Nonlinear normal modes as invariant manifolds for model order reduction

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This chapter introduces the nonlinear normal modes (NNMs) for vibrating systems as invariant manifolds of the phase space, and their use for model order reduction of nonlinear structures. NNMs are defined as the continuation of the linear normal modes by enforcing tangency to a subset of master eigenspaces for small amplitudes. Conservative and damped dynamics are considered, as well as forced systems where the NNMs are time-dependent. A systematic procedure using the parametrisation method for invariant manifolds, is devised for their computation, directly operating from the physical space, and up to arbitrary order of expansions. Applications to academic examples are shown to highlight the ability of the method to deal with hardening/softening behaviour, the presence of a folding manifold, and superharmonic resonance. In each case, reduced-order models with minimal dimensions and excellent accuracy, are derived.

1 Introduction

This chapter is devoted to a presentation of the concept of *Nonlinear Normal mode* (NNM) and its use in model order reduction for nonlinear vibrating systems with smooth nonlinearity. Since NNMs have been used in the literature with different definitions, one goal of the presentation is to unify them thanks to the latest developments that clarify the links between damped and conservative cases, and between mathematical definitions and operative techniques. The emphasis is on using the invariant manifold theory to propose an accurate and usable definition of an NNM. Since the invariance property is key to deriving efficient reduced-order models (ROMs) with minimal dimensions, NNMs are particularly well suited to propose direct and efficient computations of reduced dynamics on curved manifolds through the definition of nonlinear mappings. The scope of the presentation is also restricted to the case of smooth nonlinearity. In such a case, the geometric nonlinearity, arising in large amplitude vibrations of thin structures, is the canonical example that will be used to illustrate the concepts.

NNMs have a long history, and the notion has witnessed several different definitions. Whereas a linear normal mode (LNM) has a unique and well-posed definition, extending the notion to the nonlinear range cannot be a straightforward story, since the main properties and characteristics of LNMs cannot be all fulfilled at the nonlinear level. Therefore, and depending on the context, many different notions have been proposed to extend the idea. The very first appearance of a definition for an NNM dates back from the 1960s with the works by Rosenberg (1962, 1964a,b, 1966); Pak and Rosenberg (1968) and Kauderer (1958), and was restricted to conservative systems. In such a case the existence of a family of periodic orbits emerging in the continuation of the linear normal modes provided an effective framework that has been used in subsequent works (Rand (1974); Vakakis et al. (1996); Vakakis (1997); Kerschen et al. (2009)). The idea of using invariant manifolds tangent at origin to their linear counterpart extended the applicability range of NNMs and was first introduced by Shaw and Pierre (1991).

Most of the ideas and methods for defining NNMs are derived from the dynamical system theory, and find their roots in the works by Poincaré (1892) and Lyapunov (1907). In particular, two of the most important tools in dynamical systems theory have been used to define and compute NNMs. The center manifold reduction theorem (Carr (1981); Kelley (1967); Carr and Muncaster (1983)) has been used as the foundation to derive the invariant manifold approach for constructing NNMs proposed by Shaw and Pierre (1993, 1994); Shaw (1994). On the other hand, the normal form theory (Poincaré (1892); Dulac (1912); Elphick et al. (1987); Manneville (1990); Iooss and Adelmeyer (1998); Haragus and Iooss (2009); Iooss (1988); Murdock (2003)) allowed a different understanding and presentation for computing NNMs as invariant manifolds for reduced-order modelling (Jézéquel and Lamarque (1991); Touzé et al. (2004); Touzé and Amabili (2006)). The link between the two approaches has been made clear in the dynamical systems literature thanks to the parametrisation method for invariant manifold (Cabré et al. (2003a,b, 2005, 2003a); Haro et al. (2016)), and this method has then been translated in the nonlinear vibration framework to generalize the previous approaches (Haller and Ponsioen (2016); Ponsioen et al. (2020); Jain and Haller (2022); Vizzaccaro et al. (2022); Opreni et al. (2023b); Touzé et al. (2021)).

Two recent advances have been developed for the broad use of NNMs as a key tool for model order reduction of nonlinear vibrating structures in recent years. First, automated arbitrary-order expansions have been proposed, thus overcoming the limitations in amplitude faced by earlier developments. This advance has been first proposed by Ponsioen et al. (2018) for autonomous systems, and then extended for non-autonomous problems by Breunung and Haller (2018). The second important step ahead has been to derive direct methods, in the sense that they can be directly applied to a finite element (FE) model, without the need to first project the nonlinear problem in the modal space. Direct methods relying on the invariant manifold approach have been proposed by Veraszto et al. (2020); Vizzaccaro et al. (2021b); Jain and Haller (2022); Vizzaccaro et al. (2022); Opreni et al. (2023b). Moreover, these recent developments led to the release of open source codes integrating these features (Jain et al. (2021); Frangi et al. (2022)). Another interesting aspect of the recent developments is the numerous comparisons that have been drawn out between invariant-based methods and other techniques of current use in model order reduction, showing how the NNMs have to be considered as reference reduction method that provides the minimal dimension for a ROM (Haller and Ponsioen (2017); Shen et al. (2021a); Vizzaccaro et al. (2021a); Touzé et al. (2021)). All these developments advocate for a broader use of NNMs for model order reduction.

This chapter aims at providing proper and effective definitions of NNMs, covering the case of conservative and dissipative vibratory systems, as well as explaining in detail an effective computational framework by making use of the parametrisation method for invariant manifolds. The link with earlier methods developed in the 90s-2000 will also be clearly emphasized.

2 Nonlinear normal modes and invariant manifolds: definitions

In this section, the main definitions for nonlinear normal modes (NNMs) are reviewed, for conservative and dissipative cases. In particular, it will be shown how the notion of invariant manifold allows the unification of the different definitions that have been proposed in the literature. Nonlinear vibrating systems with smooth nonlinearity are considered. The most common case encountered in the literature is that of geometric nonlinearity, describing large amplitude oscillations of slender structures. This case will thus be often taken as a reference in the text and applicative example. The general starting point is a semi-discrete model under the form:

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} + \mathbf{K}\mathbf{X} + \mathbf{f}_{NL}(\mathbf{X}, \dot{\mathbf{X}}) = \mathbf{F}(t) \tag{1}$$

with $\mathbf{X}$ the N-dimensional vector of unknown nodal displacements (number of degrees-of-freedom, DOFs), $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ respectively the mass, damping and stiffness matrices, $\mathbf{F}$ the vector of external force applied on the body and $\mathbf{f}_{NL}(\mathbf{X}, \dot{\mathbf{X}})$ the nonlinear term. The space discretization method can be of any type (Rayleigh-Ritz, finite difference, ...). In general, it will be considered here that the Finite Element (FE)

procedure has been employed to derive Eq. (1). The notation $\mathbf{f}_{NL}$ for the nonlinear term will be used for its generality when needed. Considering geometric nonlinearity in the classical framework implementing linear elasticity with a full Lagrangian formulation, a Green-Lagrange strain measure, conjugated with the second Piola-Kirchhoff stress measure, $\mathbf{f}_{NL}$ contains only quadratic and cubic nonlinear terms and can be explicitly written as

$$\begin{aligned}\mathbf{f}_{NL}(\mathbf{X}) &= \mathbf{G}(\mathbf{X},\mathbf{X}) + \mathbf{H}(\mathbf{X},\mathbf{X},\mathbf{X}) \\ &= \sum_{r=1}^{N}\sum_{s=1}^{N}\mathbf{G}_{rs}X_rX_s + \sum_{r=1}^{N}\sum_{s=1}^{N}\sum_{t=1}^{N}\mathbf{H}_{rst}X_rX_sX_t,\end{aligned} \tag{2}$$

with $\mathbf{G}_{rs}$ and $\mathbf{H}_{rst}$ the vectors of quadratic and cubic coefficients G^p_{rs}, H^p_{rst}. Before defining NNMs, linear normal modes (LNMs) are briefly recalled.

2.1 Linear normal modes

The linear normal modes are first defined for conservative problems since leading to real modes, see for example Meirovitch (1980); Géradin and Rixen (2015). Considering Eq. (1) without external forcing, damping and nonlinear terms, the linear normal modes are the solutions of the eigenvalue problem

$$(\mathbf{K} - \omega^2\mathbf{M})\boldsymbol{\phi} = 0. \tag{3}$$

Since the dimension of the problem is N, the solutions to Eq. (3) lead to a family of N eigenvector-eigenfrequency pairs $\{\boldsymbol{\phi}_i, \omega_i\}_{i=1\ldots N}$. For the generalisation to the nonlinear case, it is important to give a geometric interpretation of the LNMs in phase space. The phase space of the problem is introduced by rewriting the equations of motion with first-order derivatives, under the form of a dynamical system. The transformation to a first-order system is not unique and many different choices are possible (see *e.g.* Tisseur and Meerbergen (2001); Jain and Haller (2022) for discussions on this point). Following the choice proposed by Vizzaccaro et al. (2024), Eq. (1) can be rewritten as

$$\mathbf{B}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{F}_{NL}(\mathbf{x}) + \mathbf{F}_{exc}(t), \tag{4}$$

with $\mathbf{x} = [\mathbf{Y} \;\; \mathbf{X}]^t$ and $\mathbf{Y} = \dot{\mathbf{X}}$ introduced as the velocity. The matrices $\mathbf{B}$ and $\mathbf{A}$ then reads:

$$\mathbf{B} = \begin{bmatrix}\mathbf{M} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}\end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix}-\mathbf{C} & -\mathbf{K}\\ \mathbf{M} & \mathbf{0}\end{bmatrix}. \tag{5}$$

In phase space, as shown in Fig. 1, the LNMs are orthogonal planes that divide the space in invariant two-dimensional subspaces, as a direct consequence of the fact that the oscillator equations are uncoupled in the modal space. Each of these subspaces shares the invariance property and is densely filled with periodic orbits corresponding to uncoupled motions. Fig. 1(a-b) shows the first two LNMs in two different 3d representations of the full phase space, for a conservative problem, where the coordinates (u_i, v_i) respectively refer to the modal displacement and velocity of mode i.

The notion can easily be generalised to damped systems. Two different cases are generally considered in the literature. The first case corresponds to a damping matrix $\mathbf{C}$ which is diagonalised by the real modes. In such case, the LNMs are real and a real-valued formalism can still be used to construct the modal basis and uncouple the linear equations of motions. The damping matrices fulfilling this property have been extensively studied and the general case has been derived by Caughey (1960); Caughey and O'Kelly (1965); Adhikari (2006). In particular, the widely used case of Rayleigh damping falls into this category. In such case, the geometrical interpretation is left unchanged and leads to Fig. 1(c): the eigenplanes are two-dimensional invariant subspaces composed of damped oscillations governed by each modal damping ratio.

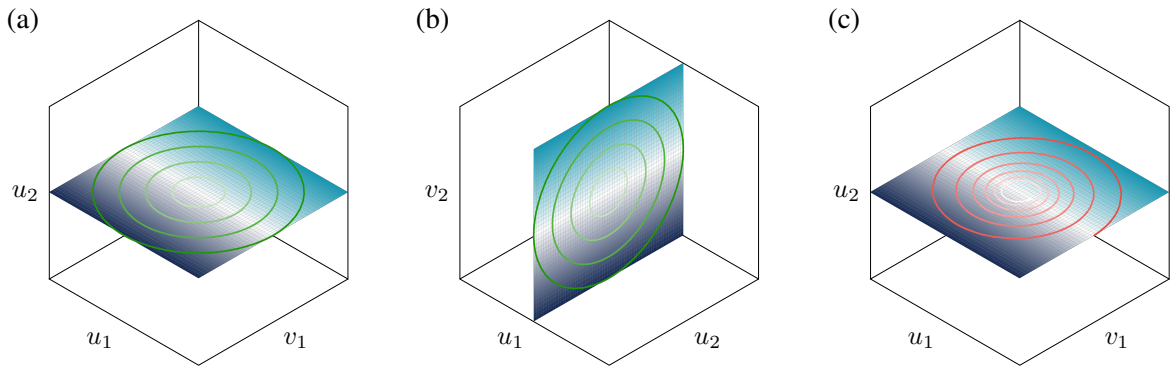

Figure 1: Representation of linear normal modes (LNMs) as eigenspaces of the phase space. (a) First LNM of the conservative problem in the space (u_1, v_1, u_2), represented as the plane $u_2 = 0$ filled with a family of periodic orbits, (b) Second LNM in the space (u_1, u_2, v_2), (c) first LNM in the dissipative case with a decaying orbit spiralling to the stable fixed point.

When the damping matrix is not made diagonal by the real normal modes, the system must be written at first order as in Eq. (4). One then needs to define the right and left eigenvectors that are generally complex. The right eigenvectors $\mathbf{Y}_s, \forall\, s \in [1, D]$ are associated to the eigenvalues λ_s such that

$$\forall s = 1, \ldots, D: \quad (\lambda_s \mathbf{B} - \mathbf{A})\, \mathbf{Y}_s = \mathbf{0}. \tag{6}$$

The left eigenvectors $\mathbf{X}_s$ are defined with

$$\forall s = 1, \ldots, D: \quad \mathbf{X}_s^\star\, (\lambda_s \mathbf{B} - \mathbf{A}) = \mathbf{0}, \tag{7}$$

where $\mathbf{X}_s^\star = \bar{\mathbf{X}}_s^t$ is the conjugate-transpose operation, also referred to as Hermitian transpose. The geometric interpretation still holds when considering complex modes obtained from the first-order system.

The LNMs have many important properties explaining their wide use in vibration theory. They constitute the only linear basis which uncouples the equations of motion, and thanks to this property, linear problems can be solved thanks to the superposition theorem. The invariance property, even if not especially mentioned when defining LNMs, is embedded in their definition and a key feature related to uncoupling. Extending this powerful notion to the nonlinear range has been a longstanding idea since one would like to enjoy the same properties, if possible, to solve out nonlinear vibration problems. Of course, this is not possible in general and one needs to let apart some properties while trying to keep some others. In this direction, the invariance property has emerged as the most important characteristic that could be kept to extend the definition of LNMs to nonlinear regimes. The next sections will review how these ideas have emerged throughout the years and specify the definitions one can use for NNMs in the conservative and dissipative cases.

2.2 Nonlinear normal modes: conservative case

In the conservative case, the eigenspectrum of the vibrating system (1) is composed of pairs of purely imaginary numbers $\{\pm i\omega_j\}_{j=1\ldots N}$, a case that has been of interest in the dynamical system community, see *e.g.* Arnold (1977); Iooss and Adelmeyer (1998). The center theorem from Lyapunov states the existence of two-dimensional manifolds densely filled with periodic orbits, for each couple of imaginary eigenvalues, under the assumption of non-resonance condition (Lyapunov (1907); Kelley (1969); Gordon (1971); Weinstein (1973); Meyer et al. (2015)). These invariant manifolds are generally referred to as *Lyapunov subcenter manifold* (LSM). In the vibration community, the LSMs have been used through their by-product definition as a family of periodic orbits for defining NNMs since the pioneering works by Rosenberg (1962, 1966) and Kauderer (1958). In this framework, an NNM is defined as the corresponding LSM

emanating from small amplitude vibrations embedded in the eigenplanes. This definition has been used by many different investigators, who also used different analytical and numerical approaches for their computation. Analytical methods have been developed by Rand (1974); Pecelli and Thomas (1979); King and Vakakis (1994, 1996); Vakakis et al. (1996); Nayfeh and Nayfeh (1994); Nayfeh (1995); Nayfeh et al. (1996); Nayfeh and Lacarbonara (1997); Nayfeh (1998); Nayfeh et al. (1999); Lacarbonara and Camillacci (2004). Most of these approaches use either perturbative techniques or energy-based considerations. In the more recent years, LSMs have been computed numerically using continuation techniques for periodic orbits (Slater (1996); Arquier et al. (2006); Noreland et al. (2009); Cochelin and Vergez (2009); Kerschen et al. (2009); Peeters et al. (2009); Renson et al. (2016)).

Restricting the definition of an NNM as a family of periodic orbits (without enforcing these orbits to be in the vicinity of the linear vibrations) extends the applicability range and leads to considering the NNM as some isolated family of periodic orbits that can be very far from the fixed point. This broader definition then led to consider that more NNMs than LNMs can exist in a nonlinear system, as exemplified by Vakakis (1997). However, restricting the definition of NNM as LSM in the conservative case allows keeping the continuity with LNMs thanks to the imposed tangency for vanishing amplitudes. Besides, it defines the backbone curve (amplitude-frequency relationship) as a by-product of this operative definition.

This is illustrated in Fig. 2(a), which shows the family of periodic orbits of a nonlinear conservative system that are embedded in the corresponding invariant manifold (LSM). The NNM is shown in the space (u_1, v_1, u_2) and thus corresponds to the first NNM, one can observe the tangency to the linear horizontal plane for small amplitude motions. The corresponding backbone curve of the first NNM is shown as a green line in Fig. 2(c): it displays the amplitude A - frequency nonlinear relationship, which is here of the hardening type. The curve starts exactly at the linear eigenfrequency ω_1 for vanishing amplitudes.

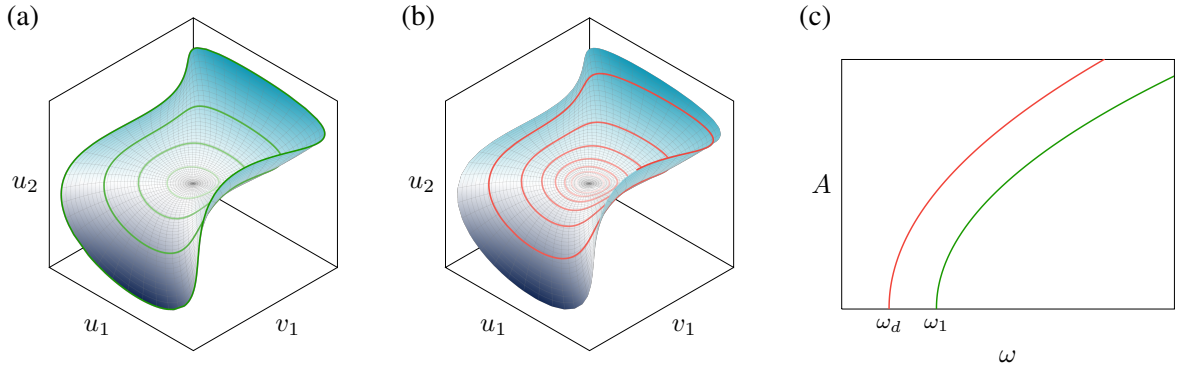

Figure 2: Illustrative representation of a nonlinear normal mode (NNM) as an invariant manifold in phase space. (a) NNM of a conservative system as an LSM, tangent at origin to the corresponding eigenplance, densely filled with periodic orbits, (b) NNM of a dissipative autonomous system as a SSM, with a decaying orbit, (c) Backbone curves of the conservative problem in green, corresponding to (a); and of the dissipative problem in red, corresponding to (b).

2.3 Nonlinear normal modes: dissipative case

In the dissipative case, the eigenvalues have both real and imaginary parts and read, for $j = 1 \dots N$, $\lambda_{j\pm} = -\xi_j\omega_j \pm i\omega_j\sqrt{1-\xi_j^2}$, where ξ_j is the modal damping ratio. The periodic orbits do not exist anymore and each trajectory converges to the rest state. In this context, Shaw and Pierre (1991) defined an NNM as an invariant manifold of the phase space that is tangent at the origin to the linear eigenplanes, which thus reduces to LSM in the conservative case. For their computation, they used the center manifold theorem and the technique used for its demonstration (Carr (1981); Kelley (1967); Carr and Muncaster (1983)) in order to provide efficient, asymptotic techniques (Shaw and Pierre (1993); Shaw (1994);

Pesheck et al. (2001a)) as well as numerical solutions (Pesheck et al. (2002); Jiang et al. (2005a,b)). A normal form approach has also been developed to provide another parametrisation of the same invariant manifolds Touzé et al. (2004); Touzé and Amabili (2006).

However, in this context of dissipative systems, the definition of NNMs as invariant manifolds faces a stringent problem of non-uniqueness, which might lead to computational problems when resorting to numerical techniques. This problem of non-uniqueness has been underlined by Neild et al. (2015); Cirillo et al. (2016); Haller and Ponsioen (2016), highlighting that the structure of the phase space is dominated by strongly decaying modes leading to fast contraction of the flow. As a consequence, there are infinitely many invariant manifolds tangent to any subspace spanned by modes having small damping ratios (see Neild et al. (2015); Haller and Ponsioen (2016); Cirillo et al. (2016) for simple illustrations on linear and nonlinear systems).

The problem of uniqueness has been tackled by Haller and Ponsioen (2016), who introduced Spectral Submanifolds (SSMs) in order to have an unequivocal definition of NNMs for dissipative systems. SSMs are defined as the smoothest nonlinear continuation of a spectral subspace of the linearized system. Existence and uniqueness are then related to classical non-resonance conditions (as is also the case for LSM), but also to the order of the expansion used to provide an asymptotic approximation, as compared to the *spectral quotient*. The spectral quotient is defined as the ratio between the strongest decay rate of the slave modes, divided by the weakest decay rate of the master modes. The main result derived by Haller and Ponsioen (2016) demonstrates that approximations of the unique SSM can be computed by asymptotic expansion of order at least equal to the spectral quotient plus one. All lower-order truncations are approximations not only of the unique SSM but also of all the other invariant manifolds.

The method used to prove uniqueness relies on the parametrisation method for invariant manifolds, a key technique which will be specifically introduced in Section 3. The spectral quotient however imposes a stringent condition on the uniqueness. For most of the problems at hand in nonlinear vibrations of continuous media, this number is very large and might go to infinity. As a consequence, the conditions for achieving uniqueness are almost never met in practice and one shall only rely on approximations of the SSM. Nevertheless, the framework provided by spectral submanifold allows one to define, in the dissipative case, the NNM as the SSM.

An NNM in the dissipative case is illustrated in Fig. 2(b), which shows the invariant manifold (SSM), tangent at origin to the corresponding eigenplane. A damped orbit, which converges to the fixed point representing the structure at rest, is also shown. Finally, the corresponding backbone curve is represented in Fig. 2(c) in red. This damped backbone curve represents in fact the instantaneous oscillation frequency of the decaying orbit which stays in the invariant manifold. For vanishing amplitude A, the backbone starts at the oscillating frequency of the damped system, denoted as $\omega_d = \omega_1\sqrt{1-\xi_1^2}$.

Most problems in nonlinear vibrations are concerned with forced oscillations. In such cases, the NNMs become a time-dependent manifold, which follows the main motion of the fixed points together with deformations during an oscillation. These complications can be taken into account in the definition and computation of NNMs, this will be underlined in Section 3.4.

2.4 Computation of NNMs for model-order reduction: early developments

In this Section, the two methods that were introduced in the 90s-2000s to compute NNMs as invariant manifolds, and their use in the context of model order reduction, are briefly recalled. The two methods are two different parametrisation of the same manifolds, as it will be shown in Section 3, by referring to the *parametrisation method for invariant manifolds*. The first method uses a so-called *graph style* parametrisation and can be more easily introduced by applying the technique provided by the center manifold theorem. The second technique uses the *normal form style* parametrisation.

2.4.1 The graph style parametrisation

To make a simple presentation, restriction to a nonlinear dynamics governed by a single master NNM is here considered. The technique used to compute the NNMs as invariant manifolds by Shaw and Pierre (1991) relies on the center manifold theorem (Carr (1981); Guckenheimer and Holmes (1983); Manneville (1990); Wiggins (2003)), a classical tool in dynamical systems theory, that has been applied in many different fields of physics. Shaw and Pierre (1991) adapted the method to tackle nonlinear vibrating systems. To simplify the presentation, let us also assume that the dynamics of the system, Eq. (1), has been rewritten in the modal space, and let us denote as $\{x_p, y_p\}_{p=1...N}$ the modal coordinates. The system at first order with modal damping ratio reads:

$$\dot{x}_p = y_p \tag{8a}$$
$$\dot{y}_p = -\omega_p^2 x_p - 2\xi_p \omega_p x_p - f_p(x_1, ..., x_N), \tag{8b}$$

with f_p the function grouping the nonlinear terms, assumed as only depending on the displacement for the sake of simplicity. The idea is to assume the existence of a functional relationship between all the slave coordinates labelled s and the master one denoted as m, *i.e.* $\forall\, s \neq m$, there exist two functions U_s and V_s, solely depending on the displacement and velocities of the master coordinates (x_m, y_m), such that

$$x_s = U_s(x_m, y_m), \tag{9a}$$
$$y_s = V_s(x_m, y_m). \tag{9b}$$

At this stage, U_s and V_s are the unknowns. They depend on displacement and also velocity, because neglecting the velocity in such a relationship will result in losing the intrinsic two-dimensional nature of the manifolds. One can also note that Eq. (9) enforces a functional dependence between the master and slave modal variables. This first assumption explains the choice of the name as *graph style* parametrisation.

The methodology to find the unknown functions U_s and V_s consists in deriving Eqs. (9) with respect to time and substitute in the dynamical equations (8) whenever possible in order to eliminate all explicit dependence on time, yielding, $\forall\, s \neq m$:

$$\frac{\partial U_s}{\partial x_m} y_m + \frac{\partial U_s}{\partial y_m}\left(-\omega_m^2 U_m - f_m\right) = V_s(x_m, y_m), \tag{10a}$$
$$\frac{\partial V_s}{\partial x_m} y_m + \frac{\partial V_s}{\partial y_m}\left(-\omega_m^2 U_m - f_m\right) = -\omega_s^2 U_s - f_s. \tag{10b}$$

Eqs. (10) are a set of $2N-2$ partial differential equations depending on the master coordinates (x_m, y_m). They describe the geometry of the two-dimensional invariant manifold in the $2N$-dimensional phase space. The solutions to Eqs. (10) will give the $N-1$ unknown functions (U_s, V_s).

Since Eqs. (10) contains all the nonlinearities of the initial problem, no simple solutions are available. In their first papers, Shaw and Pierre proposed to solve them using asymptotic expansions. For example, a third-order expansion in the real master coordinates, writes:

$$x_s = a_{11}^s x_m^2 + a_{12}^s x_m y_m + a_{22}^s y_m^2 + b_{111}^s x_m^3 + b_{112}^s x_m^2 y_m + b_{122}^s x_m y_m^2 + b_{222}^s y_m^3, \tag{11a}$$
$$y_s = \alpha_{11}^s x_m^2 + \alpha_{12}^s x_m y_m + \alpha_{22}^s y_m^2 + \beta_{111}^s x_m^3 + \beta_{112}^s x_m^2 y_m + \beta_{122}^s x_m y_m^2 + \beta_{222}^s y_m^3. \tag{11b}$$

Explicit expressions for the introduced coefficients have been given by Pesheck et al. (2001a) for both single-mode and multi-mode expansions. The reduced dynamics on the invariant manifold is found by substituting the functional relationships (9) into the equation of motion for the master mode m :

$$\ddot{x}_m + \omega_m^2 x_m + f_m(U_1(x_m, y_m), ..., x_m, y_m, ..., U_N(x_m, y_m)) = 0. \tag{12}$$

With the expressions of the coefficients given by Pesheck et al. (2001a), Eq. (12) can be explicitly written up to order three as:

$$\ddot{x}_m + \omega_m^2 x_m + g_{mm}^m x_m^2 + h_{mmm}^m x_m^3 + x_m \left(\sum_{\substack{s=1 \\ s \neq m}}^{N} 2\, g_{ms}^m g_{mm}^s \left[\frac{2\omega_m^2 - \omega_s^2}{\omega_s^2(\omega_s^2 - 4\omega_m^2)} x_m^2 + \frac{2}{\omega_s^2(\omega_s^2 - 4\omega_m^2)} y_m^2 \right] \right) = 0. \tag{13}$$

In subsequent developments, Pesheck et al. (2002) proposed to solve (10) by ad-hoc numerical methods (see also Pesheck et al. (2001b); Jiang et al. (2005a)), and then extended the method to harmonically forced vibrating systems, thus considering time-dependent invariant manifolds (Jiang et al. (2005b)). Along the same lines, different numerical procedures have been proposed by Noreland et al. (2009); Blanc et al. (2013); Renson et al. (2014) to solve the nonlinear PDEs of the invariant manifold.

2.4.2 The normal form approach

Normal form theory has been first introduced in the pioneering works by Poincaré (1892) and Dulac (1912). It is a classical tool in the study of dynamical systems, that is most often used in order to simplify local dynamics and propose the simplest system displaying a given codimension bifurcation Guckenheimer and Holmes (1983); Manneville (1990); Wiggins (2003). It has been first introduced in the vibration community by Jézéquel and Lamarque (1991), who already underlined the link with NNMs.

The fundamental idea of the normal form approach is to simplify as much as possible a given dynamical system, in the vicinity of a fixed point (or a periodic orbit), by introducing a nonlinear change of coordinates. The calculation method is sequential and works order by order, trying to cancel as much as possible monomials of a given degree. The main outcome is summarized in the theorems by Poincaré and Poincaré-Dulac. If no resonance relationship exists between the eigenvalues of the problem, then a nonlinear change of coordinate can be derived such that the resulting problem with the new variables is linear. On the other hand, if resonances exist, then all the non-resonant monomials can be cancelled by a nonlinear mapping, and the resulting dynamical system will be simplified, containing only the resonant monomials (Poincaré (1892); Dulac (1912)).

Resonance relationships between a set of N eigenvalues $\{\lambda_p\}_{p=1\ldots N}$ are defined as commensurability relations reading:

$$\lambda_j = \sum_{i=1}^{N} m_i \lambda_i, \quad m_i \geq 0, \quad \sum_{i=1}^{N} m_i = k, \tag{14}$$

where k is the degree of the monomial retained. Such a relationship must be interpreted as a nonlinear resonance check between the linear part of the dynamics, driven by the eigenvalue λ_j, and the monomial of order k defined by $x_1^{m_1} x_2^{m_2} \ldots x_N^{m_N}$. Such an interpretation makes explicit how one defines a resonant monomial from the resonance relationship.

In most mathematical textbooks, the nonlinear change of coordinates provided by normal form is N to N, without any idea of reduction. As noted by Haro et al. (2016), partial computations of normal form with fewer new unknowns as starting coordinates can be defined, providing a parametrisation of the underlying invariant manifold. This idea has been introduced in the nonlinear vibration community to define effective model order reduction techniques with NNMs as invariant manifold computed with a normal form parametrisation (Touzé et al. (2004); Touzé and Amabili (2006)), for systems with geometric nonlinearity. Starting from the equations of motion in the modal space, written as Eq. (8), a complete nonlinear change of coordinate can be defined, $\forall\, p = 1 \ldots N$, as:

$$x_p = R_p + \mathcal{P}(R_p, S_p), \tag{15a}$$

$$y_p = S_p + \mathcal{Q}(R_p, S_p). \tag{15b}$$

The newly introduced coordinates (R_p, S_p) are called the *normal coordinates*. In the calculations shown in Touzé et al. (2004); Touzé and Amabili (2006), a specific variant of the normal form approach, later called *oscillator normal form*, is used. The idea is to keep a block-diagonal linear part without complexification, using the anti-diagonal blocks

$$\begin{pmatrix} 0 & 1 \\ -\omega_p^2 & 0 \end{pmatrix} \tag{16}$$

instead of a complete diagonalization using $\mathbf{L} = \text{diag}\{\pm i\omega_p\}$ as recommended by the mathematics. This choice comes with many implications that will be further discussed in Section 3.2. In Eqs. (15), one can note that the nonlinear change of coordinate is identity-tangent. $\mathcal{P}$ and $\mathcal{Q}$ are third-order polynomials whose explicit expressions are detailed in Touzé et al. (2004) for the conservative case, and in Touzé and Amabili (2006) for the dissipative case.

With the normal coordinates, the dynamics is expressed within an invariant-based span of the phase space, such that proper truncations can now be realized. Reducing the dynamics to a single master mode is then easy, one has just to select (R_p, S_p) as the master coordinates and cancel out all the other ones: $\forall\, j \neq p, R_j = S_j = 0$. This represents a different perspective from the techniques using the center manifold technique. In the latter case, one first selects the master coordinates, and expresses the slave modal coordinates as a function of the masters, Eq. (9). In the normal form approach, one first computes the complete nonlinear change of coordinates, then reduces by selecting a subset of master normal coordinates. A closer comparison between the two approaches is given by Touzé et al. (2021), underlining that both methods lead to an equivalent representation of the dynamics in terms of NNMs as invariant manifolds, but using different parametrisation. This link is fully formalized thanks to the parametrisation method for invariant manifold, which will be discussed in detail in the next Section.

3 The direct parametrisation method for invariant manifolds: application to vibratory systems

This Section is the core of the present chapter. The parametrisation method for invariant manifold will be first introduced in a general setting. Then, the application of the technique to nonlinear vibrating systems, taking into account the peculiarity of nonlinear vibrations, is detailed.

3.1 The parametrisation method for invariant manifolds

The parametrisation method has been first introduced in a very abstract setting by Cabré et al. (2003a,b, 2005). Generally speaking, it is a technique that allows one to prove the existence and uniqueness of invariant manifolds, which produces, at the same time, effective computational schemes based on asymptotic polynomial expansions. It generalizes the classical stable and unstable invariant manifold theorems in the neighbourhood of a fixed point or a limit cycle. For model order reduction, it is a powerful method that provides accurate high-order nonlinear mappings and reduced dynamics, and which clearly links approaches using either center manifold of normal form, showing that they correspond to two different parametrisation styles to solve out the same invariance equation. In the dynamical systems community, it has been used in many different contexts, see for example van den Berg et al. (2016); van den Berg and James (2016); Castelli et al. (2015); Gonzalez et al. (2022), to cite just a few. Haro et al. (2016), in an outstanding monograph, proposes an overview of the method, with a special emphasis on computational developments and applications of the technique. In the remainder of this Section, we will use their notation, and closely follow their presentation to explain its working principle for dynamical solutions around a given fixed point.

Let us consider the general case of an autonomous dynamical system:

$$\dot{\mathbf{z}} = \mathbf{F}(\mathbf{z}), \tag{17}$$

with $\mathbf{z} = [z_1, ..., z_n]^t$ a n-dimensional vector in the phase space $\mathbb{C}^n$. Let us assume that a fixed point $\mathbf{z}_\star$ exists, such that $\mathbf{F}(\mathbf{z}_\star) = \mathbf{0}$. The method is *local* in the sense that one is interested in analyzing the dynamics in the vicinity of the fixed point $\mathbf{z}_\star$. The linearized dynamics, assuming $\mathbf{z} = \mathbf{z}_\star + \mathbf{v}$, where $\mathbf{v}$ represents a small perturbation, writes $\dot{\mathbf{v}} = D\mathbf{F}(\mathbf{z}_\star)\mathbf{v}$. The n eigenvalues of $D\mathbf{F}(\mathbf{z}_\star)$ are denoted as $(\lambda_1, \ldots, \lambda_n)$, with $(\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_n)$ the associated eigenvectors.

The goal of the parametrisation method for invariant manifolds is to provide both a nonlinear mapping and a reduced dynamics along a selected invariant manifold $\mathcal{W}$ that can be viewed as the nonlinear continuation of an eigenspace V^L of small dimension. Let us consider that the d-dimensional linear subspace $V^L \subset \mathbb{C}^n$, from which one would like to compute its associated nonlinear invariant manifold, is spanned by d master modes:

$$V^L = \text{span}(\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_d), \quad \text{with} \quad d \ll n. \tag{18}$$

In the same lines, the associated $n \times d$ matrix of the master eigenvectors is denoted as $\mathbf{L}$:

$$\mathbf{L} = [\boldsymbol{\phi}_1 \quad \boldsymbol{\phi}_2 \ \ldots \ \boldsymbol{\phi}_d], \tag{19}$$

and $\boldsymbol{\Lambda}_L$ stands for the diagonal matrix containing the master eigenvalues

$$\boldsymbol{\Lambda}_L = \text{diag}(\lambda_1, \ldots, \lambda_d).$$

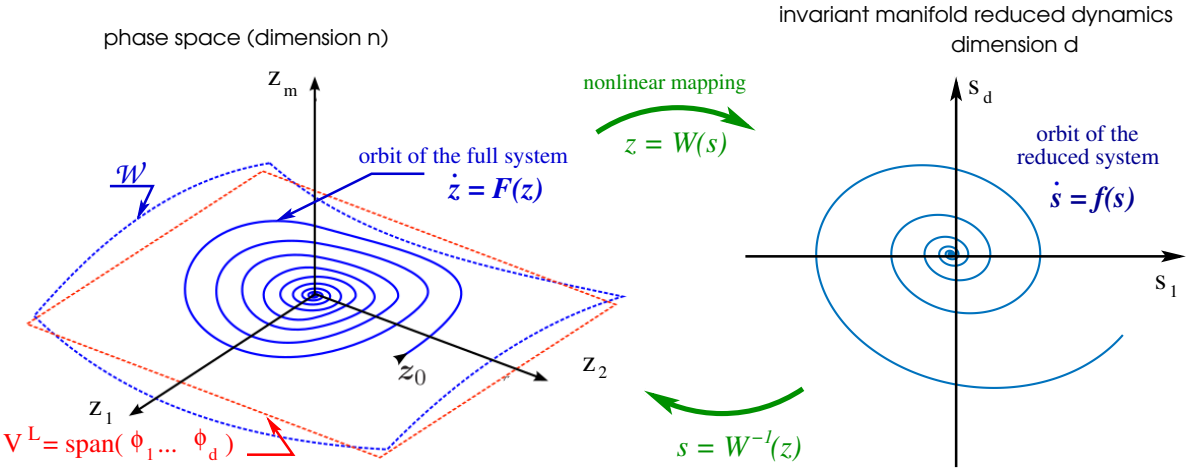

Figure 3: Schematical representation of the parametrisation method for invariant manifolds. Left: view in the n-dimensional original phase space. Right: view in the reduced subspace $\mathcal{W}$ (invariant manifold) with dimension $d \ll n$.

The situation is sketched in Fig. 3.1. A high-order approximation of the d-dimensional invariant manifold $\mathcal{W}$, that is tangent to V^L at $\mathbf{z}_\star$, is searched for, together with the corresponding reduced dynamics. To that purpose, the two main unknowns are introduced. The first one is the nonlinear mapping between the original $\mathbf{z}$ coordinate, and a new *normal* coordinate $\mathbf{s}$ (with dimension d), describing the dynamics along the embedding:

$$\mathbf{z} = \mathbf{W}(\mathbf{s}). \tag{20}$$

The second unknown is the reduced dynamics along the invariant manifold:

$$\dot{\mathbf{s}} = \mathbf{f}(\mathbf{s}). \tag{21}$$

The main feature of the parametrisation method is to introduce as a starting point the so-called *invariance equation*, which states that the sought manifold is indeed invariant: any trajectory of the

reduced dynamics (21) stays in $\mathcal{W}$ for any time t. The invariance equation is found by differentiating (20) with respect to time, replacing in the original equation (17), and using (21) to eliminate time. One easily arrives at

$$\mathbf{F}(\mathbf{W}(\mathbf{s})) = D\mathbf{W}(\mathbf{s})\mathbf{f}(\mathbf{s}). \tag{22}$$

To solve for this equation, the two unknowns are expanded as polynomial over the normal coordinate $\mathbf{s}$:

$$\mathbf{W}(\mathbf{s}) = \mathbf{z}_\star + \mathbf{L}\mathbf{s} + \sum_{k\geq 2} \mathbf{W}_k(\mathbf{s}), \tag{23a}$$

$$\mathbf{f}(\mathbf{s}) = \mathbf{\Lambda}_L\mathbf{s} + \sum_{k\geq 2} \mathbf{f}_k(\mathbf{s}). \tag{23b}$$

In Eqs. (23), the linear terms have been set apart and explicitly written at this stage since the tangency conditions impose their values. The linear part of the reduced dynamics is thus completely governed by the eigenvalues, while the linear term of the nonlinear mapping recovers the usual modal projection. The nonlinear terms in both expansions will thus take care of the departure from the linear solution that is brought about by the nonlinear dynamics of the original system, which will be computed order by order by solving for each unknown term of order k: $\mathbf{W}_k(\mathbf{s})$ and $\mathbf{f}_k(\mathbf{s})$, that gathers all the monomials of order k.

The invariance equation (22) is solved recursively. Assuming order $k-1$ is known, one can gather all the order k terms from Eq. (22) by simply rewriting it as:

$$[D\mathbf{W}(\mathbf{s})\mathbf{f}(\mathbf{s})]_k = [\mathbf{F}(\mathbf{W}(\mathbf{s}))]_k \tag{24}$$

where the notation $[\cdot]_k$ is introduced to mean that only order k of the given term is selected. Eq. (24) is generally referred to as the co-homological equation of order k (Haro et al. (2016)). Expanding the different terms, this equation can be rewritten, following Haro et al. (2016), as:

$$D\mathbf{F}(\mathbf{z}_\star)\mathbf{W}_k(\mathbf{s}) - \mathbf{L}\mathbf{f}_k(\mathbf{s}) - D\mathbf{W}_k(\mathbf{s})\mathbf{\Lambda}_L\mathbf{s} = -\mathbf{E}_k(\mathbf{s}), \tag{25}$$

where the unknown terms (namely $\mathbf{W}_k$ and $\mathbf{f}_k$) have been gathered on the left-hand side, while the known terms have been set to the right-hand side and grouped together in $\mathbf{E}_k$, which reads:

$$\mathbf{E}_k(\mathbf{s}) = [\mathbf{F}(\mathbf{W}_{<k}(\mathbf{s}))]_k - [D\mathbf{W}_{<k}(\mathbf{s})f_{<k}(\mathbf{s})]_k\,. \tag{26}$$

These known terms involve lower-order solutions that are assumed to be computed in the previous steps. This is made explicit by using the shortcut notation $\mathbf{W}_{<k}(\mathbf{s})$ which selects all terms of order strictly smaller than k.

Explicit analytical solutions to the homological equations can be written in the modal space up to arbitrary order. To that purpose, let us introduce the following notations that are needed to pass from the initial coordinates to the modal space. $\mathbf{P}$ and $\mathbf{\Lambda}$ respectively denote the full matrix of eigenvectors and the full $n \times n$ matrix of eigenvalues, such that:

$$\mathbf{\Lambda} = \mathbf{P}^{-1}D\mathbf{F}(z_\star)\mathbf{P} = \begin{bmatrix} \mathbf{\Lambda}_L & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_N \end{bmatrix}, \tag{27}$$

where $\mathbf{\Lambda}_L$ and $\mathbf{\Lambda}_N$ repsectively refers to the master and slave eigenvalues. The order-k coefficients of the nonlinear mapping in the modal space are denoted as $\boldsymbol{\xi}_k(\mathbf{s})$:

$$\boldsymbol{\xi}_k(\mathbf{s}) = \mathbf{P}^{-1}\mathbf{W}_k(\mathbf{s}). \tag{28}$$

Finally, the projection of the right-hand side known term $\mathbf{E}_k(\mathbf{s})$ is denoted as $\boldsymbol{\eta}_k(\mathbf{s}) = -\mathbf{P}^{-1}\mathbf{E}_k(\mathbf{s})$. Left-multiplying Eq. (25) by $\mathbf{P}^{-1}$ operates the projection to modal space and reads:

$$\mathbf{\Lambda}\boldsymbol{\xi}_k(\mathbf{s}) - \begin{bmatrix} \mathbf{I}_d \\ \mathbf{0} \end{bmatrix} \mathbf{f}_k(\mathbf{s}) - D\boldsymbol{\xi}_k(\mathbf{s})\mathbf{\Lambda}_L\mathbf{s} = \boldsymbol{\eta}_k(\mathbf{s}), \tag{29}$$

where $\mathbf{I}_d$ stands for the $d \times d$ identity matrix. Eq. (29) is underdetermined since two unknowns, namely the reduced dynamics coefficients $\mathbf{f}_k(\mathbf{s})$ and the nonlinear mapping term $\boldsymbol{\xi}_k(\mathbf{s})$, are present. Nevertheless, the special structure of this equation and in particular the second term which makes appear a matrix of zeros that discard the presence of $\mathbf{f}_k(\mathbf{s})$ in the last $n-d$ lines, calls for a separate solution distinguishing the d first lines to the $n-d$ last. The d first lines correspond to the master coordinates and are called the tangent part of the co-homological equation. On the other hand, the $n-d$ last lines correspond to the slave coordinates, and are called the normal part. Separating the unknown terms of the nonlinear mapping term projected in the modal space $\boldsymbol{\xi}_k(\mathbf{s})$ as

$$\boldsymbol{\xi}_k(\mathbf{s}) = \begin{bmatrix} \boldsymbol{\xi}_k^L(\mathbf{s}) \\ \boldsymbol{\xi}_k^N(\mathbf{s}) \end{bmatrix}, \tag{30}$$

such that $\boldsymbol{\xi}_k^L$ refers to the master tangent part and $\boldsymbol{\xi}_k^N$ to the normal part, one can see that the normal part of the equation is not underdetermined anymore and reads:

$$\boldsymbol{\Lambda}_N \boldsymbol{\xi}_k^N(\mathbf{s}) - D\boldsymbol{\xi}_k^N(\mathbf{s})\boldsymbol{\Lambda}_L \mathbf{s} = \boldsymbol{\eta}_k^N(\mathbf{s}). \tag{31}$$

Only the tangent part is underdetermined:

$$\boldsymbol{\Lambda}_L \boldsymbol{\xi}_k^L(\mathbf{s}) - D\boldsymbol{\xi}_k^L(\mathbf{s})\boldsymbol{\Lambda}_L \mathbf{s} - \mathbf{f}_k(\mathbf{s}) = \boldsymbol{\eta}_k^L(\mathbf{s}) \tag{32}$$

In order to derive explicit analytical expressions for the solutions to Eqs. (31)-(32), an effective representation of all the monomials of order k that constitute the polynomial terms manipulated in the equations, needs to be used. Following Haro et al. (2016); Reed and Simon (1980), the multi-index notation is introduced for each term of the unknown vectors $\boldsymbol{\xi}_k(\mathbf{s}) = \left[\xi_k^1(\mathbf{s}),\ \ldots,\ \xi_k^n(\mathbf{s})\right]^T$ and $\mathbf{f}_k(\mathbf{s}) = \left[f_k^1(\mathbf{s}),\ \ldots,\ f_k^d(\mathbf{s})\right]^T$, as well as for the known terms of the left-hand side $\boldsymbol{\eta}_k(\mathbf{s}) = \left[\eta_k^1(\mathbf{s}),\ \ldots,\ \eta_k^n(\mathbf{s})\right]^T$. For example, the polynomial term of the i-th coordinate $\xi_k^i(\mathbf{s})$, which is a polynomial term containing only order k terms in the indeterminate $\mathbf{s}$ of dimension d, can be written

$$\xi_k^i(\mathbf{s}) = \sum_{\mathbf{m}} \xi_{\mathbf{m}}^i s_1^{m_1} s_2^{m_2} \ldots s_d^{m_d}, \tag{33}$$

i.e. a summation on all the order-k monomials $\xi_{\mathbf{m}}^i s_1^{m_1} s_2^{m_2} \ldots s_d^{m_d}$, where $\mathbf{m} = [m_1\ m_2\ \ldots m_d]$ gathers the powers of each of the d variables, which is such that $|\mathbf{m}| = m_1 + m_2 + \ldots + m_d = k$; and $\xi_{\mathbf{m}}^i$ is the scalar coefficient of the monomial.

To write down the explicit solutions, a dedicated development needs to be written for the derivative term $D\boldsymbol{\xi}_k(\mathbf{s}) = \left[\frac{\partial \xi_k^i}{\partial s_j}\right]_{(i,j)}$. A generic term in this matrix reads:

$$\frac{\partial \xi_k^i}{\partial s_j} = \sum_{\mathbf{m}} \xi_{\mathbf{m}}^i m_j s_1^{m_1} s_2^{m_2} \ldots s_j^{m_j - 1} \ldots s_d^{m_d} \tag{34}$$

In the normal cohomological equation, this term is multiplied by $\lambda_j s_j$. Hence, one naturally recovers the original monomial $s_1^{m_1} s_2^{m_2} \ldots s_d^{m_d}$, such that Eq. (31) can be solved at the level of an arbitrary monomial in the summation. The line $i \in [d+1, \ldots, n]$ of Eq. (31) can thus be written easily as:

$$\left(\lambda_i - \mathbf{m}.\boldsymbol{\lambda}_L\right) \xi_{\mathbf{m}}^i = \eta_{\mathbf{m}}^i, \tag{35}$$

where the notation $\mathbf{m}.\boldsymbol{\lambda}_L = m_1\lambda_1 + \ldots + m_d\lambda_d = \sum_{j=1}^d m_j\lambda_j$ has been introduced for convenience. To solve for the unknown coefficient $\xi_{\mathbf{m}}^i$, $i \in [d+1, \ldots, n]$, the following discussion holds:

- If there is no cross-resonance (or *outer resonance*), *i.e.*:

$$\forall\, i \in [d+1, \ldots, n\,], \quad \lambda_i \neq m_1\lambda_1 + \ldots + m_d\lambda_d, \tag{36}$$

then the normal part of the co-homological equation bears a unique solution:

$$\xi^i_{\mathbf{m}} = \frac{\eta^i_{\mathbf{m}}}{\lambda_i - \mathbf{m}.\boldsymbol{\lambda}_L}.$$

This holds as long as there is no resonance relationship between a slave eigenvalue and the set of master eigenvalues.

- If there is a cross-resonance, then Eq. (35) has no solution. As a matter of fact, this is a logical situation. Indeed, the set of master modes is selected on the basis that they do not share any cross-resonance with the slaves. In such a case, one knows that there is a strong nonlinear coupling between the coordinates that cannot be neglected, such that trying to reduce the dynamics without incorporating a slave mode that has a resonance relationship with a master is meaningless. If such a resonance exists, then the slave mode must be included in the set of master modes.

Let us now solve the tangent part of the co-homological equation, Eq. (32), which can also be easily rewritten at the level of an arbitrary monomial, as:

$$\forall\, i = 1, \ldots,\ d, \qquad (\lambda_i - \mathbf{m}.\boldsymbol{\lambda}_L)\, \xi^i_{\mathbf{m}} - f^i_{\mathbf{m}} = \eta^i_{\mathbf{m}}. \tag{37}$$

This scalar equation has two unknowns and might be ill-conditioned in the case where a resonance between the master eigenvalues exists. Indeed, if for some $i \in [1, \ldots, d]$,

$$\lambda_i = \mathbf{m}.\boldsymbol{\lambda}_L = m_1\lambda_1 + \ldots + m_d\lambda_d, \tag{38}$$

then the nonlinear mapping coefficient $\xi^i_{\mathbf{m}}$ disappears from Eq. (37) such that one has no choice anymore on the coefficient of the reduced dynamics $f^i_{\mathbf{m}}$. This means in particular that the simplest choice $f^i_{\mathbf{m}} = 0$ cannot be selected and the reduced dynamics cannot be linearized.

Because of the underdeterminacy, an infinity of solutions exists. Each solution corresponds to a different parametrisation of the same invariant manifold, thus explaining the choice of the name for the method. Each of these solutions is then called a style of parametrisation. Importantly, in between all these styles, only two are relevant and meaningful. They lead to two opposite ways of solving the tangent part. All the other styles can be considered as variants. The two main styles are called the *graph style* and the *normal form style*. Let us now explain how they are derived and what are their main properties.

The graph style is characterized by vanishing the coefficients of the nonlinear mapping in the tangent part. The aim is thus to simplify as much as possible the nonlinear mapping. Setting $\forall\, i = 1 \ldots\ d, \forall\, \mathbf{m}$ such that $|\mathbf{m}| = k$: $\xi^i_{\mathbf{m}} = 0$, substantially simplifies Eq. (37), which has then an explicit solution for the coefficient of the reduced dynamics reading: $f^i_{\mathbf{m}} = -\eta^i_{\mathbf{m}}$. A direct consequence of this choice is that the possible resonances are not taken into account and disappear from the analysis.

This parametrisation has been named graph style because of the simple relationship that exists between the introduced normal coordinates and the master modal coordinates. To explain this point, let us introduce $\mathbf{y}$ the coordinate in the modal space, which is related to $\mathbf{z}$ via $\mathbf{z} = \mathbf{P}\mathbf{y}$. The nonlinear mapping equation (23a) writes in the modal space:

$$\mathbf{y} = \mathbf{y}_\star + \mathbf{P}^{-1}\mathbf{L}\mathbf{s} + \sum_{k\geq 2} \boldsymbol{\xi}_k(\mathbf{s}) \tag{39}$$

Assuming for the sake of simplicity that the fixed point under study is at the origin of the phase space, such that $\mathbf{y}_\star = 0$, and using the choice $\xi^i_{\mathbf{m}} = 0$ for $i = 1 \ldots\ d$, for all orders $k \geq 2$, one immediately understands that the first d lines of the nonlinear mapping condense to a simple identity: for $i = 1 \ldots\ d$, $y_i = s_i$. This means that the normal variables used to describe the invariant manifold and the reduced dynamics with the graph style parametrisation are nothing else than the modal master coordinates

$\{y_j\}_{j=1,\dots,d}$. Hence the nonlinear mapping is written as a graph over the modal master coordinates, and for each of the slave coordinates one can simply write:

$$\forall\, j = d+1 \dots n, \quad y_j = \sum_{k\geq 2} \xi_k^j(\mathbf{s}) = U_j(y_1, \dots, y_d). \tag{40}$$

One then understands that this style of parametrisation is completely equivalent to the calculations led in the field of reduction methods where the technique of the center manifold theorem was used. Since the first step is to assume such a relationship (compare for example Eqs. (40) and (9)), then equivalent results are derived. As a consequence in the field of nonlinear vibrations, the methods used by Shaw and Pierre in their work to compute NNMs are equivalent to using the parametrisation method for invariant manifold with the graph style. It also follows from this interpretation that such parametrisation is not able to follow a folding of the invariant manifold, by definition of searching for a graph over the master modal coordinates.

The normal form style makes the opposite selection and is characterized by vanishing the coefficients of the reduced dynamics in the tangent part as long as this is possible. Looking back to Eq. (37), one immediately sees that vanishing $f_{\mathbf{m}}^i$ is possible if and only if no resonance condition is met. Hence the normal style parametrisation is characterized by the following choice:

- if $\lambda_i \neq \mathbf{m}.\boldsymbol{\lambda}_L$, then one selects:

$$f_{\mathbf{m}}^i = 0, \quad \xi_{\mathbf{m}}^i = \frac{\eta_{\mathbf{m}}^i}{\lambda_i - \mathbf{m}.\boldsymbol{\lambda}_L}. \tag{41}$$

- if $\lambda_i \simeq \mathbf{m}.\boldsymbol{\lambda}_L$, then one selects:

$$\xi_{\mathbf{m}}^i = 0, \quad f_{\mathbf{m}}^i = -\eta_{\mathbf{m}}^i. \tag{42}$$

This style of parametrisation has been named the normal form style due to the number of similarities it shares with the original normal form theory as developed by Poincaré. Indeed, if $d = n$ (meaning that no reduction is done), the normal form style solutions lead exactly to the same computation of the classical Poincaré normal form. The normal form style can thus be interpreted as a partial normal form derivation, solely on the master coordinates, thus operating in the same calculation as a reduction. When no resonance exists, the choice (41) leads to a linear dynamics on the reduced invariant manifold, following Poincaré theorem. On the other hand, if a resonance exists, then the choice (42) translates the fact that a strong coupling between the master coordinates exists, such that the reduced dynamics cannot be linearized, following the results of the Poincaré-Dulac theorem.

In the field of nonlinear vibration, this development explicitly shows that the computation of NNMs using a normal form approach as proposed in (Touzé et al. (2004); Touzé and Amabili (2006)) is completely equivalent to using the parametrisation method with a normal form style. In particular, one can note that with this style, the normal coordinates are now nonlinearly related to the master modal coordinates. In particular, such a choice is theoretically able to pass over the potential folding of the invariant manifold.

3.2 Application to nonlinear vibrating systems

For the sake of simplicity, the starting point considered here is that of a continuous structure, discretized by the finite element procedure, and containing geometric nonlinearity. This choice is here retained in order to provide an existing framework where one can apply the parametrisation method in its full realm. However, the method can handle easily any type of smooth nonlinearity and can thus be adapted to different contexts. The equations of motion are given by Eqs. (1)-(2).

Some preliminary remarks are worth mentioning before entering the detailed calculations. Eq. (1) is second-order in time. This has important consequences. To fit with the parametrisation method developed

in Section 3.1, the problem needs to be recast to a first-order dynamical system. This is easily realized by adding the velocities as independent variables. This underlines that both displacements and velocities need to be taken into account in all dependencies, both for the nonlinear mapping and the reduced dynamics. This fact is sometimes overlooked in numerous reduction methods where static approximations are explicitly or implicitly used, see for example Vizzaccaro et al. (2021a); Shen et al. (2021a); Touzé et al. (2021) and references therein.

Another consequence of this fact is related to the appearance of eigenvalues by conjugate pairs. For a damped linear vibrating systems of dimension N, the N pairs of eigenvalues are, $\forall p \in [1, N]$: $\lambda_\pm = -\xi_p\omega_p \pm i\omega_p\sqrt{1-\xi_p^2}$, where ξ_p refers to the modal damping ratio for simplicity. For a conservative problem, eigenvalues come by pairs of purely complex conjugates $\lambda_\pm = \pm i\omega_p$. Again, the pairs of eigenvalues are the consequence of the fact that displacements and velocities are intimately related and need to be kept together to produce a modal motion in an eigenplane of dimension 2.

Most of the applications consider lightly damped systems, such that the imaginary part of the eigenvalue is more important than the real part. Most of the studies on NNMs also consider the conservative case, and losses are sometimes added as a small perturbation. This complex spectrum comes with two important consequences. First, since complex numbers are at hand, the calculations need to operate a complexification, and then at the end of the process, a realification is needed to come back to real quantities. This process is discussed for example by Haro et al. (2016); Vizzaccaro et al. (2022). It is important to note that the calculations should be transparent to this constraint. Second, trivial resonances exist for conservative problems. This problem has long been analyzed for vibrating systems, see *e.g.* Touzé et al. (2004) and references therein.

A nonlinear resonance occurs when a commensurability relationship between the eigenvalues exists taking the generic form given by Eq. (14). When the eigenspectrum is composed of pairs of purely imaginary complex conjugates, trivial resonances occur for each odd order k of the nonlinearity in (14). For example for $k = 3$, the following relationship: $+i\omega_j = +i\omega_p - i\omega_p + i\omega_j$ is always fulfilled. For lightly damped systems, it is assumed that the damping is small such that the near-zero value of such trivial resonance is sufficient to enforce resonance (small denominator problem). The main consequence is that, when operating a normal form transform, odd orders (cubic, quintic, ...) cannot be cancelled and will stay in the normal form in any case. From the mathematical point of view, this is not good news since the calculations for constructing NNMs will be more cumbersome. From the physical point of view, this is however good news since continuous structures are known to have a frequency dependence with amplitude (backbone curve), which is indeed dictated by the coefficients in front of the odd coefficients of the normal form. The interested reader is referred to Touzé (2014); Touzé et al. (2004); Touzé and Amabili (2006), for discussions related to this specific point of trivial resonance.

Due to the complexification procedure, variants of normal forms have been introduced in the past for vibrating systems. In order to explain how these variants are related and how they can be understood in the light of the parametrisation method, the case of the Duffing oscillator is considered. Even if this discussion is specific to vibratory systems with its eigenspectrum composed of purely imaginary complex conjugates, it falls in the general discussions related to the degrees of freedom one can have when deriving normal forms, see *e.g.* the discussions on *free functions* in Kahn and Zarmi (2014).

The conservative Duffing oscillator is characterized by its natural frequency ω_0 and the coefficient h of the nonlinearity:

$$\ddot{u} + \omega_0^2 u + hu^3 = 0. \tag{43}$$

To rewrite the system with a first-order diagonal linear part, the following linear change of coordinate can be applied

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ i\omega_0 & -i\omega_0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \tag{44}$$

where $v = \dot{u}$ has been introduced. Note that other linear transforms with different normalizations can be applied here. Eq. (43) then reads

$$\dot{y}_1 = i\omega_0 y_1 + i\frac{h}{2\omega_0}\left(y_1^3 + 3y_1^2 y_2 + 3y_1 y_2^2 + y_2^3\right), \tag{45a}$$

$$\dot{y}_2 = -i\omega_0 y_1 - i\frac{h}{2\omega_0}\left(y_1^3 + 3y_1^2 y_2 + 3y_1 y_2^2 + y_2^3\right). \tag{45b}$$

With this linear step, complexification has been realized, and the dynamics write as two coupled nonlinear equations for (y_1, y_2), and the only cubic monomial hu^3 in Eq. (43) gave birth to 8 monomials in Eqs. (45). Eq. (45b) is the complex conjugate of (45a): this is the direct consequence of the fact that the initial problem, Eq. (43), is second-order in time.

The normal form procedure can be unfolded on Eq. (45). It starts with the introduction of a nonlinear change of coordinates between modal coordinates (y_1, y_2) and normal coordinates (z_1, z_2) as

$$y_1 = z_1 + a_{11}z_1^3 + a_{12}z_1^2 z_2 + a_{13}z_1 z_2^2 + a_{14}z_2^3 \tag{46a}$$

$$y_2 = z_2 + a_{21}z_1^3 + a_{22}z_1^2 z_2 + a_{23}z_1 z_2^2 + a_{24}z_2^3. \tag{46b}$$

The normal dynamics is also introduced as

$$\dot{z}_1 = i\omega_1 z_1 + f_{11}z_1^3 + f_{12}z_1^2 z_2 + f_{13}z_1 z_2^2 + f_{14}z_2^3, \tag{47a}$$

$$\dot{z}_2 = -i\omega_1 z_2 + f_{21}z_1^3 + f_{22}z_1^2 z_2 + f_{23}z_1 z_2^2 + f_{24}z_2^3, \tag{47b}$$

where the linear part is preserved because the nonlinear change of coordinates (46) is identity-tangent. Details of this calculation are reported in de Figueiredo Stabile et al. (2024), here only the main conclusions needed to understand the different variants of normal form, are explained. Three different variants will be detailed. The *complex normal form* (CNF) is, in some sense, the most canonical choice, recommended by mathematicians. However, two different variants have been developed in vibration theory, mostly in order to ease the comeback to real coordinates and the interpretability of the obtained results. The *real normal form* (RNF), first introduced by Neild and Wagg (2011); Neild et al. (2015), broadens the interpretability of the resonance relationship to gain more symmetries in the normal form and ease the return to oscillator-like equations. Finally, the *oscillator normal form* (ONF), first introduced by Touzé et al. (2004), proposes a calculation that never uses complex entries, and is thus prone to keep oscillator-like equations throughout the process.

3.2.1 The complex normal form (CNF)

The complex normal form style (CNF) is the classical treatment proposed in mathematical textbooks to deal with purely imaginary complex eigenspectrum, see *e.g.* Iooss (1988); Haragus and Iooss (2009); Wiggins (2003); Jézéquel and Lamarque (1991). Referring to the simple case of the Duffing equation, it amounts to cancelling the six non-resonant monomials in Eq. (45). This follows from a strict interpretation of the resonance relationship stemming from the homological equations at each order, recalled in Eq. (14). For conservative mechanical systems, trivial resonance relationships appear at each odd order in the normal form computation. Focusing on the simple case of the Duffing equation with cubic nonlinearity, an interesting feature of the CNF is that, for each odd order, only one resonant monomial stays in the CNF, as a consequence of these trivial resonances. As an illustration, we give below the CNF for the Duffing equation up to order 7, which reads:

$$\dot{z}_1 = i\omega_0 z_1 + i\frac{3h}{2\omega_0}z_1^2 z_2 - i\frac{51h^2}{2^4\omega_0^3}z_1^3 z_2^2 + i\frac{1419h^3}{2^7\omega_0^5}z_1^4 z_2^3, \tag{48a}$$

$$\dot{z}_2 = -i\omega_0 z_2 - i\frac{3h}{2\omega_0}z_1 z_2^2 + i\frac{51h^2}{2^4\omega_0^3}z_1^2 z_2^3 - i\frac{1419h^3}{2^7\omega_0^5}z_1^3 z_2^4. \tag{48b}$$

As announced, one can observe that only one resonant monomial of the form $z_1^{p+1}z_2^p$ stays in the normal dynamics for each odd order $2p+1$. As a consequence of this particular structure, an analytical backbone curve can be derived from the CNF at any order. To that purpose, realification needs to be performed by introducing polar coordinates as:

$$z_1 = \frac{1}{2}\rho\,\mathrm{e}^{i\alpha}, \quad z_2 = \frac{1}{2}\rho\,\mathrm{e}^{-i\alpha}\,. \tag{49}$$

Then the backbone curve relating the nonlinear oscillation frequency ω_{NL}, to the amplitude ρ, can be easily derived as:

$$\omega_{NL} = \omega_0\left(1 + \frac{3h}{2^3\omega_1^2}\rho^2 - \frac{51h^2}{2^8\omega_1^4}\rho^4 + \frac{1419h^3}{2^{13}\omega_1^6}\rho^6\right). \tag{50}$$

3.2.2 The real normal form (RNF)

The main idea of the RNF consists of keeping two resonant monomials in Eq. (45a), instead of a single one for the CNF. This can be done by enlarging the interpretation of the resonance relationship, Eq. (14). Indeed, the primary goal of introducing the RNF in Neild and Wagg (2011); Wagg (2022) was to apply the normal form technique directly to second-order problems in time, typical for nonlinear oscillations. The homological equations are thus derived for the initial problem, which enforces the need to compute the second derivative of the mapping with respect to time, thus making naturally appearing squares of the eigenfrequencies in the resonance relationships. This point will be further commented on in the next section where the homological equations will also be rewritten for the displacement mapping only. Rewriting the resonance relationship (14) in such case leads to square values, reading:

$$(\lambda_k)^2 = \left(\sum_{i=1}^{n} m_i\lambda_i\right)^2, \quad \text{with} \quad m_i \geq 0 \quad \text{and} \quad \sum_{i=1}^{n} m_i = p, \tag{51}$$

which is indeed the resonance relationship used to derive the RNF. For the sake of illustration, the RNF of the Duffing equation is here given up to order 5, it reads

$$\dot{z}_1 = i\omega_1 z_1 + i\frac{3h}{2\omega_1}\left(z_1^2z_2 + z_1z_2^2\right) - i\frac{15h^2}{2^4\omega_1^3}z_1^3z_2^2 - i\frac{3h^2}{2^3\omega_1^3}z_1^2z_2^3, \tag{52a}$$

$$\dot{z}_2 = -i\omega_1 z_2 - i\frac{3h}{2\omega_1}\left(z_1^2z_2 + z_1z_2^2\right) + i\frac{3h^2}{2^3\omega_1^3}z_1^3z_2^2 + i\frac{15h^2}{2^4\omega_1^3}z_1^2z_2^3. \tag{52b}$$

As a consequence of the choice retained for fulfilling the resonance relationship, the same monomials now appear on the two lines of the normal dynamics. Interestingly, the cubic order terms share the same coefficients. This property is useful in order to retrieve an oscillator equation when coming back to real coordinates using a cartesian representation. However, retrieving an exact oscillator-like equation works only up to order 3. From order five, this is not possible anymore, as shown in (de Figueiredo Stabile et al. (2024)).

3.2.3 The oscillator normal form (ONF)

The oscillator normal form (ONF) has been first introduced in Touzé (2003); Touzé et al. (2004); Touzé and Amabili (2006); Touzé (2014), with the main idea of keeping oscillator equations without using any complex formulation. To that purpose, the linear change of coordinate given by Eq. (44) is not operated and complex quantities are not introduced. Instead, the first-order dynamical system is written with an anti-diagonal linear part as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}u\\v\end{bmatrix} = \begin{bmatrix}0 & 1\\-\omega_0^2 & 0\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix} + \begin{bmatrix}0\\-hu^3\end{bmatrix}, \tag{53}$$

As a consequence, the Duffing equation (43) is under its oscillator normal form: there is no change to be done. In ONF, the trivially resonant monomial is u^3 and cannot be cancelled. In a more general context of coupled oscillators featuring also even nonlinearities, all quadratic terms are not resonant and can be eliminated thanks to a nonlinear change of coordinate, see Touzé et al. (2004) for general discussions and Touzé (2014) for examples and classification of nonlinear terms thanks to this interpretation of the resonance relationship.

One of the main advantages of the ONF is thus to keep oscillator-like equations throughout the process. The nonlinear change of coordinate is given between two real coordinates that are homogeneous to a displacement and a velocity, whereas this interpretation is lost when using complex formulations. This choice came along with other advantages. For example, only the ONF allows drawing out a term-by-term comparison of the NNM calculation using either the center manifold technique as proposed by Shaw and Pierre, or the normal form approach, see *e.g.* Touzé et al. (2004); Touzé et al. (2021) for such discussions. A final advantage of the ONF is that it can be rewritten from physical coordinates, which allows deriving a non-intrusive version of the reduction technique using the normal form, which has been named DNF for direct normal form, see Vizzaccaro et al. (2021b).

However, numerous drawbacks are linked to this formulation. First, it is difficult to translate the choice on the resonant monomial as a broader algebraic interpretation of the resonance relationships, as it has been possible for the RNF with (51). As a consequence, it appears very difficult (and maybe not possible) to generalize the ONF to arbitrary order and automate its computation.

3.3 The direct parametrisation method for autonomous nonlinear vibrating systems

This section is devoted to the detailed presentation of the application of the parametrisation method for invariant manifolds to the case of nonlinear vibrating systems featuring geometric nonlinearity. To start with, the case of an autonomous system is considered, *i.e.* external forcing is discarded. The presentation starts with the damped case, since in the context of the method, the distinction between damped and undamped case is straightforwardly treated. Following the definitions recalled in Section 2, the NNM that will be computed here is the extension to the nonlinear regime of a given spectral subspace. Without damping, the computed NNM is the Lyapunov Subcenter Manifold (LSM), gathering families of periodic orbits emanating from free conservative oscillations. In the damped case, the NNM is the spectral submanifold (SSM), which is the smoothest invariant manifold originating from a given spectral subspace. For more detailed discussions on the link between conservative and dissipative invariant manifolds, the interested reader is referred to de la Llave and Kogelbauer (2019).

A special emphasis will be set on showing how the method can be operated directly from the physical space, (*i.e.* from the DOFs of the FE problem), without the need to compute the whole eigenbasis and first project the problem in modal space. Since the parametrisation method as shown in Section 3.1 uses the modal space to derive explicit analytical expressions, the idea is here to use the solutions known from the modal space to derive a direct method that will need as inputs only the master modes.

The starting point is a semi-discrete version of the original problem, for example derived using a finite element procedure, for a continuous mechanical structure featuring geometric nonlinearity. The equations of motion read:

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} + \mathbf{G}(\mathbf{U},\mathbf{U}) + \mathbf{H}(\mathbf{U},\mathbf{U},\mathbf{U}) = \mathbf{0} \tag{54}$$

with $\mathbf{U}$ the vector of unknown nodal displacements with dimension N. The quadratic and cubic nonlinearity writes:

$$\mathbf{G}(\mathbf{U},\mathbf{U}) = \sum_{r=1}^{N}\sum_{s=1}^{N} \mathbf{G}_{rs} U_r U_s, \tag{55a}$$

$$\mathbf{H}(\mathbf{U},\mathbf{U},\mathbf{U}) = \sum_{r=1}^{N}\sum_{s=1}^{N}\sum_{t=1}^{N} \mathbf{H}_{rst} U_r U_s U_t. \tag{55b}$$

It is worth mentioning that, in the case of three-dimensional finite elements, such a formulation is exact; this is not the case for beam and shell elements, where the presence of the rotational degrees of freedom renders cubic nonlinearities an approximation limited to the case of small rotations.

The linear normal modes of the conservative problem are the solutions of

$$(-\omega_j^2\mathbf{M}+\mathbf{K})\boldsymbol{\phi}_j=\mathbf{0}. \tag{56}$$

Assuming normalization with respect to the mass matrix, they share the classical properties, $\forall\,(j,k)\in[1,N]$:

$$\boldsymbol{\phi}_j^{\mathrm{T}}\mathbf{M}\boldsymbol{\phi}_k=\delta_{jk},\qquad \boldsymbol{\phi}_j^{\mathrm{T}}\mathbf{K}\boldsymbol{\phi}_k=\omega_j^2\delta_{jk},\qquad \boldsymbol{\phi}_j^{\mathrm{T}}\mathbf{C}\boldsymbol{\phi}_k=2\xi_j\omega_j\delta_{jk}. \tag{57}$$

In particular it has been assumed that the damping formulation is such that the real normal modes diagonalise the damping matrix $\mathbf{C}$ as well. Since this stands as a classical assumption in structural dynamics, and is used in most of the applications, it will be used in this section in order to detail how the real normal modes can be accurately followed in the construction of an NNM-based reduced-order model. The typical case of Rayleigh damping fulfils this assumption and will be used in most applications. Note however that this assumption is not restrictive for the method, which can handle any kind of damping laws as long as it is expressed via smooth linear or nonlinear functions. In particular, the next Section 3.4 will generalize the method to non-autonomous problems by also considering the more general case of complex modes.

To apply the parametrisation method, the problem needs to be rewritten as a first-order dynamical system. Many different choices are here possible, which all finally lead to the same kind of results. Here the problem is rewritten as:

$$\mathbf{M}\dot{\mathbf{V}}+\mathbf{C}\mathbf{V}+\mathbf{K}\mathbf{U}+\mathbf{G}(\mathbf{U},\mathbf{U})+\mathbf{H}(\mathbf{U},\mathbf{U},\mathbf{U})=\mathbf{0}, \tag{58a}$$

$$\mathbf{M}\dot{\mathbf{U}}=\mathbf{M}\mathbf{V}. \tag{58b}$$

The eigenvalues are sorted as follows:

$$\lambda_j=-\xi_j\omega_j+i\omega_j\sqrt{1-\xi_j^2}, \tag{59a}$$

$$\lambda_{j+N}=\bar{\lambda}_j=-\xi_j\omega_j-i\omega_j\sqrt{1-\xi_j^2}. \tag{59b}$$

This means in particular that, when complexification will be done following the same procedure as shown for the Duffing oscillator in Section 3.2, the linear part will be a diagonal matrix composed of the eigenvalues, with the second half of the eigenvalues, $j\in[N+1,2N]$ being the complex conjugate of the first half.

Finally, left and right eigenvectors of the first-order system will be needed in the calculations. They are defined as follows. The right eigenvectors $\mathbf{Y}_j$, $\forall\, j\in[1,N]$, can be written as a function of the real normal modes, as:

$$\mathbf{Y}_j=\begin{bmatrix}\boldsymbol{\phi}_j\lambda_j\\ \boldsymbol{\phi}_j\end{bmatrix},\qquad \mathbf{Y}_{j+N}=\bar{\mathbf{Y}}_j=\begin{bmatrix}\boldsymbol{\phi}_j\bar{\lambda}_j\\ \boldsymbol{\phi}_j\end{bmatrix}. \tag{60}$$

They are solutions to the eigenproblem recalled in Eq. (6). The left eigenvectors $\mathbf{X}_j$, which are the solutions to the eigenproblem given in Eq. (7), writes, $\forall\, j\in[1,N]$:

$$\mathbf{X}_j=\frac{1}{\lambda_j-\bar{\lambda}_j}\begin{bmatrix}\boldsymbol{\phi}_j\\ -\boldsymbol{\phi}_j\bar{\lambda}_j\end{bmatrix},\qquad \mathbf{X}_{j+N}=\bar{\mathbf{X}}_j=\frac{1}{\bar{\lambda}_j-\lambda_j}\begin{bmatrix}\boldsymbol{\phi}_j\\ -\boldsymbol{\phi}_j\lambda_j\end{bmatrix}, \tag{61}$$

The reduction technique starts by selecting a small subset of n master modes, $n\ll N$. The NNM emanating from this selected subset will then be computed with a high-order asymptotic development

thanks to the parametrisation method. The master mode selection is based on physical arguments. For example, if one is interested in computing the backbone curve of, say, mode j, then this mode j is the master. If, for some reason, it is awaited that a strong nonlinear interaction through an internal resonance will occur with another mode k along the backbone curve of mode j, then the two modes (j, k), will be selected as masters. For forced vibrations, the choice of the master mode is done on a frequency selection criteria, and the modes whose eigenfrequencies are in the frequency band of the forcing, will be selected as the master modes.

Let us assume that n master modes have been selected. As it will be shown in the applications, see Section 4, n typically ranges between 1 and 5. It is important to highlight that the method does not need to compute the whole eigenspectrum. Only the n master modes are needed. Since it is a nonlinear vibration problem, the reduced subspace is of dimension $2n$.

The following quantities are introduced for the computations:

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 & \dots & \mathbf{X}_n & \bar{\mathbf{X}}_1 & \bar{\mathbf{X}}_2 & \dots & \bar{\mathbf{X}}_n \end{bmatrix}, \tag{62a}$$

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_1 & \mathbf{Y}_2 & \dots & \mathbf{Y}_n & \bar{\mathbf{Y}}_1 & \bar{\mathbf{Y}}_2 & \dots & \bar{\mathbf{Y}}_n \end{bmatrix}, \tag{62b}$$

$$\mathbf{\Lambda} = \text{diag}[\lambda_1, \lambda_2, \dots, \lambda_n, \bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_n], \tag{62c}$$

$$\mathbf{\Phi} = \begin{bmatrix} \boldsymbol{\phi}_1 & \boldsymbol{\phi}_2 & \dots & \boldsymbol{\phi}_n & \boldsymbol{\phi}_1 & \boldsymbol{\phi}_2 & \dots & \boldsymbol{\phi}_n \end{bmatrix}, \tag{62d}$$

where $\mathbf{X}$ and $\mathbf{Y}$ respectively group the left and right master eigenvectors, of size $2n \times 2N$; $\mathbf{\Lambda}$ is the $2n \times 2n$ matrix of master eigenvalues, where the second half is the complex conjugate of the first; and $\mathbf{\Phi}$ is the $2n \times N$ matrix of real master modes. Note that, since we are treating with real modes for this presentation, the second half just repeats the first for $\mathbf{\Phi}$.

To work out the parametrisation method, a new *normal* variable $\mathbf{z}$, that describes the dynamics on the reduced invariant subspace, is needed. With the previous choice of n master modes and an NNM of size $2n$, $\mathbf{z}$ is a $2n$-dimensional vector reading $\mathbf{z} = [z_1\ z_2\ ...\ z_{2n}]^t$. Since the second half of the equations (and variables) will always be the complex conjugate of the first half, the normal coordinate $\mathbf{z}$ will also be denoted, for the sake of convenience, as $\mathbf{z} = [z_1\ z_2\ ...\ z_n\ z_{1^*}\ z_{2^*}\ ...\ z_{n^*}]^t$, where the following useful notation has been introduced:

$$i_k^* = \begin{cases} i_k + n & \text{if } i_k \leq n, \\ i_k - n & \text{if } i_k > n, \end{cases} \tag{63}$$

The two main unknowns which are searched for are the nonlinear mapping and the reduced dynamics. Two nonlinear mappings are simultaneously introduced as:

$$\mathbf{U} = \mathbf{\Psi}(\mathbf{z}), \tag{64a}$$

$$\mathbf{V} = \mathbf{\Upsilon}(\mathbf{z}). \tag{64b}$$

They respectively relate the displacements and velocities degrees-of-freedom of the initial FE problem to the normal coordinate. Finally, the reduced dynamics, describing the motions on the reduced NNM, reads:

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}). \tag{65}$$

To solve for these two main unknowns, the starting point is to write the invariance equation for the problem at hand, which is found by differentiating (64) with respect to time, replace in the original equations Eqs. (58), and finally eliminate time thanks to Eq. (65). One obtains the invariance equations reading:

$$\mathbf{M}\ \nabla_{\mathbf{z}}\mathbf{\Upsilon}(\mathbf{z})\ \mathbf{f}(\mathbf{z}) + \mathbf{C}\mathbf{\Upsilon}(\mathbf{z}) + \mathbf{K}\mathbf{\Psi}(\mathbf{z}) + \mathbf{G}(\mathbf{\Psi}(\mathbf{z}), \mathbf{\Psi}(\mathbf{z}) + \mathbf{H}(\mathbf{\Psi}(\mathbf{z}), \mathbf{\Psi}(\mathbf{z}), \mathbf{\Psi}(\mathbf{z})) = \mathbf{0}, \tag{66a}$$

$$\mathbf{M}\nabla_{\mathbf{z}}\mathbf{\Psi}(\mathbf{z})\ \mathbf{f}(\mathbf{z}) = \mathbf{M}\mathbf{\Upsilon}(\mathbf{z}). \tag{66b}$$

The unknowns are searched for as polynomial asymptotic expansions in the normal coordinate $\mathbf{z}$. The solution procedure operates sequentially, order by order, by solving for the homological equation of order p which is derived by selecting only the order-p terms in the invariance equations (66). The solution to the linear problem (order $p = 1$) is straightforward and one retrieves the classical linear solution (linear normal modes and linear reduced dynamics). This step is not made explicit herein but the interested reader can find the detail in Vizzaccaro et al. (2022). Using this solution, one can rewrite the unknowns as:

$$\mathbf{\Psi}(\mathbf{z}) = \mathbf{\Phi}\mathbf{z} + \sum_{p=2}^{o} [\mathbf{\Psi}(\mathbf{z})]_p \tag{67a}$$

$$\mathbf{\Upsilon}(\mathbf{z}) = \mathbf{\Phi}\mathbf{\Lambda}\mathbf{z} + \sum_{p=2}^{o} [\mathbf{\Upsilon}(\mathbf{z})]_p \tag{67b}$$

$$\mathbf{f}(\mathbf{z}) = \mathbf{\Lambda}\mathbf{z} + \sum_{p=2}^{o} [\mathbf{f}(\mathbf{z})]_p \,. \tag{67c}$$

In these equations, o stands for the maximal order of the expansion that will be computed, while the shortcut notation $[.]_p$ is used to denote that we are selecting only the order-p terms. Eqs. (67) clearly shows that we are looking for the nonlinear corrections to bring to the linear eigensolution to recover the invariant manifold. It underlines that the NNMs are the natural extension of linear normal modes to the nonlinear range.

The multi-index notation is used to express the unknown polynomials:

$$[\mathbf{\Psi}(\mathbf{z})]_p = \sum_{k=1}^{m_p} \mathbf{\Psi}^{(p,k)} \, \mathbf{z}^{\boldsymbol{\alpha}(p,k)}, \tag{68a}$$

$$[\mathbf{\Upsilon}(\mathbf{z})]_p = \sum_{k=1}^{m_p} \mathbf{\Upsilon}^{(p,k)} \, \mathbf{z}^{\boldsymbol{\alpha}(p,k)}, \tag{68b}$$

$$[\mathbf{f}(\mathbf{z})]_p = \sum_{k=1}^{m_p} \mathbf{f}^{(p,k)} \, \mathbf{z}^{\boldsymbol{\alpha}(p,k)}, \tag{68c}$$

where the unknowns are now the coefficients of the different monomials of order p: the N-dimensional vectors of coefficients for both displacement and velocity mappings, $\mathbf{\Psi}^{(p,k)}$ and $\mathbf{\Upsilon}^{(p,k)}$, and the $2n$-dimensional vector of coefficients of the reduced dynamics $\mathbf{f}^{(p,k)}$. An arbitrary monomial of order p is written as $\mathbf{z}^{\boldsymbol{\alpha}(p,k)} = z_1^{\alpha_1} z_2^{\alpha_2} \dots z_{2n}^{\alpha_{2n}}$. To each monomial is associated the set of the corresponding powers of indeterminates $\boldsymbol{\alpha}(p,k) = \{\alpha_1 \ \alpha_2 \ \dots \ \alpha_{2n}\}$. Each monomial is thus referred to with two integers: p, which is its order, and $k = 1 \dots m_p$, where m_p denotes the number of monomials of order p. The monomials need to be sorted according to a given rule and k counts the location of the given order-p monomial in this sorted list.

The order-p homological equation is found by selecting only terms of order p from the invariance equation (66). It can be simply written as:

$$\mathbf{M} \ [\nabla_{\mathbf{z}} \mathbf{\Upsilon}(\mathbf{z}) \ \mathbf{f}(\mathbf{z})]_p + \mathbf{C} \, [\mathbf{\Upsilon}(\mathbf{z})]_p + \mathbf{K} \, [\mathbf{\Psi}(\mathbf{z})]_p + [\mathbf{G}(\mathbf{\Psi}(\mathbf{z}), \mathbf{\Psi}(\mathbf{z})]_p + [\mathbf{H}(\mathbf{\Psi}(\mathbf{z}), \mathbf{\Psi}(\mathbf{z}), \mathbf{\Psi}(\mathbf{z}))]_p = \mathbf{0}, \tag{69a}$$

$$\mathbf{M} \, [\nabla_{\mathbf{z}} \mathbf{\Psi}(\mathbf{z}) \ \mathbf{f}(\mathbf{z})]_p = \mathbf{M} \, [\mathbf{\Upsilon}(\mathbf{z})]_p \,, \tag{69b}$$

using the operator $[.]_p$ already introduced in Section 3.1. Detailed calculations to make explicit each of the order-p terms in Eqs.(69) are involved and not reported here for the sake of brevity, the interested reader can find the details in Vizzaccaro et al. (2022). Each of the terms can nevertheless be expressed as

a summation over the polynomial indeterminates, reading:

$$[\nabla_{\mathbf{z}}\boldsymbol{\Psi}(\mathbf{z})\ \mathbf{f}(\mathbf{z})]_p = \sum_{k=1}^{m_p}\left[\boldsymbol{\mu}^{(p,k)} + \sigma^{(p,k)}\boldsymbol{\Psi}^{(p,k)} + \sum_{s=1}^{2n} f_s^{(p,k)}\boldsymbol{\phi}_s\right]\mathbf{z}^{\boldsymbol{\alpha}(p,k)} \tag{70a}$$

$$[\nabla_{\mathbf{z}}\boldsymbol{\Upsilon}(\mathbf{z})\ \mathbf{f}(\mathbf{z})]_p = \sum_{k=1}^{m_p}\left[\boldsymbol{\nu}^{(p,k)} + \sigma^{(p,k)}\boldsymbol{\Upsilon}^{(p,k)} + \sum_{s=1}^{2n} \lambda_s f_s^{(p,k)}\boldsymbol{\phi}_s\right]\mathbf{z}^{\boldsymbol{\alpha}(p,k)} \tag{70b}$$

$$[\mathbf{G}(\boldsymbol{\Psi}(\mathbf{z}),\boldsymbol{\Psi}(\mathbf{z})]_p = \sum_{k=1}^{m_p}\check{\mathbf{G}}^{(p,k)}\mathbf{z}^{\boldsymbol{\alpha}(p,k)} \tag{70c}$$

$$[\mathbf{H}(\boldsymbol{\Psi}(\mathbf{z}),\boldsymbol{\Psi}(\mathbf{z}),\boldsymbol{\Psi}(\mathbf{z}))]_p = \sum_{k=1}^{m_p}\check{\mathbf{H}}^{(p,k)}\mathbf{z}^{\boldsymbol{\alpha}(p,k)}. \tag{70d}$$

In these expressions, the newly introduced terms can be described and understood as follows:

- $\boldsymbol{\mu}^{(p,k)}$ and $\boldsymbol{\nu}^{(p,k)}$ gathers the known terms from the previous orders. They are equivalent to the second term included in $\mathbf{E}_p$ which has been introduced in Section 3.1, see Eq. (26). For the purpose of our computation here, one needs to understand that this term is known, and explicit expressions are given in Vizzaccaro et al. (2022).
- An important term, due to the derivatives, appears as $\sigma^{(p,k)}$, defined as:

  $$\sigma^{(p,k)} = \alpha_1\lambda_1 + ... + \alpha_{2n}\lambda_{2n} = \boldsymbol{\alpha}(p,k)\boldsymbol{\lambda}. \tag{71}$$

  It is strictly equivalent to the term $\mathbf{m}.\boldsymbol{\lambda}_L$ introduced in Eq. (35), and is of utmost importance since it conveys the nonlinear resonance relationship that is at the core of the method and drives the normal form style solution.
- $\check{\mathbf{G}}^{(p,k)}$ and $\check{\mathbf{H}}^{(p,k)}$ are the nonlinear coefficients that come through products of lower orders than the one considered, p, created through the application of the quadratic and cubic nonlinear operator. These two terms are equivalent to the first term included in $\mathbf{E}_p$ in Eq. (26). Explicit expressions are easy to write and are given in Vizzaccaro et al. (2022).

Finally, the order p homological equation can be written at the level of an arbitrary monomial $\boldsymbol{\alpha}(p,k)$, under matrix form, by letting the unknowns on the left-hand side and putting all the known terms on the right-hand side. It reads:

$$\left(\sigma^{(p,k)}\begin{bmatrix}\mathbf{M} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}\end{bmatrix} + \begin{bmatrix}\mathbf{C} & \mathbf{K}\\ -\mathbf{M} & \mathbf{0}\end{bmatrix}\right)\begin{bmatrix}\boldsymbol{\Upsilon}^{(p,k)}\\ \boldsymbol{\Psi}^{(p,k)}\end{bmatrix} + \sum_{s=1}^{2n} f_s^{(p,k)}\begin{bmatrix}\mathbf{M} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}\end{bmatrix}\mathbf{Y}_s = \begin{bmatrix}\mathbf{R}_1^{(p,k)}\\ \mathbf{R}_2^{(p,k)}\end{bmatrix}, \tag{72}$$

where the right-hand side known term has been rewritten using $\mathbf{R}_1^{(p,k)} = -\mathbf{M}\boldsymbol{\nu}^{(p,k)} - \check{\mathbf{G}}^{(p,k)} - \check{\mathbf{H}}^{(p,k)}$, and $\mathbf{R}_2^{(p,k)} = -\mathbf{M}\boldsymbol{\mu}^{(p,k)}$. Eq. (72) is a $2N$-dimensional problem that has to be solved for each monomial. Recalling the parametrisation method as exposed in Section 3.1, two problems need to be tackled simultaneously to arrive at closed-form solutions. First the problem is underdetermined: there are more unknowns than equations. Second, because of the appearance of nonlinear resonances, the problem can be ill-conditioned. To arrive at explicit expressions, projection to the modal space is needed. It also allows one to make the distinction between the normal and the tangent part of the homological equations. The normal part corresponds to the projection of Eq. (72) onto the slave modes and is not underdetermined.

Let us see how the solutions can be expressed in the modal space by left-multiplying Eq. (72) by $\mathbf{X}_r^T$, $\forall r = 1...2N$. One easily arrives at:

- in the tangent space: $\forall\, r \in [1, \ldots, 2n]$,

$$(\sigma^{(p,k)} - \lambda_r)\theta_r^{(p,k)} + f_r^{(p,k)} = g_r^{(p,k)}. \tag{73}$$

- in the normal space: $\forall\, r \in [2n+1, \ldots, 2N]$,

$$(\sigma^{(p,k)} - \lambda_r)\theta_r^{(p,k)} = g_r^{(p,k)}. \tag{74}$$

In these expressions, the unknown coefficients of the nonlinear mappings in the modal space have been named as $\theta_r^{(p,k)}$. It represents the projection of $\mathbf{\Psi}^{(p,k)}$, $\mathbf{\Upsilon}^{(p,k)}$ in the modal space:

$$\theta_r^{(p,k)} \doteq \mathbf{X}_r^{\mathrm{T}} \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{\Upsilon}^{(p,k)} \\ \mathbf{\Psi}^{(p,k)} \end{bmatrix}. \tag{75}$$

Also, the projection of the known terms has been gathered as $g_r^{(p,k)}$, simply representing the modal projection of the right-hand side of Eq. (72):

$$g_r^{(p,k)} \doteq \mathbf{X}_r^{\mathrm{T}} \begin{bmatrix} \mathbf{R}_1^{(p,k)} \\ \mathbf{R}_2^{(p,k)} \end{bmatrix} \tag{76}$$

As awaited, there is no underdeterminacy on the normal part. If no cross-resonances are assumed, which is a logical and viable assumption as already commented in Section 3.1, the solution for the unknown coefficients of the nonlinear mappings in the modal space simply read:

$$\theta_r^{(p,k)} = \frac{g_r^{(p,k)}}{\sigma^{(p,k)} - \lambda_r} \tag{77}$$

The tangent part, Eq. (73) is underdetermined and might be ill-conditioned when a resonance condition is met such that $\sigma^{(p,k)} \simeq \lambda_r$. For the case of vibrating systems, the eigenvalues read: $\lambda_{j,j^\star} = -\xi_j\omega_j \pm i\omega_j\sqrt{1-\xi_j^2}$. In general, lightly damped systems are assumed such that $\forall j, \quad \xi_j \ll 1$. Hence it is meaningful to make the resonance check on the imaginary parts only. This will also lead one to make the distinction between the trivial resonances and the internal resonances. Trivial resonances are due to the fact that one assumes a spectrum composed of pairs of purely complex conjugates, such that numerous resonances are present due to this fact, irrespective of the particular values of the eigenfrequencies. On the other hand, internal resonances are linked to specific values of the eigenfrequencies that might fulfil, in some cases, additional resonance relationships.

A parametrisation style is a way to solve the tangent part of the homological equation. Two main styles exist: the graph style and the normal form style. For application to vibrating systems, and in order to treat these resonances in an automated manner in an algorithm that will compute all the coefficients at arbitrary order, it is convenient to introduce the resonant set $\mathcal{R}$ as the set of indexes such that the choice $\theta_r^{(p,k)} = 0$, is selected. Note that such as set $\mathcal{R}$ is defined for each monomial $\boldsymbol{\alpha}(p,k) = \{\alpha_1, \ldots, \alpha_{2n}\}$, and a complete notation should have been $\mathcal{R}^{(p,k)}$. However, for the sake of simplicity, and also because the resonant set can be defined in a nested loop in an algorithmic point of view, the notation will be simplified to $\mathcal{R}$.

The generic solutions to the tangent part of the homological equation can then be easily written thanks to $\mathcal{R}$ as:

$$\begin{cases} f_r^{(p,k)} = 0, & (\sigma^{(p,k)} - \lambda_r)\theta_r^{(p,k)} = g_r^{(p,k)}, & \text{if } r \notin \mathcal{R}, \\ \theta_r^{(p,k)} = 0, & f_r^{(p,k)} = g_r^{(p,k)}, & \text{if } r \in \mathcal{R}. \end{cases}$$

From an algorithmic point of view, one then only needs to explain how the resonant set $\mathcal{R}$ is fulfilled, depending on the style of parametrisation that is selected.

If one would like to use a graph style parametrisation, then $\mathcal{R}$ is filled as:

$$\mathcal{R}_{\text{Graph}} = \{1\,2\,\ldots\,2n\} \tag{78}$$

For the graph style parametrisation, one can make the distinction between the complex normal form (CNF), and the real normal form (RNF), as introduced in Section 3.2. As explained, the CNF is the most parsimonious choice, recommended by mathematicians, that considers a strict interpretation of the resonance relationship. In that case (CNF), the set $\mathcal{R}$ is filled as:

$$\mathcal{R}_{\text{CNF}} = \{r\} \qquad \text{with } r : \sigma^{(p,k)} \approx \lambda_r. \tag{79}$$

If a resonance condition is met, then only the monomial strictly associated to that resonance, is kept in the complex expression. On the other hand, the RNF admits a broader interpretation of the resonance relationship, which can be understood as a resonance on the squares of the eigenvalues as expressed in Eq. (51). The set $\mathcal{R}$ can then be automatically filled for the RNF as:

$$\mathcal{R}_{\text{RNF}} = \{r\ r^*\} \qquad \text{with } r : \sigma^{(p,k)} \approx \lambda_r. \tag{80}$$

It is straightforward to see that these two choices generalize the discussion led in Section 3.2.

Now that the solutions have been made explicit in the modal space, it is mandatory to find a way to solve the problem without the projection to the modal space. On the contrary, a direct approach is needed such that the reduced-order model can be automatically derived only with the eigenvectors of the master modes. To that purpose, one needs to use the explicit analytical solutions that can be derived in the modal space to shed light on how the problem can be operated directly from the physical space. The main problem to solve is the fact that the homological equation Eq. (72) can be ill-conditioned. To cure this potential problem, a bordering technique is here employed in order to solve the potentially ill-conditioned system. Importantly, the vectors that need to be considered to augment the size of the system and make it solvable, depend on the resonance scenario at hand for the monomial at hand, and thus on the filling of the resonant set $\mathcal{R}$.

Let us consider a generic set $\mathcal{R}$ to explain how the system needs to be enlarged. If $r \in \mathcal{R}$, then $\theta_r^{(p,k)} = 0$, which can be written in the physical space using Eq. (75) as:

$$\mathbf{X}_r^{\mathrm{T}} \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Upsilon}^{(p,k)} \\ \boldsymbol{\Psi}^{(p,k)} \end{bmatrix} = 0. \tag{81}$$

Expanding these vectorial equations, one easily arrives to:

$$\boldsymbol{\phi}_r^T \mathbf{M} \boldsymbol{\Upsilon}^{(p,k)} = \bar{\lambda}_r \boldsymbol{\phi}_r^T \mathbf{M} \boldsymbol{\Psi}^{(p,k)}. \tag{82}$$

This last equation shows how the bordering technique can be used, the idea being of adding to the ill-conditioned problem the lines that correspond to the nonlinear resonances that are fulfilled. By doing so, the singular matrix is bordered by the eigenvectors of its kernel, and thus becomes solvable.

To show more explicitly how the problem can be rewritten for a direct solution from the physical space, let us assume that the resonant set $\mathcal{R}$ contains m terms: $\mathcal{R} = \{r_1,\, r_2,\, \ldots,\, r_m\}$, and let us introduce the shortcut notation $\mathbf{f}_{\mathcal{R}}^{(p,k)}$ to denote the m terms $f_{m_j}^{(p,k)}$, and $\mathbf{f}_{\not{\mathcal{R}}}^{(p,k)}$ the other ones. Then the problem can be written in matrix form as:

$$\begin{bmatrix} \sigma^{(p,k)}\mathbf{M} + \mathbf{C} & \mathbf{K} & \mathbf{M}\boldsymbol{\phi}_{\mathcal{R}}\boldsymbol{\lambda}_{\mathcal{R}} & \mathbf{0} \\ -\mathbf{M} & \sigma^{(p,k)}\mathbf{M} & \mathbf{M}\boldsymbol{\phi}_{\mathcal{R}} & \mathbf{0} \\ \boldsymbol{\phi}_{\mathcal{R}}^{\mathrm{T}}\mathbf{M} & -\bar{\boldsymbol{\lambda}}_{\mathcal{R}}^{\mathrm{T}}\boldsymbol{\phi}_{\mathcal{R}}^{\mathrm{T}}\mathbf{M} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Upsilon}^{(p,k)} \\ \boldsymbol{\Psi}^{(p,k)} \\ \mathbf{f}_{\mathcal{R}}^{(p,k)} \\ \mathbf{f}_{\not{\mathcal{R}}}^{(p,k)} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_1^{(p,k)} \\ \mathbf{R}_2^{(p,k)} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \tag{83}$$

where the matrix $\boldsymbol{\lambda}_{\mathcal{R}}$ having on its diagonal the m eigenvalues of the resonant modes belonging to $\mathcal{R}$, has been introduced.

Eq. (83) is now solvable directly in physical space, thanks to the knowledge of the solution in modal space, which has been used to augment the size of the problem. Since the solution is directly operated in the physical space, the method can now be termed as direct parametrisation of invariant manifold (DPIM). Importantly, the way the system is augmented depends on the parametrisation style since the filling of $\mathcal{R}$ is different. The problem to solve for each monomial, Eq. (83), is now of size $(2N+m)\times(2N+m)$. A last simplification can nevertheless be operated. Recalling that the initial problem is second-order in time, such that the second half of the equations in Eq. (58) are tautological, it is possible to halve the size of the system Eq. (83). To that purpose, the relationship between the displacement and the velocity mappings needs to be unfolded.

Taking the lines $N+1$ to N of the homological equation written for an arbitrary monomial, Eq. (72), leads to:

$$-\mathbf{M}\boldsymbol{\Upsilon}^{(p,k)}+\sigma^{(p,k)}\mathbf{M}\boldsymbol{\Psi}^{(p,k)}+\sum_{r_j\in\mathcal{R}}\mathbf{M}\boldsymbol{\phi}_{r_j}f_{r_j}^{(p,k)}=-\mathbf{M}\boldsymbol{\mu}^{(p,k)} \tag{84}$$

This equation comes from the second half of the starting equations (58). Consequently, Eq. (84) is the translation of the tautological equation that relates velocities to displacement and allows, at the level of an arbitrary monomial and for a generic order p. It allows deriving an explicit relationship, that links the coefficients of the velocity mapping to those of the displacement mapping, which is valid at any order and for any monomial $\boldsymbol{\alpha}(p,k)$:

$$\boldsymbol{\Upsilon}^{(p,k)}=\sigma^{(p,k)}\boldsymbol{\Psi}^{(p,k)}+\sum_{r_j\in\mathcal{R}}\left(\boldsymbol{\phi}_{r_j}f_{r_j}^{(p,k)}\right)+\boldsymbol{\mu}^{(p,k)} \tag{85}$$

This last equation simply translates that the velocity mapping is the time derivative of the displacement mapping.

Using Eq. (85), the size of the homological equations Eq. (72) can be halved by using the displacement mapping coefficients as sole unknowns with the reduced dynamics coefficients. Eliminating the velocity mappings from Eq. (72) leads to the following N-dimensional equations:

$$\left((\sigma^{(p,k)})^2\mathbf{M}+\sigma^{(p,k)}\mathbf{C}+\mathbf{K}\right)\boldsymbol{\Psi}^{(p,k)}+\sum_{r_j\in\mathcal{R}}(\sigma^{(p,k)}-\bar{\lambda}_{r_j})\mathbf{M}\boldsymbol{\phi}_{r_j}f_{r_j}^{(p,k)}=\mathbf{F}^{(p,k)}, \tag{86}$$

where the right-hand side term has been introduced as:

$$\mathbf{F}^{(p,k)}=-\check{\mathbf{G}}^{(p,k)}-\check{\mathbf{H}}^{(p,k)}-\mathbf{M}\boldsymbol{\nu}^{(p,k)}-(\sigma^{(p,k)}\mathbf{M}+\mathbf{C})\boldsymbol{\mu}^{(p,k)}. \tag{87}$$

Besides halving the size of the problem to be solved, which comes with important computational savings, one can also see that Eq. (86) makes appear the square of the combination of eigenvalues term $\sigma^{(p,k)}$ which is responsible for the resonance relationship. It gives a second-order view on the problem of nonlinear resonances, adapted to mechanical vibratory systems, and allows a better understanding of the broader interpretation of the resonance relationship that is used for the real normal form parametrisation. Finally, it gives an explicit expression of the homological equation that can be directly used for mechanical problems, since it calls regular outputs of a finite element approximation: mass, damping and stiffness matrices, master eigenmodes only. The only terms that need to be computed in an intrusive manner are the results of the quadratic and cubic nonlinear operators $\mathbf{G}$ and $\mathbf{H}$, see for example Vizzaccaro et al. (2022); Opreni et al. (2023b); Touzé et al. (2014), for discussions and explicit expressions for the treatment of these terms in a FE context.

The final step consists of rewriting the bordering equation, Eq. (82), as a function of the displacement mapping only, and rewrite the problem to be solved, Eq. (83), without the lines corresponding to the velocity mapping. This step is not developed here for the sake of brevity, the interested reader can find this development in Vizzaccaro et al. (2022), where the explicit expressions are given, depending on the style selected and thus on the filling of the set $\mathcal{R}$.

As a summary of this section, the direct parametrisation method for invariant manifolds, has been introduced. It is an adaptation of the general parametrisation method shown in Section 3.1 to the case of nonlinear vibrating systems featuring geometric nonlinearity expressed with quadratic and cubic nonlinearities. The rewriting of the method allows for tackling important features. First, the method is direct and can be applied to a FE discretization, without the need to compute the whole modal basis: only the master modes are needed as inputs. The method is fully rewritten by also using the specificities of the mechanical systems, and the fact that the initial problem is second-order in time has been used to halve the size of the problems to be solved. The method is general and can handle arbitrary order expansions, together with the choice of the parametrisation style. If a graph style solution is selected, then one recovers the solution strategy developed by Shaw and Pierre (1991). Two different normal form styles have been introduced, hence enlarging the range of solutions proposed with the oscillator normal form in Touzé et al. (2004) for computing NNM-based ROMs.

The development in this section is limited to autonomous problems, but can tackle conservative as well as dissipative systems. The next section is devoted to extending this solution strategy to forced problems.

3.4 The direct parametrisation method for forced nonlinear vibrating systems

In this section, the direct parametrisation method is extended in order to handle the case of mechanical systems subjected to external periodic forcing. This case is of importance in the field of nonlinear vibrations since a crucial feature is the frequency-response curve (FRC) and the bifurcations they can encounter for some parameter values. From the theoretical point of view for deriving efficient ROMs based on the invariant manifold theory, the main consequence is that one has to deal with time-dependent invariant manifolds. The NNM in such a case is oscillating with the external periodic forcing and deforms in a way that needs to be taken into account to achieve accurate ROM predictions. The NNM will also have to consider both a dependence with respect to time and forcing frequency, which makes the solution more difficult to handle. As we will show, the time dependence can be easily discarded by assuming periodic motions. On the other hand, strictly speaking, a ROM should be computed for each forcing frequency since the manifold motions and deformations are strongly dependent on that parameter.

The situation is sketched in Fig. 4, which shows an illustrative representation of the oscillations and deformations of such time-dependent NNM, by selecting four different time instants, equiparted on a period of the forcing. In the dynamical system community, such a time-dependent invariant manifold is often referred to as a whisker, see *e.g.* Haro and de la Llave (2006). Fig. 4(a) shows the four snapshots of the whisker in the same 3d representation, while Figs. 4(b-e) show the four instants separately. An orbit of the forced system is shown in yellow. Interestingly, the time instant of the point running along the orbit crosses the whisker at the corresponding time (yellow point). The motion of the fixed point is shown with a white circle. The whisker experiences both a body oscillation, following the motion of the fixed point, and also slight deformations.

To present the methodology, a first-order dynamical system is considered in this section, in order to enlarge the scope of applications. The presentation will closely follows Vizzaccaro et al. (2024). Further developments and simplifications to deal with the case of vibrating systems, as those shown for example in Section 3.3, will not be reported here for the sake of brevity, but the interested reader can refer to Vizzaccaro et al. (2024). The starting point is thus selected as the following system:

$$\mathbf{B}\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{Q}(\mathbf{y},\mathbf{y}) + \varepsilon\mathbf{C}\,\mathrm{e}^{\tilde{\lambda}t}\,. \tag{88}$$

The phase space of dimension D is here selected as $\mathbb{C}^D$. The real-valued matrix $\mathbf{B}$ is not assumed to have any general property; in particular, it might be singular, meaning that the development also considers the case of differential-algebraic equations (DAE). On the other hand, the real-valued matrix $\mathbf{A}$ is assumed to have full rank. The nonlinearity is given by a smooth analytical function in quadratic form $\mathbf{Q}(\mathbf{y},\mathbf{y})$. This choice is here retained for pedagogical purposes only. It allows however dealing with any kind of analytical nonlinearity, using the quadratic recast, see for example Cochelin and Vergez (2009); Karkar

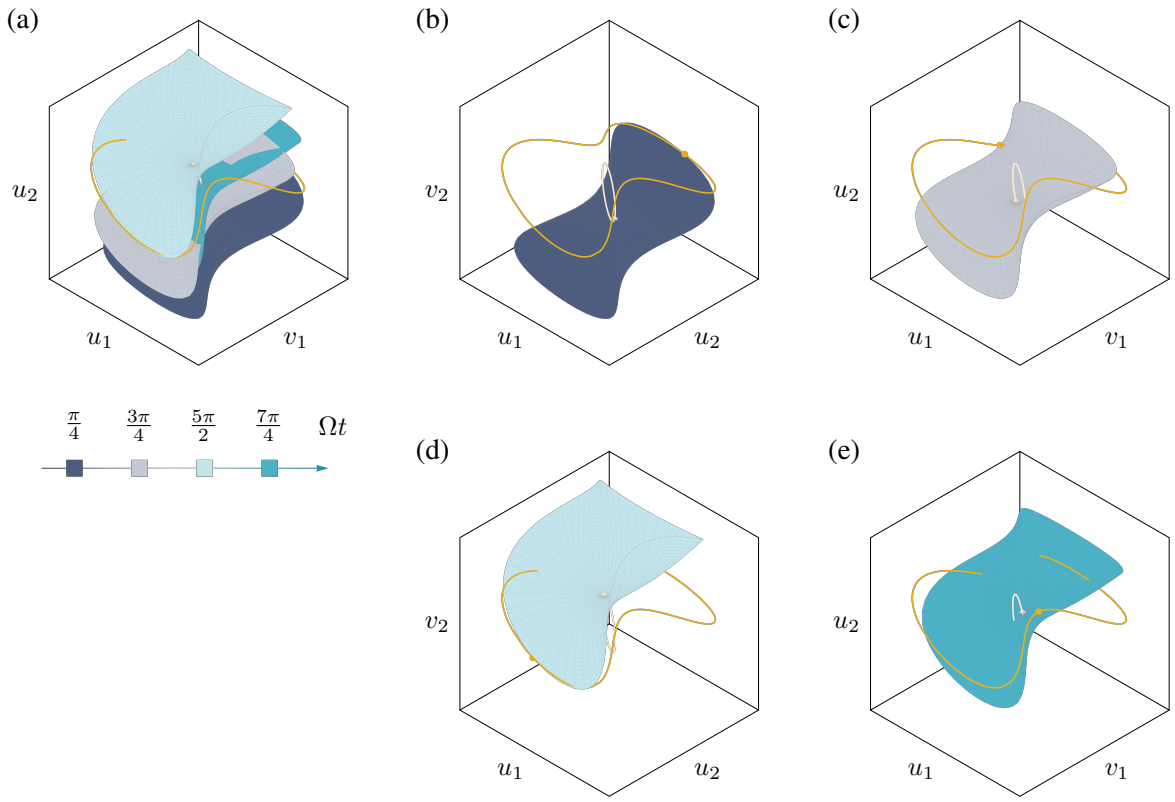

Figure 4: Schematical representation of a time-dependent invariant manifold in the case of a forced vibrating system. (a) 3d view of the NNM at four different times of the forcing period, together with an orbit of the forced system. (b-e) Separated views of the four different phases. The orbit of the forced system is in yellow, and the oscillating motion of the fixed point is marked with a white circle. The time-dependent invariant manifold is often referred to as a whisker.

et al. (2013); Guillot et al. (2019). The forcing term has a time dependence $\mathrm{e}^{\tilde{\lambda}t}$ with $\tilde{\lambda}$ the forcing value. Constant forcing term is here not considered since its effect is to change the location of the fixed point. The forcing direction is represented by the vector $\mathbf{C}$. Besides, a small forcing assumption is explicitly taken into account by multiplying the forcing amplitude with a small parameter ε. This is needed in order to fall in the case where existence and uniqueness have already been proven by considering the whisker as a small perturbation of the autonomous invariant manifold, as shown for instance in Haro and de la Llave (2006); Haller and Ponsioen (2016).

Using the small parameter ε and the existence and uniqueness results, the geometry of the whisker in phase space can be described by a time-dependent nonlinear mapping $\mathbf{W}^{(\varepsilon)}(\bar{\mathbf{z}}, t)$ that maps the initial coordinate $\mathbf{y}$ to the reduced invariant subspace described by the d-dimensional vector $\bar{\mathbf{z}}$ collecting the normal coordinates, where $d \ll D$. The reduced dynamics along the embedding can then be formally written as

$$\dot{\bar{\mathbf{z}}} = \mathbf{f}^{(\varepsilon)}(\bar{\mathbf{z}}, t). \tag{89}$$

Note that in this introductory presentation, the calculation is assumed to be done for a single value of the external forcing $\tilde{\lambda}$, such that its dependence is not reported yet. The invariance equation, which has to be solved for in order to find the unknowns $\mathbf{W}^{(\varepsilon)}(\bar{\mathbf{z}}, t)$ and $\mathbf{f}^{(\varepsilon)}(\bar{\mathbf{z}}, t)$, reads:

$$\mathbf{B}\nabla_{\bar{\mathbf{z}}}\mathbf{W}^{(\varepsilon)}\mathbf{f}^{(\varepsilon)} + \mathbf{B}\partial_t\mathbf{W}^{(\varepsilon)} = \mathbf{A}\mathbf{W}^{(\varepsilon)} + \mathbf{Q}(\mathbf{W}^{(\varepsilon)}, \mathbf{W}^{(\varepsilon)}) + \varepsilon\mathbf{C}\,\mathrm{e}^{\tilde{\lambda}t}\,. \tag{90}$$

A first strategy to solve this case is to use a first-order approximation in terms of powers of ε for the two unknowns. This method is meaningful and has been used for instance in Breunung and Haller (2018);

Jain and Haller (2022); Opreni et al. (2023b) for applications to vibrating systems. With this assumption, a two-term expansion, limited to the first perturbation brought about by the forcing, is developed as:

$$\mathbf{y} = \bar{\mathbf{W}}(\bar{\mathbf{z}}) + \varepsilon\hat{\mathbf{W}}(\bar{\mathbf{z}}, t) + \mathcal{O}(\varepsilon^2), \tag{91a}$$

$$\dot{\bar{\mathbf{z}}} = \bar{\mathbf{f}}(\bar{\mathbf{z}}) + \varepsilon\hat{\mathbf{f}}(\bar{\mathbf{z}}, t) + \mathcal{O}(\varepsilon^2). \tag{91b}$$

The unknowns have been split into two different terms denoted respectively with bar and hat variables. Substituting for (91) in the invariance equation (90), and equating the like-powers of ε, leads to two distinctive problems for the selected orders ε^0 and ε^1. Interestingly, the problem at order ε^0 is exactly the autonomous problem. Consequently, using the results of the previous section, the solutions for $\bar{\mathbf{W}}(\bar{\mathbf{z}})$ and $\bar{\mathbf{f}}(\bar{\mathbf{z}})$ can be derived thanks to arbitrary order expansions. The problem at order ε^1 makes appear the non-autonomous contribution brought about by the forcing. It can be solved for the two unknowns $\hat{\mathbf{W}}(\bar{\mathbf{z}}, t)$ and $\hat{\mathbf{f}}(\bar{\mathbf{z}}, t)$, once the autonomous problem at order ε^0 has been computed. Again, arbitrary order solutions are at reach, such that two expansions with different orders can be handled simultaneously, see *e.g.* Opreni et al. (2023b) for more details. As a matter of fact, this solution strategy uses the fact that the ε scaling allows separating the contributions of autonomous and non-autonomous terms, so they can be computed one after the other.

The main advantage of this processing, as underlined in Opreni et al. (2023b), is that the structure of the homological equations resulting from the splitting between the ε orders, is the same. Hence the computational framework developed to solve the autonomous problem can be readily extended to treat the first-order non-autonomous perturbation. However, this strategy suffers from two important limitations. First, the truncation to the ε^1 term in the forcing might lead to deterioration of the quality of the results when considering large values of the forcing in some engineering applications. Second, this assumption impedes to consider superharmonic resonances in the framework.

A different solution strategy is proposed in this section, which allows for overcoming the two underlined limitations. It is based on a rewriting of the initial problem to be solved, which has been shown to be equivalent to an all-order asymptotic expansion in the perspective of the ε development of the unknowns. The details of this particular proof are reported in Vizzaccaro et al. (2024), here only the general methodology is presented.

The first idea is to make the forced system (88) autonomous by augmenting the size of the state space variable by one unit, as it is classically proposed in all textbooks dealing with dynamical systems, see *e.g.*Guckenheimer and Holmes (1983). An additional variable $\tilde{z}$, which is such that $\tilde{z} = \varepsilon\, \mathrm{e}^{\tilde{\lambda}t}$, is introduced and the initial system is rewritten as:

$$\mathbf{B}\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{Q}(\mathbf{y}, \mathbf{y}) + \mathbf{C}\tilde{z}, \tag{92a}$$

$$\dot{\tilde{z}} = \tilde{\lambda}\tilde{z}, \tag{92b}$$

Since the added coordinate $\tilde{z}$ plays a very specific role in the dynamics and should not be mixed with the physical degrees of freedom, it is considered from this stage as directly expressed with the *normal* variables, meaning that the last line will be excluded from the unknown nonlinear mapping, and the last equation won't be changed throughout the iterative calculations of the parametrisation method, ensuring that, at the end of the process, the forcing variable will keep the same meaning. The normal variable is thus introduced as

$$\mathbf{z} = \begin{bmatrix} \bar{\mathbf{z}} \\ \tilde{z} \end{bmatrix}, \tag{93}$$

it is of dimension $d+1$ with $\bar{\mathbf{z}}$ grouping the master modes of dimension d, and the last entry corresponding to the forcing, already expressed in normal coordinates. The second advantage is that $\tilde{z}$ has amplitude ε, as ascertained by its definition as $\tilde{z} = \varepsilon\, \mathrm{e}^{\tilde{\lambda}t}$. Hence the whole normal coordinate vector $\mathbf{z}$ as defined in Eq. (93) can be considered as small with respect to 1 such that asymptotic expansions can be safely derived.

The parametrisation method can now be applied to the autonomous Eq. (92a) by searching for a nonlinear mapping relating the physical degrees of freedom $\mathbf{y}$ to the normal coordinate $\mathbf{z}$ as

$$\mathbf{y} = \mathbf{W}(\mathbf{z}), \tag{94}$$

hence ensuring that Eq. (92b) will be left unchanged by the calculation. The reduced dynamics now reads:

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}). \tag{95}$$

The invariance equation, which is found by differentiating Eq. (94) with respect to time, and then eliminating time thanks to the reduced dynamics (95), now reads:

$$\mathbf{B}\nabla_{\mathbf{z}}\mathbf{W}(\mathbf{z})\mathbf{f}(\mathbf{z}) = \mathbf{A}\mathbf{W}(\mathbf{z}) + \mathbf{Q}(\mathbf{W}(\mathbf{z}), \mathbf{W}(\mathbf{z})) + \mathbf{C}\tilde{z}. \tag{96}$$

Both unknowns $\mathbf{W}$ and $\mathbf{f}$ are searched for as polynomial expansions of arbitrary order o, and are written as

$$\mathbf{W}(\mathbf{z}) = \sum_{p=1}^{o} \left[\mathbf{W}(\mathbf{z})\right]_p, \tag{97a}$$

$$\mathbf{f}(\mathbf{z}) = \sum_{p=1}^{o} \left[\mathbf{f}(\mathbf{z})\right]_p, \tag{97b}$$

Let us now detail how the method handles the new variable due to the forcing in the calculation of the first-order terms, as well as for an arbitrary order. At order 1, Eq. (96) reads

$$\mathbf{B}\left[\nabla_{\mathbf{z}}\mathbf{W}(\mathbf{z})\mathbf{f}(\mathbf{z})\right]_1 = \mathbf{A}\left[\mathbf{W}(\mathbf{z})\right]_1 + \mathbf{C}\tilde{z}. \tag{98}$$

Since only linear terms are selected, one can write for the two unknowns:

$$\left[\mathbf{W}(\mathbf{z})\right]_1 = \mathbf{W}^{(1)}\mathbf{z}, \tag{99a}$$

$$\left[\mathbf{f}(\mathbf{z})\right]_1 = \mathbf{f}^{(1)}\mathbf{z}. \tag{99b}$$

A simple calculation detailed in Vizzaccaro et al. (2024) shows that the first d columns of the $D \times (d+1)$ matrix $\mathbf{W}^{(1)}$ are the d right master eigenvectors $\{\mathbf{Y}_k\}_{k=1,\dots,d}$ associated to the d master eigenvalues, see Eq. (6). This choice ensures the tangency of the computed NNM to the master modes, as awaited. The coefficients of the reduced linear dynamics in the $(d+1) \times (d+1)$ matrix $\mathbf{f}^{(1)}$ can be expanded as follows:

$$\mathbf{f}^{(1)} = \begin{bmatrix} \bar{\mathbf{f}}^{(1)} & \mathbf{f}^{(1,d+1)} \\ \mathbf{0} & \tilde{\lambda} \end{bmatrix}. \tag{100}$$

Indeed, the last line is enforced by the choices made at the beginning of the process. To treat separately the added coordinate that represents the forcing and let the new equation in (92) unaffected by the parametrisation algorithm, one necessarily has to impose that the last line of Eq. (95) is linear and reads $\mathbf{f}_{d+1}(\mathbf{z}) = \tilde{\lambda}\tilde{z}$. This choice translates naturally on the treatment of the order-1 homological equation and enforces the last line in the matrix $\mathbf{f}^{(1)}$. Following the choice that imposes tangency to linear master modes, the upper left $d \times d$ block $\bar{\mathbf{f}}^{(1)}$ in Eq. (100) is easily found to be:

$$\bar{\mathbf{f}}^{(1)} = \mathrm{diag}(\lambda_1, \dots, \lambda_d). \tag{101}$$

Finally, the calculation proceeds without changing the awaited result for the physical degrees of freedom, and retrieves the tangency to the linear master modes. The last unknowns to be solved for are: the last

column of $\mathbf{W}^{(1)}$, which is denoted here as $\mathbf{W}^{(1,d+1)}$, and the d-dimensional vector $\mathbf{f}^{(1,d+1)}$ introduced in (100). Expanding the order-1 homological equation by isolating the remaining unknowns, yields:

$$\left(\tilde{\lambda}\mathbf{B}-\mathbf{A}\right)\mathbf{W}^{(1,d+1)}=\mathbf{C}-\mathbf{B}\mathbf{Y}\mathbf{f}^{(1,d+1)}. \tag{102}$$

The solution to this last equation is made difficult by the fact that there are two unknowns for a single equation, as well as by the fact that the left-hand side term can become singular in case of a primary resonance, if the forcing value $\tilde{\lambda}$, is aligned with one master eigenvalue. This problem is however classical in the context of the parametrisation method for invariant manifold, generalises to arbitrary order, and is solved by using the different styles of solution.

To set apart the case where the singularity might appear, one needs to introduce $\mathcal{R}^{(1,d+1)}$ as the set of modes that are in primary resonance with the forcing value $\tilde{\lambda}$. In short, $\mathcal{R}^{(1,d+1)}$ contains any r mode $\mathbf{Y}_r$ such that $\lambda_r \simeq \tilde{\lambda}$. For all $r \in \mathcal{R}^{(1,d+1)}$, the matrix $\tilde{\lambda}\mathbf{B}-\mathbf{A}$ is nearly singular, and its kernel has the dimension of the cardinality of $\mathcal{R}^{(1,d+1)}$. The components of $\mathbf{W}^{(1,d+1)}$ that are parallel to the kernel subspace must be set to zero, which generates an additional set of equations that have to be appended to Eq. (102) in order to make it solvable while imposing this vanishing condition:

$$\forall\, r \in \mathcal{R}^{(1,d+1)}, \quad \mathbf{X}_r^{\star}\mathbf{B}\mathbf{W}^{(1,d+1)}=0. \tag{103}$$

For the other non-resonant components, $r \notin \mathcal{R}^{(1,d+1)}$, the retained choice is to set

$$\forall\, r \in \mathcal{R}^{(1,d+1)}, \quad f_r^{(1,d+1)}=0. \tag{104}$$

The augmented solvable system combining Eqs. (102)-(103)-(104) finally reads:

$$\begin{bmatrix}\tilde{\lambda}\mathbf{B}-\mathbf{A} & \mathbf{B}\mathbf{Y}_{\mathcal{R}} & \mathbf{0}\\ \mathbf{X}_{\mathcal{R}}^{\star}\mathbf{B} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{W}^{(1,d+1)}\\ \mathbf{f}_{\mathcal{R}}^{(1,d+1)}\\ \mathbf{f}_{\not{\mathcal{R}}}^{(1,d+1)}\end{bmatrix}=\begin{bmatrix}\mathbf{C}\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix}. \tag{105}$$

where the two matrices $\mathbf{Y}_{\mathcal{R}}$ and $\mathbf{X}_{\mathcal{R}}$ containing respectively the resonant left and right eigenvectors have been introduced to simplify notations as

$$\mathbf{Y}_{\mathcal{R}}=\begin{bmatrix}\mathbf{Y}_{r_1} & \dots & \mathbf{Y}_{r_p}\end{bmatrix}, \quad \forall\, r_j \in \mathcal{R}^{(1,d+1)}, \tag{106a}$$

$$\mathbf{X}_{\mathcal{R}}=\begin{bmatrix}\mathbf{X}_{r_1} & \dots & \mathbf{X}_{r_p}\end{bmatrix}, \quad \forall\, r_j \in \mathcal{R}^{(1,d+1)}, \tag{106b}$$

and the notation for $\mathcal{R}^{(1,d+1)}$ has been abbreviated to $\mathcal{R}$ in the subscripts. Besides, the vector of unknown coefficients of the reduced dynamics has been split into two parts, by separating the resonant ones collecting all the indices belonging to $\mathcal{R}^{(1,d+1)}$, and denoted as $\mathbf{f}_{\mathcal{R}}^{(1,d+1)}$, from the non-resonant ones collecting the indices that do not belong to $\mathcal{R}^{(1,d+1)}$, and denoted as $\mathbf{f}_{\not{\mathcal{R}}}^{(1,d+1)}$.

Selecting order-p from the invariance equation, one obtains the associated homological equation of order p as:

$$\mathbf{B}\left[\nabla_{\mathbf{z}}\mathbf{W}(\mathbf{z})\mathbf{f}(\mathbf{z})\right]_p=\mathbf{A}\left[\mathbf{W}(\mathbf{z})\right]_p+\left[\mathbf{Q}(\mathbf{W},\mathbf{W})\right]_p. \tag{107}$$

The idea is to write this homological equation at the level of an arbitrary monomial $\mathbf{z}^{\boldsymbol{\alpha}(p,k)}$ defined by the vector of integers $\boldsymbol{\alpha}(p,k)=\{\alpha_1,\ \dots,\ \alpha_{d+1}\}$. To do so we need to isolate the unknown vectors $\mathbf{W}^{(p,k)}$ and $\mathbf{f}^{(p,k)}$ from the known ones calculated in previous instances of the iterative procedure. The method follows the previous developments reported in Sections 3.1 and 3.3, and is not detailed here again, see Vizzaccaro et al. (2024). Following the general guidelines, each homological equation of order p at the level of the arbitrary monomial (p,k) has the same structure, which reads:

$$\left(\sigma^{(p,k)}\mathbf{B}-\mathbf{A}\right)\mathbf{W}^{(p,k)}+\sum_{s=1}^{d}\mathbf{B}\mathbf{Y}_s f_s^{(p,k)}=\mathbf{R}^{(p,k)}, \tag{108}$$

where $\mathbf{R}^{(p,k)}$ aggregates all quantities generated by lower order monomials, and its full expression is not reported here for the sake of brevity, see Vizzaccaro et al. (2024). The second term appearing in Eq. (108) is $\sigma^{(p,k)}$, defined as:

$$\sigma^{(p,k)} = \sum_{s=1}^{d+1} \alpha_s \lambda_s. \tag{109}$$

Let us introduce $\mathcal{R}^{(p,k)}$ the resonant set, which collects all the r indexes such that the nonlinear resonance relationship $\lambda_r \simeq \sigma^{(p,k)}$ is fulfilled:

$$\mathcal{R}^{(p,k)} = \{r \in [1,d] \mid \lambda_r \simeq \sigma^{(p,k)}\}. \tag{110}$$

Importantly, the index r here covers only the master modes and does not contain the $(d+1)$-th term related to the forcing, following the choices made at the beginning to solve the non-autonomous problem. For an index $r \in \mathcal{R}^{(p,k)}$, the matrix $(\sigma^{(p,k)}\mathbf{B} - \mathbf{A})$ is nearly singular and its kernel has the same dimension as the cardinality of $\mathcal{R}^{(p,k)}$. The components of $\mathbf{W}^{(p,k)}$ parallel to the kernel subspace cannot be found from Eq. (108). For this reason, they must be set to zero, whatever the style used.This leads to considering the added equations:

$$\mathbf{X}_r^\star \mathbf{B} \mathbf{W}^{(p,k)} = 0, \quad \forall r \in \mathcal{R}^{(p,k)}. \tag{111}$$

This condition imposes $\mathbf{W}^{(p,k)}$ to be orthogonal to the kernel of $(\sigma^{(p,k)}\mathbf{B} - \mathbf{A})$. In a graph style parametrisation, these equations are sufficient since the choice of vanishing the coefficients of the nonlinear transform in the modal space is always retained. This is not the case in a normal form style parametrisation, since in this case, the idea is to simplify as much as possible the reduced dynamics by vanishing the coefficients for the non-resonant monomials, leading to

$$f_r^{(p,k)} = 0, \forall r \notin \mathcal{R}^{(p,k)}. \tag{112}$$

Finally, in order to propose direct computations that can be done from the physical space, and using a bordering technique that augments the size of the system to avoid singularities, leads to the following problem to be solved for a given monomial with arbitrary order p, by grouping Eqs. (108), (111) and (112):

$$\begin{bmatrix} \sigma^{(p,k)}\mathbf{B} - \mathbf{A} & \mathbf{B}\mathbf{Y}_\mathcal{R} & \mathbf{0} \\ \mathbf{X}_\mathcal{R}^\star \mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{W}^{(p,k)} \\ \mathbf{f}_\mathcal{R}^{(p,k)} \\ \mathbf{f}_{\not\mathcal{R}}^{(p,k)} \end{bmatrix} = \begin{bmatrix} \mathbf{R}^{(p,k)} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \tag{113}$$

where the two matrices $\mathbf{Y}_\mathcal{R}$ and $\mathbf{X}_\mathcal{R}$ have been introduced following Eq. (106), while $\mathbf{f}_\mathcal{R}^{(p,k)}$ contains the coefficients $f_r^{(p,k)}$, $\forall r \in \mathcal{R}^{(p,k)}$, and $\mathbf{f}_{\not\mathcal{R}}^{(p,k)}$ the coefficients $f_r^{(p,k)}$, $\forall r \notin \mathcal{R}^{(p,k)}$.

In summary, the DPIM has been here extended to handle the case of forced systems. A specific strategy has been detailed that allows to give a unified and unrestricted treatment of the non-autonomous term, whilst enforcing only slight modifications to the general algorithm. Indeed, most changes happen at the linear level to solve for the added coordinate representing the forcing, while generic order terms are treated in a very similar way, once the modifications in the representations (normal variable, nonlinear mappings) are done. The solution has been here derived for a generic first-order problem, and can be detailed for mechanical systems, see Vizzaccaro et al. (2024).

3.5 The `MORFE` project

The algorithmic developments associated to the direct parametrisation method for invariant manifolds, have been written in an open-source code called `MORFE` (Model Order Reduction for Finite Element

structures). It is an evolving research code that gathers the main features of the method, made available such that anybody can use the technique for its own purposes. It offers direct efficient and simulation-free reduction techniques for nonlinear vibrating structures.

The finite element part of the code has been derived for classical three-dimensional elements implementing geometric nonlinearity in the general context. It has been decided from the beginning to rely on free softwares only, and the `julia` language has been selected for the project. Different versions are now available at the github repository:
`https://github.com/MORFEproject/`.

The first version was released in 2022, and has been mainly coded by Andrea Opreni during his PhD thesis, and conceptualised by Andrea Opreni and Alessandra Vizzaccaro. The main `julia` function has been called `MORFEInvariantManifold.jl`. It has been used in order to derive the numerical results shown in Vizzaccaro et al. (2022); Opreni et al. (2023b); Martin et al. (2023), and the files to launch these examples can be found in the repository.

An important improvement has been brought to the code and led to a new version called `MORFE2.0`, which has been uploaded and publicly released in 2023. Fundamental changes as polynomial representations have been done, and the non-autonomous general treatment shown in Section 3.4 to handle forced system, has been implemented. This version has been mainly coded by Attilio Frangi, based on a preliminary version of Alessandra Vizzaccaro, and run for examples by Giorgio Gobat and Alessio Colombo.

The last update in the repository was brought about in 2024, by uploading a symbolic version called `MORFE_Symbolic`. The idea is to show the realm of the method in terms of predictions and analytical developments. The results obtained using such treatment are shown in de Figueiredo Stabile et al. (2024). This code has been mainly developed by Alessandra Vizzaccaro and André Stabile. Note that a `Mathematica` version is also made available.

4 Applications

In this Section, numerical results on different academic test cases are reported. The aim is to illustrate the potential of the reduction method in providing accurate predictions with a minimal number of master modes. Indeed, in the three selected cases, reduction to a single NNM is performed.

4.1 A clamped-clamped arch with increasing curvature

The first selected example is a clamped-clamped beam whose curvature will be increased in order to transform it into an arch with a given rise. It is well known that systems like straight beams and plates have a symmetric restoring force with respect to the mid-line or mid-plane, and that they display a hardening behaviour in the nonlinear regime (Nayfeh and Pai (2004); Amabili (2008); Touzé et al. (2002)). For systems with initial curvature such as arches and shells, the restoring force is no longer symmetric and presents a quadratic term, which is known for producing a softening characteristic. The balance between these two nonlinear effects depends both on the curvature and the mode studied, such that predicting the type of nonlinearity of curved structure has long been a difficult task in nonlinear vibrations, see for examples the discussions reported in Rega et al. (2000); Pellicano et al. (2002); Arafat and Nayfeh (2003); Touzé and Thomas (2006). Fortunately, the normal form approach allows to offer a correct prediction by an efficient reduction, as shown for example in Touzé and Thomas (2006); Touzé et al. (2008).

This example aims at showing that such a difficult case with a change in nonlinear behaviour, can be very efficiently computed using the reduction to a single master mode. The selected clamped-clamped arch for the numerical computations has the following dimensions: a length L = 640 μm, a thickness h=6.4 μm, and a width b=32 μm. It is made of isotropic polycrystalline silicon with Young modulus 160 GPa, Poisson ratio 0.22, and density 2320 kg/m^3. The structure is meshed with 15-nodes quadratic wedge elements. The mesh contains 1161 nodes and 3 degrees-of-freedom per node resulting in 3483 DOFs.

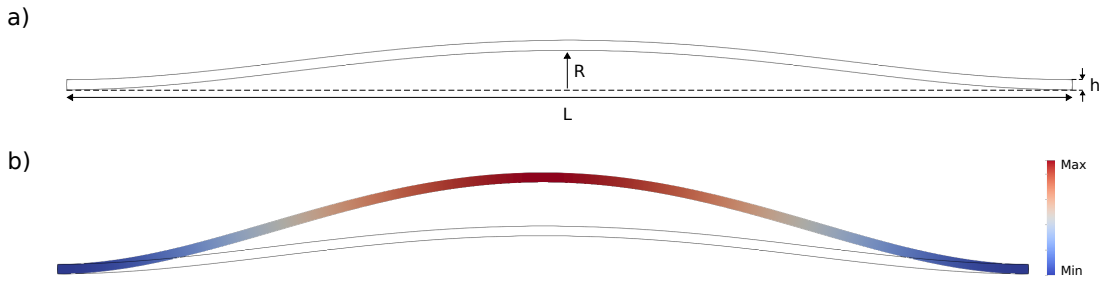

Figure 5: A clamped-clamped arch of length L and thickness h with increasing curvature parametrized by the rise R of the midpoint. First eigenmode shape.

The curvature is handled by increasing the rise parameter R that measures the deformation of the center point of the beam. It is assumed that the beam is statically deformed with the shape of the first vibration mode, see Fig. 5, which also shows the first vibration mode. The master mode in each of the cases studied is selected as the fundamental mode. Four test cases corresponding to different values of the rise R are selected and reported in Table 1. The first case, denoted as `Ref`, corresponds to the straight beam. The three other cases, I, II and II, have an increasing rise with a maximum for case III as $R = 3.36$ μm. Note that in all cases, the rise parameter is small and the arches are shallow. The most important curvature selected corresponds to a rise that is close to half the thickness of the beam only.

Geometry	`Ref`	I	II	III
Rise [μm]	0.00	2.4	2.88	3.36

Table 1: Selected values of the arch rise parameter R selected for the four test cases.

The results are reported in Fig. 6. The backbone curves for the four cases are computed, and reduction to a single NNM in the autonomous case, following the method presented in Section 3.3, are compared to a reference solution obtained by numerical continuation directly implemented on the full order FE model with 3483 DOFs, thus resulting in an important computational time. The ROMs are computed using a real normal form style (RNF).

The reference case reported in Fig. 6(a) displays a hardening behaviour. In this case, the third-order expansion already gives an excellent prediction, and convergence up to half the beam's thickness is obtained with an order 5. On the other hand, third-order expansions fail for all the arch structures under consideration. For instance, even for the purely hardening arch I, whose backbone curve is reported in Fig. 6(b), the presence of quadratic terms makes low-order expansions deviate from the reference solution.

The effectiveness of low-order expansions is further reduced for arches with higher rise values. Indeed, arches II and III actually show an initially softening behaviour, followed by a hardening response at higher amplitudes, a behaviour that is totally missed by a third-order parametrisation, which diverges towards smaller frequency values. On the other hand, higher-order parametrisations can correctly capture this transition and a very good agreement with the reference HBFEM solution is achieved with asymptotic expansions of order 13-15.

This example highlights the efficacy of the direct parametrisation method for invariant manifolds in producing accurate predictions for nonlinear vibrating systems. Reduction to a single NNM is particularly meaningful in this case and the transition from hardening to softening behaviour is perfectly recovered thanks to the arbitrary order expansion.

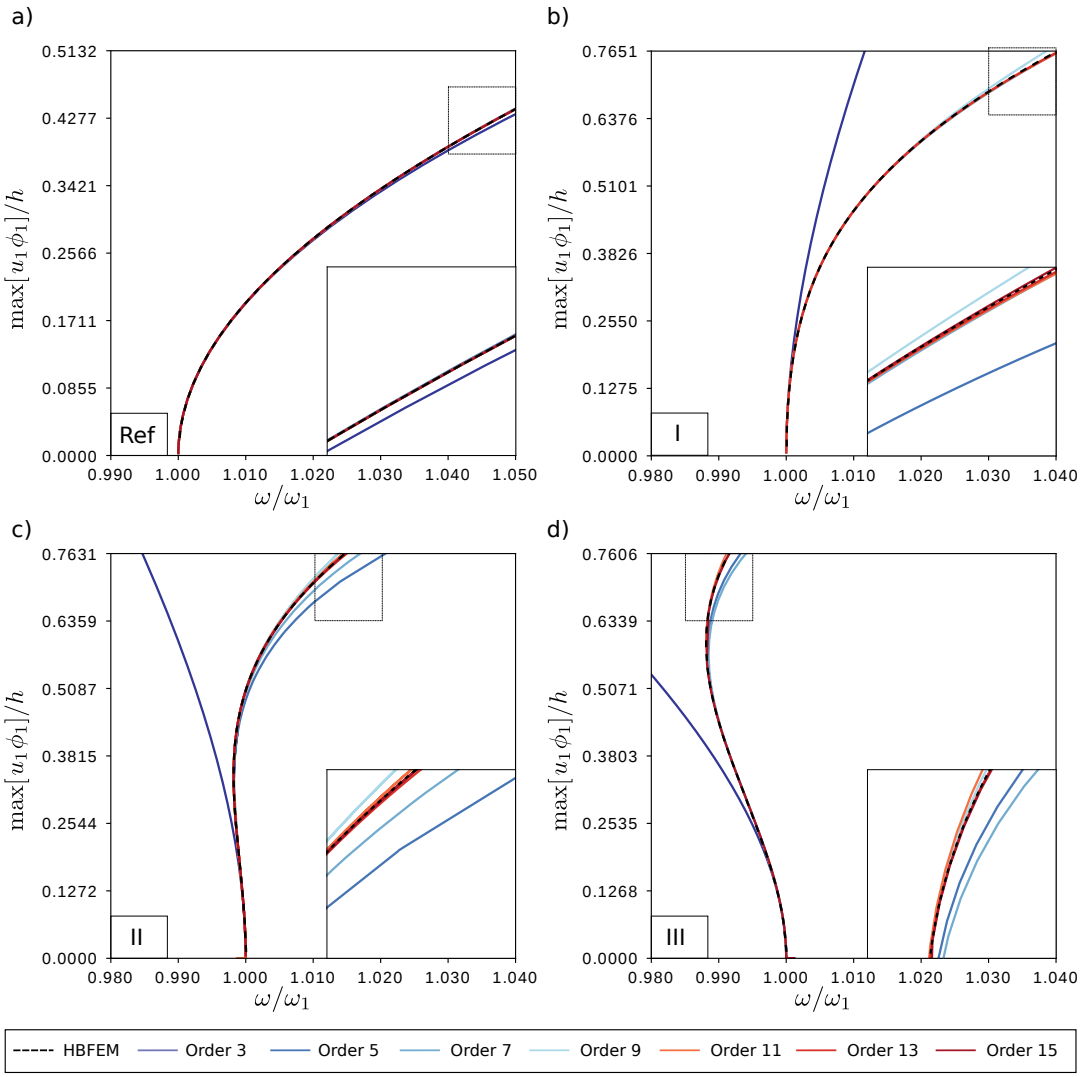

Figure 6: Backbone curves for the four arch structures with increasing curvature. The reference solution is denoted as HBFEM (Harmonic Balance technique with numerical continuation procedure, directly on the full-order FE model). It is compared to ROMs using a single NNM with increasing orders. Adapted from Vizzaccaro et al. (2022).

4.2 The cantilever beam

The second example is a cantilever beam. It will be used in order to illustrate the differences between the two main styles of parametrisation, the graph style and the normal form style. Clamped-free beams are known for displaying inertia nonlinearity, see *e.g.* Crespo da Silva and Glynn (1978); Touzé et al. (2021); Debeurre et al. (2023). Besides, since the axial motion is left unconstrained, the geometric nonlinearity comes mainly from the large rotations of the cross-section, and thus has a significant contribution to the dynamics once the transverse displacement is of the order of the length (first nonlinear effects are observable for, say, 1/20 to 1/10 the length). The cantilever beam has also been known to be a difficult case for model order reduction techniques, see *e.g.* Shen et al. (2021b) and references therein. The results shown in this section also explain the failure of numerous methods in this specific case.

The selected beam is one meter long, with a thickness of 2 cm and a width of 5 cm. It is made of Titanium with Young modulus 104 GPa, Poisson ratio 0.3, and density 4400 kg/m^3. The beam is

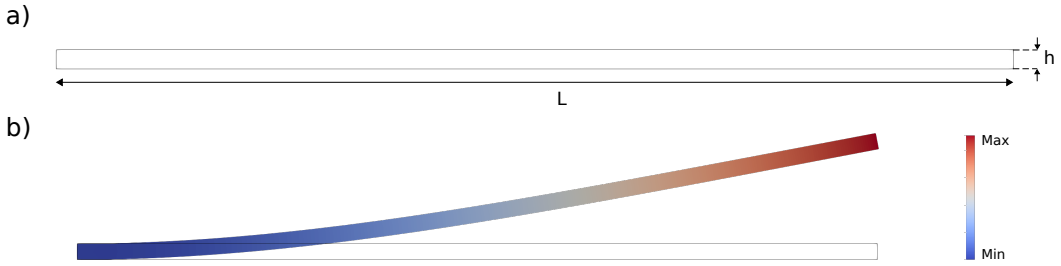

Figure 7: A cantilever beam of length L and thickness h. First eigenmode shape.

discretized with 3D finite elements that are 15-node quadratic wedge elements. The mesh contains 621 nodes, resulting in a model with 1863 DOFs. The interest is in the computation of the backbone curve of the fundamental mode, whose deformed vibration shape is shown in Fig. 7. Therefore reduction to a single NNM corresponding to the lowest frequency mode, is used to propose a ROM, using the DPIM for autonomous mechanical system as proposed in Section 3.3.

The computation of the backbone curves is reported in Fig. 8. A reference solution is obtained by using numerical continuation on the full-order model. This solution is compared to ROMs constructed with a single NNM and increasing orders of the asymptotic expansion. Orders ranging from 3 to 25 are shown. Finally, three different styles are compared: the graph style parametrisation, and two variants of the normal form style: CNF and RNF (see Section 3.2). The main observation from the obtained results is the very slow convergence of the graph style solution as compared to both CNF and RNF. Besides, it appears that there is a limit in terms of amplitude, where the graph style solution is not able to converge anymore and instead starts abruptly diverging. On the other hand, the normal form style solutions smoothly converge to the actual solution with increasing orders, as awaited.

In order to better understand the limitation faced by the graph style in this case, Fig. 9 shows the shape of the NNM in phase space, computed by both the graph style (first line) and real normal form style (second line). The results given using CNF is not shown for the sake of brevity, being equivalent to the RNF case (see Vizzaccaro et al. (2022)). The striking observation is that the manifold corresponding to the first NNM encounters a folding at large amplitude. This folding point constitutes an intrinsic limitation for the graph style parametrisation. Due to the choice retained for the coordinates used to parameterize the manifold, it cannot pass over foldings. This explains the incorrect behaviour of the solution provided by the graph style and the limitation observed on the backbone curves. On the other hand, the normal form style can pass over the folding and ensures accurate and converged results up to larger amplitudes. This folding of the manifold also explains why the cantilever beam has long been a difficult case for model reduction techniques. Indeed, whatever the technique used, it has to face the complex geometry of the associated LSM in phase space with this folding point. Most of the methods used to tackle this difficulty resorts to an important augmentation of the size of the ROM, in order to achieve a sufficiently large phase space reduced model that is able to unfold the solutions and approximate them accurately. The normal form style is in this case perfectly suited to provide excellent results up to large amplitudes, while maintaining the dimension of the ROM at its smallest value (only one master mode).

4.3 Superharmonic resonance

The last case under study considers a non-autonomous system. A clamped-clamped beam, harmonically forced around one third the eigenfrequency of the fundamental mode, is considered. This case corresponds to a 3:1 superharmonic resonance. Solutions at primary resonance could have also been shown, and examples are reported *e.g.* in Opreni et al. (2023b); Martin et al. (2023). The 3:1 superharmonic resonance is here selected because only the high-order treatment on the forcing term, as proposed in Section 3.4

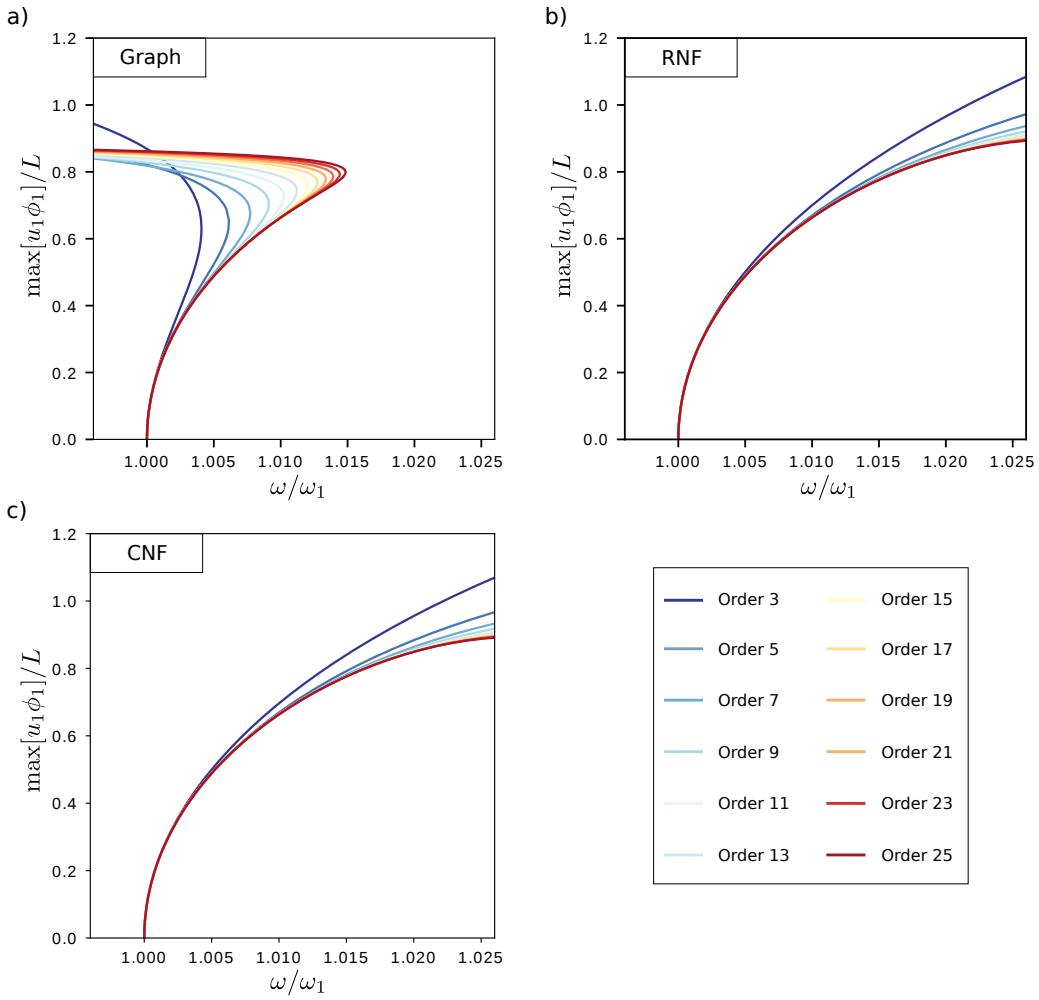

Figure 8: Backbone curves for the cantilever beam. Comparisons of the results obtained with different styles of parametrisation: (a) graph style, (b) real normal form style, (c) complex normal form style; for increasing orders. Reference full-order solution in black. Adapted from Vizzaccaro et al. (2022).

can handle such a case. Indeed, earlier treatments of the non-autonomous case assumed a first-order development on the forcing term, see *e.g.* Jain and Haller (2022); Opreni et al. (2023b). With such an assumption, superharmonic resonances are not considered in the possible resonance scenario, such that non-autonomous ROMS for those cases are not available.

The selected beam has the following dimensions: $L = 1000$ μm, width $B = 24$ μm, and thickness $H = 10$ μm. It is made of polycrystalline silicon, which is modelled as an isotropic material with density $\rho = 2320$ kg/m^3, Young's modulus $E = 160$ GPa and Poisson ratio $\nu = 0.22$. The reference finite element model consists of a mesh of 15-nodes quadratic wedge elements with 2607 nodes, yielding a system with 7821 DOFs. The forcing frequency Ω is selected in the vicinity of one-third the fundamental bending eigenfrequency, denoted as ω_{B1}. A reference solution is obtained thanks to a direct numerical continuation of the FE model. It is compared to the solutions obtained by reducing the problem to a single NNM, where the resonance relationship $\Omega = \omega_{B1}/3$ is taken into account to compute the time-dependent invariant manifold. Note that, in the present case, the coefficients of the ROM are computed for a single value of the forcing frequency. This ROM is then used for numerical continuation with small variations of the excitation frequency in the vicinity of the superharmonic response. A Rayleigh dissipation model is assumed for the damping matrix, and mass-proportional damping is selected as

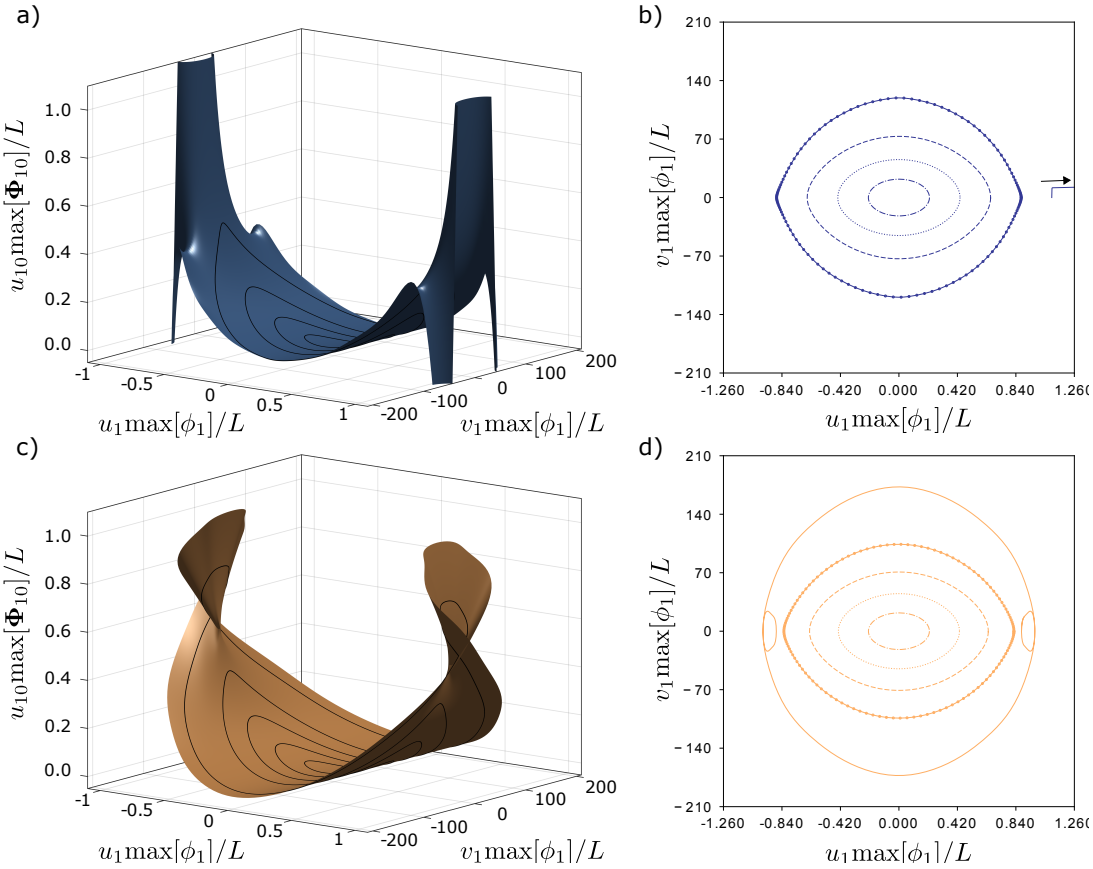

Figure 9: First NNM (invariant manifold) in phase space for the cantilever beam. First line: results obtained using a graph style parametrisation. (a) 3-d view in modal coordinates (u_1, v_1, u_{10}), where mode 10 is the first axial model. (b) 2-d representation of periodic orbits. The arrow indicates the failure of the method with a diverging orbit. Second line: real normal form style, showing that the folding of the manifold is correctly reproduced by the method. Adapted from Vizzaccaro et al. (2022).

$\mathbf{C} = \alpha\mathbf{M}$, with $\alpha = \omega_{B1}/500$. Finally, the spatial distribution of the forcing is assumed to have the shape of the fundamental eigenmode.

The results for the frequency response curves are reported in Fig. 10. ROMS with increasing orders are compared to the reference solution, which shows the typical loop of the superharmonic resonance. The orders of the truncations for the ROMs are denoted as $\mathcal{O}(z^p, \varepsilon^q)$, in order to distinguish the order used for the asymptotic expansion in the normal coordinate z, and the order of the non-autonomous coordinate $\tilde{z}$, that also corresponds to the order of the assumption in terms of asymptotic expansion along the ε forcing term. For example, $\mathcal{O}(z^3, \varepsilon^1)$ corresponds to a maximum order of 3 for the normal coordinate z, together with a first-order assumption on the forcing term, such that $\tilde{z}$ has a maximum order of 1 in all the monomials of the ROM. This specific case with an ε^1 truncation on the non-autonomous terms, corresponds to the ROMS derived for example using the methods proposed in Jain and Haller (2022); Opreni et al. (2023b). As announced, they are not able to catch the superharmonic resonance.

On the other, from an order $\mathcal{O}(z^3, \varepsilon^3)$, meaning that up to the third order is selected for the forcing, is sufficient to recover the 3:1 superharmonic resonance. This ROM is thus able to retrieve the loop of the superharmonic resonance, nevertheless, it is not converged in terms of accuracy of amplitudes as compared to the reference solution. Increasing the orders to $\mathcal{O}(z^5, \varepsilon^5)$, $\mathcal{O}(z^7, \varepsilon^7)$ and $\mathcal{O}(z^9, \varepsilon^9)$, shows how the ROM solution converges to the reference.

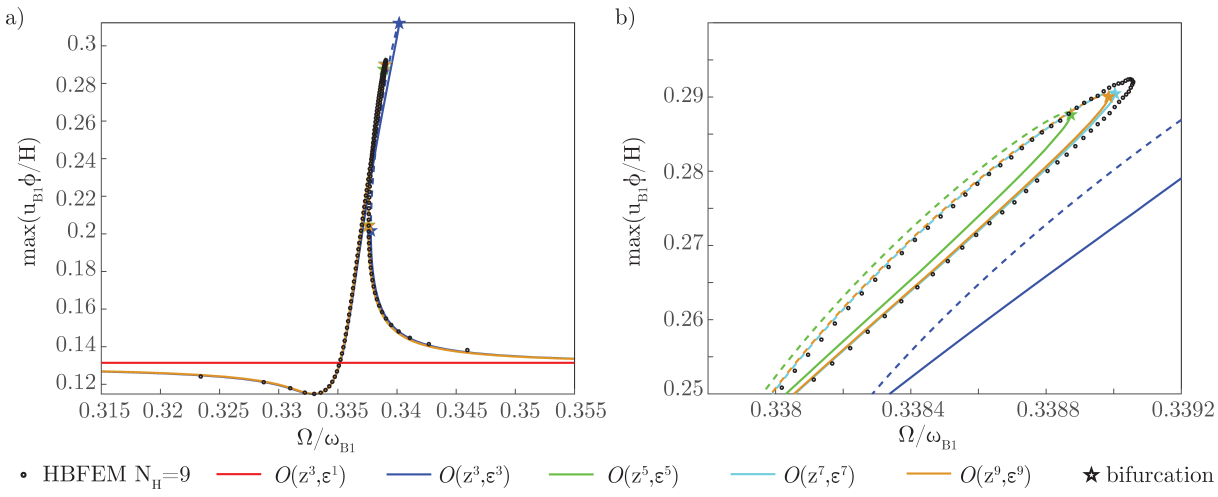

Figure 10: Superharmonic resonance for a clamped-clamped beam with harmonic forcing in the vicinity of 1/3 the fundamental bending eigenfrequency ω_{B1}. Full-order solution (HBFEM) compared to single-NNM ROM with increasing orders. For the ROMs, stable solutions are reported with plain lines while unstable solutions are plotted with dashed lines. Adapted from Vizzaccaro et al. (2024).

4.4 Computational performance

This Section aims to give an idea of the computational gains that are obtained thanks to using the DPIM for producing ROMs for finite element problems. The comparisons are drawn out on the computing times needed to obtain a frequency response curve (or a backbone curve), on one side for the reference solution, implementing HBFEM directly on all the DOFs of the FE model, as compared to the ROMs. The number of degrees of freedom in each case gives a first idea of the impressive gains that are awaited when reducing the dynamical solutions to invariant manifolds. Indeed, the number of DOFs of the FE models shown in Sections 4.1, 4.2 and 4.3 are respectively 3483, 1863 and 7821. On the other hand, each ROM considered has been obtained thanks to the reduction to a single NNM and thus only contains two DOFs. Even if the meshes employed here on academic examples are coarse, resulting in a small number of DOFs as compared to industrial practices, the method has been tested with refined mesh up to millions of DOFs.

The computing times need to distinguish the time needed to construct the ROM, and the simulation time for one run of numerical continuation. Since the DPIM is a simulation-free method that directly works on the mesh, the constructing time is once and for all. On the other hand, numerical continuations can be performed for varying values of bifurcation parameter and many different queries can be done.

Numerical values of computing times are reported in Fig. 11. The mesh used for this test is a MEMS (Micro Electro Mechanical System) structure, analyzed and tested in Vizzaccaro et al. (2022); Opreni et al. (2023a). In Fig. 11(a), the time needed to construct a ROM with 9732 DOFs and increasing orders in the polynomial expansion of the autonomous variable, is shown. All the analyses were performed on a desktop workstation with an AMD Ryzen 5950X processor at 4.9 GHz and 128GB RAM. One can see that order 9 needs a bit more than 2 minutes, which is still small for a ROM obtained once and for all, given the fact that all the results presented underline that order 9 is generally a good value that offers a comfortable accuracy and convergence. Increasing the orders again, the computing times also increase with for example around 15 minutes for an order 13 expansion and 32 minutes for order 15.

Fig. 11(b) shows both the constructing time and the memory requirement for order 5 truncation, while increasing the number of DOFs of the mesh. For this example, meshes with up to 5 millions DOFs have been tested. One can observe an increasing computational burden, which remains however accessible on a standard workstation.

Once the ROM was obtained (in a generally named *offline stage*, which is done only once for such

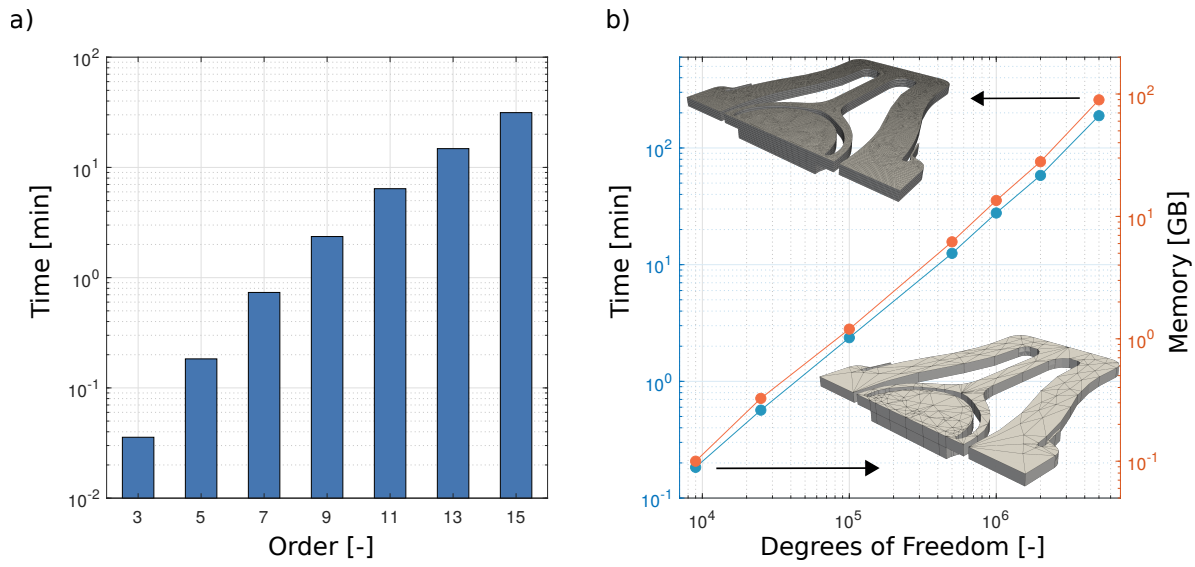

Figure 11: Computational times illustrating the DPIM performance. (a) Computing time to construct an autonomous ROM while increasing the order, for a mesh having 9732 DOFs. (b) computing construction time and memory requirements for an order 5 ROM with increasing number of DOFs in the mesh.

a simulation-free technique), the online stage consists in using the ROM to produce FRCs or backbone curves. For the reference solution obtained with HBFEM, the order of magnitude of the computing times for a model with approximately 10 000 DOFs is more than one full day, and can be one day and half, and sometimes two days for difficult cases with bifurcation points where the continuation technique might slow down. On the other hand, running continuation on the ROM leads to almost immediate solutions. If a backbone curve is sought and CNF has been used, there is no computing time since the backbone curve is analytic from the coefficients of the ROM. To obtain FRCs in different loading scenarios, our experience shows that the typical computing time is of the order of 1 minute, thus resulting in an impressive gain as compared to the full-order HBFEM solution.

5 Conclusions

In this chapter, nonlinear normal modes, defined as invariant manifolds tangent to their linear counterpart at origin, have been introduced for model-order reduction. Definitions for conservative and dissipative vibrating systems have been reminded. The direct parametrisation method for invariant manifold has been shown as an efficient simulation-free method allowing one to obtain accurate ROMs with minimal dimensions.

The techniques employed make extensive use of the tools provided by the dynamical system theory and the geometry in phase space. In this realm, the parametrisation method proved to be a fundamental tool, that allows linking previously developed techniques, together with deriving high order algorithms. This method has already been used in many different contexts, see *e.g.* van den Berg and James (2016); Gonzalez et al. (2022); Bihan et al. (2017). In vibration theory, it has been used for beams and plates in Jain et al. (2018); Ponsioen et al. (2020); Li et al. (2022); Li and Haller (2022); and for rotor-foundation systems exhibiting instabilities in Mereles et al. (2023). Examples on beams and realistic MEMS structures are reported in Vizzaccaro et al. (2022); Opreni et al. (2023b), while the case of rotating structures including a twisted plate mimicking a fan blade is investigated in Martin et al. (2023). All these results advocate for a broad use of the technique as it provides accurate ROMs for nonlinear vibrating structures.

Importantly, other reduction techniques used in the realm of geometric nonlinearity, have been compared to invariant manifold-based techniques in different papers, underlining that using the NNMs achieves the best accuracy while providing the smallest dimensional ROMs. In particular, the Proper Orthogonal Decomposition (POD) has been compared to NNMs in Amabili and Touzé (2007), underlining that the curvature of the NNMs allows decreasing the size of the reduced subspace. The modal derivative (MD) approach has been compared to NNM-based reduction techniques in Vizzaccaro et al. (2021a); Touzé et al. (2021). Using the oscillator normal form, it has been emphasized that modal derivatives are a simplification of the more general results given using normal form approach, and that MD can produce inaccurate results if a slow-fast assumption is not fulfilled between the slave and master eigenfrequencies. Finally, the implicit condensation has been compared to NNM-based ROMs in Shen et al. (2021a,b), underlining also that IC fails in case the slow/fast assumption is not met. IC and MD generally use static assumptions at some point of their derivation, which in turn lead to limitations that are overcome by invariant manifold techniques.

Next developments of the method should handle different physical phenomena, to extend the applicability range of the method to other problems in engineering. In the field of MEMS, the first promising results have been obtained in Opreni et al. (2023a); Frangi et al. (2023) (see also chapter 2 by A. Frangi presenting further developments), showing how one can couple the method with piezoelectric terms, or take into account the electro-mechanical coupling, in order to offer accurate ROMs for real physical structures.

Acknowledgments

The authors would like to thank their close collaborators who provided precious contributions in many of the applicative examples shown in this chapter. In particular, Attilio Frangi is thanked for providing an efficient finite element environment to apply the parametrisation method, as well as providing challenging testbeds with MEMS structures. Andrea Opreni is thanked for all the first developments he conducted on the coding of the method, which gave rise to the first version of `MORFE`. The numerical examples shown in Sections 4.1-4.2 have been run by Andrea who made an excellent work to provide such efficient codings. Giorgio Gobat is thanked for his precious help in developing all the numerical simulations related to the example shown in Section 4.3. Olivier Thomas, Loic Salles and Aurélien Grolet are thanked for uncountable discussions related to nonlinear dynamics and reduction techniques.

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