x_2, \Psi^{-1}(\gamma(t)))$$

$\geq \|\Psi(x_1) - \gamma(t)\| + \|\Psi(x_2) - \gamma(t)\|$

$\geq (\|x - \gamma(t)\| - \|\Psi(x_1) - x\|) + (\|x - \gamma(t)\| - \|\Psi(x_2) - x\|) + 2\left(\frac{1}{\kappa_{\max}} - r\right)$

$\geq \frac{1}{\kappa_{\max}}$. 

As a consequence, the open geodesic balls $B_{\mathcal{M}}(x_i, 1/(2\kappa_{\max}))$ are mutually disjoint. Using Lemma A.2, we find that $\text{vol}(B_{\mathcal{M}}(x_i, 1/(2\kappa_{\max}))) \geq 2^{-d} \omega_d(2\kappa_{\max})^{-d}$. By compactness of $\mathbb{M}$, this implies that $\mathcal{U}$ is of finite cardinality $L$ and that

$$L \omega_d(4\kappa_{\max})^{-d} \leq \text{vol}(\mathbb{M}),$$

which yields the bound on $L$.

To show that the $U_i$’s in $\mathcal{U}$ have geodesic diameter less than $4r$, let us take $U \in \mathcal{U}$ and $y, z \in U$ be fixed. Because $U$ is path-connected, there exists a continuous path $\gamma : [0, 1] \to U$ (not necessarily geodesic) such that $\gamma(0) = y$ and $\gamma(1) = z$. We let

$$\tau := \sup \mathcal{T} \quad \text{with} \quad \mathcal{T} := \{ t \in [0, 1] \mid d_{\mathcal{M}}(y, \gamma(t)) < 1/\kappa_{\max}\}.$$  

We clearly have $0 \in \mathcal{T}$, and $\mathcal{T}$ is open in $[0, 1]$ since $\gamma$ is continuous. Furthermore, by continuity, $d_{\mathcal{M}}(y, \gamma(\tau)) \leq 1/\kappa_{\max}$. Using Lemma A.1, we find that

$$\frac{1}{2} d_{\mathcal{M}}(y, \gamma(\tau)) \leq \left(1 - \frac{\kappa_{\max}}{2} d_{\mathcal{M}}(y, \gamma(\tau))\right) d_{\mathcal{M}}(y, \gamma(\tau)) \leq \|\Psi(y) - \Psi(z)\| \leq 2r,$$

so that $d_{\mathcal{M}}(y, \gamma(\tau)) \leq 4r < 1/\kappa_{\max}$, meaning that $\tau \in \mathcal{T}$ which can only be the case if $\tau = 1$. As a result we have $d_{\mathcal{M}}(y, z) \leq 4r$, hence the result. \qed

A.3 Geometric well-posedness of tangents and curvature

Proof of Lemma 2.1. Let $(\mathbb{M}, \Psi) \in \mathcal{I}(M)$, $x \in M$ and $r < 1/(16\kappa_{\max})$. From Lemma A.3, we can write $\Psi^{-1}(M \cap B(x, r))$ as a finite disjoint union $\bigcup_{\ell=1}^L U_{\ell}$ of path-connected $U_1, \ldots, U_L \subset \mathbb{M}$,
each of which is a closed neighborhood of \( x_{0,\ell} \in \Psi^{-1}(x) \) with geodesic diameter less than 4\( r \). In particular, \( U_\ell \subset \text{exp}_{x_{0,\ell}}(B_{T_{x_{0,\ell}}}(0, 4r)) \).

On the other hand if \( x_\ell \in \Psi^{-1}(x) \) is fixed, then Lemma A.1 asserts that the map \( \Psi \circ \exp_{x_0} : \Phi_{x_0}^M : B_{T_{x_0}}(0, 4r) \to M \) is a \( C^2 \)-diffeomorphism onto its image. Furthermore, the first order Taylor expansion of \( \Phi_{x_0}^M \) near \( 0 \in T_{x_0}M \) writes as
\[
\Phi_{x_0}^M(v) = x + d\Phi_{x_0}^M(0)[v] + o(\|v\|)
\]
since \( \Phi_{x_0}^M(0) = \Psi(x_0) = x \). Write \( V_{\ell} := \Psi(U_\ell) \). Then for all \( y \in V_\ell \),
\[
y - x = d\Phi_{x_0,\ell}^M(0)[(\Phi_{x_0,\ell}^M)^{-1}(y)] + o(\| (\Phi_{x_0,\ell}^M)^{-1}(y) \|)
\]
Furthermore, points 1. and 3. of Lemma A.2 assert that \( \| y - x \| \leq \| (\Phi_{x_0,\ell}^M)^{-1}(y) \| \leq (1 + 2\kappa_{\text{max}} \| y - x \| )\| y - x \| \), so that
\[
y - x = d\Phi_{x_0,\ell}^M(0)[(\Phi_{x_0,\ell}^M)^{-1}(y)] + o(\| y - x \|).
\]
To show that \( T_x^{(\Psi)}M \) does not depend on \( (M, \Psi) \), note that by construction, \( d\Psi(x_0)[T_{x_0}M] = d\Phi_{x_0}^M(0)[T_{x_0}M] \). At first order, the above expansion hence yields
\[
d\Psi(x_0)[T_{x_0}M] = \left\{ v \in \mathbb{R}^D \mid \forall \varepsilon > 0, \exists y_\varepsilon \in (V_\ell \cap B(x, \varepsilon)) \setminus \{ x \} \text{ s.t. } \frac{y_\varepsilon - x}{\| y_\varepsilon - x \|} - \frac{v}{\|v\|} < \varepsilon \right\}.
\]
As a consequence, since \( M \cap B(x, \varepsilon) = \bigcup_{\ell=1}^L (V_\ell \cap B(x, \varepsilon)) \) for all \( \varepsilon \leq r \),
\[
T_x^{(\Psi)}M := \bigcup_{\ell=1}^L d\Psi(x_0,\ell)[T_{x_0,\ell}M]
\]
\[
= \left\{ v \in \mathbb{R}^D \mid \forall \varepsilon > 0, \exists y_\varepsilon \in (M \cap B(x, \varepsilon)) \setminus \{ x \} \text{ s.t. } \frac{y_\varepsilon - x}{\| y_\varepsilon - x \|} - \frac{v}{\|v\|} < \varepsilon \right\}
\]
does not depend on the chosen parametrisation \( (M, \Psi) \). We note \( T_xM := T_x^{(\Psi)}M \) from now on.

To show that \( \kappa^{(\Psi)}(M) \) does not depend on \( (M, \Psi) \), proceed using the Taylor expansion of \([3, \text{Lemma B.5.}]\), and get
\[
\| \Pi_{x_0,\ell}^{(\Psi)} \|_{\text{op}} = \limsup_{y_0,\ell \in M} \frac{2 \text{ dist} \left( \Phi_{x_0,\ell}^M(y_0,\ell) - x \mid d\Psi_{x_0,\ell} \Psi[T_{x_0,\ell}M] \right) \| \Phi_{x_0,\ell}^M(y_0,\ell) - x \|^2}{2 \text{ dist} \left( y_\ell - x \mid T_\ell \right) \| y_\ell - x \|^2},
\]
where \( T_\ell := d\Psi(x_0,\ell)[T_{x_0,\ell}M] \). Hence, writing \( \kappa^{(\Psi)}(x) := \sup_{x_\ell \in M} \| \Pi_{x_0}^{(\Psi)} \|_{\text{op}} \), we get
\[
\kappa^{(\Psi)}(x) = \max_{1 \leq \ell \leq L} \limsup_{y_\ell \in V_\ell} \frac{2 \text{ dist} \left( y_\ell - x \mid T_\ell \right) \| y_\ell - x \|^2}{\text{ dist} \left( y_\ell - x \mid T_\ell \right) \| y_\ell - x \|^2},
\]
This latter quantity might still depend on the immersion \( \Psi \) since the decomposition \( V_1, \ldots, V_L \) depend on it. To show that \( \kappa^{(\Psi)}(M) = \sup_{x_\ell \in M} \kappa^{(\Psi)}(x) \) does not depend on \( \Psi \), we will show that \( \kappa^{(\Psi)}(x) \) does not depend on \( \Psi \). For this, note that for all \( \ell \) and \( y_\ell \in V_\ell \), we have trivially
\[
\text{dist}(y_\ell - x \mid T_\ell) \geq \text{dist}(y_\ell - x \mid \bigcup_{1 \leq \ell' \leq L} T_{\ell'}),
\]
23
so that we easily get that

\[ \kappa(\Phi)(x) \geq \limsup_{y \in M} \frac{2 \text{dist}(y - x \mid T_x M)}{\| y - x \|^2}. \]

We will show the converse inequality to obtain the result. We let \( \ell \in \{1, \ldots, L\} \) be such that

\[ \kappa(\Phi)(x) = \limsup_{y \in V_\ell} \frac{2 \text{dist}(y_\ell - x \mid T_{\ell})}{\| y_\ell - x \|^2}. \]  \hspace{1cm} (9)

By extraction of sub-sequences on the unit sphere of \( T_\ell \), consider a unit direction \( v \in T_\ell \) achieving the lim sup. That is, there exists \( \varepsilon_n \searrow 0 \) such that if \( y_n = \exp_{x_n}^{M}(\varepsilon_n v) \), then

\[ \text{dist} (y_n - x \mid T_\ell) = \frac{1}{2} \| y_n - x \|^2 \kappa(\Phi)(x) + o(\|y_n - x\|^2) \]

\[ = \frac{1}{2} \varepsilon_n^2 \kappa(\Phi)(x) + o(\varepsilon_n^2), \]

where we used that \( y_n - x = \varepsilon_n v + O(\varepsilon_n^2) \) from Lemma A.2 2. We can then find a direction \( w \in T_\ell \) such that for all \( \eta > 0 \) sufficiently small, \( v + \eta w \notin T_{\ell'} \) for all \( T_{\ell'} \neq T_\ell \): that’s because

\[ \bigcup_{\ell' \mid \ell', \ell \in T_\ell} \bigcup \]

is non empty so we can pick a \( w \) in it. If there exists two distinct values \( \eta_1, \eta_2 \) such that \( v + \eta_1 w, v + \eta_2 w \in T_{\ell'} \), then \( (\eta_1 - \eta_2) w \in T_{\ell'} \) which is a contradiction. Hence, for all \( \ell' \), there exists at most one \( \eta_{\ell'} > 0 \) such that \( v + \eta_{\ell'} w \) is in \( T_{\ell'} \) and thus \( v + \eta w \) is in none of these subspaces for \( \eta < \min_{\ell'} \eta_{\ell'} \). We then consider the small perturbation of \( y_n \) in direction \( w \) given by

\[ z_n := \exp_{y_n}^{V_\ell}(\varepsilon_n \eta \pi_{T_{\ell}} v_{\ell}(w)), \]

Thanks to Lemma A.2 again, we know that

\[ \pi_{T_{\ell}} v_{\ell}(w) = w + O(\varepsilon_n), \]

\[ y_n - x = \varepsilon_n v + O(\varepsilon_n^2), \]

\[ z_n - y_n = \varepsilon_n \eta \pi_{T_{\ell}} y_{\ell}(w) + O(\eta^2 \varepsilon_n^2) \]

\[ = \varepsilon_n \eta w + O(\eta^2 \varepsilon_n^2). \]

As a result, \( \| z_n - x \| = \varepsilon_n + O(\eta \varepsilon_n) \) and hence

\[ \text{dist}(z_n - x \mid T_\ell)^2 = \| \pi_{T_{\ell}} (z_n - x) \|^2 \]

\[ = \| \pi_{T_{\ell}} (y_n - x) \|^2 + \| \pi_{T_{\ell}} (z_n - y_n) \|^2 + 2 \| \pi_{T_{\ell}} (z_n - y_n), \pi_{T_{\ell}} (y_n - x) \| \]

\[ = \frac{1}{4} \kappa(\Phi)(x)^2 \varepsilon_n^4 + O(\eta \varepsilon_n^4). \]

All in all, this gives

\[ \frac{2 \text{dist}(z_n - x \mid T_\ell)}{\| z_n - x \|^2} = \frac{\kappa(\Phi)(x) \varepsilon_n^2 + O(\eta \varepsilon_n^2)}{\varepsilon_n^2 + O(\eta \varepsilon_n^2)} \]

\[ = \kappa(\Phi)(x) + O(\eta), \]
so that the sequence \( z_n \in V_\ell \) achieves the \( \limsup \) up to \( O(\eta) \). Now for all \( \ell' \) such that \( T_{\ell'} \neq T_\ell \), there holds,
\[
\text{dist}(z_n - x \mid T_{\ell'}) = \| \pi_{T_{\ell'}}(z_n - x) \| = \| \varepsilon_n \pi_{T_{\ell'}}(v + \eta w) + O(\varepsilon_n^2) \|
= \varepsilon_n \| \pi_{T_{\ell'}}(v + \eta w) \| + O(\varepsilon_n^2),
\]
and because \( v + \eta w \notin T_{\ell'} \) for all small \( \eta > 0 \), there holds that \( \| \pi_{T_{\ell'}}(v + \eta w) \| > 0 \). As a result, for \( n \) large enough, for all \( \ell' \) such that \( T_{\ell'} \neq T_\ell \), we have
\[
\text{dist}(z_n - x \mid T_{\ell'}) > \text{dist}(z_n - x \mid T_\ell),
\]
and in particular
\[
\text{dist}(z_n - x \mid T_\ell M) = \text{dist}(z_n - x \mid T_\ell),
\]
since \( T_\ell M = \cup_{\ell'\neq\ell} T_{\ell'} \). We thus conclude that
\[
\frac{2 \text{dist}(z_n - x \mid T_\ell M)}{\| z_n - x \|^2} = \frac{2 \text{dist}(z_n - x \mid T_\ell)}{\| z_n - x \|^2} = \kappa(\Theta)(x) + O(\eta),
\]
so that for all \( \eta > 0 \),
\[
\limsup_{y \in M} \frac{2 \text{dist}(y - x \mid T_\ell M)}{\| y - x \|^2} \geq \kappa(\Theta)(x) + O(\eta).
\]
The proof is then complete after taking \( \eta \to 0 \).

A.4 Volume estimates

This section is devoted to prove the heuristic developed in Section 3.1. To obtain fine bounds, we will need the following notion of generalized angles.

Definition A.4 (Angles between subspaces). Let \( T, T' \subset \mathbb{R}^D \) be linear subspaces of respective dimensions \( d \geq d' \). The principal angles between \( T \) and \( T' \) is the sequence of numbers \( \Theta(T, T') := (\theta_1, \ldots, \theta_d') \subset [0, \pi/2]^d \) defined by
\[
\Theta(T, T') := \left( \arccos \sigma_1(U^\top U'), \ldots, \arccos \sigma_d(U^\top U') \right),
\]
where
- \( U \in \mathbb{R}^{D \times d} \) and \( U' \in \mathbb{R}^{D \times d'} \) are orthonormal bases of \( T \) and \( T' \) respectively;
- \( \sigma_k(U^\top U') \) is the \( k \)-th largest singular value of \( U^\top U' \).

We extend \( \Theta(T, T') \) as a vector of size \( d \) by setting \( \theta_k = \pi/2 \) for \( k \geq d' + 1 \).

When \( \dim T = \dim T' \), the angles \( \Theta(T, T') \) are linked to \( \kappa(T, T') = \| \pi_T - \pi_{T'} \|_{\text{op}} \) through
\[
\kappa(T, T') = \sqrt{1 - \cos^2 \Theta_d(T, T')} = \sin \Theta_d(T, T'),
\]
see for instance [36, Proof of Thm 2.5.1]. When \( d > d' \), we have \( \kappa(T, T') = 1 \) and \( \Theta_d(T, T') = \pi/2 \) so that (10) still holds for flats of different dimension.

Lemma A.5. Let \( T, T' \subset \mathbb{R}^D \) be linear subspaces of respective dimensions \( d \geq d' \). For all \( h_\ell \geq h_1 > 0 \), there holds
\[
\mathcal{H}^d \left( T \cap S_{T'}(0, h_\ell, h_1) \right) \leq 4^d \prod_{k=1}^d \left\{ h_\ell \wedge \frac{h_1}{\sin \Theta_k(T, T')} \right\},
\]
where \( \Theta(T, T') \in [0, \pi/2]^d \) is the sequence of principal angles between \( T \) and \( T' \) (Definition A.4).
Proof. Notice that since $h_\perp \leq h_\parallel$, $S_{T'}(0, h_\parallel, h_\perp) \subseteq B(0, 2h_\parallel)$. Furthermore, there holds

$$\{ v \in T \mid \|\pi_{T''}(v)\| \leq h_\parallel \} = \{ v \in T \mid \langle \pi_{T''}(v), \pi_{T''}(v) \rangle \leq h_\parallel^2 \} \subseteq \{ v \in T \mid \langle v, \mathcal{H}v \rangle \leq h_\parallel^2 \},$$

where $\mathcal{H} := \pi_T \circ \pi_{T''} \circ \pi_T$ is a self-adjoint operator of $T$. Let $e_1, \ldots, e_d$ be an orthonormal diagonalizing basis for $\mathcal{H}$, with associated eigenvalues $\lambda_1, \ldots, \lambda_d$. Writing $v_i := \langle e_i, v \rangle$ for $v \in \mathbb{R}^D$, one gets

$$\mathcal{H}^d \{ v \in B(0, 2h_\parallel) \cap T \mid \langle v, \mathcal{H}v \rangle \leq h_\parallel^2 \} = \mathcal{H}^d \left\{ v \in B(0, 2h_\parallel) \cap T \mid \sum_{i=1}^d \lambda_i v_i^2 \leq h_\parallel^2 \right\} \leq \mathcal{H}^d \left\{ v \in B(0, 2h_\parallel) \cap T \mid \forall i, v_i^2 \leq h_\parallel^2 / \lambda_i \right\} \leq 4^d \prod_{i=1}^d \left( \frac{h_\parallel / \sqrt{\lambda_i}}{\lambda_i} \right).$$

To obtain an explicit expression of the $\lambda_i$’s, let $U \in \mathbb{R}^{D \times d}$ and $V \in \mathbb{R}^{D \times d'}$ be orthonormal bases of $T$ and $T'$ respectively. Then there holds

$$\mathcal{H} = U U^\top (\text{Id}_D - V V^\top) U U^\top = U(\text{Id}_d - (U^\top V)(U^\top V)^\top) U^\top,$$

so that as an endomorphism of $T$, the spectrum of $\mathcal{H}$ is given by

$$\text{spec } \mathcal{H} = 1 - \text{spec } \{(U^\top V)(U^\top V)^\top\} = 1 - \cos^2 \Theta(T, T') = \sin^2 \Theta(T, T'),$$

which ends the proof. \qed

From Lemma A.5, we deduce the following immediate corollary.

**Corollary A.6.** Let $T, T' \subseteq \mathbb{R}^D$ be linear subspaces of respective dimensions $d$ and $d'$. For all $h_\parallel \geq h_\perp > 0$, there holds

(i) If $d = d'$, then

$$\mathcal{H}^d (T \cap S_{T'}(0, h_\parallel, h_\perp)) \leq 4^d h_\parallel^{d-1} \left\{ \frac{h_\parallel}{\delta(T, T')} \wedge h_\parallel \right\}.$$

(ii) If $d > d'$, then

$$\mathcal{H}^d (T \cap S_{T'}(0, h_\parallel, h_\perp)) \leq 4^d h_\parallel^{d'} h_\parallel^{d-d'}.$$

**Proof.** This is a direct application of Lemma A.5 and Definition A.4. \qed

Corollary A.6 extends to the non-linear case in the following sense.

**Lemma A.7.** Let $M \in \mathcal{M}^{(d)}(\kappa_{\max}, V_{\max})$ and $T \subseteq \mathbb{R}^D$ be a linear subspace of dimension $d' \leq d$. Take $h_\parallel < 1/(8\kappa_{\max})$ and $h_\perp := \eta \kappa_{\max} h_\parallel^2$ for some constant $0 < \eta < 1/(\kappa_{\max} h_\parallel)$. Then for all $x \in \mathbb{R}^D$,

(i) If $d' = d$, there holds

$$\mathcal{H}^d (M \cap S_T(x, h_\parallel, h_\perp)) \leq C_d V_{\max} \kappa_{\max}^d h_\parallel^{d-1} \times \left\{ \frac{(1 + \eta) \kappa_{\max} h_\parallel^2}{\theta} \wedge h_\parallel \right\},$$

where $\theta := \inf_{x \in M \cap S_T(x, h_\parallel, h_\perp)} \delta(T \mid T_x M)$. 26
(ii) If \( d' < d \), there holds
\[
\mathcal{H}^d(M \cap S_T(x, h_\ell, h_\perp)) \leq C_d (1 + \eta)^{d-d'} V_{\max}^{2d-d'} h_\perp^{2d-d'}.
\]

(iii) If \( d' = d \) and \( \mathcal{H}^d(M \cap S_T(x, h_\ell, h_\perp)) \geq \beta h_\ell^d \) for some \( \beta > 0 \), then there exists \( y \in M \) such that
\[
\mathcal{H}^d(M \cap S_T(x, h_\ell, h_\perp)) \leq C_d (1 + \eta) \kappa_{\max} h_\ell.
\]

Proof. Let \((M, \Psi) \in \mathcal{I}(M)\) be such that \( \text{vol}(M) < 2V_{\max} \). Since \( h_\perp \geq h_\ell \), Lemma A.3 yields
\[
M \cap S_T(x, h_\ell, h_\perp) \subset M \cap B(x, 2h_\ell) = \bigcup_{j=1}^N \Psi(U_j)
\]
with \( \text{diam}_{\mathcal{M}}(U_j) \leq 8h_\ell \) and \( N \leq N_0 \) :: \( \text{vol}(M)(4\kappa_{\max})^d/\omega_d \). A union bound hence yields, letting \( V_j = \Psi(U_j) \),
\[
\mathcal{H}^d(M \cap S_T(x, h_\ell, h_\perp)) \leq \sum_{j=1}^N \mathcal{H}^d(V_j \cap S_T(x, h_\ell, h_\perp)) \leq N_0 \max_{1 \leq j \leq N} \mathcal{H}^d(V_j \cap S_T(x, h_\ell, h_\perp)). \tag{11}
\]
If \( V_j \cap S_T(x, h_\ell, h_\perp) = \emptyset \) then \( \mathcal{H}^d(V_j \cap S_T(x, h_\ell, h_\perp)) = 0 \). Otherwise, take \( x_j \in V_j \cap S_T(x, h_\ell, h_\perp) \), and use triangle inequality to get
\[
V_j \cap S_T(x, h_\ell, h_\perp) \subset V_j \cap S_T(x_j, 2h_\ell, 2h_\perp).
\]
Let us write \( x_{0,j} := (\Psi^{-1})(x_j) \). From Lemma A.2 3., we have that for all \( y \in V_j \cap S_T(x_j, 2h_\ell, 2h_\perp) \), the tangent vector \( v = (\exp_{x_{0,j}}^{-1})(y) \in T_{x_{0,j}}(\Psi)M \) satisfies
\[
\|v - x_j - y\| \leq 2\kappa_{\max} \|x_j - y\|^2 \leq 32\kappa_{\max} h_\ell^2,
\]
where the last bound comes from \( S_T(x_j, 2h_\ell, 2h_\perp) \subset B(x_j, 4h_\ell) \). By triangle inequality, we hence get that
\[
v \in S_T(x_j, 2h_\ell, 32\kappa_{\max} h_\ell, 2h_\perp, 32\kappa_{\max} h_\ell) \subset S_T(x_j, 6h_\ell, (2\eta + 32)\kappa_{\max} h_\ell).
\]
In summary, we have proven that
\[
V_j \cap S_T(x_j, 2h_\ell, 2h_\perp) \subset \exp_{x_{0,j}}^{\mathcal{M}}(T_{x_{0,j}}^{\Psi}(M \cap S_T(x_j, 6h_\ell, (2\eta + 32)\kappa_{\max} h_\ell)) \bigcap S_T(x_j, 6h_\ell, (2\eta + 32)\kappa_{\max} h_\ell))
\]

To prove point (i), assume that \( d = d' \) and use Lemma A.2 4. and Corollary A.6 (i) to get
\[
\mathcal{H}^d(V_j \cap S_T(x, h_\ell, h_\perp)) \leq 2^d \mathcal{H}^d(T_{x_{0,j}}^{\Psi}(M \cap S_T(x_j, 6h_\ell, (2\eta + 32)\kappa_{\max} h_\ell)))
\]
\[
\leq 48^d h_\ell^{d-1} \left\{ \begin{array}{l}
(\eta + 1)\kappa_{\max} h_\ell^2 \\
\angle(T_{x_{0,j}}^{\Psi}(M, T) \wedge 6h_\ell)
\end{array} \right\} \tag{12}
\]
The last inequality rewrites as
\[
\mathcal{H}^d(V_j \cap S_T(x, h_\ell, h_\perp)) \leq C_d h_\ell^{d-1} \left\{ \begin{array}{l}
(\eta + 1)\kappa_{\max} h_\ell^2 \\
\angle(T_{x_{0,j}}^{\Psi}(M, T) \wedge h_\ell)
\end{array} \right\}, \tag{13}
\]
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which combined with (11) concludes the proof of (i).

For point (ii), use point (ii) of Corollary A.6 in the bound (12) to get

\[ \mathcal{H}^d(V_j \cap S_T(x, h_T, h_1)) \leq 8^d (6h_T)^d \left( (2\eta + 32)\kappa_{\text{max}}h_T^d \right)^{d-d'} \leq C_d(\eta + 1)^{d-d'}\kappa_{\text{max}}h_T^{2d-d'}, \]

and combine it with (11).

For point (iii), item (i) and its proof yield the existence of some \( j \leq N \) such that \( \mathcal{H}^d(V_j \cap S_T(x, h_T, h_1)) \geq \beta h_T^d / N_0 \). If \( (\eta + 1)\kappa_{\text{max}}h_T^2 / \angle(T_{x_0}, M, T) \geq h_T \), then \( \angle(T_{x_0}, M, T) \leq (\eta + 1)\kappa_{\text{max}}h_T \). If not, then using (13), we get that

\[ \angle(T_{x_0}, M, T) \leq C_d V_{\text{max}}\kappa_{\text{max}}^{d+1}(\eta + 1)\beta h_T. \]

Furthermore, writing \( y := \exp_{x_0}^M(v) \) with \( v := \pi_{T_{x_0}}(x-x_j) \), we get that \( \|y-x_j-v\| \leq 2\kappa_{\text{max}}\|v\|^2 \). In addition,

\[
\|v - (x - x_j)\| \leq \|\pi_{T_{x_0}}(\pi_T(x-x_j)) - \pi_T(x-x_j)\| + 2h_1 \\
\leq \angle(T_{x_0}, M, T) \times \|\pi_T(x-x_j)\| + 2h_1 \\
\leq (\beta^{-1}C_d V_{\text{max}}\kappa_{\text{max}}^d + 1)(\eta + 1)\kappa_{\text{max}}h_T^2 + 2\kappa_{\text{max}}h_T^2,
\]

which together with \( \|v\|^2 \leq 8h_T^2 \) yields

\[ \|y - x\| \leq C_d(\beta^{-1}V_{\text{max}}\kappa_{\text{max}}^d + 1)(\eta + 1)\kappa_{\text{max}}h_T^2 \]

Finally, using Lemma A.2 6, we find that

\[ \angle(T|T_j M) \leq \angle(T_{x_0}, M, T) + 5\kappa_{\text{max}}\|y-x_j\| \\
\leq C_d(\beta^{-1}V_{\text{max}}\kappa_{\text{max}}^d + 1)(\eta + 1)\kappa_{\text{max}}h_T,
\]

ending the proof. \( \square \)

Point (ii) of Lemma A.7 takes the simple, weaker form,

\[ \mathcal{H}^d(M \cap S_T(x, h_T, h_1)) \leq C_d(\eta + 1)^{d+1}V_{\text{max}}\kappa_{\text{max}}^{d+1}h_T^{d+1}, \tag{14} \]

which corresponds to the limiting case where \( d = d' + 1 \) and which will be of use in the forthcoming proofs.

**Lemma A.8.** Let \( M \in M^{(d)}(\kappa_{\text{max}}, V_{\text{max}}) \), \( x \in M \), and \( T \subset \mathbb{R}^D \) be a linear subspace of dimension \( d \). Take \( h_T < 1/(8\kappa_{\text{max}}) \) and \( h_1 := \eta\kappa_{\text{max}}h_T^2 \) for some constant \( 0 < \eta < 1/(\kappa_{\text{max}}h_T) \). If \( \angle(T|T_j M) \leq b\kappa_{\text{max}}h_T \) with \( b \geq \sqrt{\eta} \), then

\[ \mathcal{H}^d(M \cap S_T(x, h_T, h_1)) \geq c_d((\eta/b) \wedge 1)^{d}h_T^{d}. \]

**Proof.** Writing \( \gamma := (\eta/(2b)) \wedge 1 \), let us show that there exists \( x_0 \in \Psi^{-1}(x) \) such that the inclusion \( \exp_{x_0}^M(T_{x_0} \cap B(x, \gamma h_T)) \subset M \cap S_T(x, h_T, h_1) \) holds. Combined with the lower bound of Lemma A.2 4., this inclusion will be sufficient to conclude.

To prove the announced inclusion, take \( x_0 \in \Psi^{-1}(x) \) such that \( \angle(T, T_{x_0}) M \leq b\kappa_{\text{max}}h_T \). Given arbitrary \( v \in T_{x_0} \cap B(x, \gamma h_T) \) write \( y := \exp_{x_0}^M(v) \). Thanks to Lemma A.2 1., we have \( \|y - x\| \leq \|v\| \leq \gamma h_T \leq h_T \). Furthermore, Lemma A.2 5. yields that

\[
\|\pi_{T_{x_0}}(y - x)\| \leq \|y - x - \pi_{T_{x_0}}(y - x)\| \leq \frac{K_{\text{max}}}{2}\|y - x\|^2 \leq \frac{K_{\text{max}}}{2}\eta h_T^2,
\]

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where we used that $\gamma^2 \leq \eta^2/b^2 \leq \eta$ since $b \geq \sqrt{n}$. Thus,
\[
\| \pi_{T^i}(y - x) \| \leq \| \pi_{T^i(\varphi)}(y - x) \| + \zeta(T_{x_0}(\varphi) M, T) \| y - x \|= \frac{K_{\text{max}}}{2} \eta h^2 + b K_{\text{max}} \gamma h^2_{\text{p}}
\]
which concludes the proof. \(\square\)

## B Concentration bounds

To link slab counting and (integrated) mass at the population level, we will use extensively the following concentration bound or the $K$ empirical distributions $P_{1,n}, \ldots, P_{K,n}$ (see Section 2.3 for notation).

**Lemma B.1.** Let $S$ denote the set of all slabs of $\mathbb{R}^D$:
\[
S := \{ S_T(x, h, h) \mid x \in \mathbb{R}^D, T \subset \mathbb{R}^D \text{ linear, and } h, h \in \mathbb{R}_+ \}.
\]
For $n$ large enough so that \((800D^2 \log D)^{\log(n/\alpha_{\min})} \leq 1, \text{ then for all } \sqrt{(800D^2 \log D)^{\log(n/\alpha_{\min})}} \leq \varepsilon \leq 1,
\[
\mathbb{P}\left( \max_{k \in \{1, \ldots, K\}} \sup_{S \in S} \frac{P_k(S) - P_{k,n}(S)}{P_k(S)} \vee \frac{P_{k,n}(S) - P_k(S)}{P_{k,n}(S)} \geq \varepsilon \right) \leq 4K \exp\left( -\varepsilon^2 n/16 \right).
\]
As a result, if we let
\[
\varepsilon := \left( \frac{\log n}{n} \right)^{1/2} \text{ with } \gamma \geq \frac{16q}{\alpha_{\min}} \sqrt{\frac{800D^2 \log D}{\alpha_{\min}}}
\]
for some $q > 0$, then for $n$ large enough so that $\varepsilon \leq 1$, the probability of the event
\[
E_1 := \left\{ \max_{k \in \{1, \ldots, K\}} \sup_{S \in S} \frac{P_k(S) - P_{k,n}(S)}{P_k(S)} \vee \frac{P_{k,n}(S) - P_k(S)}{P_{k,n}(S)} < \varepsilon \right\}
\]
is at least $1 - 4K n^{-q}$.

The proof is based on a Vapnik-Chernovenkis argument on the class of slabs, combined with concentration of multinomial distributions.

**Proof.** Write $\rho_0(t) := 1_{t \geq 0}$ for the Heaviside function on $\mathbb{R}$, and $\rho_2(t) := t^2$ for the square function. Let $T \subset \mathbb{R}^D$ be a linear subset of dimension $d \in \{1, \ldots, D\}$. If $e_1, \ldots, e_d$ denotes an orthogonal basis of $T$ and $e_{d+1}, \ldots, e_D$ an orthogonal basis of $T^\perp$, then we have
\[
1_{S_T(x, h, h)}(z) = 1 \Leftrightarrow \begin{cases} \| \pi_T(z - x) \|^2 \leq h^2_y \\ \| \pi_{T^\perp}(z - x) \|^2 \leq h^2_h \end{cases} \Leftrightarrow \begin{cases} \sum_{j=1}^d \rho_2((e_j, z - x)) \leq h^2_y \\ \sum_{j=d+1}^D \rho_2((e_j, z - x)) \leq h^2_h \end{cases} \Leftrightarrow \begin{cases} \rho_0(h^2_y - \sum_{j=1}^d \rho_2((e_j, z - x))) = 1 \\ \rho_0(h^2_h - \sum_{j=d+1}^D \rho_2((e_j, z - x))) = 1 \end{cases} \Leftrightarrow \rho_0\left(h^2_y - \sum_{j=1}^d \rho_2((e_j, z - x))\right) + \rho_0\left(h^2_h - \sum_{j=d+1}^D \rho_2((e_j, z - x))\right) - 2 \geq 0.
\]
This formulation shows that indicators of slabs can be written the sign of a neural network with piecewise polynomial activation functions of degree at most 2 and 2 pieces, 2 hidden layers, $D + 2$ non-linear units, and at most $2(D \times (2D) + 1) + 1 = 4D^2 + 3$ weights. Momentarily using notation from [13, Theorem 7], we have $p = 1, d = 2, R = 3D + 2, L = 2, W = 4D^2 + 3$ and $R = 3D + 2$. As a result, because $D \geq 2$, we get that the class $S$ of all possible slabs has Vapnik-Chernovenkis dimension at most

$$VC(S) \leq L + LW \log_2(4eR \log_2(2eR)) \leq 100D^2 \log D.$$ 

On the other hand, for all $k \in \{1, \ldots, K\}$, conditionally on $N_k$, $P_{k,n} := \frac{1}{N_k} \sum_{i=1}^n 1_{Y_i = k} \delta_{X_i}$ is the empirical distribution of an i.i.d. sample $X'_1, \ldots, X'_N$ with common distribution $P_k$. Hence, from [18, Theorem 5.1] and Sauer’s lemma, we get that for all $\varepsilon > 0$,

$$P \left( \sup_{S \in S} \frac{P_k(S) - P_{k,n}(S)}{\sqrt{P_k(S)}} \geq \varepsilon \bigg| N_k \right) \leq 4 \left( \frac{2\varepsilon N_k}{v} \right)^v \exp \left(-\varepsilon^2 N_k/4\right),$$

where $v := 100D^2 \log D$. In particular, for all $\varepsilon \geq \sqrt{(800D^2 \log D) \log N_k / N_k}$,

$$P \left( \sup_{S \in S} \frac{P_k(S) - P_{k,n}(S)}{\sqrt{P_k(S)}} \geq \varepsilon \bigg| N_k \right) \leq \exp \left(-\varepsilon^2 N_k/8\right).$$

To obtain an unconditional bound, we apply Bernstein’s inequality [19, Corollary 2.11 and (2.10)] to the Binomial variable $N_k$ with parameters $n$ and $\alpha_k > 0$, which yields

$$P \left( N_k \leq n\alpha_k/2 \right) \leq \exp(-3n\alpha_k/28).$$

As a result,

$$P \left( \sup_{S \in S} \frac{P_k(S) - P_{k,n}(S)}{\sqrt{P_k(S)}} \geq \varepsilon \right) \leq P \left( N_k \leq -n\alpha_k/2 \right) + P \left( \sup_{S \in S} \frac{P_k(S) - P_{k,n}(S)}{\sqrt{P_k(S)}} \geq \varepsilon \bigg| N_k > n\alpha_k/2 \right) \leq \exp(-3n\alpha_k/28) + \exp(-\varepsilon^2 n\alpha_k/16) \leq 2 \exp(-\varepsilon^2 n\alpha_k/16),$$

as soon as $\varepsilon \leq 1$ and $\varepsilon \geq \sqrt{(800D^2 \log D) \log(n\alpha_k/\min_{n\alpha_k})}$. A union bound over $k \in \{1, \ldots, K\}$ allows to conclude that

$$P \left( \max_{k=1,\ldots,K} \sup_{S \in S_d} \frac{P_k(S) - P_{k,n}(S)}{\sqrt{P_k(S)}} \geq \varepsilon \right) \leq 2K \exp(-\varepsilon^2 n\alpha_{\min}/16).$$

To conclude the proof, do the exact same reasoning for the relative deviation $(P_{k,n}(S) - P_k(S))/\sqrt{P_k(S)}$ by using the other inequality of [18, Theorem 5.1].

We deduce the following thresholded version of Lemma B.1, which roughly asserts that slabs $S$ with $d_k$-dimensional mass $P_k(S)$ of order $\log n/n$ have can be distinguished through $P_{k,n}(S)$.

**Corollary B.2.** On the event $E_1$ of Lemma B.1, there holds, for all $c > 0$,

1. If $P_k(S) \geq c^2 \varepsilon^2$, then $P_{k,n}(S) \geq (1 - 1/c) P_k(S)$.
2. If $P_k(S) \leq c^2 \varepsilon^2$, then $P_{k,n}(S) \leq (1 + c)^2 \varepsilon^2$. 

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Proof. If \( P_k(S) \geq c^2 \varepsilon^2 \), then
\[
P_{k,n}(S) \geq P_k(S) - \varepsilon \sqrt{P_k(S)} \geq (1 - 1/c) P_k(S).
\]

Reciprocally, if \( P_k(S) \leq c^2 \varepsilon^2 \), then \( P_{k,n}(S) - P_k(S) \leq \varepsilon \sqrt{P_{k,n}(S)} \) which implies
\[
\sqrt{P_{k,n}(S)} \leq \frac{1}{2} \left( \sqrt{\varepsilon^2 + 4P_k(S)} + \varepsilon \right) \leq \frac{1}{2} \left( \sqrt{1 + 4c^2 + 1} \varepsilon \right) \leq (c + 1) \varepsilon,
\]
which ends the proof. □

Similarly, the following concentration bound asserts that enough geodesic balls of radii \( (\log n/n)^{1/d_k} \) all contain at least approximately \( \log n \) points drawn from the \( d_k \)-dimensional layer.

**Lemma B.3.** Given \( k \in \{1, \ldots, K\} \) and \( \rho_{d_k} > 0 \), let \( \mathcal{P}_k \) denote a maximal \((2\rho_{d_k})\)-packing of \( M_k \), in the sense that \( \|p - p'\| > 2\rho_{d_k} \) for all \( p \neq p' \in \mathcal{P}_k \). Let \( \mathcal{B}_k \) denote the class of subsets of \( \mathbb{R}^D \) defined by
\[
\mathcal{B}_k := \{ \Psi_k(B_M(p, \rho_{d_k})) \mid p \in \mathcal{P}_k \}.
\]

Let \( q > 0 \), \( v \geq 0 \) and
\[
\varepsilon := \left( \frac{\log n}{n} \right)^{1/2} \text{ and } \rho_d := \left( \frac{\Upsilon_{d} \varepsilon^2}{a_{\min} \gamma_{d_{\max}}} \right)^{1/d} \text{ with } \gamma \geq \frac{56q}{3a_{\min}} \text{ and } \Upsilon_{d} \geq 2(1 \vee v) \frac{2^d}{\omega_d}.
\]

Then for \( n \) large enough, the probability of the event
\[
\mathcal{E}_2 := \left\{ \min_{1 \leq k \leq K} \min_{B \in \mathcal{B}_k} P_{k,n}(B) \geq v \varepsilon^2 \right\}
\]
is at least \( 1 - 2Kn^{-q} \).

**Proof.** Let first \( k \in \{1, \ldots, K\} \) and \( B \in \mathcal{B}_k \) be fixed. As soon as \( \varepsilon^2 \leq a_{\min}/(4d_k \Upsilon_{d_k}) \), Lemma A.2 4. applies and yields
\[
P_k(B) \geq a_{\min} \kappa_{d_{\max}} 2^{-d_k} \omega_{d_k} \rho_{d_k}^{d_k} \geq 2^{-d_k} \omega_{d_k} \Upsilon_{d_k} \varepsilon^2 \geq 2(1 \vee v) \varepsilon^2.
\]

Therefore, from Bernstein’s bound for binomial distributions, we have
\[
\mathbb{P}\left( P_{k,n}(B) \leq \frac{1}{2} P_k(B) \middle| N_k \right) \leq \exp\left( -\frac{3}{28} N_k P_k(B) \right) \leq \exp\left( -\frac{3}{14} N_k \varepsilon^2 \right).
\]

Using the same bound as in the proof of Lemma B.1, we proceed with
\[
\mathbb{P}\left( P_{k,n}(B) \leq \frac{1}{2} P_k(B) \right) \leq \mathbb{P}(N_k \leq n\alpha_k/2) + \mathbb{P}\left( P_{k,n}(B) \leq \frac{1}{2} P_k(B) \middle| N_k > \alpha_k n/2 \right)
\]
\[
\leq \exp\left( -3n\alpha_k/28 \right) + \exp\left( -3\alpha_k n\varepsilon^2/28 \right)
\]
\[
\leq 2 \exp\left( -\frac{3}{28} \alpha_k n\varepsilon^2 \right),
\]

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for \( n \) large enough such that \( \varepsilon \ll 1 \). On the other hand, the cardinal of \( \mathcal{B}_k \) can be bounded from above thanks to Lemma A.2 4. by

\[
\text{Card}(\mathcal{B}_k) \leq \frac{\text{vol} \mathcal{M}_k}{2^{-d_k} \omega d_k \rho_{d_k}} \leq \frac{2 \nu_{\text{max}} k^{d_k}}{2^{-d_k} \omega d_k \left( \frac{1}{\alpha_{\text{min}} k^{d_k}} \right)} \leq \frac{\nu_{\text{max}}}{\varepsilon^2} \leq \exp \left( \frac{3}{56} \alpha_k n \varepsilon^2 \right),
\]

where the last inequality holds for \( n \) large enough. As a result, for \( n \) large enough, a union bound over the elements of \( \mathcal{B}_k \) yields

\[
\mathbb{P} \left( \min_{B \in \mathcal{B}_k} \frac{P_{k,n}(B)}{P_k(B)} \leq \frac{1}{2} \right) \leq 2 \exp \left( -\frac{3}{28} \alpha_k n \varepsilon^2 \right) \text{Card}(\mathcal{B}_k)
\]

\[
\leq 2 \exp \left( -\frac{3}{56} \alpha_k n \varepsilon^2 \right).
\]

By a union bound over \( k \in \{1, \ldots, K\} \), we conclude that

\[
\mathbb{P} \left( \min_{1 \leq k \leq K} \min_{B \in \mathcal{B}_k} \frac{P_{k,n}(B)}{P_k(B)} \leq \frac{1}{2} \right) \leq 2K \exp \left( -\frac{3}{56} \alpha_{\text{min}} n \varepsilon^2 \right) \leq 2Kn^{-q},
\]

This ends the proof since on the complement of this event, (15) asserts that

\[
\min_{1 \leq k \leq K} \min_{B \in \mathcal{B}_k} P_{k,n}(B) \geq \frac{1}{2} \min_{1 \leq k \leq K} \min_{B \in \mathcal{B}_k} P_k(B) \geq \nu \varepsilon^2.
\]

\[ \square \]

C Algorithmic guarantees

To prove the performances of Slabelling, the following notation for thickenings of sets will be useful.

**Definition C.1** (Offset). Given \( A \subset \mathbb{R}^D \) and \( r \geq 0 \), the \( r \)-offset of \( A \) is

\[
A^r := \{ x \in \mathbb{R}^D \mid \text{dist}(x, A) \leq r \}.
\]

We are now in position to prove the main technical result on Slabelling. Write:

- \( \mathcal{X}^{(d)} \) for the current value of set \( \mathcal{X} \) at the start of step \( d \in \{1, \ldots, d_{\text{max}}\} \);
- \( \mathcal{X}^{(d)} \) for the set of labeled points at the end of step \( d \).

In particular, \( \mathcal{X}^{(1)} = \mathcal{X} \) is the input point cloud and \( \mathcal{X}^{(d+1)} = \mathcal{X}^{(d)} \setminus \mathcal{X}^{(d)} \).

**Lemma C.2.** There exist constants \( \Upsilon^*, \sigma^* \) depending on \( D \), and \( \zeta_d^* \) depending on \( d \) such that, if we choose

\[
\Upsilon \geq \nu_{\text{max}} \Upsilon^*, \quad \gamma \geq \frac{600D^2 \log D}{\alpha_{\text{min}}} \sqrt{\frac{56q}{3\alpha_{\text{min}}}}, \quad 4D \leq \sigma \leq \sigma^*, \quad \zeta_d \geq \zeta_d^*,
\]

and

\[
\varepsilon := \left( \frac{\gamma \log n}{n} \right)^{1/2}, \quad \rho_d := \left( \frac{\Upsilon \varepsilon^2}{\alpha_{\text{min}} k_d^{d_k}} \right)^{1/d}, \quad h_d := 48 \left( 1 + \frac{1}{8d} \right) \alpha_{d_k}, \quad k_d := \kappa_{d_k} \leq \kappa_{\text{max}},
\]

\[
r_d := h_d, \quad \delta_d := \zeta_d \kappa_{\text{max}} k_d^2, \quad n_d := \sigma \varepsilon_n^2,
\]

the following holds.

For \( n \) large enough, when runned with a i.i.d. point cloud \( \mathcal{X}_n = \{X_1, \ldots, X_n\} \) from some unknown distribution \( \hat{P} = \sum_{k=1}^K \alpha_k P_k \in \hat{P} \) and parameters (16) and (17), the algorithm Slabelling operates in a way such that with probability at least \( 1 - 6Kn^{-q} \), at all steps \( d \in \{1, \ldots, d_{\text{max}}\} \), we have:
1. If a \((d + 1)\)-tuple of points \(x = (x_1, \ldots, x_{d+1})\) in \(\tilde{X}(d)\) is co-detected, then \(d = d_k\) for some \(k \in \{1, \ldots, K\}\) and
\[
d_H(\text{conv}(x_1, \ldots, x_{d+1}) \mid M_k) \leq \xi_{d_k} \kappa_{\text{max}} h_{d_k}^2 \quad \text{where} \quad \xi_{d_k} = C_{d_k} \nu_{\text{max}} \frac{a_{\text{max}}}{a_{\text{min}}}.
\] (18)

Furthermore, for all \(y \in \text{conv}(x_1, \ldots, x_{d+1})\), there exists \(x \in B(y, \xi_{d_k} \kappa_{\text{max}} h_{d_k}^2) \cap M_k\) such that
\[
\angle (\text{span}(x_1, \ldots, x_{d+1}) \mid T_x M_k) \leq \xi_{d_k} \kappa_{\text{max}} h_{d_k}.
\]

2. If \(d = d_k\), then for all \(x \in M_k\) and all \(d_k\)-dimensional subspace \(T \subset T_x M_k\), there exists a \((d_k + 1)\)-tuple \((x_1, \ldots, x_{d+1})\) of \(\tilde{X}(d_k)\) co-detected such that,
\[
\text{dist}(x \mid \text{conv}(x_1, \ldots, x_{d+1})) \leq \zeta_{d_k} \kappa_{\text{max}} h_{d_k}
\]
and
\[
\angle (T, \text{span}(x_1, \ldots, x_{d+1})) \leq C_{d_k} \kappa_{\text{max}} h_{d_k}.
\]

For the proof of Lemma C.2, we set ourselves on the intersection of the events \(\mathcal{E}_1\) and \(\mathcal{E}_2\) of Lemma B.1 and Lemma B.3, which hold simultaneously with these parameters with probability higher than \(1 - 6Kn^{-\vartheta}\), as soon as we take
\[
\Upsilon^* \geq \nu^* \max_{1 \leq d \leq D-1} \frac{2d+1}{\omega_d}
\]
for some \(\nu^* \geq 2\). We also take \(\Upsilon^* \geq 1\). The value of \(\nu^*\) will be specified later in the proof, see (23). The Lemma C.2 is proven by induction on \(d \in \{1, \ldots, d_{\text{max}}\}\). The initialization step is essentially the same as the induction step but easier, since we do not have to handle the points that have been removed from \(\tilde{X}(d)\) in previous steps. All steps are thus treated identically. We let \(d \in \{1, \ldots, d_{\text{max}}\}\) and we assume that points 1 and 2 of Lemma C.2 holds for all \(d' \in \{1, \ldots, d-1\}\) (perhaps empty). In particular, point 1 and 2 respectively yield the two inclusions
\[
\begin{aligned}
\tilde{X}(d) & \supset X_n \setminus \bigcup_{d_k < d} M_k^{\xi_{d_k} \kappa_{\text{max}} h_{d_k}^2 + \delta_{d_k}} \\
\tilde{X}(d) & \subset X_n \setminus \bigcup_{d_k < d} M_k,
\end{aligned}
\] (19)
which will be the only thing we need for the induction to work — whence the uselessness of treating the initialization step separately since (19) holds automatically true for \(d = 1\).

**Remark C.3** (On the chosen immersions below). In what follows, the \(d_k\)-dimensional support \(M_k = \text{support}(P_k)\) of the unknown distribution \(P = \sum_{k=1}^K \alpha_k P_k \in \tilde{P}\) is parametrized by an arbitrary immersion \((\Psi_k, M_k) \in \Pi(M_k)\) such that
\[
\text{vol}(M_k) \leq 2 \nu_{\text{max}} (\kappa_{\text{max}})^{-d_k},
\]
chosen once for all.

**Proof of Lemma C.2 — point 1, induction step.** Let \(x = (x_1, \ldots, x_{d+1})\) be a \((d + 1)\)-tuple of \(\tilde{X}(d)\) co-detected by the algorithm, meaning that \(\text{rad}(x) \leq r_{d_k}\) and \(\text{Card}(S \cap \tilde{X}(d)) \geq \sigma n \varepsilon^2\), where \(S := S(x, h_{d_k}, \kappa_{\text{max}} h_{d_k}^2)\) is the slab associated to \(x\).

- **Proof that** \(d \in \{d_1, \ldots, d_K\}\). Let us write \(k := \inf \{k' \mid d_k' \geq d\}\). From the second inclusion of the inductive hypothesis (19), \(\tilde{X}(d) \cap X_{k',n} \subset \tilde{X}(d) \cap M_{k'} = \emptyset\) for all \(k' < k\). Hence, we have
\[
\sum_{k' < k} P_{k',n}(S) \geq \frac{1}{n} \sum_{k' < k} \text{Card}(S \cap \tilde{X}(d) \cap X_{k',n}) = \frac{1}{n} \text{Card}(S \cap \tilde{X}(d)) \geq \frac{n_{d}}{n} \sigma \varepsilon^2.
\]
As a result, there exists $k_0 \geq k$ such that $P_{k_0,n}(S) \geq \sigma \varepsilon^2 / K$. Using Corollary B.2, and the fact that $\sigma \geq 4D \geq 4K$, we find that $P_{k_0}(S) \geq \varepsilon^2$. On the other hand, letting $C_{d_k}$ denote a generic constant depending on $d_k$ only, point (ii) of Lemma A.7 yields that for $n$ large enough

$$
\varepsilon^2 \leq P_{k_0}(S) \leq C_{d_k} \left( a_{\max} \kappa_{\max} \right)^{d_{k_0}} \left( \nu_{\max} / \kappa_{\max} \right)^{2d_{k_0}} d_{k_0} \left( \kappa_{\max} h_{d_k} \right)^{2d_{k_0} - d_{k_0}}
$$

and since $h_d = (\gamma \varepsilon^2 / (a_{\min} \kappa_{\max}^d))^{1/d}$, it yields

$$
a_{\min} \kappa_{\max}^d / \gamma \leq C_{d_k} a_{\max} \nu_{\max} \left( \kappa_{\max} h_{d_k} \right)^{2(d_{k_0} - d)/d}.
$$

This inequality can only hold if $2(d_{k_0} - d)/d = 0$ for $n$ large enough. Therefore $d_{k_0} = d$. As $d \leq d_k \leq d_{k_0}$ by construction, we hence have proven that $d = d_k$.

- **Proof of the Hausdorff distance bound.** As $P_k(S) = P_{k_0}(S) \geq \varepsilon^2$, we get

$$
\mathcal{H}(d_k) (M_k \cap S) \geq \frac{\varepsilon^2}{a_{\max} \kappa_{\max} d_{k_0}} = \frac{c_{d_k}}{\gamma} a_{\min} \frac{h_{d_k}}{\kappa_{\max}}.
$$

Hence, point (iii) of Lemma A.7 yields the existence of some $x \in M_k$ such that $\| \bar{x} - x \| \leq \mu_k \kappa_{\max} h_{d_k}^2$ and $\varepsilon(T \mid T_k) \leq \mu_k \kappa_{\max} h_{d_k}$, where $T := \text{span}(\bar{x})$ is the affine span of $\bar{x} = (x_1, \ldots, x_{d+1})$, and

$$
\mu_k := C_{d_k} (1 + \nu_{\max} \beta^{-1}) \vee 1 \quad \text{with} \quad \beta := \frac{c_{d_k}}{\gamma} a_{\min} \frac{h_{d_k}}{\kappa_{\max}}.
$$

In particular, we get

$$
1 \leq \mu_k \leq C'_{d_k} \nu_{\max} \gamma \frac{a_{\max}}{a_{\min}},
$$

where we used that $\gamma$ and $\nu_{\max}$ were lower-bounded by numeric constants. Because $\text{rad}(\bar{x}) \leq r_{d_k}$, we have in particular that $\| x_j - x \| \leq r_{d_k} + \mu_k \kappa_{\max} h_{d_k}^2$ for all $j \in \{1, \ldots, d_k + 1\}$. Recall that $T_k M_k = \bigcup_{x_0 \in \Psi_k(x)} T(x_0) M_k$. Write $x_0 \in \Psi_k^{-1}(x)$ for any pre-image of $x$ such that $\varepsilon(T, T(x_0) M_k) \leq \mu_k \kappa_{\max} h_{d_k}$. For all $j \in \{1, \ldots, d_k\}$, consider $y_j = \text{exp}_{x_0}^{M_k}(v_j)$ where $v_j := \pi_{T(x_0)} M_k(x_j - x)$. Then triangle inequality and Lemma A.2. (applicable to $v$ provided that $n$ is large enough so that $\mu_k \kappa_{\max} h_{d_k} \leq 1/8$) lead to

$$
\| y_j - x_j \| \leq \| (y_j - x) - \pi_{T(x_0)} M_k(x_j - x) \| + \| (x_j - x) - \pi_{T(x_0)} M_k(x_j - x) \|
$$

$$
\leq \kappa_{\max} \| \pi_{T(x_0)} M_k(x_j - x) \|^2 + \| (x_j - \bar{x}) - \pi_{T(x_0)} M_k(x_j - \bar{x}) \| + \| (x - \bar{x}) - \pi_{T(x_0)} M_k(x - \bar{x}) \|
$$

$$
\leq \kappa_{\max} \left( r_{d_k} + \mu_k \kappa_{\max} h_{d_k}^2 \right)^2 + \| \pi_{T(x_0)} M_k \|_{op} \| x_j - \bar{x} \| + \| x - \bar{x} \|
$$

$$
\leq \kappa_{\max} \left( r_{d_k} + \mu_k \kappa_{\max} h_{d_k}^2 \right)^2 + \mu_k \kappa_{\max} h_{d_k} r_{d_k} + \mu_k \kappa_{\max} h_{d_k}^2,
$$

where we used that $\mu_k \geq 1$. In particular, $\text{rad}(y_1, \ldots, y_{d+1}) \leq r_{d_k} + C \mu_k \kappa_{\max} h_{d_k}^2 \leq 2h_{d_k}$ for $n$ large enough. Applying [10, Lem 12] to $V_{j,k} = \text{exp}_{x_0}^{M_k}(B_{T(x_0)} M_k(0, 2h_{d_k}))$, we get that as soon as $n$ is large enough so that $2h_{d_k} \leq 1/(2\kappa_{\max})$, there holds

$$
\text{d}_H(\text{conv}(y_1, \ldots, y_{d+1}) \mid M_k) \leq \frac{1}{\kappa_{\max}} \left( 1 - \sqrt{1 - (2\kappa_{\max} h_{d_k})^2} \right) \leq 4\kappa_{\max} h_{d_k}^2.
$$
But since we have
\[ d_H(\text{conv}(x_1, \ldots, x_{d_k+1}), \text{conv}(y_1, \ldots, y_{d+1})) \leq \max_{1 \leq i \leq d+1} \|x_i - y_i\|, \]
we deduce that
\[ d_H(\text{conv}(x_1, \ldots, x_{d_k+1}) \mid M_k) \leq C\mu_k\kappa_{\text{max}}h_{d_k}^2, \]
which is what we wanted to show.

- **Proof of the angle bound.** Finally, for any \( z \in \text{conv}(x_1, \ldots, x_{d_k+1}) \), we can let \( w = \exp_{x_0}^{M_k}(\pi_{T_{x_0}^{(\Psi_k)}}(z-x)) \in M_k \). The same computations as above show that \( \|z-w\| \leq C\mu_k\kappa_{\text{max}}h_{d_k}^2 \) and that
\[
\angle(T \mid T_wM_k) \leq \angle(T, T_{x_0}^{(\Psi_k)}M_k) + \angle(T_{x_0}^{(\Psi_k)}M_k \mid T_wM_k) \\
\leq \mu_k\kappa_{\text{max}}h_{d_k} + 5\kappa_{\text{max}}\|w-x\| \\
\leq \mu_k\kappa_{\text{max}}h_{d_k} + 5\kappa_{\text{max}}r_{d_k} + 5\mu_k\kappa_{\text{max}}h_{d_k}^2 \\
\leq C\mu_k\kappa_{\text{max}}h_{d_k},
\]
which concludes the proof.

**Proof of Lemma C.2 — point 2, induction step.** We let \( x_0 \in (\Psi_k)^{-1}(x) \) be fixed. We identify \( T_{x_0}^{(\Psi_k)}M_k \) to \( \mathbb{R}^d_k \) with its canonical basis \((e_1, \ldots, e_{d_k})\). Define \( e_{d_k+1} \) to be the unit vector
\[ e_{d_k+1} := -\frac{1}{\sqrt{d_k}} \sum_{i=1}^{d_k} e_i. \]
Then, for all \( i \in \{1, \ldots, d_k+1\} \) and \( s > 0 \), let us write
\[ Q_i(s) := \{v \in \mathbb{R}^d_k \mid \|v - e_i\| \leq s\} \quad \text{and} \quad \bar{Q}_i(s) := \bar{h}_{d_k} Q_i(s) \quad \text{with} \quad \bar{h}_{d_k} := \frac{1}{2(1+1/(8d_k))} h_{d_k}. \]
Given any family of vectors \( v_i \in \bar{Q}_i(1/8d_k) \) for \( 1 \leq i \leq d_k+1 \) with associated renormalized \( \bar{v}_i := v_i/\|v_i\| \), we will show that
1. For all \( i \in \{1, \ldots, d_k+1\} \), \( \bar{v}_i \in Q_i(1/4d_k) \);
2. For all \( i \neq j \in \{1, \ldots, d_k+1\} \), there holds that \( |\langle \bar{v}_i, \bar{v}_j \rangle| \leq 1/2d_k \);
3. There holds that \( 0 \in \text{conv}(v_1, \ldots, v_{d_k+1}) \);
4. For all \( i \in \{1, \ldots, d_k+1\} \),
\[ X_{k,n} \cap \left( \exp_{x_0}^{M_k}(\bar{Q}_i(1/8d_k)) \setminus \bigcup_{k' < k} M_{k'}^{\xi_{d_k+1},\kappa_{\text{max}}h_{d_k}^2,s_{d_k'}} \right) \neq \emptyset. \]
For point 1, simply notice that for all \( \lambda > 0 \), \( \bar{v}_i \) is the projection of \( \lambda v_i \) onto the unit sphere. Triangle inequality and this fact applied to \( \lambda = 1/\bar{h}_{d_k} \) hence yield
\[
\|\bar{v}_i - e_i\| \leq \|\bar{v}_i - v_i/\bar{h}_{d_k}\| + \|v_i/\bar{h}_{d_k} - e_i\| \leq 2\|v_i/\bar{h}_{d_k} - e_i\| \\
\leq \frac{1}{4d_k},
\]
where the last inequality comes from the fact that $\|v_i - \tilde{h}_{d_k}e_i\| \leq \tilde{h}_{d_k}/(8d_k)$ by construction.

For point 2, use point 1 to get
\[
|\langle \tilde{v}_i, \tilde{v}_j \rangle| \leq |\langle \tilde{v}_i - e_i, \tilde{v}_j \rangle| + |\langle e_i, \tilde{v}_j - e_j \rangle| \\
\leq 1/2d_k.
\]

For point 3, let $V := (\langle \tilde{v}_j, e_i \rangle)_{1 \leq i,j \leq d_k} \in \mathbb{R}^{d_k \times d_k}$ and $A := I - V$. We have $\|A\|_\infty \leq 1/4d$ so that $\|A^{\ell}\|_\infty \leq d_k^{-1}/(4d_k)^{\ell} = 1/(d_k4^\ell)$ for all $\ell \geq 1$. We deduce that $V$ is invertible, with the decomposition
\[
V^{-1} = I + R \quad \text{and} \quad \|R\|_\infty \leq \sum_{\ell=1}^{\infty} \frac{1}{d_k4^\ell} \leq \frac{1}{2d_k}.
\]

From the invertibility of $V$, one can write
\[
-\tilde{v}_{d_k+1} = \sum_{i=1}^{d_k} (\langle \tilde{v}_{d_k+1}, e_i \rangle e_i = \sum_{j=1}^{d_k} a_j \tilde{v}_j \quad \text{with} \quad a_j = \sum_{i=1}^{d_k} (V^{-1})_{i,j} \langle -\tilde{v}_{d_k+1}, e_i \rangle.
\]

Furthermore, point 1 yields that
\[
\left| \langle -\tilde{v}_{d_k+1}, e_i \rangle \right| - \frac{1}{\sqrt{d_k}} \leq 1/(4d_k)
\]
for all $i \in \{1, \ldots, d_k\}$. As a result,
\[
a_j = (V^{-1})_{j,j} \langle -\tilde{v}_{d_k+1}, e_j \rangle + \sum_{i \neq j} (V^{-1})_{i,j} \langle -\tilde{v}_{d_k+1}, e_i \rangle \\
\geq \left(1 - \frac{1}{2d_k}\right) \left(\frac{1}{\sqrt{d_k}} - \frac{1}{4d_k}\right) - (d_k - 1) \times \frac{1}{2d_k} \times \left(1 + \frac{1}{4d_k}\right) \\
\geq \frac{1}{2\sqrt{d_k}} - \frac{3}{8d_k} > 0.
\]

From which we conclude that the convex coefficients $\tilde{a}_j := a_j/(\sum_{m=1}^{d_k+1} a_m)$ for $j \in \{1, \ldots, d_k + 1\}$ with $a_{d_k+1} := 1$ satisfy
\[
0 = \sum_{j=1}^{d_k+1} \tilde{a}_j \tilde{v}_j \in \text{conv} \{v_1, \ldots, v_{d_k+1}\}.
\]

For point 4, write
\[
\lambda_{k'} := \xi_{d_{k'}} \kappa_{\max}^2 h_{d_{k'}}^2 + \delta_{d_{k'}} =: \tau_{k'} \kappa_{\max}^2 h_{d_{k'}}^2
\]
for all $k' < k$. By construction, $\tau_{k'}$ is greater than some numeric constant, i.e. $\tau_{k'} \geq c$. Let also
\[
B_i := \exp_{X_{d_k}}(Q_i(1/8d_k)) \quad \text{for} \quad i \in \{1, \ldots, d_k + 1\}.
\]

Then we have
\[
P_{k,n} \left( B_i \cap \bigcup_{k' < k} M_{k'}^{\lambda_{k'}} \right) \geq P_{k,n} (B_i) - \sum_{k' < k} P_{k,n} \left( B_i \cap M_{k'}^{\lambda_{k'}} \right).
\]

Let now $k' < k$ be fixed. Notice that $B_i \cap M_{k'}^{\lambda_{k'}} \subset (B_i^{\lambda_{k'}} \cap M_{k'}^{\lambda_{k'}})^\lambda_{k'}$. For $n$ large enough, we have $\lambda_{k'} < \tilde{h}_{d_k}$, and hence
\[
\text{rad}(B_i^{\lambda_{k'}}) \leq \frac{\tilde{h}_{d_k}}{4d_k} + \lambda_{k'} \\
\leq \frac{1}{2} \tilde{h}_{d_k} < \frac{1}{4\kappa_{\max}}.
\]
As a result, Lemma A.3 yields the existence of \( z_1, \ldots, z_{N'} \) in \( M_{k'} \) with \( N' \leq N_0^{(d_{k'})} \) such that
\[
B_i^{\lambda_{k'}} \cap M_{k'} \subset \bigcup_{\ell=1}^{N'} \Psi_k( B_{M_{k'}}(z_\ell, 2\tilde{h}_{d_k}) ).
\]

With points 1 and 5 of Lemma A.2, we further find that for all \( \ell \in \{1, \ldots, N'\} \),
\[
\Psi_k( B_{M_{k'}}(z_\ell, 4\tilde{h}_{d_k}) ) \subset S_{T_\ell^{(\phi_{k'})}}(x_\ell, 2\tilde{h}_{d_k}, 2\kappa_{\max} \tilde{h}_{d_k}^2),
\]
where \( x_\ell := \Psi_k(z_\ell) \), so that in the end
\[
B_i \cap M_{k'}^{\lambda_{k'}} \subset \bigcup_{\ell=1}^{N'} S_{T_\ell^{(\phi_{k'})}}(x_\ell, 3\tilde{h}_{d_k}, (2 + \tau_{k'}) \kappa_{\max} \tilde{h}_{d_k}^2). \tag{22}
\]

Using point (ii) of Lemma A.7, we find that
\[
P_k\left( S_{T_\ell^{(\phi_{k'})}}(x_\ell, 3\tilde{h}_{d_k}, (2 + \tau_{k'}) \kappa_{\max} \tilde{h}_{d_k}^2) \right) \leq (a_{\max} \kappa_{\max}^{d_k}) \times (C_{d_k} \tau_{k'}^{d_k-d_{k'}} \nu_{\max}^{d_k-d_{k'}} \tilde{h}_{d_k}^{2d_k-d_{k'}}).
\]
Because \( d_k > d_{k'} \), we can put the latter bound in the form \( c_n \varepsilon^2 \) where \( c_n = o(1) \) so that using Corollary B.2, we get that
\[
P_{k,n}\left( S_{T_\ell^{(\phi_{k'})}}(x_\ell, 3\tilde{h}_{d_k}, (2 + \tau_{k'}) \kappa_{\max} \tilde{h}_{d_k}^2) \right) \leq (1 + c_n) \varepsilon^2 \leq 2\varepsilon^2,
\]
for \( n \) large enough. This latter bound for all \( k' < k \) and all \( \ell \in N' \) together with (22) yield the final bound
\[
\sum_{k' < k} P_{k,n}(B_i \cap M_{k'}^{\lambda_{k'}}) \leq C_D \nu_{\max} \varepsilon^2,
\]
where we used that \( K \leq D \) and \( N_0^{(d_{k'})} \leq C_D \nu_{\max} \) (see Lemma A.3). On the other hand, because \( \tilde{h}_{d_k}/8d_k \geq 3\rho_{d_k} \), we know that must \( B_1 \) contain a ball of the form \( \Psi_k( B_{d_k}(p, \rho_{d_k}) ) \) with center \( p \in \mathcal{P}_k \) from Lemma B.3. As \( Y \geq \nu_{\max} Y^* \), it must contain \( \nu_{\max} \nu^* n^2 \) points from \( \mathcal{X}_{k,n} \), and thus (21) yields
\[
P_{k,n}(B_i \setminus \bigcup_{k' < k} M_{k'}^{\lambda_{k'}}) \geq (\nu^* - C_D) \nu_{\max} \varepsilon^2, \tag{23}
\]
and it is sufficient to take \( \nu^* > C_D \) to make sure that \( P_{k,n}(B_i \setminus \bigcup_{k' < k} M_{k'}^{\lambda_{k'}}) > 1/n \) for \( n \) large enough, which ends the proof of point 4.

- **Proof of the distance bound.** For all \( 1 \leq i \leq d+1 \) we let \( x_i \in \mathcal{X}_{k,n} \) be a point that belongs to \( \exp_{x_0}^M( \bar{\mathcal{Q}}_i(1/8d_k) ) \setminus \bigcup_{k' < k} M_{k'}^{\lambda_{k'}} \). We denote \( v_i := \exp_{x_0}^{-1}(x_i) \) and let \( \lambda_i \in [0,1] \) summing to 1 such that \( 0 = \sum_{i=1}^{d+1} \lambda_i v_i \) (point 2). From Lemma A.2, we have \( \|x_i - x - v_i\| \leq \kappa_{\max} \tilde{h}_{d_k}^2 / (64d_k^2) \), from which we deduce
\[
\|x - \sum_{i=1}^{d+1} \lambda_i x_i\| = \left\| \sum_{i=1}^{d+1} \lambda_i (x + v_i - x_i) \right\| \leq \sum_{i=1}^{d+1} \lambda_i \kappa_{\max} \tilde{h}_{d_k}^2 / 64d_k^2 \]
\[= \frac{\kappa_{\max} \tilde{h}_{d_k}^2}{64d_k^2}, \]
so that
\[
\text{dist}(x \mid \text{conv}(x_1, \ldots, x_{d+1})) \leq \kappa_{\max} \tilde{h}_{d_k}^2 / (64d_k^2) = \zeta_{d_k} \kappa_{\max} \tilde{h}_{d_k}^2,
\]
where \( \zeta_{d_k} \) is exactly the constant appearing before \( \kappa_{\max} \tilde{h}_{d_k}^2 \) in the last expression.
– **Proof of the angle bound.** We now let $T := \text{span}(x_1, \ldots, x_{d_k+1})$. Write $w_i := x_i - \pi_T(x)$, so that $T = \text{Vect}\{w_1, \ldots, w_d\}$. We have
\[
\|w_i - v_i\| = \|x_i - \pi_T(x) - v_i\|
\leq \|x - \pi_T(x)\| + \|x_i - x - v_i\|
\leq \frac{\kappa_{\text{max}} h^2}{32d_k}
\]
so that
\[
\|\ddot{w}_i - \ddot{v}_i\| \leq \frac{\|w_i - v_i\|}{\|v_i\|} + \frac{\|w_i\|}{\|v_i\|} - 1 \leq \frac{2}{\|v_i\|} \leq \kappa_{\text{max}} h_d.
\]
We complete $(e_i)_{1 \leq i \leq d_k}$ to a full orthonormal basis of $\mathbb{R}^D$ and extend $V \in \mathbb{R}^{D \times d_k}$ (see point 3) with 0. Let also $W := (\langle \ddot{w}_j, e_i \rangle) \in \mathbb{R}^{D \times d_k}$. From the last bound, we have $\|W - V\|_{\text{op}} \leq \sqrt{d_k} \kappa_{\text{max}} h_d$, and thanks to points 2 and the Gershgorin circle theorem,
\[
\|(V^TV)^{-1}\|_{\text{op}} \leq \left(1 - \frac{d_k - 1}{2d_k}\right)^{-1} \leq 2.
\]
We thus have that
\[
\|W(W^TW)^{-1}W^T - V(V^TV)^{-1}V^T\|_{\text{op}} \leq C\sqrt{d_k} \kappa_{\text{max}} h_d
\]
for some explicit numerical constant $C$, whence $\langle T, T_{x_0}^{(s_k)} M_k \rangle \leq C\sqrt{d_k} \kappa_{\text{max}} h_d$.

– **Proof that $x_1, \ldots, x_{d+1}$ are codetected.** First, we know that
\[
\|x - x_i\| \leq \|v_i\| \leq \frac{1}{8d_k} \tilde{h}_{d_k} \leq \frac{h_{d_k}}{2} \leq \frac{r_{d_k}}{2},
\]
so that $\text{rad}(x_1, \ldots, x_{d_k+1}) \leq r_{d_k}$. Second, since
\[
\text{dist}(x \mid \text{conv}(x_1, \ldots, x_{d_k+1})) \leq \kappa_{\text{max}} \frac{\tilde{h}_{d_k}^2}{64d_k^2} \leq \frac{1}{2} \kappa_{\text{max}} h_{d_k}^2 \leq \frac{1}{2} h_{d_k},
\]
we know that the slab defined by $(x_1, \ldots, x_{d_k+1})$ contains the slab
\[
S := S_T(x, h_{d_k}/2, \kappa_{\text{max}} h_{d_k}^2/2).
\]
Using Lemma A.8 and the angle bound we just showed, we get that
\[
P_k(S) \geq C_{d_k} a_{\text{min}}^{d_k} \tilde{h}_{d_k}^2 \geq C_D \nu_{\text{max}}^* \varepsilon^2.
\]
Assuming that we took $\nu^*$ large enough so that $4 \leq C_D \nu_{\text{max}}^*$, which we can (recall that $\nu_{\text{max}}$ is lowered-bounded by a numeric constant without loss of generality, see Remark 2.4), Corollary B.2 yields
\[
P_{\kappa,n}(S) \geq \frac{1}{2} P_k(S) \geq C_D \nu_{\text{max}}^* \varepsilon^2.
\]
On the other hand, in the same spirit as what as been done above, one can show that
\[ P_{k,n}(S \cap \bigcup_{k' < k} M_{k'}^{\lambda_{k'}}) \leq C_D' \nu_{\max}^2 \varepsilon^2. \]

In the end, we thus have
\[ \text{Card} \left( S \cap \widehat{X}^{(d_k)} \right) \geq \text{Card} \left( X_n \cap \left( S \setminus \bigcup_{k' < k} M_{k'}^{\lambda_{k'}} \right) \right) \]
\[ \geq (C_D \nu^* - C_D') \nu_{\max}^2 n \varepsilon^2. \]

Using again the fact that \( \nu_{\max} \) is lower-bounded by a numeric constant (see Remark 2.4), we can write
\[ \text{Card} \left( S \cap \widehat{X}^{(d_k)} \right) \geq (C_D \nu^* - C_D') \nu_{\max}^2 n \varepsilon^2 \geq \frac{1}{2} C_D \nu^* n \varepsilon^2, \]

as soon as \( \nu^* \geq 2C_D'/C_D \). We conclude by setting \( \sigma^* := \frac{1}{2} C_D \nu^* \) and by taking \( \nu^* \) large enough so that \( \sigma^* \geq 4D \). The tuple \((x_1, \ldots, x_{d_k+1})\) is then co-detected by the algorithm at step \( d_k \), which ends the proof.

\[ \square \]

D Proof of the main results

The proofs of Theorem 4.2, Theorem 4.3, Theorem 4.5 and Theorem 4.6 are straightforward applications of Lemma C.2. It only remains to prove Theorem 4.4. The proof relies on the following lemma.

Lemma D.1. Let \( k' < k \). Then for all \( \lambda \leq 1/16 \kappa_{\max} \)
\[ P_k(M_{k'}^{\lambda}) \leq C_{d_k} a_{\max}^2 \nu_{\max}^2 \kappa_{\max}^{d_k - d_{k'}} \lambda^{d_{k'} - d_{k'}}. \]

Proof. We first write that
\[ P_k(M_{k'}^{\lambda}) \leq (a_{\max} \kappa_{\max})^{d_k} (M_k \cap M_{k'}^{\lambda}), \]

hence reducing the result to a volume bound. From the volume bound \( \text{vol}(M_{k'}) \leq 2 \nu_{\max} \kappa_{\max}^{d_{k'}} \), we can cover \( M_{k'} \) with at most
\[ N' \leq C_{d_{k'}} \nu_{\max} \kappa_{\max}^{d_{k'}} \left( \sqrt{\frac{\lambda}{\kappa_{\max}}} \right)^{d_{k'}} = C_{d_{k'}} \nu_{\max} (\kappa_{\max} \lambda)^{-d_{k'}/2} \]
balls \( B_{M_{k'}}(x_{0,j}, \sqrt{\frac{\lambda}{\kappa_{\max}}}) \) with \( x_{0,j} \in M_k \). Let also write \( T_j := T_{x_{0,j}}(M_{k'}) \) and \( y_j := \Psi_{k'}(x_{0,j}) \). Then from Lemma A.2 5.,
\[ \Psi_{k'}(B_{M_{k'}}(x_{j}, \sqrt{\frac{\lambda}{\kappa_{\max}}})) \subset S_{T_j} \left( y_j, \sqrt{\frac{\lambda}{\kappa_{\max}}}, \lambda \right) \]
for all \( j \in \{1, \ldots, N'\} \). Furthermore, using the fact that \( \lambda \leq \sqrt{\frac{\lambda}{\kappa_{\max}}} \), we get
\[ M_k \cap M_{k'}^{\lambda} \subset \bigcup_{j=1}^{N'} M_k \cap S_{T_j} \left( y_j, \sqrt{\frac{\lambda}{\kappa_{\max}}}, \lambda \right)^\lambda \]
\[ \subset \bigcup_{j=1}^{N'} M_k \cap S_{T_j} \left( y_j, 2\sqrt{\frac{\lambda}{\kappa_{\max}}}, 2\lambda \right). \]
Applying point (ii) of Lemma A.7 to each element of the union, we get
\[ \mathcal{H}(d_k) \left( M_k \cap S_{T_j} \left( y_j, 2\sqrt{\lambda/k_{\max}} \right) \right) \leq C_{d_k} \nu_{\max}^2 k_{\max}^{-d_k - d_{d_k'}} \left( \frac{\lambda}{k_{\max}} \right)^{2d_k - d_{d_k'}} \]
so that using an union bound over \( j \in \{1, \ldots, N'\} \) we find
\[ \mathcal{H}(d_k) (M_k \cap M_{k'}^{\lambda}) \leq (C_{d_k} \nu_{\max}(k_{\max} \lambda)^{-d_{d_k}/2}) \times C_{d_k} \nu_{\max}^2 k_{\max}^{-d_k - d_{d_k'}} \]
\[ = C_{d_k} \nu_{\max}^2 k_{\max}^{-d_k - d_{d_k'}} \lambda_k^{d_k - d_{d_k'}} \]
which ends the proof.

**Proof of Theorem 4.4.** As in the proof of Lemma C.2, let us write
\[ \lambda_k := \xi_{d_k} k_{\max} \theta_{d_k}^2 + \delta_{d_k}, \]
where \( \xi_{d_k} \) is given by (18). We write \( E := E_1 \cap E_2 \), where \( E_1 \) and \( E_2 \) are the events of Lemma B.1 and Lemma B.3 respectively. In particular, we have \( \mathbb{P}(E^c) \leq 6Kn^{-q/ r} \), and Lemma C.2 yields that on the event on \( E \),
\[ X_{k,n} \setminus \bigcup_{k'<k} M_{k'}^{\lambda_{k'}} \subset \overline{X}_{k,n} \subset X_{k,n} \cup \bigcup_{k'>k} X_{k',n} \cap M_{k'}^{\lambda_{k'}}. \]
Hence,
\[ \# \left\{ X_{k,n} \Delta \overline{X}_{k,n} \right\} \mathbb{P} \leq \# \left\{ X_{k,n} \setminus \overline{X}_{k,n} \right\} \mathbb{P} + \# \left\{ \overline{X}_{k,n} \setminus X_{k,n} \right\} \mathbb{P} \leq \sum_{k'<k} \# \left\{ X_{k,n} \cap M_{k'}^{\lambda_{k'}} \right\} \mathbb{P} + \sum_{k'>k} \# \left\{ X_{k',n} \cap M_{k'}^{\lambda_{k'}} \right\} \mathbb{P}. \]
Taking the expectation, we find that
\[ \mathbb{E} \left[ \frac{\# \left\{ X_{k,n} \Delta \overline{X}_{k,n} \right\}}{1 \lor N_k} \mathbb{P} \right] \leq \sum_{k'<k} \mathbb{E} \left[ \frac{\# \left\{ X_{k,n} \cap M_{k'}^{\lambda_{k'}} \right\}}{1 \lor N_k} \mathbb{P} \right] + \sum_{k'>k} \mathbb{E} \left[ \frac{\# \left\{ X_{k',n} \cap M_{k'}^{\lambda_{k'}} \right\}}{1 \lor N_k} \mathbb{P} \right]. \]
For \( k' < k \), it is easy to see that
\[ \mathbb{E} \left[ \frac{\# \left\{ X_{k,n} \cap M_{k'}^{\lambda_{k'}} \right\}}{1 \lor N_k} \mathbb{P} \right] = \mathbb{E} \left[ \frac{1}{1 \lor N_k} \mathbb{E} \left[ \frac{\# \left\{ X_{k,n} \cap M_{k'}^{\lambda_{k'}} \right\}}{1 \lor N_k} \mathbb{P} \mathbb{E} \left[ N_k \right] \right] \right] \]
\[ \leq \mathbb{E} \left[ \frac{N_k}{1 \lor N_k} \mathbb{E} \left[ X_{k,n} \cap M_{k'}^{\lambda_{k'}} \mathbb{P} \mathbb{E} \left[ N_k \right] \right] \right] \leq C_{d_k} a_{\max}^2 k_{\max}^{-d_k - d_{d_k'}} \lambda_k^{d_k - d_{d_k'}} \]
where we used Lemma D.1. For \( k' > k \), we use Lemma D.1 again, together with an extra Bernstein concentration for the binomial random variable \( N_k \) (see proof of Lemma B.1), to get
\[ \mathbb{E} \left[ \frac{\# \left\{ X_{k,n} \cap M_{k'}^{\lambda_{k'}} \right\}}{1 \lor N_k} \mathbb{P} \right] \leq n \mathbb{P} \left( N_k \leq \alpha_k n/2 \right) + \frac{1}{\alpha_k n/2} \mathbb{E} \left[ \# \left\{ X_{k,n} \cap M_{k'}^{\lambda_{k'}} \right\} \mathbb{P} \mathbb{E} \left[ N_k \right] \right] \]
\[ \leq n e^{-3\alpha_k n/28} + \frac{1}{\alpha_k n/2} \mathbb{P} \left( M_k^{\lambda_{k'}} \mathbb{E} \left[ N_k \right] \right) \]
\[ \leq n e^{-3\alpha_k n/28} + \frac{\alpha_k}{\alpha_k} C_{d_{d_k}} a_{\max}^2 k_{\max}^{-d_k - d_{d_k'}} \lambda_k^{d_k - d_{d_k'}}. \]
The leading terms in the two sums on the right hand side of (25) are thus the ones corresponding to \( k' = k - 1 \) and \( k' = k + 1 \) respectively, hence concluding the proof. \( \square \)
References


