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Persistence Diagram Estimation : Beyond Plug-in Approaches

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Abstract

Persistent homology is a tool from Topological Data Analysis (TDA) used to summarize the topology underlying data. It can be conveniently represented through persistence diagrams. Observing a noisy signal, common strategies to infer its persistence diagram involve plug-in estimators, and convergence properties are then derived from sup-norm stability. This dependence on the sup-norm convergence of the preliminary estimator is restrictive, as it essentially imposes to consider regular classes of signals. Departing from these approaches, we design an estimator based on image persistence. In the context of the Gaussian white noise model, and for large classes of piecewise-Hölder signals, we prove that the proposed estimator is consistent and achieves minimax rates. Notably, these rates coincide with the well known minimax rates for Hölder continuous signals.

Introduction

Topological Data Analysis (TDA) is a field that aims to provide representations to describe the “shape” of data. A central tool in TDA is persistent homology and its representation through persistence diagrams. It offers a practical, multiscale, summary of the topology underlying data. Although in its early years research has primarily focused on deterministic settings, it didn’t take long for statistical questions to emerge. A prominent topic is the estimation of persistent diagram and, more specifically, the investigation of convergence properties of considered estimators. A first historical success in this direction is [Bubenik and Kim \(2006\)](#), that formalizes the problem of persistence diagram estimation from a density in a parametric setting. Following this initial work, in wider non-parametric settings, [Fasy et al. \(2014\)](#) provide confidence sets for persistence diagrams from a sampled density and [Chazal et al. \(2014\)](#) provide (nearly) minimax estimators of the density support controlling its regularity. In the slightly different setting of non-parametric regression, [Bubenik et al. \(2009\)](#) study a plug-in estimator based on kernel estimation and show that it achieves minimax rates over Hölder spaces. Most of these works, consider plug-in estimators and the proofs for convergence properties then exploit stability theorem ([Chazal et al., 2009a](#)) for sup norm, lifting results from non-parametric estimation or geometric inference. Although this approach can be proved to be minimax over regular classes (e.g. Hölder spaces), it falls short for function spaces where sup norm convergence is not assured. This raises the issue of whether consistent persistence diagram estimation is achievable outside these regular classes and at which rates. A first step in this direction is the work of [Bobrowski et al. \(2017\)](#). This work proposes a general, non-plug-in

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approach in both the context of non-parametric regression and the density model. Under a weak condition (q -tameness) they show some near-consistency properties of their estimators. But they do not provide convergence rates. The challenge likely stems from the framework being too broad to quantify convergence rates or to demonstrate proper consistency. Hence, it is interesting to identify narrower classes on which we can provide stronger and more precise results. Recently, in [Henneuse \(2024\)](#), we proposed a new approach that also departs from the sup norm stability. This work study the inference of persistence diagram, from a minimax perspective, for Gaussian white noise model and non-parametric regression. Although we consider a plug-in estimator, our analysis of convergence properties did not rely on sup-norm stability (but on a weaker notion of stability). We introduced and studied classes of piecewise-Hölder continuous function with discontinuities set having a positive reach. The reach ([Federer, 1959](#)) is a popular curvature measure in geometric inference, that can be thought as a way to describe the geometric regularity of a set. Under this reach assumption, we manage to show that a histogram plug-in estimator permits to achieve the well known minimax rates for Hölder-continuous functions. Still, the positive reach assumption imposes some limitation on the shape of discontinuity sets, typically it does not allow corners or multiple points (i.e. self intersections of the discontinuities set). Relaxation of this assumption was already discussed in this earlier work. We highlighted that it would imply to consider other estimators, as, even in the noiseless setting, the histogram approximation falls short on simple examples (see [Figure 6](#)). This serves as a motivation to depart from plug-in approach for persistence diagram inference and as starting point for this new work.

We consider the Gaussian white noise model given by the following stochastic equation,

$$dX_{t_1, \dots, t_d} = f(t_1, \dots, t_d) dt_1 \dots dt_d + \theta dW_{t_1, \dots, t_d}$$

with W a d -parameters Wiener field, $f : [0, 1]^d \rightarrow \mathbb{R}$ a signal and $\theta \geq 0$ the level of noise. In this context, our goal is to estimate $\text{dgm}(f)$, the persistence diagram coming from the sublevel sets of f .

For a set $A \subset [0, 1]^d$, \bar{A} we denote its adherence and ∂A its boundary. We suppose that f verifies the following assumptions :

- **A1.** f is a piecewise (L, α) -Hölder-continuous function, i.e. there exist M_1, \dots, M_l open sets of $[0, 1]^d$ such that,

$$\bigcup_{i=1}^l \bar{M}_i = [0, 1]^d$$

and $f|_{M_i}$ is in $H(L, \alpha)$, $\forall i \in \{1, \dots, l\}$, with,

$$H(L, \alpha) = \left\{ f : [0, 1]^d \rightarrow \mathbb{R} \text{ s.t. } |f(x) - f(y)| \leq L \|x - y\|_2^\alpha, \forall x, y \in [0, 1]^d \right\}.$$

- **A2.** f verifies, $\forall x_0 \in [0, 1]^d$,

$$\liminf_{x \in \bigcup_{i=1}^l M_i \rightarrow x_0} f(x) = f(x_0).$$

In this context, two signals, differing only on a null set, are statistically undistinguishable. And persistent homology is sensitive to point-wise irregularity, two signals differing only on a null set can have persistence diagrams that are arbitrarily far. Assumption **A2** prevents such scenario.

- **A3.** Let $1 \geq \mu > 0$ and $R_\mu > 0$. For all $I \subset \{1, \dots, l\}$ and $x \in \bigcap_{i \in I} \overline{B}_2(\partial M_i, R_\mu)$, there exists a half cone C_x with apex x of angle θ verifying $\cos(\theta/2) \geq \mu$ and,

$$\bigcup_{i \in I} \Gamma_{\partial M_i}(x) \subset C_x$$

with $\Gamma_{\partial M_i}(x)$ the set of closest point of x in ∂M_i . This assumption gives control over the geometry of the discontinuities and can be compared to a positive μ -reach assumptions (Chazal et al., 2006). The link with μ -reach is discussed in Section 1.2.

The class of function verifying **A1**, **A2** and **A3** is denoted $S_d(L, \alpha, \mu, R_\mu)$. For our purpose, we show in Appendix A that persistence diagrams of signals in $S_d(L, \alpha, \mu, R_\mu)$ are well-defined. This class is significantly larger than the one considered in Henneuse (2024), allowing multiple points and corners in the discontinuities set.

Contribution

We extend the main result of Henneuse (2024). Unlike the classes considered there, over the weaker classes $S_d(L, \alpha, \mu, R_\mu)$, a plug-in estimator from histogram estimation is no longer consistent (see Figure 6). To overcome this issue, we propose, $\widehat{\text{dgm}}(f)$ a non-plug-in estimators of the persistence diagram of f . Over $S_d(L, \alpha, \mu, R_\mu)$, we show that this estimator is consistent and achieves convergence rates coinciding with known minimax rates for Hölder-continuous signals. More precisely, our main result is the following theorem.

Theorem 1. *There exists \tilde{C}_0 and \tilde{C}_1 such that, for all $t > 0$,*

$$\mathbb{P} \left(\sup_{f \in S_d(L, \alpha, \mu, R_\mu)} d_b \left(\widehat{\text{dgm}}(f), \text{dgm}(f) \right) \geq t \left(\theta^2 \log \left(\frac{1}{\theta} \right) \right)^{\frac{\alpha}{d+2\alpha}} \right) \leq \tilde{C}_0 \exp \left(-\tilde{C}_1 t^2 \right).$$

As these rates are minimax under stronger conditions (see Henneuse, 2024, Theorem 3), they are evidently minimax over $S_d(L, \alpha, \mu, R_\mu)$.

The paper is organized as follows. Section 1 recalls the necessary background on geometric measure theory and persistent homology, Section 2 describes our estimation procedure, and Section 3 is dedicated to the proof of Theorem 1. Secondary results as long as proofs of technical lemmas can be found in appendix.

1 Background

This section provides the necessary background to follow this paper.

1.1 Persistent Homology

We here present briefly some notions related to persistence homology. Persistent homology permits to encode the evolution of topological features (in the homology sense) along a family of nested spaces, called filtration. Moving along indices, topological features (connected components, cycles, cavities, ...) can appear or die (existing connected components merge, cycle or cavities are filled, ...). For a broader overview and visual illustrations of persistent homology, we recommend Chazal and Michel (2021). For detailed and rigorous constructions, see Chazal et al. (2016). Additionally, since the construction discussed here involves (singular) homology, the reader can refer to Hatcher (2000).

Definition 1. Let $\Lambda \subset \mathbb{R}$ be a set of indices. A **filtration** over Λ is a family $(\mathcal{K}_\lambda)_{\lambda \in \Lambda}$ of topological spaces satisfying, $\forall \lambda, \lambda' \in \Lambda, \lambda \leq \lambda'$,

$$\mathcal{K}_\lambda \subset \mathcal{K}_{\lambda'}.$$

A typical filtration that we will consider in this paper is, for a function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, the family of sublevel sets $(\mathcal{F}_\lambda)_{\lambda \in \mathbb{R}} = (f^{-1}([-\infty, \lambda]))_{\lambda \in \mathbb{R}}$. The associated family of homology group of degree $s \in \mathbb{N}$, $\mathbb{V}_{f,s} = (H_s(\mathcal{F}_\lambda))_{\lambda \in \mathbb{R}}$, equipped with $v_\lambda^{\lambda'}$ the linear application induced by the inclusion $\mathcal{F}_\lambda \subset \mathcal{F}_{\lambda'}$, for all $\lambda \leq \lambda'$, forms a persistence module. To be more precise, in this paper, $H_s(\cdot)$ is the singular homology functor in degree s with coefficient in a field (typically $\mathbb{Z}/2\mathbb{Z}$). Hence, $H_s(\mathcal{F}_\lambda)$ is a vector space.

Still, the estimator we propose in Section 2, relies on image module (Cohen-Steiner et al., 2009), that does not come (directly) from sublevel sets filtration. Hence, we need the following, more general, definition.

Definition 2. Let $\Lambda \subset \mathbb{R}$ be a set of indices. A **persistence module** over λ is a family $\mathbb{V} = (\mathbb{V}_\lambda)_{\lambda \in \Lambda}$ of vector spaces equipped with linear application $v_\lambda^{\lambda'} : \mathbb{V}_\lambda \rightarrow \mathbb{V}_{\lambda'}$ such that, $\forall \lambda \leq \lambda' \leq \lambda'' \in \Lambda$,

$$v_\lambda^{\lambda'} = id$$

and

$$v_{\lambda'}^{\lambda''} \circ v_\lambda^{\lambda'} = v_\lambda^{\lambda''}.$$

Under q -tameness of the persistence module, it is possible to show that the algebraic structure of the persistence module encodes exactly the evolution of the topological features along the indices Λ .

Definition 3. A persistence module \mathbb{V} is said to be **q -tame** if $\forall \lambda < \lambda' \in \Lambda, \text{rank}(v_\lambda^{\lambda'})$ is finite. By extension, when considering the persistence modules $(\mathbb{V}_{f,s})_{s \in \mathbb{N}}$ coming from the sublevels sets filtration of a real functions f , we say that f is q -tame if $\mathbb{V}_{f,s}$ is for all $s \in \mathbb{N}$.

Furthermore, the algebraic structure of the persistence module can be summarized by a collection $\{(b_i, d_i), i \in I\} \subset \overline{\mathbb{R}}^2$, which defined the **persistence diagram**. Following previous remarks, b_i corresponds to the birth time of a topological feature, d_i corresponds to its death time and $d_i - b_i$ to its lifetime. For a detailed construction of persistence diagrams, see Chazal et al. (2016).

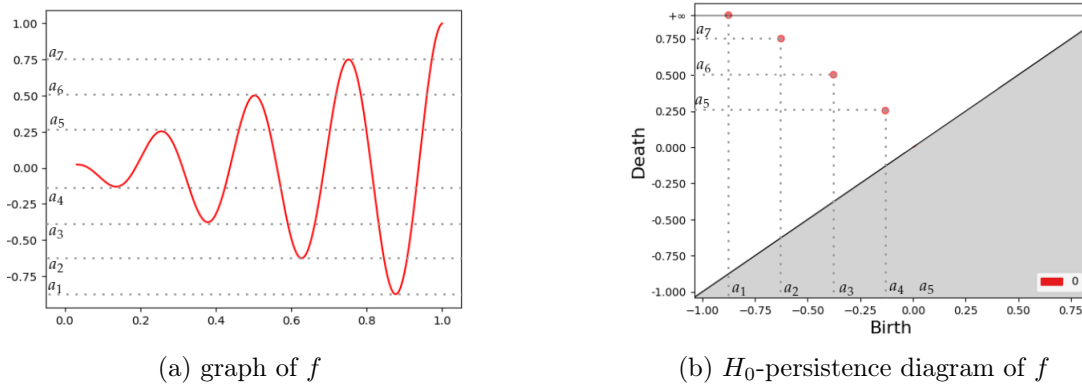


Figure 1: Graph of $f(x) = x \cos(8\pi x)$ over $[0, 1]$ and the persistence diagram associated to its sublevel sets filtration. a_1, \dots, a_4 correspond to local minima of f and thus birth times in $\text{dgm}(f)$. a_5, \dots, a_7 correspond to local maxima of f and thus death times in $\text{dgm}(f)$.

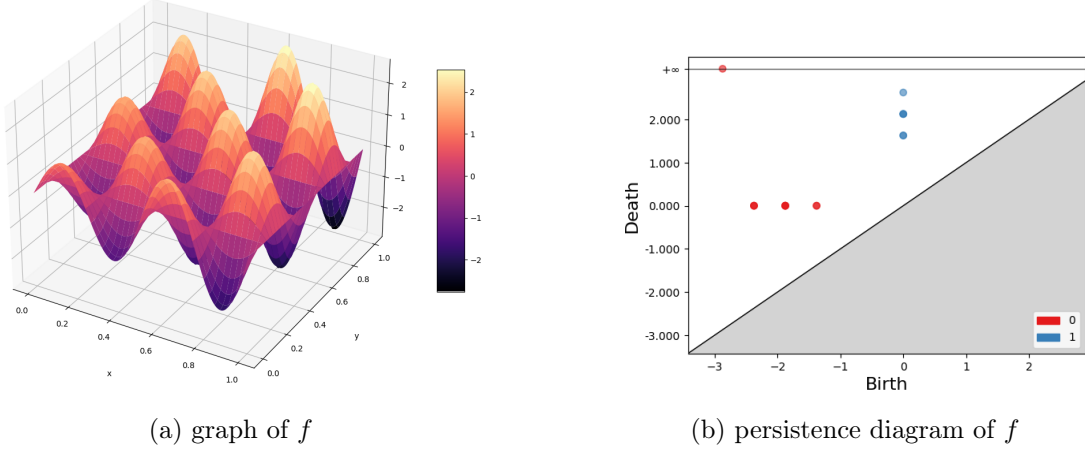


Figure 2: Graph of $f(x) = \sin(4\pi x) \cos(4\pi y)(x + y + 1)$ over $[0, 1]^2$ and the persistence diagram associated to its sublevel sets filtration, red points corresponds to the H_0 -persistence diagram and blue points to the H_1 -persistence diagram.

To compare persistence diagrams, a popular distance, especially in statistical context, is the bottleneck distance.

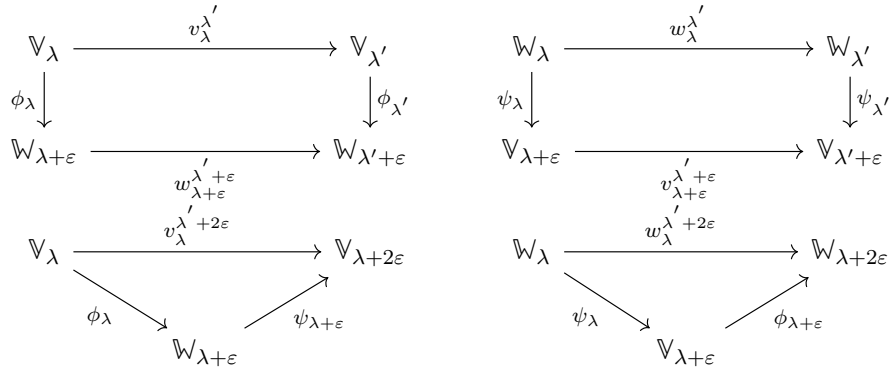
Definition 4. The **bottleneck distance** between two persistence diagrams D_1 and D_2 is,

$$d_b(D_1, D_2) = \inf_{\gamma \in \Gamma} \sup_{p \in D_1} \|p - \gamma(p)\|_\infty$$

with Γ the set of all bijection between D_1 and D_2 (both enriched with the diagonal).

We now introduce the algebraic stability theorem for the bottleneck distance. This theorem was the key for proving upper bounds in [Henneuse \(2024\)](#), it is also the case in this work to prove [Theorem 1](#). This theorem relies on interleaving between modules, a notion we extensively exploit.

Definition 5. Two persistence modules $\mathbb{V} = (\mathbb{V}_\lambda)_{\lambda \in \Lambda \subset \mathbb{R}}$ and $\mathbb{W} = (\mathbb{W}_\lambda)_{\lambda \in \Lambda \subset \mathbb{R}}$ are said to be ε -**interleaved** if there exists two families of applications $\phi = (\phi_\lambda)_{\lambda \in \Lambda \subset \mathbb{R}}$ and $\psi = (\psi_\lambda)_{\lambda \in \Lambda \subset \mathbb{R}}$ where $\phi_\lambda : \mathbb{V}_\lambda \rightarrow \mathbb{W}_{\lambda+\varepsilon}$, $\psi_\lambda : \mathbb{W}_\lambda \rightarrow \mathbb{V}_{\lambda+\varepsilon}$, and for all $\lambda < \lambda'$ the following diagrams commutes,



Theorem ([Chazal et al. 2009a](#), "algebraic stability"). Let \mathbb{V} and \mathbb{W} two q -tame persistence modules. If \mathbb{V} and \mathbb{W} are ε -interleaved then,

$$d_b(\text{dgm}(\mathbb{V}), \text{dgm}(\mathbb{W})) \leq \varepsilon.$$

We now give a corollary of this result, proved earlier in special cases (Barannikov, 1994; Cohen-Steiner et al., 2005). We insist on the fact that this is a strictly weaker result than algebraic stability.

Theorem ("sup norm stability"). *Let f and g two real-valued q -tame function, for all $s \in \mathbb{N}$*

$$d_b(\text{dgm}(\mathbb{V}_{f,s}), \text{dgm}(\mathbb{V}_{g,s})) \leq \|f - g\|_\infty.$$

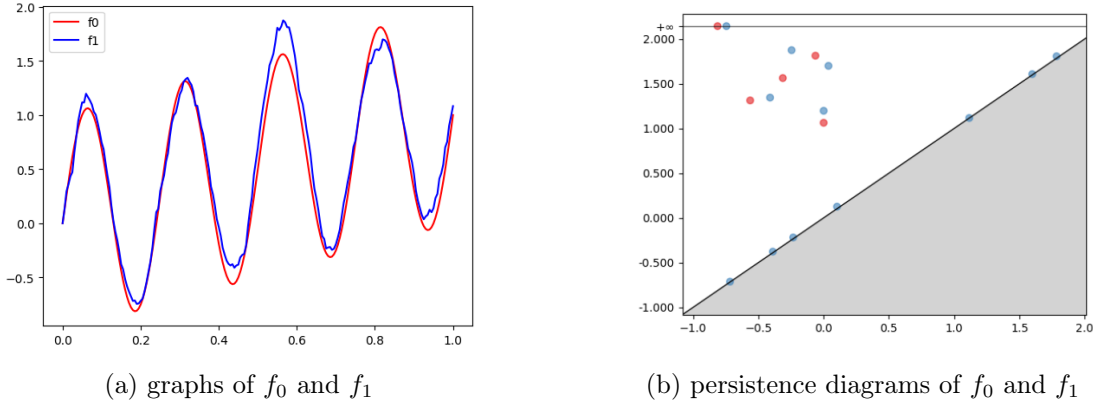


Figure 3: 1D Illustration of stability theorems.

This last property is often used to upper bound the errors (in bottleneck distance) of "plug-in" estimators of persistence diagrams. It enables the direct translation of convergence rates in sup-norm to convergence rates in bottleneck distance, which for regular classes of signals provide minimax upper bounds. Still, for wider classes, this approach falls short. The alternative approach proposed in this work focuses instead on the estimation of the persistent modules $\mathbb{V}_{f,s}$, in the sense of interleaving, and then exploits algebraic stability.

1.2 Generalized gradient

We here present some concepts from geometric measure theory, extensively used in geometric inference and TDA. The first notion used in this paper is the distance function to compact.

Definition 6. Let $K \subset [0, 1]^d$ a compact set, the **distance function** d_K is given by,

$$d_K : x \mapsto \min_{y \in K} \|x - y\|_2.$$

Generally the distance function is not differentiable everywhere, still it is possible to define a generalized gradient function that matches with the gradient at points where the distance function is differentiable.

Definition 7. Let,

$$\Gamma_K(x) = \{y \in K \mid \|x - y\|_2 = d_K(x)\}$$

the set of closest point to x in K . For $x \in [0, 1]^d \setminus K$, let denote $\Theta_K(x)$ the center of the unique smallest ball enclosing $\Gamma_K(x)$, the **generalized gradient function** $\nabla_K(x)$ is defined as,

$$\nabla_K(x) = \frac{x - \Theta_K(x)}{d_K(x)}.$$

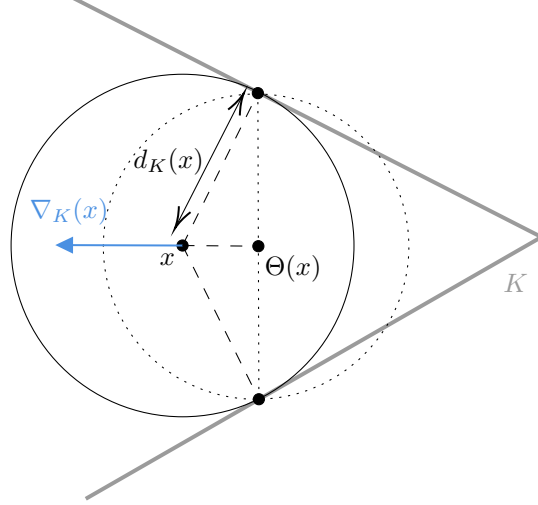


Figure 4: 2D example with 2 closest points

With these definitions we can now introduce the notion of μ -reach (Chazal et al., 2006).

Definition 8. Let $K \subset [0, 1]^d$ a compact set, its μ -reach $\text{reach}_\mu(K)$ is defined by,

$$\text{reach}_\mu(K) = \inf \left\{ r \mid \inf_{d_K^{-1}(r) \setminus K} \|\nabla_K\|_2 < \mu \right\}.$$

The 1-reach corresponds simply to the reach, a curvature measure introduced by Federer (1959). $\|\nabla_K(x)\|$ can be thought as the cosine of the (half) angle of the smallest half cone with apex x that contains $\Gamma_K(x)$ (see Chazal et al., 2009b, Section 2.1). We then have immediately the following lemma.

Lemma 1. If M_1, \dots, M_l verifies Assumption **A3**, then, for all $I \subset \{1, \dots, l\}$,

$$\text{reach}_\mu \left(\bigcup_{i \in I} \partial M_i \right) \geq R_\mu.$$

Note that we do not have the converse implication, as illustrated in Figure 5. Assumption **A3** is slightly stronger than having a positive μ -reach.

We now state a particular property of sets with positive μ -reach from Chazal et al. (2007). This property is used in the proof of Theorem 1 to quantify approximation errors. For a set $K \subset [0, 1]^d$ and $r \geq 0$ we denote $B_2(K, r) = \{x \in [0, 1]^d \text{ s.t. } d_K(x) < r\}$ and $\overline{B}_2(K, r) = \{x \in [0, 1]^d \text{ s.t. } d_K(x) \leq r\}$.

Lemma 2. (Chazal et al., 2007, Lemma 3.1.) Let $K \subset [0, 1]^d$ a compact set and let $\mu > 0, r > 0$ be such that $r < \text{reach}_\mu(K)$. For any $x \in \overline{B}_2(K, r) \setminus K$, one has

$$d_{\partial \overline{B}_2(K, r)}(x) \leq \frac{r - d_K(x)}{\mu} \leq \frac{r}{\mu}.$$

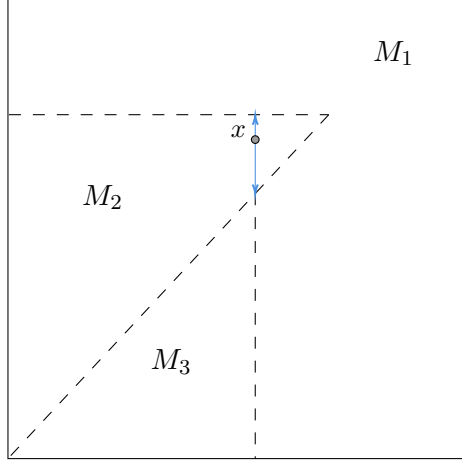


Figure 5: The discontinuities set $\bigcup_{i \in \{1,2,3\}} \partial M_i$ has a positive μ -reach, for small μ but the smallest half cone C_x containing $\bigcup_{i \in \{1,2,3\}} \Gamma_{\partial M_i}(x)$ is of angle $\theta = \pi$.

2 Procedure description

We here propose a two-steps estimation procedure.

The first step is a "rough" estimation of sublevel sets via local averaging on a regular grid plus a thickening. We call these estimators rough, as they are not meant to perform well for the estimation of sublevel sets evaluated with standard metrics. Typically, over $S_d(L, \alpha, \mu, R_\mu)$, for a fixed $\lambda \in \mathbb{R}$, our estimator $\widehat{\mathcal{F}}_\lambda$ can be (arbitrarily) far in Hausdorff distance from \mathcal{F}_λ . But, it captures well the topological features of the true signals.

Still, cubical approximation and noise may create cycles lying around the boundaries of sublevel sets estimators that do not correspond to any cycles of true signal (as in Figure 6.b). These cycles are pretty benign when appearing in regular regions (they will have "short" lifetime), but may have arbitrarily long lifetime when appearing around the discontinuities of the signal. Hence, for the second step, instead of considering directly the modules coming from this estimated filtration, we construct an image persistence module (Cohen-Steiner et al., 2009). This can be seen as a topological regularization step. The proposed image module aims to eliminate such cycles (or at least make their lifetime short) without damaging (too much) the information inferred about true cycles. The use of image modules to filter out topological noise is not a new idea, for example Cohen-Steiner et al. (2009); Chazal et al. (2011) used it to estimate persistence diagrams of a signal while working on noisy domains or closer to our context Bobrowski et al. (2017) that used the image modules between slightly shifted sublevel sets filtration to infer persistence diagrams in the non-parametric regression and the sampled density settings.

In the following, for a set $A \subset \mathbb{R}^d$ and $b \geq 0$, we denote,

$$A^b = \left\{ x \in \mathbb{R}^d \text{ s.t. } d_\infty(x, A) \leq b \right\}$$

with

$$d_\infty(x, A) = \inf_{y \in A} \|x - y\|_\infty.$$

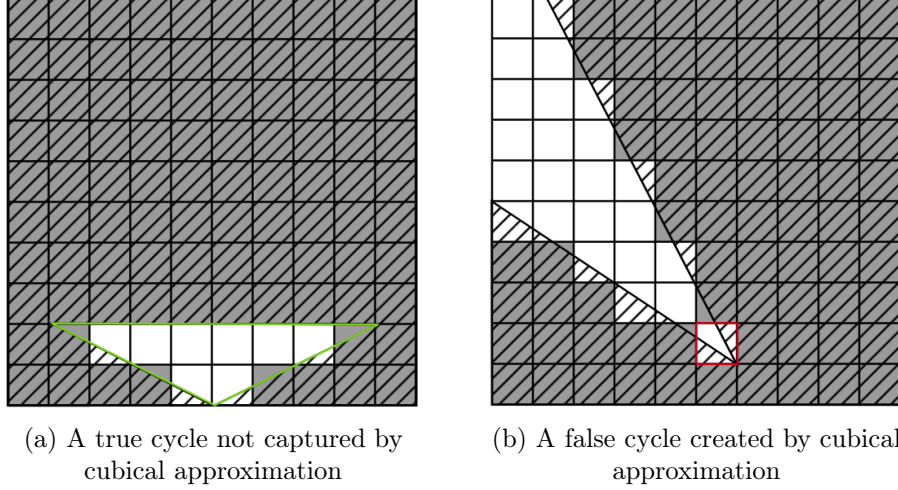


Figure 6: λ -sublevel cubical approximation for f the function defined as 0 on the hatched area and K outside (for arbitrarily large K). (a) display a case where the histogram approximation fails to capture true cycle in green, for, at least, all $0 < \lambda < K/2$. (b) displays a case where the histogram approximation create a cycle in red, not corresponding to any true cycle, with an arbitrarily long lifetime.

Step 1 : rough sublevel sets estimation. Let $h > 0$ such that $1/h$ is an integer, consider G_h the regular orthogonal grid over $[0, 1]^d$ of step h and C_h the collection of all the hypercubes of side h composing G_h . Let $r_1 > 0$, we define, $\forall \lambda \in \mathbb{R}$, the "rough" λ -sublevel estimator,

$$\widehat{\mathcal{F}}_\lambda = \left(\bigcup_{H \in C_{h,\lambda}} H \right)^{\lceil r_1 \rceil h} \quad \text{with } C_{h,\lambda} = \left\{ H \in C_h \text{ such that } \int_H dX - \int_H \lambda \leq 0 \right\}$$

with $\lceil \cdot \rceil$ the ceiling function. This thickening of the sublevel sets obtained by histogram estimation of f aims to solve the problem exposed in Figure 6.a. The choice of r_0 follows from Lemma 2 (see also the proof of Proposition 1)

Step 2 : construction of the image persistence module and associated diagram. Let, $r_2 > 0$ and,

$$\rho_\lambda : H_s \left(\widehat{\mathcal{F}}_\lambda \right) \rightarrow H_s \left(\widehat{\mathcal{F}}_\lambda^{\lceil r_2 \rceil h} \right)$$

the map induced by the inclusion $\widehat{\mathcal{F}}_\lambda \subset \widehat{\mathcal{F}}_\lambda^{\lceil r_2 \rceil h}$. For all $\lambda < \lambda'$, we have $\widehat{\mathcal{F}}_\lambda \subset \widehat{\mathcal{F}}_{\lambda'}$ and $\widehat{\mathcal{F}}_\lambda^{\lceil r_2 \rceil h} \subset \widehat{\mathcal{F}}_{\lambda'}^{\lceil r_2 \rceil h}$. Then denote,

$$\hat{v}_\lambda^{\lambda'} : H_s \left(\widehat{\mathcal{F}}_\lambda^{\lceil r_2 \rceil h} \right) \rightarrow H_s \left(\widehat{\mathcal{F}}_{\lambda'}^{\lceil r_2 \rceil h} \right)$$

the map induced by the inclusion. $\widehat{\mathcal{F}}_\lambda^{\lceil r_2 \rceil h} \subset \widehat{\mathcal{F}}_{\lambda'}^{\lceil r_2 \rceil h}$. Note that the following diagram commutes (all

maps are induced by inclusions),

$$\begin{array}{ccc} H_s(\widehat{\mathcal{F}}_\lambda) & \longrightarrow & H_s(\widehat{\mathcal{F}}_{\lambda'}) \\ \downarrow \rho_\lambda & & \downarrow \rho_{\lambda'} \\ H_s(\widehat{\mathcal{F}}_\lambda^{\lceil r_2 \rceil h}) & \xrightarrow{\widehat{v}_{\lambda'}^{\lambda'}} & H_s(\widehat{\mathcal{F}}_{\lambda'}^{\lceil r_2 \rceil h}) \end{array}$$

Thus for all $\lambda < \lambda'$,

$$\widehat{v}_{\lambda',h}^{\lambda'}(\text{Im}(\rho_\lambda)) \subset \text{Im}(\rho_{\lambda'}).$$

We now introduce $\widehat{\mathbb{V}}_{f,s}$ the persistence module associated to $(\text{Im}(\rho_\lambda))_{\lambda \in \mathbb{R}}$ equipped with the collection of maps $(\widehat{v}_{\lambda',h}^{\lambda'})_{\lambda < \lambda'}$ for the s -th order homology, $\widehat{\text{dgm}}_s(f)$ the associated persistence diagram, and $\widehat{\text{dgm}}(f)$ the collection of such persistence diagrams for all $s \in \mathbb{N}$. This diagram is well-defined, as we prove in Appendix A that $\widehat{\mathbb{V}}_{f,s}$ is q -tame. This second step aims to solve the problem exposed in Figure 6.b.

Calibration. A natural question is how to choose the parameters h , r_1 and r_2 . Lemma 2 from Henneuse (2024) (see also Section 3.2) suggests that a good choice is taking h such that,

$$\frac{h^{d+\alpha}}{\sqrt{h^d \log(1 + \frac{1}{h^d})}} > \theta \tag{1}$$

which implies that we can take,

$$h \simeq \left(\theta^2 \log\left(\frac{1}{\theta}\right) \right)^{\frac{1}{d+2\alpha}}.$$

Lemma 2 (see also the proof of Proposition 1) suggests taking,

$$r_1 = \sqrt{d}/\mu$$

and Proposition 2 suggests taking,

$$r_2 = \sqrt{d}(1 + 2/\mu^2)(\sqrt{d} + \lceil r_1 \rceil).$$

Computation. By construction, for all $\lambda \in \mathbb{R}$, $\widehat{\mathcal{F}}_\lambda$ and $\widehat{\mathcal{F}}_\lambda^{\lceil r_2 \rceil h}$ are simply unions of cubes from the regular grid G_h , thus can be thought as (geometric realization of) cubical complexes or even simplicial complexes. Hence, the computation of $\widehat{\text{dgm}}(f)$ is made possible by the algorithm for image persistence gave in Cohen-Steiner et al. (2009).

Another strategy, would be to consider,

$$\widehat{\mathcal{F}}_\lambda = \overline{B}_2 \left(\bigcup_{H \in C_{h,\lambda}} c_H, r_1 h \right)$$

with c_H the center of the hypercube H and $\widehat{\mathbb{V}}_{f,s}$, the image module induced by,

$$\rho_\lambda : H_s(\widehat{\mathcal{F}}_\lambda) \rightarrow H_s(\overline{B}_2(\widehat{\mathcal{F}}_\lambda, r_2 h)).$$

One can show, adapting the proofs of Section 3, that the subsequent estimator achieve the same convergence rates (up to slight change in constant). As $\widehat{\mathcal{F}}_\lambda$ and $B_2(\widehat{\mathcal{F}}_\lambda, r_2 h)$ are union of Euclidean balls, they can be replaced by Čech complexes. To justify properly that it lead to the same estimated persistence diagram, one can adapt the "parameter nerve theorem" (Lemma 3.4) from [Chazal and Oudot \(2008\)](#). Again, the computation of this persistence diagram is then made possible by the algorithm provided in [Cohen-Steiner et al. \(2009\)](#).

3 Convergence results

This section is dedicated to the proof of Theorem 1.

3.1 Main proof

The idea of proof follows similarly to the proof for upper bounds in [Henneuse \(2024\)](#). The general idea is to construct an interleaving between the module $\mathbb{V}_{f,s}$ and $\widehat{\mathbb{V}}_{f,s}$ and apply the algebraic stability theorem. This implies to construct two morphism, $\overline{\psi} : \mathbb{V}_{f,s} \rightarrow \widehat{\mathbb{V}}_{f,s}$ and $\overline{\phi} : \widehat{\mathbb{V}}_{f,s} \rightarrow \mathbb{V}_{f,s}$ satisfying definition 5. Propositions 1 and 2 give the necessary ingredients.

We recall some useful notations from [Henneuse \(2024\)](#). Let,

$$\|W\|_{G(h)} = \sup_{H \in C_h} \frac{|W(H)|}{\omega(h^d)}$$

with $\omega(r) = \sqrt{r \log(1 + 1/r)}$, $W(H) = \int_H dW$.

Proposition 1. *Let $f \in S_d(L, \alpha, \mu, R_\mu)$, $h \simeq (\theta^2 \log(\frac{1}{\theta}))^{\frac{1}{d+2\alpha}}$ and $r_1 = \sqrt{d}/\mu$. For θ sufficiently small such that $h < R_\mu/\sqrt{d}$, we have, for all $\lambda \in \mathbb{R}$,*

$$\mathcal{F}_{\lambda - (\|W\|_{G(h)} + L([\sqrt{d}/\mu] + \sqrt{d})^\alpha)h^\alpha} \subset \widehat{\mathcal{F}}_\lambda \subset \mathcal{F}_{\lambda + \|W\|_{G(h)}h^\alpha}^{(\sqrt{d} + [\sqrt{d}/\mu])h}.$$

The proof of Proposition 1 can be found in Section 3.2. The lower inclusion given by this proposition induced directly a morphism from $\mathbb{V}_{f,s}$ to $\widehat{\mathbb{V}}_{f,s}$, this will be our $\overline{\psi}$. But the upper inclusion is not sufficient to induce a converse morphism, the construction of $\overline{\phi}$ require additional work. To be able to use this upper inclusion, we need a map from the homology groups of $(\mathcal{F}_\lambda^h)_{\lambda \in \mathbb{R}}$, $h > 0$ into the homology groups of $(\mathcal{F}_\lambda)_{\lambda \in \mathbb{R}}$. Proposition 2 provides such a map, exploiting the geometrical properties of the discontinuities set imposed by Assumption **A3** to construct a deformation retract.

Definition 9. A subspace A of X is called a **deformation retract** of X if there is a continuous $F : X \times [0, 1] \rightarrow X$ (called a homotopy) such that for all $x \in X$ and $a \in A$,

- $F(x, 0) = x$
- $F(x, 1) \in A$
- $F(a, 1) = a$.

The function F is called a **deformation retraction** from X to A .

Homotopy, and thus homology, is invariant under deformation retract. Thus, a deformation retrac-

tion from X to A induces isomorphism between homology groups. More precisely, for all $t \in [0, 1]$, $F(\cdot, t)$ induces a morphism $F(\cdot, t)^\# : C_s(X) \rightarrow C_S(X)$ between s -cycles of X defined by composing each singular s -simplex $\sigma : \Delta^s \rightarrow X$ with $F(\cdot, t)$ to get a singular s -simplex $F^\#(\sigma, t) = F(\cdot, t) \circ \sigma : \Delta^s \rightarrow A$, then extending $F^\#(\cdot, t)$ linearly via $F^\#(\sum_i n_i \sigma_i, t) = \sum_i n_i F^\#(\sigma_i, t) = \sum_i n_i F(\cdot, t) \circ \sigma_i$. Then, $F^*(\cdot, t) : H_s(X) \rightarrow H_S(X)$ defined by $[C] \mapsto [F^\#(C, t)]$ can be shown to be an isomorphism for all $t \in [0, 1]$ (see [Hatcher, 2000](#), pages 110-113). In particular, $F^*(\cdot, 1) : H_s(X) \rightarrow H_s(A)$ is an isomorphism.

Proposition 2. *For all $0 < h < \frac{R_\mu}{2}$ and all $\lambda \in \mathbb{R}$ there exists $\mathcal{G}_{\lambda,h}, \mathcal{K}_{\lambda,h}$ and $F_{\lambda,h} : \mathcal{K}_{\lambda,h} \times [0, 1] \rightarrow \mathcal{K}_{\lambda,h}$ such that,*

- $\mathcal{F}_\lambda \subset \mathcal{G}_{\lambda,h} \subset \mathcal{F}_{\lambda+L(8(1+2/\mu^2))^\alpha h^\alpha}$
- $\overline{B}_2(\mathcal{F}_\lambda, h) \subset \mathcal{K}_{\lambda,h} \subset \overline{B}_2(\mathcal{F}_\lambda, (1+2/\mu^2)h)$
- $F_{\lambda,h}$ is a deformation retraction of $\mathcal{K}_{\lambda,h}$ onto $\mathcal{G}_{\lambda,h}$
- for all $x \in \mathcal{K}_{\lambda,h}$ and all $t \in [0, 1]$, $F_{\lambda,h}(x, t) \in \overline{B}_2\left(x, 2d_{\bigcup_{i \in \{1, \dots, l\}} \partial M_i}(x)/\mu^2\right)$.

The proof of Proposition 2 can be found in Section 3.3. Using the inclusion $\mathcal{F}_\lambda^h \subset \overline{B}_2(\mathcal{F}_\lambda, h) \subset \mathcal{K}_{\lambda,h}$, $\mathcal{G}_{\lambda,h} \subset \mathcal{F}_{\lambda+L(8(1+2/\mu^2))^\alpha h^\alpha}$ and the homology map induced by the retraction of $\mathcal{K}_{\lambda, \sqrt{d}h}$ onto $\mathcal{G}_{\lambda, \sqrt{d}h}$ we get the desired maps from the homology groups of $(\mathcal{F}_\lambda^h)_{\lambda \in \mathbb{R}}$ (for all $R_\mu > h > 0$) into the homology groups of $(\mathcal{F}_\lambda)_{\lambda \in \mathbb{R}}$. Combine with the upper inclusion of Proposition 1, we have all the necessary ingredients to construct $\overline{\phi}$ and prove Theorem 1.

Theorem 1. *Let $h \simeq (\theta^2 \log(\frac{1}{\theta}))^{\frac{1}{d+2\alpha}}$, $r_1 = \sqrt{d}/\mu$ and $r_2 = \sqrt{d}(1+2/\mu^2)(\sqrt{d} + \lceil r_1 \rceil)$. There exists \tilde{C}_0 and \tilde{C}_1 such that, for all $t > 0$,*

$$\mathbb{P} \left(\sup_{f \in S_d(L, \alpha, \mu, R_\mu)} d_b \left(\widehat{\text{dgm}}(f), \text{dgm}(f) \right) \geq t \left(\theta^2 \log \left(\frac{1}{\theta} \right) \right)^{\frac{\alpha}{d+2\alpha}} \right) \leq \tilde{C}_0 \exp \left(-\tilde{C}_1 t^2 \right).$$

Proof. It suffices to show the result for (arbitrarily) small θ (up to rescaling \tilde{C}_0). Hence, we suppose that θ is sufficiently small for the application of Propositions 1 and 2 used in this proof.

Denote $k_1 = \sqrt{d} + \lceil r_1 \rceil$ and define the morphism,

$$\overline{\psi}_\lambda : H_s(\mathcal{F}_\lambda) \rightarrow H_s \left(\widehat{\mathcal{F}}_{\lambda + (\|W\|_{G(h)} + Lk_1^\alpha)h^\alpha} \right)$$

following from the inclusion $\mathcal{F}_\lambda \subset \widehat{\mathcal{F}}_{\lambda + (\|W\|_{G(h)} + Lk_1^\alpha)h^\alpha}$ given by Proposition 1.

For the second morphism, denote $k_2 = k_1 + \lceil r_2 \rceil$ and consider,

$$j_{1,\lambda} : \text{Im}(\rho_\lambda) \rightarrow H_s \left(\mathcal{K}_{\lambda + \|W\|_{G(h)}h^\alpha, k_2h} \right)$$

the map induced by the inclusion $\widehat{\mathcal{F}}_\lambda^{\lceil r_2 \rceil h} \subset \mathcal{F}_{\lambda + \|W\|_{G(h)}h^\alpha}^{k_2h} \subset \mathcal{K}_{\lambda + \|W\|_{G(h)}h^\alpha, k_2h}$ given by Proposition 1 and Proposition 2,

$$j_{2,\lambda} : H_s \left(\mathcal{K}_{\lambda + \|W\|_{G(h)}h^\alpha, k_2h} \right) \rightarrow H_s \left(\mathcal{G}_{\lambda + \|W\|_{G(h)}h^\alpha, k_2h} \right)$$

induced by the deformation retract of Proposition 2, and,

$$j_{3,\lambda} : H_s \left(\mathcal{G}_{\lambda+\|W\|_{G(h)}h^\alpha, k_2h} \right) \rightarrow H_s \left(\mathcal{F}_{\lambda+(\|W\|_{G(h)}+L(8(1+2/\mu^2)k_2)^\alpha)h} \right)$$

induced by the inclusion $\mathcal{G}_{\lambda+\|W\|_{G(h)}h^\alpha, k_2h} \subset \mathcal{F}_{\lambda+(\|W\|_{G(h)}+L(8(1+2/\mu^2)k_2)^\alpha)h}$ given by Proposition 2.

We then define,

$$\begin{cases} \bar{\phi}_\lambda : \text{Im}(\rho_\lambda) \rightarrow H_2 \left(\mathcal{F}_{\lambda+(\|W\|_{G(h)}+L(8(1+2/\mu^2)k_2)^\alpha)h} \right) \\ \bar{\phi}_\lambda = j_{3,\lambda} \circ j_{2,\lambda} \circ j_{1,\lambda} \end{cases}$$

We now show that $\bar{\psi}$ and $\bar{\phi}$ induce an interleaving between $\widehat{\mathbb{V}}_{f,s}$ and $\mathbb{V}_{s,f}$. More precisely, we show that the following diagrams commute, for all $\lambda < \lambda'$. For compactness of notations, we denote $K_1 = \|W\|_{G(h)} + Lk_1^\alpha$ and $K_2 = \|W\|_{G(h)} + L(8(1+2/\mu^2)k_2)^\alpha$.

$$\begin{array}{ccc} \text{Im}(\rho_\lambda) & \xrightarrow{\hat{v}_\lambda^{\lambda'}} & \text{Im}(\rho_{\lambda'}) \\ \downarrow \bar{\phi}_\lambda & & \downarrow \bar{\phi}_{\lambda'} \\ H_s(\mathcal{F}_{\lambda+K_2h^\alpha}) & \xrightarrow{v_{\lambda+K_2h^\alpha}^{\lambda'+K_2h^\alpha}} & H_s(\mathcal{F}_{\lambda'+K_2h^\alpha}) \end{array} \quad (2)$$

$$\begin{array}{ccc} H_s(\mathcal{F}_\lambda) & \xrightarrow{v_\lambda^{\lambda'}} & H_s(\mathcal{F}_{\lambda'}) \\ \downarrow \bar{\psi}_\lambda & & \downarrow \bar{\psi}_{\lambda'} \\ \text{Im}(\rho_{\lambda+K_1h^\alpha}) & \xrightarrow{\hat{v}_{\lambda+K_1h^\alpha}^{\lambda'+K_1h^\alpha}} & \text{Im}(\rho_{\lambda'+K_1h^\alpha}) \end{array} \quad (3)$$

$$\begin{array}{ccc} \text{Im}(\rho_\lambda) & \xrightarrow{\hat{v}_\lambda^{\lambda+(K_1+K_2)h^\alpha}} & \text{Im}(\rho_{\lambda+(K_1+K_2)h^\alpha}) \\ \searrow \bar{\phi}_\lambda & & \nearrow \bar{\psi}_{\lambda+K_2h^\alpha} \\ & H_s(\mathcal{F}_{\lambda+K_2h^\alpha}) & \end{array} \quad (4)$$

$$\begin{array}{ccc} H_s(\mathcal{F}_\lambda) & \xrightarrow{v_\lambda^{\lambda+(K_1+K_2)h^\alpha}} & H_s(\mathcal{F}_{\lambda+(K_1+K_2)h^\alpha}) \\ \searrow \bar{\psi}_\lambda & & \nearrow \bar{\phi}_{\lambda+K_1h^\alpha} \\ & \text{Im}(\rho_{\lambda+K_1h^\alpha}) & \end{array} \quad (5)$$

- **Diagram 2** : We can rewrite the diagram as (unspecified maps are simply induced by set

inclusion),

$$\begin{array}{ccc}
\mathrm{Im}(\rho_\lambda) & \longrightarrow & \mathrm{Im}(\rho_{\lambda'}) \\
\downarrow & & \downarrow \\
H_s\left(\mathcal{K}_{\lambda+\|W\|_{G(h)}h^\alpha, k_2h}\right) & \longrightarrow & H_s\left(\mathcal{K}_{\lambda'+\|W\|_{G(h)}h^\alpha, k_2h}\right) \\
\updownarrow j_{2,\lambda} & & \updownarrow j_{2,\lambda'} \\
H_s\left(\mathcal{G}_{\lambda+\|W\|_{G(h)}h^\alpha, k_2h}\right) & \longrightarrow & H_s\left(\mathcal{G}_{\lambda'+\|W\|_{G(h)}h^\alpha, k_2h}\right) \\
\downarrow & & \downarrow \\
H_s\left(\mathcal{F}_{\lambda+K_2h^\alpha}\right) & \longrightarrow & H_s\left(\mathcal{F}_{\lambda'+K_2h^\alpha}\right)
\end{array}$$

By inclusions the upper and lower faces commutes and the central faces commutes as for all $\lambda \in \mathbb{R}$, $j_{2,\lambda}$ comes from a deformation retract, thus the diagram is commutative.

- **Diagram 3:** One can check that it can be decomposed as diagram 2 and thus the same reasoning applies.
- **Diagram 4:** Let $[C] \in \mathrm{Im}(\rho_\lambda)$. By construction, we can suppose $C \in C_s(\widehat{\mathcal{F}}_\lambda)$. The map $\bar{\phi}_\lambda$ maps $[C]$ to $[C']$ with,

$$C' = F_{\lambda+\|W\|_{G(h)}h^\alpha, k_2h}^\#(C, 1).$$

Combining the upper inclusion of Proposition 1 and the last assertion of Proposition 2, we know that, for all $x \in \widehat{\mathcal{F}}_\lambda$,

$$F_{\lambda+\|W\|_{G(h)}h^\alpha, k_2h}(x, [0, 1]) \subset \widehat{\mathcal{F}}_{\lambda+(K_1+K_2)h^\alpha}^{[r_2]h}$$

and thus,

$$F_{\lambda+\|W\|_{G(h)}h^\alpha, k_2h}^\#(C, [0, 1]) \subset C_s\left(\widehat{\mathcal{F}}_{\lambda+(K_1+K_2)h^\alpha}^{[r_2]h}\right).$$

Thus, C and C' are homologous in $\widehat{\mathcal{F}}_{\lambda+(K_1+K_2)h^\alpha}^{[r_2]h}$. As $\bar{\psi}_{\lambda+K_1h^\alpha}$ is simply an inclusion map, it maps $[C']$ to $[C]$, and hence, $[C] = [C'] = \bar{\psi}_{\lambda+K_2h^\alpha}(\bar{\phi}_\lambda([C]))$, which proves the commutativity.

- **Diagram 5:** Let $C \in C_s(\mathcal{F}_\lambda)$, $\bar{\psi}_\lambda([C]) = [C]$ as it is simply an inclusion induced map. By Proposition 2, we have $\mathcal{F}_\lambda \subset \mathcal{G}_{\lambda,h}$, thus,

$$\bar{\phi}_{\lambda+K_1}([C]) = \left[F_{\lambda+(K_1+\|W\|_{G(h)}h^\alpha, k_2h)}^\#(C, 1)\right] = [C]$$

which establish the commutativity.

The commutativity of diagrams 2,3,4 and 5 means that $\widehat{\mathbb{V}}_{f,s}$ and $\mathbb{V}_{f,s}$ are $(K_1 + K_2)h^\alpha$ interleaved, and thus we get from the algebraic stability theorem (Chazal et al., 2009a) that,

$$d_b\left(\mathrm{dgm}\left(\widehat{\mathbb{V}}_{f,s}\right), \mathrm{dgm}\left(\mathbb{V}_{f,s}\right)\right) \leq (K_1 + K_2)h^\alpha$$

and as it holds for all $s \in \mathbb{N}$ and $f \in S_d(L, \alpha, \mu, R_\mu)$,

$$\sup_{f \in S_d(L, \alpha, \mu, R_\mu)} d_b\left(\widehat{\mathrm{dgm}}(f), \mathrm{dgm}(f)\right) \leq (K_1 + K_2)h^\alpha.$$

We then conclude, using the concentration of $\|W\|_{G(h)}$. From Lemma 4 of [Henneuse \(2024\)](#), we know that there exist two constant C_0 and C_1 (depending only on d and α) such that,

$$\mathbb{P}(\|W\|_{G(h)} \leq t) \leq C_0 \exp(-C_1 t^2).$$

Then,

$$\begin{aligned} & \mathbb{P}\left(\sup_{f \in S_d(L, \alpha, \mu, R_\mu)} d_b(\widehat{\text{dgm}}(f), \text{dgm}(f)) \geq th\right) \\ & \leq \mathbb{P}(K_1 + K_2 \geq t) \\ & = \mathbb{P}\left(\|W\|_{G(h)} \geq \frac{t - L(k_1^\alpha + (8(1 + 2/\mu^2)k_2)^\alpha)}{2}\right) \\ & \leq C_0 \exp\left(-C_1 \left(\frac{t - L(k_1^\alpha + (8(1 + 2/\mu^2)k_2)^\alpha)}{2}\right)^2\right) \\ & \leq C_0 \exp\left(-\frac{C_1}{4}t^2\right) \exp(C_1 L(k_1^\alpha + (8(1 + 2/\mu^2)k_2)^\alpha) t) \\ & \quad \times \exp\left(-C_1 \left(\frac{L(k_1^\alpha + (8(1 + 2/\mu^2)k_2)^\alpha)}{2}\right)^2\right) \end{aligned}$$

and the result follows. \square

From this result, we can derive from this result bounds in expectation.

Corollary 1. *Let $p \geq 1$,*

$$\sup_{f \in S_d(L, \alpha, \mu, R_\mu)} \mathbb{E}\left(d_b(\widehat{\text{dgm}}(f), \text{dgm}(f))^p\right) \lesssim \left(\theta^2 \log\left(\frac{1}{\theta}\right)\right)^{\frac{p\alpha}{d+2\alpha}}$$

Proof. The sub-Gaussian concentration provided by Theorem 1, gives that, for all $t > 0$,

$$\mathbb{P}\left(\frac{d_b(\widehat{\text{dgm}}(f), \text{dgm}(f))}{h^\alpha} \geq t\right) \leq \tilde{C}_0 \exp(-\tilde{C}_1 t^2).$$

Now, we have,

$$\begin{aligned} & \mathbb{E}\left(\frac{d_b(\widehat{\text{dgm}}(f), \text{dgm}(f))^p}{h^{p\alpha}}\right) \\ & = \int_0^{+\infty} \mathbb{P}\left(\frac{d_b(\widehat{\text{dgm}}(f), \text{dgm}(f))^p}{h^{p\alpha}} \geq t\right) dt \\ & \leq \int_0^{+\infty} \tilde{C}_0 \exp(-\tilde{C}_1 t^{2/p}) dt < +\infty. \end{aligned}$$

\square

3.2 Proof of Proposition 1

This section is dedicated to the proof of Proposition 1. It relies on Lemma 2 (stated in Section 1) and on the following lemma from Henneuse (2024).

Lemma 3. (Henneuse, 2024, Lemma 2) Let $f : [0, 1]^d \rightarrow \mathbb{R}$ and $h > 0$ verifying (1). Let $H \subset \mathcal{F}_{\lambda + \|W\|_{G(h)} h^\alpha}^c \cap C_h$ and $H' \subset \mathcal{F}_{\lambda - \|W\|_{G(h)} h^\alpha} \cap C_h$. We then have that,

$$\int_H dX - \int_H \lambda > 0 \text{ and } \int_{H'} dX - \int_{H'} \lambda < 0.$$

Proof. Let consider here the case where in $H' \subset \mathcal{F}_{\lambda - \|W\|_{G(h)} h^\alpha}$ (The proof being the same in both cases). Note that,

$$\begin{aligned} & \int_{H'} dX - \int_{H'} \lambda \\ &= \int_{H'} (f - \lambda) + \theta \int_{H'} dW \\ &\leq -\|W\|_{\text{cube}, h} h^\alpha |H'| + \|W\|_{\text{cube}, h} \theta \omega(h^d) \\ &\leq \|W\|_{\text{cube}, h} \left(-h^{d+\alpha} + \theta \omega(h^d) \right) < 0 \end{aligned}$$

by the choice made for h . □

Proof of Proposition 1. We begin by proving the lower inclusion, let $x \in \mathcal{F}_{\lambda - (\|W\|_{G(h)} + L(\lceil \sqrt{d}/\mu \rceil + \sqrt{d})^\alpha) h^\alpha}$. Without loss of generality, let suppose $x \in \overline{M}_i$. If,

$$\overline{B}_2(x, \sqrt{dh}) \subset \left(\bigcup_{i=1}^l \partial M_i \right)^c$$

then, $H_{x,h}$, the hypercube of $C_{h,\lambda}$ containing x is included in \overline{M}_i . Assumption **A1** and **A2** then give $H_{x,h} \subset \mathcal{F}_{\lambda - \|W\|_{G(h)} h^\alpha}$. Hence, it follows from Lemma 3 that $H_{x,h} \in C_{h,\lambda}$ and consequently $x \in \widehat{\mathcal{F}}_\lambda$. Else, by Lemma 2 (which we can apply thanks to Lemma 1 and Assumption **A3**), there exists,

$$y \in \left(\left(\bigcup_{i=1}^l \partial M_i \right)^{\sqrt{dh}} \right)^c \cap M_i \text{ such that } \|x - y\|_2 \leq \sqrt{dh}/\mu.$$

Let $H_{y,h}$ the closed hypercube of C_h containing y . Hence, $H_{y,h} \subset \overline{M}_i$. and $d_{H_{y,h}}(x) \leq \sqrt{dh}(1 + 1/\mu)$. Then, Assumption **A1** and **A2** ensure that,

$$H_{y,h} \subset \mathcal{F}_{\lambda - \|W\|_{G(h)} h^\alpha}.$$

Then, Lemma 3 gives $H_{y,h} \in C_{h,\lambda}$ and thus, as $x \in H_{y,h}^{\sqrt{dh}/\mu}$, $x \in \widehat{\mathcal{F}}_\lambda$, which proves the lower inclusion.

For the upper inclusion, let $x \in \left(\mathcal{F}_{\lambda + \|W\|_{G(h)} h^\alpha} \right)^c$, and $H_{x,h}$ the hypercube of C_h containing x . We then have, $H_{x,h} \subset \mathcal{F}_{\lambda + \|W\|_{G(h)} h^\alpha}^c$. Hence, Lemma 3 gives that,

$$H_{x,h} \subset \left(\bigcup_{H \in C_{h,\lambda}} H \right)^c$$

and thus,

$$\bigcup_{H \in C_{h,\lambda}} H \subset \mathcal{F}_{\lambda + \|W\|_{G(h)} h^\alpha}^{\sqrt{d}h}.$$

Consequently,

$$\left(\bigcup_{H \in C_{h,\lambda}} H \right)^{\lceil \sqrt{d}/\mu \rceil h} = \widehat{\mathcal{F}}_\lambda \subset \mathcal{F}_{\lambda + \|W\|_{G(h)} h^\alpha}^{(\sqrt{d} + \lceil \sqrt{d}/\mu \rceil)h}$$

and the proof is complete. \square

3.3 Proof of Proposition 2

To construct the desired deformation retract, we follow the idea of proof of Proposition 2 in [Henneuse \(2024\)](#). In the proof of this proposition, the deformation retract involved the projection on the discontinuities set. Under the positive reach assumption on sufficiently small neighborhoods of the discontinuities set, the closest point is unique. Here, we do not have such a nice property. To overcome this issue, inspired by the proof of Theorem 12 from [Kim et al. \(2020\)](#), we construct a smooth flow from the generalized gradient of the discontinuities set, that essentially play the same role as the projection in the positive reach case.

Let $h < R_\mu/2$ and $x \in \overline{B}_2(\mathcal{F}_\lambda, 2(1 + 2/\mu^2)h) \cap B_2(\bigcup_{i=1}^l \partial M_i, h)$. We denote,

$$I_x = \{i \in \{1, \dots, l\} \text{ s.t. } B_2(x, 4(1 + 2/\mu^2)h) \cap M_i \cap \mathcal{F}_\lambda \neq \emptyset\}.$$

For all $x \in \overline{B}_2(\mathcal{F}_\lambda, 2(1 + 2/\mu^2)h) \cap B_2(\bigcup_{i=1}^l \partial M_i, h) \setminus \bigcup_{i=1}^l \partial M_i$, note that $I_x \neq \emptyset$. By assumption **A3** there exist $v(x) \in C_x$, verifying, for all $i \in \{1, \dots, l\}$ such that $x \in B_2(\partial M_i, R_\mu)$, and for all $y \in \Gamma_{\partial M_i}(x)$,

$$\cos(\langle v(x) - x, y - x \rangle) \geq \mu. \quad (6)$$

We then define the following vector field,

$$W(x) = \frac{v(x) - x}{\|v(x) - x\|_2}.$$

Unfortunately, this field is not continuous, and thus we cannot derive a continuous flow from it. The idea is then to approximate it by a continuous field \overline{W} that shares essentially the same dynamic. To do so, W needs to be not too wild, which is ensured by the following lemmas.

Lemma 4. *For all $x \in \overline{B}_2(\mathcal{F}_\lambda, 2(1 + 2/\mu^2)h) \cap B_2(\bigcup_{i=1}^l \partial M_i, h) \setminus \bigcup_{i=1}^l \partial M_i$, there exists U_x a neighborhood of x , such that for all $y \in U_x$,*

$$I_x \subset I_y.$$

Proof. For all $i \in I_x$, there exists $\varepsilon_i > 0$ such that,

$$d_{M_i \cap \mathcal{F}_\lambda}(x) \leq 4(1 + 2/\mu^2)h - \varepsilon_i.$$

Let $0 < \varepsilon < \min_{i \in I_x} (\varepsilon_i)/2$ and $y \in B_2(x, \varepsilon)$, for all $i \in I_x$, we have,

$$d_{M_i \cap \mathcal{F}_\lambda}(y) \leq 4(1 + 2/\mu^2)h - \varepsilon_i + \varepsilon \leq 4(1 + 2/\mu^2)h - \varepsilon.$$

Thus, $I_x \subset I_y$. \square

Lemma 5. For all $x \in \overline{B}_2(\mathcal{F}_\lambda, 2(1+h/\mu) \cap B_2(\bigcup_{i=1}^l \partial M_i, h) \setminus \bigcup_{i=1}^l \partial M_i$ there exists V_x a neighborhood of x such that for all $y \in V_x$,

$$\langle W(x), -\nabla_{\bigcup_{i \in I_y} \partial M_i}(y) \rangle \geq \mu^2/2.$$

Proof. Let $I \subset \{1, \dots, l\}$, consider a sequence $(y_n)_{n \in \mathbb{N}}$ in $[0, 1]^d$ converging to x and take $z_n \in \Gamma_{\bigcup_{i \in I} \partial M_i}(y_n)$. The sequence $(z_n)_{n \in \mathbb{N}}$ take its value in the compact $[0, 1]^d$, hence up to extracting a subsequence we can suppose it converges, we denote z its limit. By continuity of the distance function we got $\lim_{n \rightarrow \infty} d_2(z_n, x) = d_2(z, x)$ and $\lim_{n \rightarrow \infty} d_2(z_n, y_n) = d_{\bigcup_{i \in I} \partial M_i}(x)$. Thus, $z \in \Gamma_{\bigcup_{i \in I} \partial M_i}(x)$. Consequently, as there is of finite number of different I , for all $\varepsilon > 0$, there exists $V_x \subset B_2(x, h)$ a neighborhood of x such that, for all $y \in V_x$,

$$\Gamma_{\bigcup_{i \in I_y} \partial M_i}(y) \subset B_2(\Gamma_{\bigcup_{i \in I_y} \partial M_i}(x), \varepsilon).$$

Thus, let $\varepsilon > 0$, for all $z \in \Gamma_{\bigcup_{i \in I_y} \partial M_i}(y)$ there exists $\bar{z} \in \Gamma_{\bigcup_{i \in I_y} \partial M_i}(x)$ such that $\|y - z - (x - \bar{z})\| \leq 2\varepsilon$. As for all $y \in B_2(x, h)$, and $i \in I_y$, $x \in B_2(\partial M_i, 2h) \subset B_2(\partial M_i, R_\mu)$, we have by (6),

$$\begin{aligned} \left\langle \frac{z - y}{\|z - y\|}, \frac{v(x) - x}{\|v(x) - x\|} \right\rangle &\geq \left\langle \frac{\bar{z} - x}{\|\bar{z} - x\|}, \frac{v(x) - x}{\|v(x) - x\|} \right\rangle - \left\| \frac{z - y}{\|z - y\|} - \frac{\bar{z} - x}{\|\bar{z} - x\|} \right\|_2 \\ &= \cos(z - y, v(x) - x) - \left\| \frac{z - y}{\|z - y\|} - \frac{\bar{z} - x}{\|\bar{z} - x\|} \right\|_2 \\ &\geq \mu - \frac{\|y - z - (x - \bar{z})\|}{d_{\bigcup_{i \in I_y} \partial M_i}(x)} - \|z - x\| \left| d_{\bigcup_{i \in I_y} \partial M_i}(x) - d_{\bigcup_{i \in I_y} \partial M_i}(y) \right| \\ &\geq \mu - \frac{2 + \sqrt{d}}{d_{\bigcup_{i \in I_y} \partial M_i}(x)} \varepsilon \\ &\geq \mu - \frac{2 + \sqrt{d}}{\min_{I \subset \{1, \dots, l\}} d_{\bigcup_{i \in I} \partial M_i}(x)} \varepsilon \end{aligned}$$

Taking $\varepsilon > 0$ sufficiently small we have, for all $y \in V_x$ and $z \in \Gamma_{\bigcup_{i \in I_y} \partial M_i}(y)$,

$$\left\langle \frac{z - y}{\|z - y\|}, \frac{v(x) - x}{\|v(x) - x\|} \right\rangle \geq \mu/2.$$

and hence,

$$C_y \subset \left\{ u - y \text{ with } u \in [0, 1]^d \text{ s.t. } \left\langle \frac{u - y}{\|u - y\|}, \frac{v(x) - x}{\|v(x) - x\|} \right\rangle \geq \mu/2 \right\}.$$

By definition, $\nabla_{\bigcup_{i \in I_y} \partial M_i}(y)$ belongs to the cone C_y . Consequently, as $\left\| \nabla_{\bigcup_{i \in I_y} \partial M_i}(y) \right\|_2 \geq \mu$ by Assumption **A3** and Lemma 1, we have,

$$\left\langle W(x), -\nabla_{\bigcup_{i \in I_y} \partial M_i}(y) \right\rangle = \left\langle W(x), -\frac{\nabla_{\bigcup_{i \in I_y} \partial M_i}(y)}{\left\| \nabla_{\bigcup_{i \in I_y} \partial M_i}(y) \right\|_2} \right\rangle \left\| \nabla_{\bigcup_{i \in I_y} \partial M_i}(y) \right\|_2 \geq \mu^2/2.$$

□

With this two lemmas we can now prove Proposition 2.

Proof of Proposition 2. Let $W_x = U_x \cap V_x$ and extract a locally finite open cover $\{W_{x_j}, j \in \mathbb{N}\}$ from $\{W_x, x \in \overline{B}_2(\mathcal{F}_\lambda, 2(1+2/\mu^2)h) \cap \overline{B}_2(\bigcup_{i=1}^l \partial M_i, h) \setminus \bigcup_{i=1}^l \partial M_i\}$. Let $\{\rho_j, j \in \mathbb{N}\}$ an associated smooth partition of unity, and define the C^∞ vector field,

$$\overline{W} = \sum_{j \in \mathbb{N}} \rho_j W(x_j).$$

By the fundamental theorem of flow (see e.g. [Kim et al., 2020](#), Theorem 25), there exists a maximal open domain,

$$\mathbb{D} \subset \overline{B}_2(\mathcal{F}_\lambda, 2(1+2/\mu^2)h) \cap \overline{B}_2\left(\bigcup_{i=1}^l \partial M_i, h\right) \setminus \bigcup_{i=1}^l \partial M_i \times [0, +\infty[$$

and a unique smooth flow,

$$C : \mathbb{D} \rightarrow \overline{B}_2(\mathcal{F}_\lambda, 2(1+2/\mu^2)h) \cap \overline{B}_2\left(\bigcup_{i=1}^l \partial M_i, h\right) \setminus \bigcup_{i=1}^l \partial M_i.$$

verifying $\frac{d}{dt}C(x, t) = \overline{W}(C(x, t))$. Let $(x, s) \in \mathbb{D}$, by Lemma 4 and continuity of C , for t sufficiently small $I_C(x, s) \subset I_C(x, s+t)$. Hence, from Lemma 5, it follows that,

$$\begin{aligned} & \lim_{t \rightarrow 0} \frac{d_{\bigcup_{i \in I_C(x, s+t)} \partial M_i}(C(x, s+t)) - d_{\bigcup_{i \in I_C(x, s)} \partial M_i}(C(x, s))}{t} \\ & \leq \lim_{t \rightarrow 0} \frac{d_{\bigcup_{i \in I_C(x, s)} \partial M_i}(C(x, s+t)) - d_{\bigcup_{i \in I_C(x, s)} \partial M_i}(C(x, s))}{t} \\ & = \left\langle \nabla_{\bigcup_{i \in I_C(x, s)} \partial M_i}(C(x, s)), \overline{W}(C(x, s)), \right\rangle \\ & = \sum_{j \in \mathbb{N}} \rho_j(C(x, s)) \left\langle \nabla_{\bigcup_{i \in I_C(x, s)} \partial M_i}(C(x, s)), W(x_j), \right\rangle \\ & \leq -\mu^2/2. \end{aligned}$$

Let $x \in \overline{B}_2(\mathcal{F}_\lambda, h) \cap B_2(\bigcup_{i=1}^l \partial M_i, h) \setminus \bigcup_{i=1}^l \partial M_i$ and $s \in [0, +\infty[$ such that $(x, s) \in \mathbb{D}$, the foregoing implies,

$$d_{\bigcup_{i \in I_C(x, s)} \partial M_i}(C(x, s)) \leq d_{\bigcup_{i \in I_x} \partial M_i}(x) - s\mu^2/2 \quad (7)$$

Let $s_x = \sup \{s \in [0, \infty[\text{ s.t. } (x, s) \in \mathbb{D}\}$, by (7), for such x ,

$$s_x \leq 2d_{\bigcup_{i \in \{1, \dots, l\}} \partial M_i}(x)/\mu^2 \leq 2h/\mu^2.$$

Denote d_C the arc length distance along C , as $\|\overline{W}(x)\| \leq 1$, we then have,

$$\begin{aligned} d_C(x, C(x, s_x)) &= \int_0^{s_x} \left| \frac{\partial}{\partial t} C(x, t) \right| dt \\ &\leq \int_0^{s_x} \|\overline{W}(C(x, t))\|_2 dt \end{aligned}$$

$$\leq s_x \leq 2d_{\bigcup_{i \in \{1, \dots, l\}} \partial M_i}(x)/\mu^2 \leq \frac{2h}{\mu^2}. \quad (8)$$

Thus $\|x - C(x, s_x)\|_2 \leq 2d_{\bigcup_{i \in \{1, \dots, l\}} \partial M_i}(x)/\mu^2 \leq 2h/\mu^2$. Now, If $C(x, s_x) \notin \bigcup_{s \in [0, s_x[} \bigcup_{i \in I_C(x, t)} \partial M_i$ then $C(x, s_x) \in \overline{B}_2(\mathcal{F}_\lambda, (1 + 2/\mu^2)h) \cap B_2(\bigcup_{i=1}^l \partial M_i, R_\mu) \setminus \bigcup_{i=1}^l \partial M_i$ and by (7) the flow can be extended which contradicts the definition of s_x . Thus,

$$C(x, s_x) \in \bigcup_{s \in [0, s_x[} \bigcup_{i \in I_C(x, t)} \partial M_i.$$

From the foregoing we can extend continuously $C(x, \cdot)$ on $[0, +\infty[$, taking $C(x, s) = C(x, s_x)$ for all $s > s_x$.

Now, let

$$\mathcal{K}_{\lambda, h} := \overline{B}_2(\mathcal{F}_\lambda, h) \cup \left(\bigcup_{x \in S_{\lambda, h}} \{C(x, t), \quad t \in [0, s_x]\} \right)$$

with

$$S_{\lambda, h} = \left(\bigcup_{i=1}^l \overline{B}_2(\mathcal{F}_\lambda \cap M_i, h) \cap M_i \right)^c \cap \overline{B}_2(\mathcal{F}_\lambda, h).$$

Using C we then define the following retract : if

$$x \in \bigcup_{x \in S_{\lambda, h}} \{C(x, t), \quad t \in [0, s_x]\} \cap M_i \quad (9)$$

and

$$d_{M_i \cap \mathcal{F}_{\lambda + Lh^\alpha}}(C(x, s_x)) \geq 4h/\mu^2 - d_C(C(x, s_x), x) \quad (10)$$

then,

$$F_{\lambda, h}(x, t) = C(x, t\tilde{s}_x)$$

with \tilde{s}_x verifying,

$$d_C(C(x, s_x), C(x, \tilde{s}_x)) = (4h/\mu^2 - d_{M_i \cap \mathcal{F}_{\lambda + Lh^\alpha}}(C(x, s_x)))_+.$$

Else,

$$F_{\lambda, h}(x, t) = x.$$

Let $\mathcal{G}_{\lambda, h} = \text{Im}(x \mapsto F_{\lambda, h}(x, 1))$. Then, by definition, $F_{\lambda, h}(x, 1) \in \mathcal{G}_{\lambda, h}$, for all $x \in \mathcal{K}_{\lambda, h}$. As $C(x, 0) = x$ then $F_{\lambda, h}(x, 0) = x$, for all $x \in \mathcal{K}_{\lambda, h}$. Remark that, for all x verifying (9) and (10),

$$\begin{aligned} d_C(C(C(x, \tilde{s}_x), s_{C(x, \tilde{s}_x)}), C(C(x, \tilde{s}_x), \tilde{s}_{C(x, \tilde{s}_x)})) &= (4h/\mu^2 - d_{M_i \cap \mathcal{F}_{\lambda + Lh^\alpha}}(C(C(x, \tilde{s}_x), s_{C(x, \tilde{s}_x)})))_+ \\ &= (4h/\mu^2 - d_{M_i \cap \mathcal{F}_{\lambda + Lh^\alpha}}(C(x, s_x)))_+ \\ &= d_C(C(x, s_x), C(x, \tilde{s}_x)) \end{aligned}$$

and

$$d_C(C(C(x, \tilde{s}_x), s_{C(x, \tilde{s}_x)}), C(C(x, \tilde{s}_x), \tilde{s}_{C(x, \tilde{s}_x)})) = d_C(C(x, s_x), C(x, \tilde{s}_x + \tilde{s}_{C(x, \tilde{s}_x)})).$$

Hence, $d_C(C(x, s_x), C(x, \tilde{s}_x + \tilde{s}_{C(x, \tilde{s}_x)})) = d_C(C(x, s_x), C(x, \tilde{s}_x))$. If $\tilde{s}_x < s_x$ then $\left\| \frac{d}{dt} C(x, \tilde{s}_x + t) \right\| > 0$ and thus $\tilde{s}_{C(x, \tilde{s}_x)} = 0$. In this case, we then have,

$$C(C(x, \tilde{s}_x), \tilde{s}_{C(x, \tilde{s}_x)}) = C(x, \tilde{s}_x).$$

In the case where $\tilde{s}_x = s_x$, directly, we have,

$$C(C(x, s_x), \tilde{s}_{C(x, \tilde{s}_x)}) = C(x, s_x + \tilde{s}_{C(x, \tilde{s}_x)}) = C(x, s_x).$$

It follows, by construction of $F_{\lambda, h}$, that $F_{\lambda, h}(x, 1) = x$, for all $x \in \mathcal{G}_{\lambda, h}$. The proof for the continuity of $F_{\lambda, h}$ can be found in Appendix B. Hence, $F_{\lambda, h}$ is a deformation retract onto $\mathcal{G}_{\lambda, h}$. Furthermore, by (8), we have, for all $x \in \mathcal{K}_{\lambda, h}$ and all $t \in [0, 1]$,

$$F_{\lambda, h}(x, t) \in \overline{B}_2(x, 2d_{\bigcup_{i \in \{1, \dots, l\}} \partial M_i}(x)/\mu) \subset \overline{B}_2(x, 2h/\mu^2).$$

Let's now prove that $\mathcal{G}_{\lambda, h} \subset \mathcal{F}_{\lambda+L(8(1+2/\mu^2))^\alpha h^\alpha}$. Suppose that $x \in \overline{M}_i \cap \mathcal{K}_{\alpha, h}$ and let's look at the following different cases.

- If $x \notin \bigcup_{x \in S_{\lambda, h}} \{C(x, t), t \in [0, s_x]\} \cap \overline{M}_i$, then directly $F_{\lambda, h}(x, 1) = x \in \mathcal{F}_{\lambda+Lh^\alpha}$, by Assumptions **A1** and **A2**.
- If $x \in \bigcup_{x \in S_{\lambda, h}} \{C(x, t), t \in [0, s_x]\} \cap M_i$ and $d_{\mathcal{F}_{\lambda+Lh^\alpha} \cap M_i}(C(x, s_x)) \leq 4h/\mu^2$, then, as $F_{\lambda, h}(x, 1) \in \{C(x, t), t \in [0, s_x]\}$, $F_{\lambda, h}(x, 1) \in \overline{M}_i$ and,

$$d_{\mathcal{F}_{\lambda+Lh^\alpha} \cap M_i}(F_{\lambda, h}(x, 1)) \leq \frac{4h}{\mu^2} + \|x - C(x, s_x)\|_2 \leq \frac{6h}{\mu^2}.$$

By Assumptions **A1** and **A2**, it then follows that $F_{\lambda, h}(x, 1) \in \mathcal{F}_{\lambda+L(1+6/\mu^2)^\alpha h^\alpha}$.

- Else, $F_{\lambda, h}(x, 1) = C(x, s_x) \in \bigcup_{s \in [0, s_x]} \bigcup_{i \in I_{C(x, t)}} \partial M_i$. Suppose that $C(x, s_x) \in \partial M_i$, $i \in \bigcup_{s \in [0, s_x]} I_{C(x, t)}$, thus there exist $s \in [0, s_x]$ such that $C(x, s) \cap B_2(\mathcal{F}_\lambda, 4(1+2/\mu^2)) \cap M_j \neq \emptyset$. Hence, we have,

$$\begin{aligned} d_{M_i \cap \mathcal{F}_\lambda}(C(x, s_x)) &\leq 4(1+2/\mu^2) + \|C(x, s_x) - C(x, s)\| \\ &\leq 4(1+2/\mu^2) + \|C(x, s_x) - x\| \\ &\leq 4(1+2/\mu^2) + 2h/\mu^2. \end{aligned}$$

By Assumptions **A1** and **A2**, it then follows that $F_{\lambda, h}(x, 1) \in \mathcal{F}_{\lambda+L(4(1+2/\mu^2)+2h/\mu^2)^\alpha h^\alpha}$.

Hence, in all cases we have $F_{\lambda, h}(x, 1) \in \mathcal{F}_{\lambda+L(8(1+2/\mu^2))^\alpha h^\alpha}$ and the result is proved. \square

4 Discussion

In this work, we extend main results of Henneuse (2024), relaxing the positive reach condition over the discontinuities into **A3**. It allows considering signals with pretty wild discontinuities, for examples, exhibiting multiple points or (arbitrarily narrow) corners, cases that were excluded by the positive reach assumption. Once again, the convergence rates obtained here coincides with the ones known for Hölder-continuous signals. This means that there is (constant-wise) no additional cost to tackle such irregular signals. This highlights the robustness to signal irregularity of the persistence diagram inference and the interest of breaking free from analysis relying on sup norm stability.

A main difference with this previous work, is that under our relaxed conditions, a histogram plug-in estimator fails to converge consistently to the true persistence diagram. Consequently, we had to depart from usual plug-in approaches and propose a new estimator based on image module. This highlights the appeal of non-plug-in methods in TDA.

Similarly to what is done in Appendix C of [Henneuse \(2024\)](#), one can extend the method and results obtained here in the setting of non-parametric regression. Motivated by applications to modes detection, we are also interested in extension to the density model, which will be the object of forthcoming works.

A practical limitation of the proposed method is that it requires some knowledge on the parameters α to calibrate h and on μ to calibrate r_1 and r_2 . The dependence on α can be handled (while preserving the convergence rates), as in Section 2.4 of [Henneuse \(2024\)](#), via the Lespki's method ([Lepskii, 1991, 1992](#)). The dependence on μ is more problematic but rather common in homology/homotopy inference. Exploring methodologies to develop adaptive procedures requires further dedicated research.

Also, from a computational perspective (and in order to perform a numerical evaluation of this method), an important step forward would be to provide an efficient algorithm to compute the image persistence for general filtrations. This would permit to compare this method to plug-in approach, specifically to the histogram estimator. While [Bauer and Schmahl \(2022\)](#) offers an efficient such algorithm, it is tailored for Rips complexes, complicating its direct application to our context. In particular, this would permit to investigate practical selection of the parameters h , r_1 and r_2 .

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A Proof for q -tameness

This section is devoted to prove the claim that the persistence diagrams we consider and estimated persistence diagrams we propose are well-defined, by proving that the underlying persistence modules are q -tame.

Lemma 6. *Let $f \in S_d(L, \alpha, \mu, R_\mu)$. $\forall s \in \mathbb{N}, \forall h < \frac{R_\mu}{2}$, there exist a morphism ϕ such that, $\forall \lambda \in \mathbb{R}$,*

$$\begin{array}{ccc} H_s(\mathcal{F}_\lambda) & \xrightarrow{\quad\quad\quad} & H_s(\mathcal{F}_{\lambda+L(8(1+2/\mu^2))^{\alpha}h\alpha}) \\ & \searrow & \nearrow \phi_\lambda \\ & H_s(\mathcal{F}_\lambda^h) & \end{array} \quad (11)$$

is a commutative diagram (unspecified map come from set inclusions).

Proof. Let $\tilde{\phi}_\lambda : H_s(\mathcal{K}_{\lambda,h}) \rightarrow H_s(\mathcal{G}_{\lambda,h})$ the morphism associated to the deformation retract from Proposition 2. We also denote $i_{1,\lambda} : H_s(\mathcal{F}_\lambda^h) \rightarrow H_s(\mathcal{K}_{\lambda,h})$ the morphism induced by the inclusion $\mathcal{F}_\lambda^h \subset \mathcal{K}_{\lambda,h}$ and $i_{2,\lambda} : H_s(\mathcal{G}_{\lambda,h}) \rightarrow H_s(\mathcal{F}_{\lambda+L(8(1+2/\mu^2))^{\alpha}h\alpha})$ the morphism induced by the inclusion $\mathcal{G}_{\lambda,h} \subset \mathcal{F}_{\lambda+L(8(1+2/\mu^2))^{\alpha}h\alpha}$, also provided by Proposition 2. We take $\phi_\lambda = i_{2,\lambda} \circ \tilde{\phi}_\lambda \circ i_{1,\lambda}$. Diagram 11 then is (unspecified maps are the one induced by set inclusion),

$$\begin{array}{ccccc} H_s(\mathcal{F}_\lambda) & \xrightarrow{\quad\quad\quad} & H_s(\mathcal{F}_{\lambda+L(8(1+2/\mu^2))^{\alpha}h\alpha}) & & \\ \downarrow (F1) & \searrow (F2) & \searrow (F3) & & \uparrow i_{2,\lambda} \\ H_s(\mathcal{F}_\lambda^h) & \xrightarrow{i_{1,\lambda}} & H_s(\mathcal{K}_{\lambda,h}) & \xleftarrow{\tilde{\phi}_\lambda} & H_s(\mathcal{G}_{\lambda,h}) \end{array} \quad (12)$$

Faces (F1) and (F3) simply commute by inclusion. Face (F2) commutes as $\tilde{\phi}_\lambda$ is an isomorphism. All faces of diagram 12 are commutative, hence diagram 12 (and equivalently diagram 11) is commutative. \square

Proposition 3. *Let $f \in S_d(L, \alpha, \mu, R_\mu)$ then f is q -tame.*

Proof. Let $s \in \mathbb{N}$ and $\mathbb{V}_{s,f}$ the persistence module (for the s -th homology) associated to the sublevel filtration, \mathcal{F} and for fixed levels $\lambda < \lambda'$ let denote $v_\lambda^{\lambda'}$ the associated map. Let $\lambda \in \mathbb{R}$ and $h < R_\mu/2$. By Lemma 6, $v_\lambda^{\lambda+L(8(1+2/\mu^2))^{\alpha}h\alpha} = \phi_\lambda \circ \tilde{i}_\lambda$, with $\tilde{i}_\lambda : H_s(\mathcal{F}_\lambda) \rightarrow H_s(\mathcal{F}_\lambda^h)$. By assumption A1

and **A2**, \mathcal{F}_λ is compact. As $[0, 1]^d$ is triangulable, \mathcal{F}_λ is covered by finitely many cells of the triangulation, and so there is a finite simplicial complex K such that $\overline{\mathcal{F}_\lambda} \subset K \subset \mathcal{F}_\lambda^h$. Consequently, \tilde{i}_λ factors through the finite dimensional space $H_s(K)$ and is then of finite rank by Theorem 1.1 of [Crawley-Boevey \(2012\)](#). Thus, $v_\lambda^{\lambda+L(8(1+2/\mu^2))^{\alpha}h^\alpha}$ is of finite rank for all $0 < h < \frac{R}{2}$. As for any $\lambda < \lambda' < \lambda''$, $v_\lambda^{\lambda''} = v_{\lambda'}^{\lambda''} \circ v_\lambda^{\lambda'}$ we then have that $v_\lambda^{\lambda'}$ is of finite rank for all $\lambda < \lambda'$. Hence, f is q -tame. \square

Proposition 4. *Let $f \in S_d(R, L, \alpha)$ and $h > 0$ then, for all $s \in \mathbb{N}$, $\widehat{\mathcal{V}}_{s,f}$ is q -tame.*

Proof. The results also follows from by Theorem 1.1 of [Crawley-Boevey \(2012\)](#). Let $h > 0$ and $\lambda \in \mathbb{R}$. $\widehat{\mathcal{F}}_\lambda$ and $\widehat{\mathcal{F}}_\lambda^{\lceil r_2 \rceil h}$ are unions of cubes of C_h and thus finite dimensional. Hence, $\text{Im}(\rho_\lambda)$ is finite dimensional. Thus $\widehat{\mathcal{V}}_{s,f}$ is q -tame. \square

B Proof of the continuity of $F_{\lambda,h}$

This section is devoted to the proof of the continuity of the deformation retract $F_{\lambda,h}$, claimed in the proof of Proposition 2.

Lemma 7. *Let $h > 0$ and $\lambda \in \mathbb{R}$, $F_{\lambda,h}$ is continuous.*

Proof. Let $\delta, \delta' > 0$, $x, y \in \mathcal{K}_{\lambda,h}$ such that $\|x - y\|_2 \leq \delta$ and $t, s \in [0, 1]$ a such that $|t - s| \leq \delta'$. Let's look at the different cases.

First let's tackle the cases where $x \in \overline{M}_i$ and $y \in \overline{M}_j$, $i \neq j$. In this case, we have $d_{\bigcup_{i=1}^t \partial M_i}(x) \leq \|x - y\|_2 \leq \delta$ and $d_{\bigcup_{i=1}^t \partial M_i}(y) \leq \|x - y\|_2 \leq \delta$.

- If x verifies (9) and (10), by (8), we have

$$\|F_{\lambda,h}(x, t) - x\|_2 = \|x - C(x, t\tilde{s}_x)\|_2 \leq d_C(x, C(x, s_x)) \leq 2\delta/\mu^2$$

- If x does not verify (9) or (10), $F_{\lambda,h}(x, t) = x$, and directly,

$$\|F_{\lambda,h}(x, t) - x\|_2 = 0.$$

following the same reasoning we also have, in both cases for y , $\|F_{\lambda,h}(y, s) - y\|_2 \leq 2\delta/\mu^2$. Consequently, in any cases,

$$\|F_{\lambda,h}(x, t) - F_{\lambda,h}(y, s)\|_2 \leq \|F_{\lambda,h}(x, t) - x\|_2 + \|F_{\lambda,h}(y, s) - y\|_2 + \|x - y\|_2 \leq \delta(1 + 4/\mu^2).$$

This concludes for the cases where $x \in \overline{M}_i$ and $y \in \overline{M}_j$, $i \neq j$. From now, we suppose that $x, y \in M_i$.

- If x and y do not verify (9) or (10), then directly,

$$\|F_{\lambda,h}(x, t) - F_{\lambda,h}(y, s)\|_2 = \|x - y\|_2 \leq \delta.$$

- If x verifies (9) and (10) and y does not verify (9). Then, $y \in \mathcal{F}_{\lambda+Lh^\alpha}$. Thus,

$$d_{M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}}(C(x, s_x)) \leq d_C(x, C(x, s_x)) + \|x - y\|_2 \leq 2h/\mu^2 + \delta$$

and, by (10),

$$d_C(x, C(x, s)) \geq 4h/\mu^2 - d_{M_i \cap \mathcal{F}_{\lambda+Lh\alpha}}(C(x, s_x)) \geq 2h/\mu^2 - \delta \geq d_C(x, C(x, s)) - \delta.$$

Consequently, $F_{\lambda, h}(x, t) \in \{C(x, t), t \in [0, \delta]\}$. As,

$$\|F_{\lambda, h}(x, t) - F_{\lambda, h}(y, s)\|_2 = \|F_{\lambda, h}(x, t) - y\|_2 \leq \|F_{\lambda, h}(x, t) - x\|_2 + \delta$$

we can conclude in this case by continuity of C .

- If x verifies (9) and (10) and y verifies (9) but not (10), then,

$$\begin{aligned} d_C(x, C(x, s_x)) &\geq 4h/\mu^2 - d_{M_i \cap \mathcal{F}_{\lambda+Lh\alpha}}(C(x, s_x)) \\ &= 4h/\mu^2 - d_{M_i \cap \mathcal{F}_{\lambda+Lh\alpha}}(C(y, s_y)) + d_{M_i \cap \mathcal{F}_{\lambda+Lh\alpha}}(C(y, s_y)) - d_{M_i \cap \mathcal{F}_{\lambda+Lh\alpha}}(C(x, s_x)) \\ &\geq 4h/\mu^2 - d_{M_i \cap \mathcal{F}_{\lambda+Lh\alpha}}(C(y, s_y)) - \|C(x, s_x) - C(y, s_y)\|_2 \\ &\geq d_C(y, C(y, s_y)) - \|C(x, s_x) - C(y, s_y)\|_2 \\ &\geq d_C(x, C(x, s_x)) - |d_C(x, C(x, s_x)) - d_C(y, C(y, s_y))| - \|C(x, s_x) - C(y, s_y)\|_2. \end{aligned}$$

Hence, by definition of \tilde{s}_x , $F_{\lambda, h}(x, t) \in \{C(x, t), t \in [0, |d_C(y, C(y, s_y)) - d_C(x, C(x, s_x))| + \|C(x, s_x) - C(y, s_y)\|_2]\}$. Suppose without loss of generality that, $s_x \geq s_y$, then we can write,

$$C(x, s_x) = C(C(x, s_y), s_{C(x, s_y)}) = C(x, s_y + s_{C(x, s_y)}).$$

By (7), $s_{C(x, s_y)} \leq 2\|C(x, s_y) - C(y, s_y)\|_2/\mu^2$. And by continuity of C , $\lim_{\delta \rightarrow 0} \|C(x, s_y) - C(y, s_y)\|_2 = 0$, thus, $\lim_{\delta \rightarrow 0} |s_x - s_y| = 0$. Hence, by continuity of C again, $\lim_{\delta \rightarrow 0} \|F_{\lambda, h}(x, t) - x\|_2 = 0$. As,

$$\|F_{\lambda, h}(x, t) - F_{\lambda, h}(y, s)\|_2 = \|F_{\lambda, h}(x, t) - y\|_2 \leq \|F_{\lambda, h}(x, t) - x\|_2 + \|x - y\|_2 \xrightarrow[\delta \rightarrow 0]{} 0$$

we conclude in this case.

- Finally, If both x and y verifies (9) and (10), then,

$$\|F_{\lambda, h}(x, t) - F_{\lambda, h}(y, s)\|_2 = \|C(x, t\tilde{s}_x) - C(y, s\tilde{s}_y)\|_2$$

As in the previous case, it follows again in this case by continuity of C , that $\lim_{\delta \rightarrow 0} |s_x - s_y| = 0$. Consequently, by continuity of C and definition of \tilde{s}_x and \tilde{s}_y , $\lim_{\delta \rightarrow 0} |\tilde{s}_x - \tilde{s}_y| = 0$. And using again the continuity of C , we then have,

$$\lim_{\delta, \delta' \rightarrow 0} \|C(x, t\tilde{s}_x) - C(y, s\tilde{s}_y)\|_2 = 0$$

which gives the conclusion in this case.

Combining all the previous cases, we proved that $F_{\lambda, h}$ is continuous. \square