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# Persistence Diagram Estimation : Beyond Plug-in Approaches

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## Abstract

Persistent homology is a tool from Topological Data Analysis (TDA) used to summarize the topology underlying data. It can be conveniently represented through persistence diagrams. Observing a noisy signal, common strategies to infer its persistence diagram involve plug-in estimators, and convergence properties are then derived from sup-norm stability. This dependence on the sup-norm convergence of the preliminary estimator is restrictive, as it essentially imposes to consider regular classes of signals. Departing from these approaches, we design an estimator based on image persistence. In the context of the Gaussian white noise model, and for large classes of piecewise-Hölder signals, we prove that the proposed estimator is consistent and achieves minimax rates. Notably, these rates coincide with the well known minimax rates for Hölder continuous signals.

## Introduction

### Motivation

Topological Data Analysis (TDA) is a field that aims to provide representations to describe the “shape” of data. A central tool in TDA is persistent homology and its representation through persistence diagrams. It offers a practical, multiscale, summary of the topology underlying data. Although in its early years research has primarily focused on deterministic settings, it didn’t take long for statistical questions to emerge. A prominent topic is the estimation of persistent diagram and, more specifically, the investigation of convergence properties of considered estimators. A first historical success in this direction is [Bubenik and Kim \(2006\)](#), that formalizes the problem of persistence diagram estimation from a density in a parametric setting. Consistent efforts have been made to extend this model to wider, non-parametric settings. This includes the work of [Balakrishnan et al. \(2012\)](#), which addresses the estimation of Betty numbers for smooth manifolds with different noise models, as long as the work of [Fasy et al. \(2014\)](#), who provide confidence sets for persistence diagrams in a similar context and [Chazal et al. \(2014\)](#) that provide a minimax estimator while controlling the regularity of the density support. Closer to the context considered in this paper, [Bubenik et al. \(2009\)](#) and [Bobrowski et al. \(2017\)](#), tackle the problem of persistence diagram estimation in the context of the non-parametric regression settings. Most of these works, consider plug-in estimators and the proofs for convergence properties then exploit stability theorem ([Chazal et al., 2009](#)) for usual norm (typically sup norm or Gromov-Hausdorff norm), lifting results from

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non-parametric estimation or geometric inference. Although this approach can be proved to be minimax over regular classes (Hölder, Sobolev, Besov), it falls short for signals where sup norm convergence is not assured. A natural question is then, outside this regular classes, is consistent persistence diagram estimation achievable and at which rates ?

Bobrowski et al. (2017) open the door to break free from this approach, making use of image persistence (Cohen-Steiner et al., 2009). This work proposes a general approach in both the context of non-parametric regression and the density model. Under a really weak condition ( $q$ -tameness) they show some near-consistency properties of their estimator. But they do not provide convergence rates. A difficulty is maybe that this framework is too wide to quantify convergence rates or even prove proper consistency. Hence, it is interesting to identify narrower classes on which we can provide stronger and more precise results. Recently, in Henneuse (2024), we proposed a new approach that also break free from the sup norm stability. This work study the inference of persistence diagram, from a minimax perspective, for Gaussian white noise model and non-parametric regression. Although we consider a plug-in estimator, our analysis of convergence properties did not rely on sup-norm stability (but on a weaker notion of stability). We introduced and studied classes of piecewise-Hölder continuous function with discontinuities set having a positive reach. The reach (Federer, 1959) is a popular curvature measure in geometric inference, that can be thought as a way to describe the geometric regularity of a set. Under this reach assumption, we manage to show that a histogram plug-in estimator permits to achieve the well known minimax rates for Hölder-continuous functions. Relaxation of this assumption into a  $\mu$ -reach (generalization of the reach, Chazal et al., 2006) assumption was already discussed in this earlier work. We highlighted that it would imply to consider other estimators, as, even in the noiseless setting, the histogram approximation falls short (this is also illustrated here in Figure 5). Still, being able to infer the persistence diagram under this relaxation is interesting as it allows considering considerably wider classes of signal, as illustrated in Figure 2. This serves as a motivation to depart from plug-in approach for persistence diagram inference and as starting point for this new work.

## Framework

Our goal is to investigate the estimation of persistence diagrams in the following setting. It is similar to the framework of Henneuse (2024), the only difference is assumption **A3** involving the  $\mu$ -reach instead of the reach. For the reader unfamiliar with persistent homology, we provide a brief introduction, recalling some basic definitions and key properties in Section 1.2.

**Statistical model.** We considered the Gaussian white noise model given by the following stochastic equation,

$$dX_{t_1, \dots, t_d} = f(t_1, \dots, t_d) dt_1 \dots dt_d + \theta dW_{t_1, \dots, t_d}$$

with  $W$  a  $d$ -parameters Wiener field,  $f : [0, 1]^d \rightarrow \mathbb{R}$  a signal and  $\theta \geq 0$  the level of noise.

**Regularity assumptions.** Let  $f : [0, 1]^d \rightarrow \mathbb{R}$ , we make the following assumptions over  $f$  :

- **A1.**  $f$  is a piecewise  $(L, \alpha)$ -Hölder-continuous function, i.e. there exist  $M_1, \dots, M_l$  open sets of  $[0, 1]^d$  such that,

$$\bigcup_{i=1}^l \overline{M_i} = [0, 1]^d$$

and  $f|_{M_i}$  is in  $H(L, \alpha)$ ,  $\forall i \in \{1, \dots, l\}$ , with,

$$H(L, \alpha) = \left\{ f : [0, 1]^d \rightarrow \mathbb{R} \text{ s.t. } |f(x) - f(y)| \leq L\|x - y\|_2^\alpha, \forall x, y \in [0, 1]^d \right\}.$$

- **A2.**  $f$  verifies,  $\forall x_0 \in [0, 1]^d$ ,

$$\liminf_{x \in \bigcup_{i=1}^l M_i \rightarrow x_0} f(x) = f(x_0).$$

In this context, two signals, differing only on a null set, are statistically undistinguishable. And persistent homology is sensitive to point-wise irregularity, two signals differing only on a null set can have persistence diagrams that are arbitrarily far. Assumption **A2** prevents such scenario.

- **A3.** For  $\mu \in ]0, 1]$  and  $R_\mu > 0$ ,

$$\text{reach}_\mu \left( \bigcup_{i=1}^l \partial M_i \right) \geq R_\mu.$$

Here,  $\text{reach}_\mu$  denotes the  $\mu$ -reach, the generalization of the reach defined in [Chazal et al. \(2006\)](#). We invite the reader to see Section 1.1, where we recall the definition of the  $\mu$ -reach and make a few remarks about Assumption **A3**.

The class of function verifying **A1**, **A2** and **A3** is denoted  $S_d(L, \alpha, \mu, R_\mu)$ . For our purpose, we show in Appendix A that signals in  $S_d(L, \alpha, \mu, R_\mu)$  are  $q$ -tame, which ensures that their persistence diagram is well-defined.

## Contribution

We extend the main result of [Henneuse \(2024\)](#). Unlike the classes considered there, over the weaker classes  $S_d(L, \alpha, \mu, R_\mu)$ , a histogram estimator may fail to capture the topological features of the signal (see Figure 5.a), or may display additional false topological features due to cubical approximation and/or noise (See Figure 5.b). To overcome this issue, we propose,  $\widehat{\text{dgm}}(f)$  an estimator of the persistence diagram of  $f$ , making use of image persistence, as described in Section 2. We show that this estimator achieves, over  $S_d(L, \alpha, \mu, R_\mu)$ , rates coinciding with known minimax rates for Hölder-continuous signals. More precisely, our main result is the following theorem.

**Theorem 1.** *There exists  $\tilde{C}_0$  and  $\tilde{C}_1$  such that, for all  $t > 0$ ,*

$$\mathbb{P} \left( \sup_{f \in S_d(L, \alpha, \mu, R_\mu)} d_b \left( \widehat{\text{dgm}}(f), \text{dgm}(f) \right) \geq t \left( \theta^2 \log \left( \frac{1}{\theta} \right) \right)^{\frac{\alpha}{d+2\alpha}} \right) \leq \tilde{C}_0 \exp \left( -\tilde{C}_1 t^2 \right).$$

As these rates are minimax under stronger conditions (see [Henneuse, 2024](#), Theorem 3), they are evidently minimax over  $S_d(L, \alpha, \mu, R_\mu)$ . Section 3 is dedicated to the proof of Theorem 1.

## 1 Background

This section provides the necessary background to follow this paper.

## 1.1 Distance function, generalized gradient and $\mu$ -reach

We here present some concepts from geometric measure theory, extensively used in geometric inference and TDA. The first notion used in this paper is the distance function to compact.

**Definition 1.** Let  $K \subset [0, 1]^d$  a compact set, the **distance function**  $d_K$  is given by,

$$d_K : x \mapsto \min_{y \in K} \|x - y\|_2.$$

Generally the distance function is not differentiable everywhere, still it is possible to define a generalized gradient function that match with the gradient at points where the distance function is differentiable.

**Definition 2.** Let,

$$\Gamma_K(x) = \{y \in K \mid \|x - y\|_2 = d_K(x)\}$$

the set of closest point to  $x$  in  $K$ . For  $x \in [0, 1]^d \setminus K$ , let denote  $\Theta_K(x)$  the center of the unique smallest ball enclosing  $\Gamma_K(x)$ , the **generalized gradient function**  $\nabla_K(x)$  is defined as,

$$\nabla_K(x) = \frac{x - \Theta_K(x)}{d_K(x)}.$$

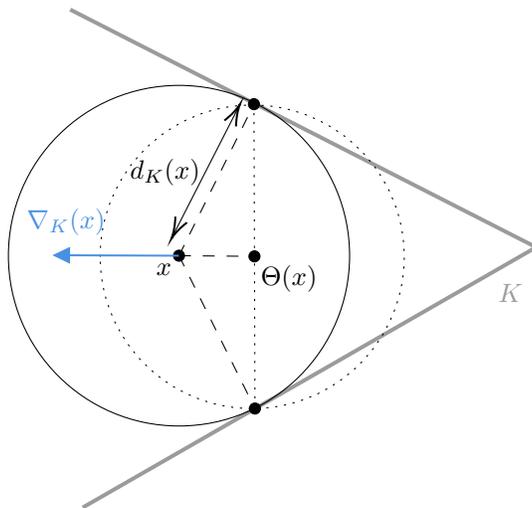


Figure 1: 2D example with 2 closest points

With this definitions we can now introduce the notion of  $\mu$ -reach (Chazal et al., 2006), on which this paper relies.

**Definition 3.** Let  $K \subset [0, 1]^d$  a compact set, its  **$\mu$ -reach**  $\text{reach}_\mu(K)$  is defined by,

$$\text{reach}_\mu(K) = \inf \left\{ r \mid \inf_{d_K^{-1}(r) \setminus K} \|\nabla_K\|_2 < \mu \right\}.$$

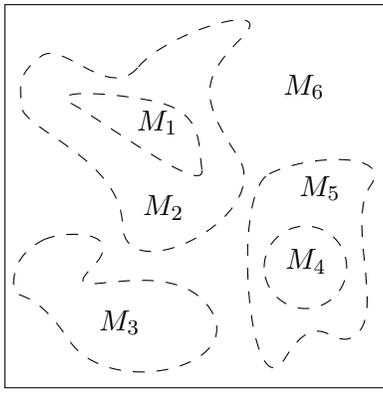
The 1-reach corresponds simply to the reach, a curvature measure introduced by Federer (1959). The  $\mu$ -reach is positive for a large class of sets. In deed, any compact set is arbitrarily close to a

positive  $\mu$ -reach set. For example, all piecewise linear surfaces have positive  $\mu$ -reach. In our context, considering the  $\mu$ -reach instead of the reach in assumption **A3** allow considering a significantly wider class of irregular signals, tolerating cusps and multiple points (for an illustration, see Figure 2).

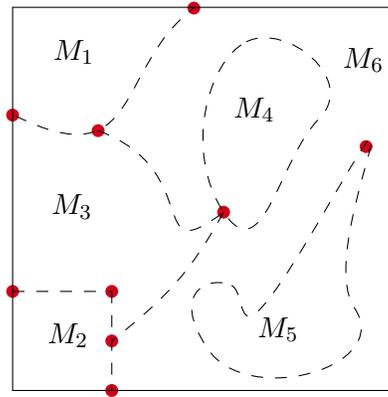
We now state a particular property of sets with positive  $\mu$ -reach from Chazal et al. (2007). This property is used in the proof of Theorem 1 to quantify approximation errors.

**Lemma 1.** (Chazal et al., 2007, Lemma 3.1.) *Let  $K \subset [0, 1]^d$  a compact set and let  $\mu > 0, r > 0$  be such that  $r < \text{reach}_\mu(K)$ . For any  $x \in K^r \setminus K$ , one has*

$$d_2(x, \partial K_r) \leq \frac{r - d_K(x)}{\mu} \leq \frac{r}{\mu}.$$



(a) Positive (1-)reach



(b) Positive  $\mu$ -reach ( $\mu$  small)

Figure 2: (a) display a partition  $M_1, \dots, M_6$  such that  $\text{reach}_1(\partial M_1 \cup \dots \cup \partial M_6) > 0$ . (b) display a partition  $M_1, \dots, M_6$  such that  $\text{reach}_1(\partial M_1 \cup \dots \cup \partial M_6) = 0$  (in red are highlighted problematic points) but for sufficiently small  $\mu > 0$ ,  $\text{reach}_\mu(\partial M_1 \cup \dots \cup \partial M_6) > 0$ .

## 1.2 Filtration, persistence module and persistence diagram

We here present briefly some notions related to persistence homology. Persistent homology permits to encode the evolution of topological features (in the homology sense) along a family of nested spaces, called filtration. Moving along indices, topological features (connected components, cycles, cavities, ...) can appear or die (existing connected components merge, cycle or cavities are filled, ...). The construction presented here involve singular homology. For an introduction to (singular) homology, the reader can refer to Hatcher (2000).

**Definition 4.** Let  $\Lambda \subset \mathbb{R}$  be a set of indices. A **filtration** over  $\Lambda$  is a family  $(\mathcal{K}_\lambda)_{\lambda \in \Lambda}$  of topological spaces satisfying,  $\forall \lambda, \lambda' \in \Lambda, \lambda \leq \lambda'$ ,

$$\mathcal{K}_\lambda \subset \mathcal{K}_{\lambda'}.$$

The typical filtration that we will consider in this paper is, for a function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ , the family of sublevel sets  $(\mathcal{F}_\lambda)_{\lambda \in \mathbb{R}}$ . The associated family of homology group of degree  $s \in \mathbb{N}$ ,  $\mathbb{V}_{f,s} = (H_s(\mathcal{F}_\lambda))_{\lambda \in \mathbb{R}}$ , equipped with  $v_\lambda^{\lambda'}$  the linear application induced by the inclusion  $\mathcal{F}_\lambda \subset \mathcal{F}_{\lambda'}$ , forms a persistence module. To be more precise, in this paper,  $H_s(\cdot)$  is the singular homology

functor in degree  $s$  with coefficient in a field (typically  $\mathbb{Z}/2\mathbb{Z}$ ). Hence,  $H_s(\mathcal{F}_\lambda)$  is a vector space. persistence modules play a key role in this paper and can be defined more generally.

**Definition 5.** Let  $\Lambda \subset \mathbb{R}$  be a set of indices. A **persistence module** over  $\lambda$  is a family  $\mathbb{V} = (\mathbb{V}_\lambda)_{\lambda \in \Lambda}$  of vector spaces equipped with linear application  $v_\lambda^{\lambda'} : \mathbb{V}_\lambda \rightarrow \mathbb{V}_{\lambda'}$  such that,  $\forall \lambda \leq \lambda' \leq \lambda'' \in \Lambda$ ,

$$v_\lambda^\lambda = id$$

and

$$v_{\lambda'}^{\lambda''} \circ v_\lambda^{\lambda'} = v_\lambda^{\lambda''}.$$

Under  $q$ -tameness of the persistence module, it is possible to show that the algebraic structure of the persistence module encodes exactly the evolution of the topological features along the indices  $\Lambda$ . For details, we encourage the reader to look at [Chazal et al. \(2016\)](#).

**Definition 6.** A persistence module  $\mathbb{V}$  is said to be  **$q$ -tame** if  $\forall \lambda < \lambda' \in \Lambda, \text{rank}(v_\lambda^{\lambda'})$  is finite. By extension, when considering the persistence modules  $(\mathbb{V}_{f,s})_{s \in \mathbb{N}}$  coming from the sublevel sets filtration of a real functions  $f$ , we say that  $f$  is  $q$ -tame if  $\mathbb{V}_{f,s}$  is for all  $s \in \mathbb{N}$ .

Furthermore, the algebraic structure of the persistence module can be summarized by a collection  $\{(b_i, d_i), i \in I\} \subset \mathbb{R}^2$ , which defined the **persistence diagram**. Following previous remarks,  $b_i$  corresponds to the birth time of a topological feature,  $d_i$  corresponds to its death time and  $d_i - b_i$  to its lifetime.

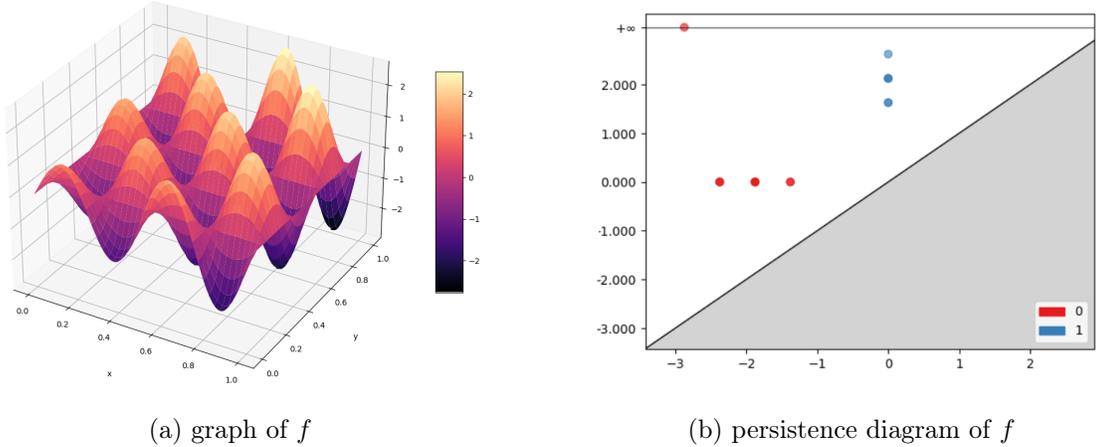


Figure 3: Graph of  $f(x) = \sin(4\pi x) \cos(4\pi y)(x + y + 1)$  over  $[0, 1]^2$  and the persistence diagram associated to its sublevel sets filtration, horizontal axis corresponds to birth times, vertical axis to death times, red points corresponds to the  $H_0$  persistence diagram and blue points to the  $H_1$  persistence diagram.

To compare persistence diagrams, a popular distance, especially in statistical context, is the bottleneck distance.

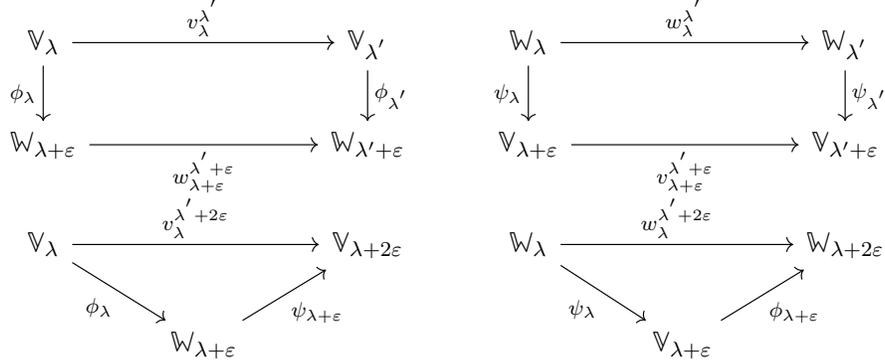
**Definition 7.** The **bottleneck distance** between two persistence diagrams  $D_1$  and  $D_2$  is,

$$d_b(D_1, D_2) = \inf_{\gamma \in \Gamma} \sup_{p \in D_1} \|\gamma(p) - p\|_\infty$$

with  $\Gamma$  the set of all bijection between  $D_1$  and  $D_2$  (both enriched with the diagonal).

We now present the algebraic stability theorem for the bottleneck distance. This theorem was the key for proving upper bounds in [Henneuse \(2024\)](#), it is also the case in this work to prove Theorem 1. This theorem relies on interleaving between modules, a notion we extensively exploit.

**Definition 8.** Two persistence modules  $\mathbb{V} = (\mathbb{V}_\lambda)_{\lambda \in I \subset \mathbb{R}}$  and  $\mathbb{W} = (\mathbb{W}_\lambda)_{\lambda \in I \subset \mathbb{R}}$  are said to be  $\varepsilon$ -**interleaved** if there exists two families of applications  $\phi = (\phi_\lambda)_{\lambda \in I \subset \mathbb{R}}$  and  $\psi = (\psi_\lambda)_{\lambda \in I \subset \mathbb{R}}$  where  $\phi_\lambda : \mathbb{V}_\lambda \rightarrow \mathbb{W}_{\lambda+\varepsilon}$ ,  $\psi_\lambda : \mathbb{W}_\lambda \rightarrow \mathbb{V}_{\lambda+\varepsilon}$ , and for all  $\lambda < \lambda'$  the following diagrams commutes,



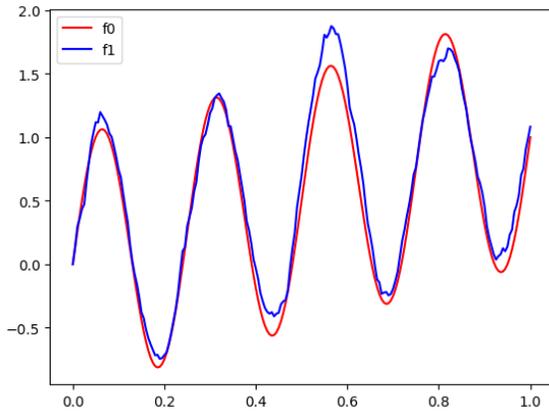
**Theorem (Chazal et al. 2009, "algebraic stability").** Let  $\mathbb{V}$  and  $\mathbb{W}$  two  $q$ -tame persistence modules. If  $\mathbb{V}$  and  $\mathbb{W}$  are  $\varepsilon$ -interleaved then,

$$d_b(\text{dgm}(\mathbb{V}), \text{dgm}(\mathbb{W})) \leq \varepsilon.$$

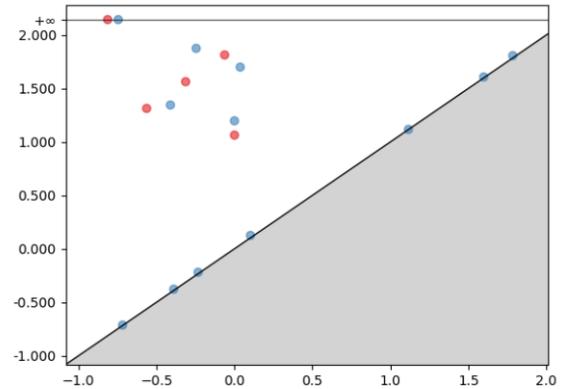
We now give a corollary of this result, proved earlier in special cases ([Barannikov, 1994](#); [Cohen-Steiner et al., 2005](#)). We insist on the fact that this is a strictly weaker result than algebraic stability.

**Theorem ("sup norm stability").** Let  $f$  and  $g$  two real-valued  $q$ -tame function, for all  $s \in \mathbb{N}$

$$d_b(\text{dgm}(\mathbb{V}_{f,s}), \text{dgm}(\mathbb{V}_{g,s})) \leq \|f - g\|_\infty.$$



(a) graphs of  $f_0$  and  $f_1$



(b) persistence diagrams of  $f_0$  and  $f_1$

Figure 4: 1D Illustration of stability theorems.

This last property is often used to upper bounds the errors (in bottleneck distance) of "plug-in" estimators of persistence diagrams. It enables the direct translation of convergence rates in sup-norm to convergence rates in bottleneck distance, which for regular classes of signals provide minimax upper bounds. Still, for wider classes, this approach falls short.

## 2 Procedure description

We here propose a two-steps estimation procedure.

The first step is a "rough" estimation of sublevel sets via local averaging on a regular grid plus a thickening. We call these estimators rough, as they are not meant to perform well for the estimation of sublevel sets evaluated with standard metrics. Typically, over  $S_d(L, \alpha, \mu, R_\mu)$ , for a fixed  $\lambda \in \mathbb{R}$ , our estimator  $\widehat{\mathcal{F}}_\lambda$  can be (arbitrarily) far in Hausdorff distance from  $\mathcal{F}_\lambda$ . But, it captures well the topological features of the true signals.

Still, cubical approximation and noise may create cycles lying around the boundaries of sublevel sets estimators that do not correspond to any cycles of true signal (as in Figure 5.b). These cycles are pretty benign when appearing in regular regions (they will have "short" lifetime), but may have arbitrarily long lifetime when appearing around the discontinuities of the signal. Hence, for the second step, instead of considering directly the modules coming from this estimated filtration, we construct an image persistence module (Cohen-Steiner et al., 2009). This can be seen as a topological regularization step. The proposed image module aims to eliminate such cycles (or at least make their lifetime short) without damaging (too much) the information inferred about true cycles. The use of image modules to filter out topological noise is not a new idea, for example Cohen-Steiner et al. (2009); Chazal et al. (2011) used it to estimate persistence diagrams of a signal while working on noisy domains or closer to our context Bobrowski et al. (2017) that used the image modules between slightly shifted filtrations to infer the persistence diagram in the non-parametric regression settings.

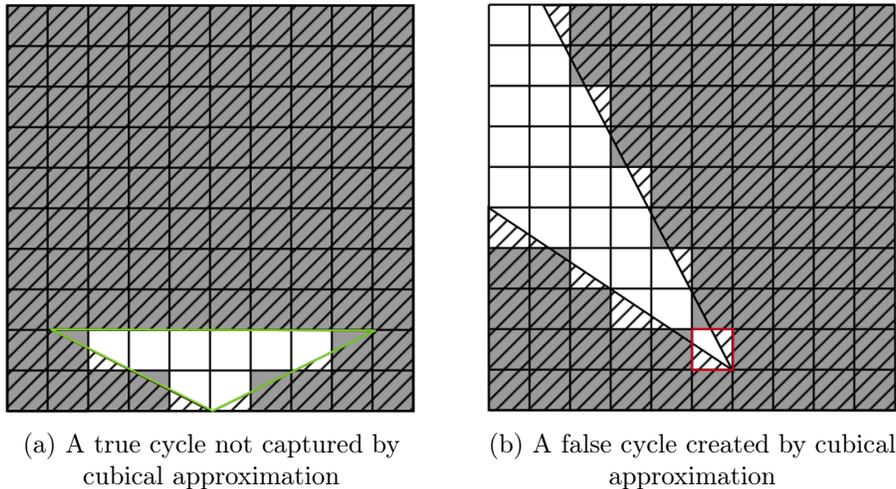


Figure 5:  $\lambda$ -sublevel cubical approximation for  $f$  the function defined as 0 on the hatched area and  $K$  outside (for arbitrarily large  $K$ ). (a) display a case where the histogram approximation fails to capture true cycle in green, for, at least, all  $0 < \lambda < K/2$ . (b) displays a case where the histogram approximation create a cycle in red, not corresponding to any true cycle, with an arbitrarily long lifetime.

In the following, for a set  $A \subset \mathbb{R}^d$  and  $b \geq 0$ , we denote,

$$A^b = \left\{ x \in \mathbb{R}^d \text{ s.t. } d_2(x, A) \leq b \right\}$$

with

$$d_2(x, A) = \inf_{y \in A} \|x - y\|_2.$$

**Step 1 : rough sublevel sets estimation.** Let  $h > 0$  such that  $1/h$  is an integer, consider  $G_h$  the regular orthogonal grid over  $[0, 1]^d$  of step  $h$  and  $C_h$  the collection of all the hypercubes of side  $h$  composing  $G_h$ . We define,  $\forall \lambda \in \mathbb{R}$ , the "rough"  $\lambda$ -sublevel estimator,

$$\widehat{\mathcal{F}}_{\lambda, h} = \left( \bigcup_{H \in C_{h, \lambda}} H \right)^{\sqrt{d}h/\mu} \quad \text{with } C_{h, \lambda} = \left\{ H \in C_h \text{ such that } \int_H dX - \int_H \lambda \leq 0 \right\}.$$

**Step 2 : construction of the image persistence module and associated diagram.** Let,

$$\rho_{\lambda, h} : H_s \left( \widehat{\mathcal{F}}_{\lambda, h} \right) \rightarrow H_s \left( \widehat{\mathcal{F}}_{\lambda, h}^{2\sqrt{d}(1+1/\mu)h/\mu} \right)$$

the map induced by the inclusion  $\widehat{\mathcal{F}}_{\lambda, h} \subset \widehat{\mathcal{F}}_{\lambda, h}^{2\sqrt{d}(1+1/\mu)h/\mu}$ . For all  $\lambda < \lambda'$ , we have  $\widehat{\mathcal{F}}_{\lambda, h} \subset \widehat{\mathcal{F}}_{\lambda', h}$  and  $\widehat{\mathcal{F}}_{\lambda, h}^{2\sqrt{d}(1+1/\mu)h/\mu} \subset \widehat{\mathcal{F}}_{\lambda', h}^{2\sqrt{d}(1+1/\mu)h/\mu}$ . Then denote,

$$\widehat{v}_{\lambda, h}^{\lambda'} : H_s \left( \widehat{\mathcal{F}}_{\lambda, h}^{2\sqrt{d}(1+1/\mu)h/\mu} \right) \rightarrow H_s \left( \widehat{\mathcal{F}}_{\lambda', h}^{2\sqrt{d}(1+1/\mu)h/\mu} \right)$$

the map induced by the inclusion.  $\widehat{\mathcal{F}}_{\lambda, h}^{2\sqrt{d}(1+1/\mu)h/\mu} \subset \widehat{\mathcal{F}}_{\lambda', h}^{2\sqrt{d}(1+1/\mu)h/\mu}$ . Note that the following diagram commute (all maps are induced by inclusions),

$$\begin{array}{ccc} H_s \left( \widehat{\mathcal{F}}_{\lambda, h} \right) & \xrightarrow{\quad\quad\quad} & H_s \left( \widehat{\mathcal{F}}_{\lambda', h} \right) \\ \downarrow \rho_{\lambda, h} & & \downarrow \rho_{\lambda', h} \\ H_s \left( \widehat{\mathcal{F}}_{\lambda, h}^{2\sqrt{d}(1+1/\mu)h/\mu} \right) & \xrightarrow{\widehat{v}_{\lambda, h}^{\lambda'}} & H_s \left( \widehat{\mathcal{F}}_{\lambda', h}^{2\sqrt{d}(1+1/\mu)h/\mu} \right) \end{array}$$

Thus for all  $\lambda < \lambda'$ ,

$$\widehat{v}_{\lambda, h}^{\lambda'} \left( \text{Im}(\rho_{\lambda, h}) \right) \subset \text{Im}(\rho_{\lambda', h}).$$

We now introduce  $\widehat{\mathbb{V}}_{f, s}^h$  the persistence module associated to  $(\text{Im}(\rho_{\lambda, h}))_{\lambda \in \mathbb{R}}$  equipped with the collection of maps  $(\widehat{v}_{\lambda, h}^{\lambda'})_{\lambda < \lambda'}$  for the  $s$ -th order homology,  $\widehat{\text{dgm}}_s(f)$  the associated persistence diagram, and  $\widehat{\text{dgm}}(f)$  the collection of such persistence diagrams for all  $s \in \mathbb{N}$ . This diagram is well-defined, as we prove in Appendix A that  $\widehat{\mathbb{V}}_{f, s}^h$  is  $q$ -tame. The computation of this persistence diagram is (theoretically) made possible by the algorithm for image persistence gave in [Cohen-Steiner et al. \(2009\)](#).

Interestingly, if we are only interested in estimating  $H_0$ -persistence diagrams, the second step is unnecessary, as shown in Appendix C.

**Calibration.** A natural question is how to calibrate the window-size  $h$ . Lemma 2 from [Henneuse](#)

(2024) suggests that a good choice is taking  $h_{\theta,\alpha}$  such that,

$$\frac{h_{\theta,\alpha}^{d+\alpha}}{\sqrt{h_{\theta,\alpha}^d \log\left(1 + \frac{1}{h_{\theta,\alpha}^d}\right)}} > \theta$$

which implies that we can take,

$$h_{\theta,\alpha} \simeq \left(\theta^2 \log\left(\frac{1}{\theta}\right)\right)^{\frac{1}{d+2\alpha}}.$$

### 3 Proof of Theorem 1

The proof follows similarly to the proof for upper bounds in Henneuse (2024), the structure of our estimator even allows avoiding some technical consideration. The general idea is to construct an interleaving between the true persistence module and the estimated persistence module. The following two propositions give the necessary ingredients. The morphism from true to estimated module is simply induced by the lower inclusion of Proposition 1. The morphism from estimated to true module is build, combining the upper inclusion of Proposition 1 and the deformation retract of Proposition 2.

Before proving those two propositions, we recall Lemma 2 and some useful notations from Henneuse (2024). Let,

$$\|W\|_{cube,h} = \sup_{H \in C_h} \frac{|W(H)|}{\omega(L(H))}$$

with  $\omega(r) = \sqrt{r \log(1 + 1/r)}$ ,  $W(H) = \int_H dW$  and  $L$  the Lebesgue measure.

**Lemma 2** (Henneuse 2024). *Let  $f : [0, 1]^d \rightarrow \mathbb{R}$ . Let  $H \subset \mathcal{F}_{\lambda + \|W\|_{cube,h_{\theta,\alpha}} h_{\theta,\alpha}^\alpha}^c \cap C_{h_{\theta,\alpha}}$  and  $H' \subset \mathcal{F}_{\lambda - \|W\|_{cube,h_{\theta,\alpha}} h_{\theta,\alpha}^\alpha} \cap C_{h_{\theta,\alpha}}$ . We then have that,*

$$\int_H dX - \int_H \lambda > 0 \text{ and } \int_{H'} dX - \int_{H'} \lambda < 0.$$

**Proposition 1.** *Let  $f \in S_d(L, \alpha, \mu, R_\mu)$ , and  $\theta$  sufficiently small, For all  $\lambda \in \mathbb{R}$ ,*

$$\mathcal{F}_{\lambda - (\|W\|_{cube,h_{\theta,\alpha}} + Ld^{\alpha/2}(1/\mu+1)^\alpha) h_{\theta,\alpha}^\alpha} \subset \widehat{\mathcal{F}}_{\lambda,h_{\theta,\alpha}} \subset \mathcal{F}_{\lambda + \|W\|_{cube,h_{\theta,\alpha}} h_{\theta,\alpha}^\alpha}^{\sqrt{d}(1+1/\mu)h_{\theta,\alpha}}.$$

*Proof.* We begin by proving the lower inclusion, let  $x \in \mathcal{F}_{\lambda - (\|W\|_{cube,h_{\theta,\alpha}} + Ld^{\alpha/2}(1/\mu+1)^\alpha) h_{\theta,\alpha}^\alpha}$ . Without loss of generality, let suppose  $x \in \overline{M}_i$ . If,

$$B_2\left(x, \sqrt{d}h_{\theta,\alpha}\right) \subset \left(\bigcup_{i=1}^l \partial M_i\right)^c$$

then,  $H_{x,h_{\theta,\alpha}}$ , the hypercube of  $C_{h_{\theta,\alpha},\lambda}$  containing  $x$  is included in  $\overline{M}_i$ . Assumption **A1** and **A2** then give  $H_{x,h_{\theta,\alpha}} \subset \mathcal{F}_{\lambda - \|W\|_{cube,h_{\theta,\alpha}} h_{\theta,\alpha}^\alpha}$ . Hence, it follows from Lemma 2 that  $H_{x,h_{\theta,\alpha}} \in C_{h_{\theta,\alpha},\lambda}$  and

consequently  $x \in \widehat{\mathcal{F}}_\lambda$ . Else, by Lemma 1, there exists,

$$y \in \left( \left( \bigcup_{i=1}^l \partial M_i \right)^{\sqrt{d}h_{\theta,\alpha}} \right)^c \cap M_i \text{ such that } \|x - y\|_2 \leq \sqrt{d}h_{\theta,\alpha}/\mu.$$

Let  $H_{y,h_{\theta,\alpha}}$  the closed hypercube of  $C_{h_{\theta,\alpha}}$  containing  $y$ . Hence,  $H_{y,h_{\theta,\alpha}} \subset \overline{M}_i$ . and  $d_2(x, H_{y,h_{\theta,\alpha}}) \leq \sqrt{d}h(1 + 1/\mu)$ . Then, Assumption **A1** and **A2** ensure that,

$$H_{y,h_{\theta,\alpha}} \subset \mathcal{F}_{\lambda - \|W\|_{\text{cube}, h_{\theta,\alpha}}} h_{\theta,\alpha}^\alpha.$$

Then, Lemma 2 gives  $H_{y,h_{\theta,\alpha}} \in C_{h_{\theta,\alpha}, \lambda}$  and thus, as  $x \in H_{y,h_{\theta,\alpha}}^{\sqrt{d}h_{\theta,\alpha}/\mu}$ ,  $x \in \widehat{\mathcal{F}}_\lambda$ , which proves the lower inclusion.

For the upper inclusion, let  $x \in \left( \mathcal{F}_{\lambda + \|W\|_{\text{cube}, h_{\theta,\alpha}}} h_{\theta,\alpha}^\alpha \right)^c$ , and  $H_{x,h_{\theta,\alpha}}$  the hypercube of  $C_{h_{\theta,\alpha}}$  containing  $x$ . We then have,  $H_{x,h_{\theta,\alpha}} \subset \mathcal{F}_{\lambda + \|W\|_{\text{cube}, h_{\theta,\alpha}}}^c h_{\theta,\alpha}^\alpha$ . Hence, Lemma 2 gives that,

$$H_{x,h_{\theta,\alpha}} \subset \left( \bigcup_{H \in C_{h,\lambda}} H \right)^c$$

and thus,

$$\bigcup_{H \in C_{h_{\theta,\alpha}, \lambda}} H \subset \mathcal{F}_{\lambda + \|W\|_{\text{cube}, h_{\theta,\alpha}}} h_{\theta,\alpha}^\alpha.$$

Consequently,

$$\left( \bigcup_{H \in C_{h_{\theta,\alpha}, \lambda}} H \right)^{\sqrt{d}h_{\theta,\alpha}/\mu} = \widehat{\mathcal{F}}_{\lambda, h_{\theta,\alpha}} \subset \mathcal{F}_{\lambda + \|W\|_{\text{cube}, h_{\theta,\alpha}}}^{\sqrt{d}(1+1/\mu)h_{\theta,\alpha}} h_{\theta,\alpha}^\alpha$$

and the proof is complete.  $\square$

**Proposition 2.** For all  $0 < h < \frac{R_\mu}{2}$  and  $A \subset \mathbb{R}$  there exists two collections of spaces  $\mathcal{G} = (\mathcal{G}_{\lambda,h})_{\lambda \in \mathbb{R}}$  and  $\mathcal{K} = (\mathcal{K}_{\lambda,h})_{\lambda \in \mathbb{R}}$  such that  $\forall \lambda \in \mathbb{R}$ ,

$$\mathcal{F}_\lambda \subset \mathcal{G}_{\lambda,h} \subset \mathcal{F}_{\lambda + L(4(1+2/\mu))^\alpha h^\alpha}$$

and

$$\mathcal{F}_\lambda^h \subset \mathcal{K}_{\lambda,h} \subset \mathcal{F}_\lambda^{(1+2/\mu)h}$$

and  $\mathcal{K}_{\lambda,h}$  retracts by deformation onto  $\mathcal{G}_{\lambda,h}$ .

*Proof.* To construct the desired deformation retract, we follow the idea from the proof of Theorem 12 in Kim et al. (2020) combine with the idea of Proposition 1 in Henneuse (2024). In Proposition 1 in Henneuse (2024), the deformation retract involved the projection on the discontinuities set, as on sufficiently small neighborhood of the discontinuities set, the closest point is unique thanks to the positive reach assumption. Here, we do not have such nice property. Still, from the generalized gradient of the discontinuities set, we construct a smooth flow, following Kim et al. (2020), that essentially play the same role as the projection in the positive reach case.

Let denote,

$$\partial M_{\lambda,h} = \bigcup_{i=1}^l \left\{ x \in \partial M_i \cap ]0, 1[^d \text{ s.t. } B_2(x, 4(1+2/\mu)h) \cap M_i \cap \mathcal{F}_\lambda \neq \emptyset \right\}.$$

Let  $x \in \mathcal{F}_\lambda^{2(1+2/\mu)h} \cap \left( \bigcup_{i=1}^l \partial M_i \right)^h \setminus \bigcup_{i=1}^l \partial M_i$ , and,

$$I_{\lambda,h}^x = \left\{ i \in 1, \dots, l \text{ s.t. } B_2(x, h) \cap \overline{M}_i \cap \mathcal{F}_\lambda \neq \emptyset \right\}.$$

Assumption **A3** ensures that,

$$\text{reach}_\mu \left( \bigcup_{i \in I_{\lambda,h}^x} \partial M_i \right) \geq R_\mu$$

and thus,

$$\|\nabla_{\partial M_{\lambda,h}}(x)\| = \left\| \nabla_{\bigcup_{i \in I_{\lambda,h}^x} \partial M_i}(x) \right\| \geq \mu.$$

By Lemma 38 of [Kim et al. \(2020\)](#), there exists  $U_x$  a neighborhood of  $x$  such that, for all  $y \in U_x$ ,

$$\langle \nabla_{\partial M_{\lambda,h}}(x), \nabla_{\partial M_{\lambda,h}}(y) \rangle \geq \mu/2.$$

We then define,

$$W_x(y) = \begin{cases} -\nabla_{\partial M_{\lambda,h}}(x) & \text{if } y \in U_x \\ 0 & \text{Else} \end{cases}$$

consider the open cover,

$$\left\{ U_x, x \in \mathcal{F}_\lambda^h \cap \left( \bigcup_{i=1}^l \partial M_i \right)^h \setminus \bigcup_{i=1}^l \partial M_i \right\}$$

and extract a locally finite open cover  $\{U_{x_i}, i \in \mathbb{N}\}$ . Let  $\{\rho_i, i \in \mathbb{N}\}$  an associated smooth partition of unity, and define the vector field  $W = \sum_{i \in \mathbb{N}} \rho_i W_{x_i}$ .

By the fundamental theorem of flow (see [Kim et al., 2020](#), Theorem 25), there exists a domain

$\mathbb{D} \subset \mathcal{F}_\lambda^{2(1+2/\mu)h} \cap \left( \bigcup_{i=1}^l \partial M_i \right)^h \setminus \bigcup_{i=1}^l \partial M_i \times ]0, +\infty[$  and the unique smooth flow  $C : \mathbb{D} \rightarrow \mathcal{F}_\lambda^{2(1+2/\mu)h} \cap \left( \bigcup_{i=1}^l \partial M_i \right)^h \setminus \bigcup_{i=1}^l \partial M_i$ . Let  $(x, s) \in \mathbb{D}$  and remark that,

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{d_{\partial M_{\lambda,h}}(C(x, s+h)) - d_{\partial M_{\lambda,h}}(C(x, s))}{h} &\leq \left\langle \nabla_{\partial M_{\lambda,h}}, \frac{d}{ds} C(x, s) \right\rangle \\ &= \langle \nabla_{\partial M_{\lambda,h}}, W(C(x, s)) \rangle \\ &\leq -\mu/2 \end{aligned} \tag{1}$$

Let  $x \in \mathcal{F}_\lambda^h$ , and  $s_x = \sup \{s \in ]0, \infty[ \text{ s.t. } (x, s) \in \mathbb{D}\}$ . By [1](#), we have  $C(x, s_x) \notin \partial M_{\lambda,h}^h$  and  $s_x \leq 2h/\mu^2$ . Denote  $d_C$  the arc length distance along  $C$ , as  $\|W\| \leq \mu$ , we then have,

$$d_C(x, C(x, s_x)) = \int_0^{s_x} \left| \frac{\partial}{\partial t} C(x, t) \right| dt$$

$$\begin{aligned}
&\leq \int_0^{s_x} \|W(C(x, t))\|_2 dt \\
&\leq s_x \mu \leq \frac{2h}{\mu}
\end{aligned} \tag{2}$$

Thus  $\|x - C(x, s_x)\|_2 \leq 2h/\mu$ . Hence, if  $C(x, s_x) \notin \partial M_{\lambda, h}$  then  $C(x, s_x) \in \mathcal{F}_\lambda^{(1+2/\mu)h}$  and thus the flow can be extended, which is contradictory. Consequently,  $C(x, s_x) \in \partial M_{\lambda, h}$ .

From the foregoing we can extend continuously  $C(x, \cdot)$  on  $[0, +\infty[$ , taking  $C(x, s) = C(x, s_x)$  for all  $s > s_x$ .

Let

$$\mathcal{K}_{\lambda, h} := \mathcal{F}_\lambda^h \cup \left( \bigcup_{x \in S_{\lambda, h}} \{C(x, t), \quad t \in [0, s_x]\} \right)$$

with

$$S_{\lambda, h} = \left( \bigcup_{i=1}^l (\mathcal{F}_\lambda \cap M_i)^h \cap M_i \right)^c \cap \mathcal{F}_\lambda^h.$$

Using  $C$  we then define the deformation retract : if

$$x \in \bigcup_{x \in S_{\lambda, h}} \{C(x, t), \quad t \in [0, s_x]\} \cap M_i \tag{3}$$

and

$$d_2(C(x, s_x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) \geq 4h/\mu - d_C(C(x, s_x), x) \tag{4}$$

then,

$$F_{\lambda, h}(x, t) = C(x, t\tilde{s}_x)$$

with  $\tilde{s}_x$  verifying,

$$d_C(C(x, s_x), C(x, \tilde{s}_x)) = (4h/\mu - d_2(C(x, s_x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}))_+.$$

Else,

$$F_{\lambda, h}(x, t) = x.$$

Let  $\mathcal{G}_{\lambda, h} = \text{Im}(x \mapsto F_{\lambda, h}(x, 1))$ . Then, by definition,  $F_{\lambda, h}(x, 1) \in \mathcal{G}_{\lambda, h}$ , for all  $x \in \mathcal{K}_{\lambda, h}$ . As  $C(x, 0) = x$  then  $F_{\lambda, h}(x, 0) = x$ , for all  $x \in \mathcal{K}_{\lambda, h}$ . Remark that, by construction,  $\tilde{s}_{C(x, s_x)} = \tilde{s}_x$ . Then, we have  $F_{\lambda, h}(x, 1) = x$ , for all  $x \in \mathcal{G}_{\lambda, h}$ . The proof for the continuity of  $F_{\lambda, h}$  can be found in Appendix B. Hence,  $F_{\lambda, h}$  is a deformation retract onto  $\mathcal{G}_{\lambda, h}$ .

Let's now prove that  $\mathcal{G}_{\lambda, h} \subset \mathcal{F}_{\lambda+L(4(1+2/\mu))^\alpha h^\alpha}$ . Suppose that  $x \in \overline{M}_i \cap \mathcal{K}_{\alpha, h}$  and let's look at the following different cases.

- If  $x \notin \bigcup_{x \in S_{\lambda, h}} \{C(x, t), \quad t \in [0, s_x]\} \cap \overline{M}_i$ , then directly  $F_{\lambda, h}(x, 1) = x \in \mathcal{F}_{\lambda+Lh^\alpha}$ , by Assumptions **A1** and **A2**.
- If  $x \in \bigcup_{x \in S_{\lambda, h}} \{C(x, t), \quad t \in [0, s_x]\} \cap M_i$  and  $d_2(C(x, s_x), \mathcal{F}_{\lambda+Lh^\alpha} \cap M_i) \leq 4h/\mu$ , then, as

$F_{\lambda,h}(x, 1) \in \{C(x, t), t \in [0, s_x]\}$ ,  $F_{\lambda,h}(x, 1) \in \overline{M}_i$  and

$$d_2(F_{\lambda,h}(x, 1), \mathcal{F}_{\lambda+Ld^{\alpha/2}h^{\alpha}} \cap M_i) \leq \frac{4h}{\mu} + \|x, C(x, s_x)\|_2 \leq \frac{6h}{\mu}.$$

By Assumptions **A1** and **A2**, it then follows that  $F_{\lambda,h}(x, 1) \in \mathcal{F}_{\lambda+L(\frac{6}{\mu})^{\alpha}h^{\alpha}}$ .

- Else,  $F_{\lambda,h}(x, 1) = C(x, s_x) \in \partial M_{\lambda,h}$  or  $F_{\lambda,h}(x, 1) = x$  and  $x \in \partial M_{\lambda,h}$ . By definition of  $\partial M_{\lambda,h}$  there exists  $j \in \{1, \dots, l\}$  such that  $C(x, s_x) \in \partial M_j$  and  $y \in M_j$  such that  $\|C(x, s_x) - y\|_2 \leq 4(1+2/\mu)h$ . By Assumptions **A1** and **A2**, it then follows that  $F_{\lambda,h}(x, 1) \in \mathcal{F}_{\lambda+L(4(1+2/\mu))^{\alpha}h^{\alpha}}$ .

Hence, in all cases we have  $F_{\lambda,h}(x, 1) \in \mathcal{F}_{\lambda+L(4(1+2/\mu))^{\alpha}h^{\alpha}}$  and the result is proved.  $\square$

Now equipped with these two lemmas, we can prove Theorem 1.

**Theorem 1.** *There exists  $\tilde{C}_0$  and  $\tilde{C}_1$  such that, for all  $t > 0$ ,*

$$\mathbb{P} \left( \sup_{f \in S_d(L, \alpha, \mu, R_{\mu})} d_b(\widehat{\text{dgm}}(f), \text{dgm}(f)) \geq t \left( \theta^2 \log \left( \frac{1}{\theta} \right) \right)^{\frac{\alpha}{d+2\alpha}} \right) \leq \tilde{C}_0 \exp(-\tilde{C}_1 t^2).$$

*Proof.* The strategy here is to exploit the inclusions from Proposition 1 and the deformation retracts from Proposition 2 to construct an interleaving between  $\hat{\mathbb{V}}_{f,s}^{h_{\theta,\alpha}}$  and  $\mathbb{V}_{s,f}$  and then apply the algebraic stability theorem (Chazal et al., 2009). The first step is the construction of two persistence module homomorphisms,  $\bar{\phi} : \hat{\mathbb{V}}_{f,s}^{h_{\theta,\alpha}} \rightarrow \mathbb{V}_{s,f}$  and  $\bar{\psi} : \mathbb{V}_{s,f} \rightarrow \hat{\mathbb{V}}_{f,s}^{h_{\theta,\alpha}}$ . The second step is checking that  $\bar{\phi}$  and  $\bar{\psi}$  verify the commutativity conditions required to induce an interleaving. It suffices to show the result for (arbitrarily) small  $\theta$  (up to rescaling  $\tilde{C}_0$ ), then suppose that  $\theta$  is sufficiently small for the application of Propositions 1 and 2 used in this proof.

Denote  $k_1 = \sqrt{d}(1 + 1/\mu)$ , the morphism,

$$\bar{\psi}_{\lambda} : H_s(\mathcal{F}_{\lambda}) \rightarrow H_s \left( \hat{\mathcal{F}}_{\lambda + (\|W\|_{\text{cube}, h_{\theta,\alpha}} + Lk_1^{\alpha})h_{\theta,\alpha}^{\alpha}} \right)$$

follows directly from the inclusion  $\mathcal{F}_{\lambda} \subset \hat{\mathcal{F}}_{\lambda + (\|W\|_{\text{cube}, h_{\theta,\alpha}} + Lk_1^{\alpha})h_{\theta,\alpha}^{\alpha}}$  given by Proposition 1.

For the second morphism, denote  $k_2 = \sqrt{d}(1 + 1/\mu)(1 + 2/\mu)$  and consider,

$$j_{1,\lambda} : \text{Im}(\rho_{\lambda,h}) \rightarrow H_s \left( \mathcal{K}_{\lambda + \|W\|_{\text{cube}, h_{\theta,\alpha}} h_{\theta,\alpha}^{\alpha}, k_2 h_{\theta,\alpha}} \right)$$

the map induced by the inclusion  $\hat{\mathcal{F}}_{\lambda, h_{\theta,\alpha}}^{2k_1 h_{\theta,\alpha}/\mu} h_{\theta,\alpha} \subset \mathcal{F}_{\lambda + \|W\|_{\text{cube}, h_{\theta,\alpha}} h_{\theta,\alpha}^{\alpha}, k_2 h_{\theta,\alpha}} \subset \mathcal{K}_{\lambda + \|W\|_{\text{cube}, h_{\theta,\alpha}} h_{\theta,\alpha}^{\alpha}, k_2 h_{\theta,\alpha}}$  given by Proposition 1 and Proposition 2,

$$j_{2,\lambda} : H_s \left( \mathcal{K}_{\lambda + \|W\|_{\text{cube}, h_{\theta,\alpha}} h_{\theta,\alpha}^{\alpha}, k_2 h_{\theta,\alpha}} \right) \rightarrow H_s \left( \mathcal{G}_{\lambda + \|W\|_{\text{cube}, h_{\theta,\alpha}} h_{\theta,\alpha}^{\alpha}, k_2 h_{\theta,\alpha}} \right)$$

induced by the deformation retracts of Proposition 2, and,

$$j_{3,\lambda} : H_s \left( \mathcal{G}_{\lambda + \|W\|_{\text{cube}, h_{\theta,\alpha}} h_{\theta,\alpha}^{\alpha}, k_2 h_{\theta,\alpha}} \right) \rightarrow H_s \left( \mathcal{F}_{\lambda + (\|W\|_{\text{cube}, h_{\theta,\alpha}} + L(4(1+2/\mu)k_2)^{\alpha})h_{\theta,\alpha}^{\alpha}} \right)$$

induced by the inclusion  $\mathcal{G}_{\lambda+\|W\|_{cube,h_{\theta,\alpha}}h_{\theta,\alpha}^\alpha,k_2h_{\theta,\alpha}} \subset \mathcal{F}_{\lambda+(\|W\|_{cube,h_{\theta,\alpha}}+L(4(1+2/\mu)k_2)^\alpha)h_{\theta,\alpha}}$  given by Proposition 2. We then define,

$$\begin{cases} \bar{\phi}_\lambda : \text{Im}(\rho_{\lambda,h_{\theta,\alpha}}) \rightarrow H_2\left(\mathcal{F}_{\lambda+(\|W\|_{cube,h_{\theta,\alpha}}+L(4(1+2/\mu)k_2)^\alpha)h_{\theta,\alpha}}\right) \\ \bar{\phi}_\lambda = j_{3,\lambda} \circ j_{2,\lambda} \circ j_{1,\lambda} \end{cases}$$

We now show that  $\bar{\psi}$  and  $\bar{\phi}$  induced an interleaving between  $\hat{\mathbb{V}}_{f,s}^{h_{\theta,\alpha}}$  and  $\mathbb{V}_{s,f}$ . More precisely, we show that the following diagrams commute, for all  $\lambda < \lambda'$  (for compactness of notation we denote  $K_1 = \|W\|_{cube,h_{\theta,\alpha}} + Lk_1^\alpha$  and  $K_2 = \|W\|_{cube,h_{\theta,\alpha}} + L(4(1+2/\mu)k_2)^\alpha$ ),

$$\begin{array}{ccc} \text{Im}(\rho_{\lambda,h_{\theta,\alpha}}) & \xrightarrow{\hat{v}_{\lambda,h_{\theta,\alpha}}^{\lambda'}} & \text{Im}(\rho_{\lambda',h_{\theta,\alpha}}) \\ \downarrow \bar{\phi}_\lambda & & \downarrow \bar{\phi}_{\lambda'} \\ H_s(\mathcal{F}_{\lambda+K_2h_{\theta,\alpha}^\alpha}) & \xrightarrow{v_{\lambda+K_2h_{\theta,\alpha}^\alpha}^{\lambda'+K_2h_{\theta,\alpha}^\alpha}} & H_s(\mathcal{F}_{\lambda'+K_2h_{\theta,\alpha}^\alpha}) \end{array} \quad (5)$$

$$\begin{array}{ccc} H_s(\mathcal{F}_\lambda) & \xrightarrow{v_\lambda^{\lambda'}} & H_s(\mathcal{F}_{\lambda'}) \\ \downarrow \bar{\psi}_\lambda & & \downarrow \bar{\psi}_{\lambda'} \\ \text{Im}(\rho_{\lambda+K_1h_{\theta,\alpha}^\alpha,h_{\theta,\alpha}}) & \xrightarrow{\hat{v}_{\lambda+K_1h_{\theta,\alpha}^\alpha,h_{\theta,\alpha}}^{\lambda'+K_1h_{\theta,\alpha}^\alpha}} & \text{Im}(\rho_{\lambda'+K_1h_{\theta,\alpha}^\alpha,h_{\theta,\alpha}}) \end{array} \quad (6)$$

$$\begin{array}{ccc} \text{Im}(\rho_{\lambda,h_{\theta,\alpha}}) & \xrightarrow{\hat{v}_{\lambda,h_{\theta,\alpha}}^{\lambda+(K_1+K_2)h_{\theta,\alpha}^\alpha}} & \text{Im}(\rho_{\lambda+(K_1+K_2)h_{\theta,\alpha}^\alpha,h_{\theta,\alpha}}) \\ \searrow \bar{\phi}_\lambda & & \nearrow \bar{\psi}_{\lambda+K_2h_{\theta,\alpha}^\alpha} \\ & H_s(\mathcal{F}_{\lambda+K_2h_{\theta,\alpha}^\alpha}) & \end{array} \quad (7)$$

$$\begin{array}{ccc} H_s(\mathcal{F}_\lambda) & \xrightarrow{v_\lambda^{\lambda+(K_1+K_2)h_{\theta,\alpha}^\alpha}} & H_s(\mathcal{F}_{\lambda+(K_1+K_2)h_{\theta,\alpha}^\alpha}) \\ \searrow \bar{\psi}_\lambda & & \nearrow \bar{\phi}_{\lambda+K_1h_{\theta,\alpha}^\alpha} \\ & \text{Im}(\rho_{\lambda+K_1h_{\theta,\alpha}^\alpha,h_{\theta,\alpha}}) & \end{array} \quad (8)$$

• **Diagram 5** : We can rewrite the diagram as (unspecified maps are simply induced by set

inclusion),

$$\begin{array}{ccc}
\text{Im}(\rho_{\lambda, h_{\theta, \alpha}}) & \longrightarrow & \text{Im}(\rho_{\lambda', h_{\theta, \alpha}}) \\
\downarrow & & \downarrow \\
H_s(\mathcal{K}_{\lambda + \|W\|_{\text{cube}, h_{\theta, \alpha}} h_{\theta, \alpha}^\alpha, k_2 h_{\theta, \alpha}}) & \longrightarrow & H_s(\mathcal{K}_{\lambda' + \|W\|_{\text{cube}, h_{\theta, \alpha}} h_{\theta, \alpha}^\alpha, k_2 h_{\theta, \alpha}}) \\
\uparrow j_{2, \lambda} & & \uparrow j_{2, \lambda'} \\
H_s(\mathcal{G}_{\lambda + \|W\|_{\text{cube}, h_{\theta, \alpha}} h_{\theta, \alpha}^\alpha, k_2 h_{\theta, \alpha}}) & \longrightarrow & H_s(\mathcal{G}_{\lambda' + \|W\|_{\text{cube}, h_{\theta, \alpha}} h_{\theta, \alpha}^\alpha, k_2 h_{\theta, \alpha}}) \\
\downarrow & & \downarrow \\
H_s(\mathcal{F}_{\lambda + K_2 h_{\theta, \alpha}^\alpha}) & \longrightarrow & H_s(\mathcal{F}_{\lambda' + K_2 h_{\theta, \alpha}^\alpha})
\end{array}$$

By inclusions the upper and lower faces commutes and the central faces commutes as for all  $\lambda \in \mathbb{R}$ ,  $j_{2, \lambda}$  come from a deformation retract, thus the diagram is commutative.

- **Diagram 6:** One can check that it can be decomposed as diagram 5 and thus the same reasoning applies.
- **Diagram 7:** Let  $[C] \in \text{Im}(\rho_{\lambda, h_{\theta, \alpha}})$ . By construction, we can suppose  $C \in C_s(\widehat{\mathcal{F}}_{\lambda, h_{\theta, \alpha}})$ . The map  $\bar{\phi}_\lambda$  maps  $[C]$  to  $[C']$  with  $C' \in C_s(\mathcal{F}_{\lambda + K_2 h_{\theta, \alpha}^\alpha})$ . By construction,  $C'$  is the retraction of  $C$  in  $\mathcal{K}_{\lambda + \|W\|_{\text{cube}, h_{\theta, \alpha}} h_{\theta, \alpha}^\alpha, k_2 h_{\theta, \alpha}}$  via the deformation retract constructed in the proof of Proposition 2. From the proof of Proposition 2 (more precisely the bound on the arc length distance 2) we know that the support of this retraction is included in  $\widehat{\mathcal{F}}_{\lambda, h_{\theta, \alpha}}^{2k_1 h_{\theta, \alpha} / \mu}$ . Thus,  $C$  and  $C'$  are homologous in  $\widehat{\mathcal{F}}_{\lambda, h_{\theta, \alpha}}^{2k_1 h_{\theta, \alpha} / \mu}$  and consequently they are homologous in  $\widehat{\mathcal{F}}_{\lambda + (K_1 + K_2) h_{\theta, \alpha}^\alpha, h_{\theta, \alpha}}^{2k_1 h_{\theta, \alpha} / \mu}$ . As  $\bar{\psi}_{\lambda + K_1 h_{\theta, \alpha}^\alpha}$  is simply an inclusion map, it maps  $[C']$  to  $[C']$ , and hence,  $[C] = [C'] = \bar{\psi}_{\lambda + K_2 h_{\theta, \alpha}^\alpha}(\bar{\phi}_\lambda([C]))$ , which proves the commutativity.
- **Diagram 8:** Let  $C \in C_s(\mathcal{F}_\lambda)$ ,  $\bar{\psi}_\lambda([C]) = [C]$  as it is simply an inclusion induced map and by construction of the deformation retract in the proof of Proposition 2,  $\bar{\phi}_{\lambda + K_1}([C]) = [C]$ , which establish the commutativity.

The commutativity of diagrams 5, 6, 7 and 8 means that  $\widehat{\mathbb{V}}_{f, s}^{h_{\theta, \alpha}}$  and  $\mathbb{V}_{f, s}$  are  $(K_1 + K_2)h_{\theta, \alpha}^\alpha$  interleaved, and thus we get from the algebraic stability theorem (Chazal et al., 2009) that,

$$d_b(\text{dgm}(\widehat{\mathbb{V}}_{f, s}^{h_{\theta, \alpha}}), \text{dgm}(\mathbb{V}_{f, s})) \leq (K_1 + K_2)h_{\theta, \alpha}^\alpha$$

and as it holds for all  $s \in \mathbb{N}$  and  $f \in S_d(L, \alpha, \mu, R_\mu)$ ,

$$\sup_{f \in S_d(L, \alpha, \mu, R_\mu)} d_b(\widehat{\text{dgm}}(f), \text{dgm}(f)) \leq (K_1 + K_2)h_{\theta, \alpha}^\alpha.$$

We then conclude, using the concentration of  $\|W\|_{\text{cube}, h_{\theta, \alpha}}$ , as in the proof proposition 5 in Henneuse (2024). More precisely, from Lemma 4 of Henneuse (2024), we have,

$$\mathbb{P}(\|W\|_{\text{cube}, h} \leq t) \leq C_0 \exp(-C_1 t^2).$$

Then,

$$\begin{aligned}
& \mathbb{P} \left( \sup_{f \in S_d(L, \alpha, \mu, R_\mu)} d_b \left( \widehat{\text{dgm}}(f), \text{dgm}(f) \right) \geq th_{\theta, \alpha} \right) \\
& \leq \mathbb{P} (K_1 + K_2 \geq t) \\
& = \mathbb{P} \left( \|W\|_{\text{cube}, h_{\theta, \alpha}} \geq \frac{t - L(k_1^\alpha + (4(1 + 2/\mu)k_2)^\alpha)}{2} \right) \\
& \leq C_0 \exp \left( -C_1 \left( \frac{t - L(k_1^\alpha + (4(1 + 2/\mu)k_2)^\alpha)}{2} \right)^2 \right) \\
& \leq C_0 \exp \left( -\frac{C_1}{4} t^2 \right) \exp (C_1 L (k_1^\alpha + (4(1 + 2/\mu)k_2)^\alpha) t) \\
& \quad \times \exp \left( -C_1 \left( \frac{L(k_1^\alpha + (4(1 + 2/\mu)k_2)^\alpha)}{2} \right)^2 \right)
\end{aligned}$$

and the result follows.  $\square$

From this result, we can derive from this result bounds in expectation.

**Corollary 1.** *Let  $p \geq 1$ ,*

$$\sup_{f \in S_d(L, \alpha, \mu, R_\mu)} \mathbb{E} \left( d_b \left( \widehat{\text{dgm}}(f), \text{dgm}(f) \right)^p \right) \lesssim \left( \theta^2 \log \left( \frac{1}{\theta} \right) \right)^{\frac{p\alpha}{d+2\alpha}}$$

*Proof.* The sub-Gaussian concentration provided by Theorem 1, gives that, for all  $t > 0$ ,

$$\mathbb{P} \left( \frac{d_b \left( \widehat{\text{dgm}}(f), \text{dgm}(f) \right)}{h_{\theta, \alpha}^\alpha} \geq t \right) \leq \tilde{C}_0 \exp \left( -\tilde{C}_1 t^2 \right).$$

Now, we have,

$$\begin{aligned}
& \mathbb{E} \left( \frac{d_b \left( \widehat{\text{dgm}}(f), \text{dgm}(f) \right)^p}{h_{\theta, \alpha}^{p\alpha}} \right) \\
& = \int_0^{+\infty} \mathbb{P} \left( \frac{d_b \left( \widehat{\text{dgm}}(f), \text{dgm}(f) \right)^p}{h_{\theta, \alpha}^{p\alpha}} \geq t \right) dt \\
& \leq \int_0^{+\infty} \tilde{C}_0 \exp \left( -\tilde{C}_1 t^{2/p} \right) dt < +\infty.
\end{aligned}$$

$\square$

## 4 Discussion

In this work, we extend main results of [Henneuse \(2024\)](#), relaxing the positive reach condition over the discontinuities set to a  $\mu$ -reach condition. It allows considering signals with pretty wild discontinuities, for examples, exhibiting multiple points or cusps, cases that were excluded by the positive

reach assumption. More precisely, we show that persistence diagrams of piecewise Hölder-continuous signals with sets of discontinuities having a positive  $\mu$ -reach can be inferred consistently, with identical convergence rates to the ones known for Hölder-continuous signals. This means that there is (constant-wise) no additional cost to tackle such irregular signals. Once again, this highlights the robustness to signal irregularity of the persistence diagram inference and the interest of breaking free from analysis relying on sup norm stability.

Under these relaxed conditions, as illustrated in Figure 5, a histogram plug-in estimator fails to converge consistently to the true persistence diagram. Departing from the usual plug-in approach, we propose a new estimator, to overcome this issue. The significant improvements gained by relaxing assumptions call for exploration of non-plug-in methods in future research.

Similarly to what is done in Appendix C of Henneuse (2024), one can extend the method and results obtained here in the setting of non-parametric regression. Motivated by applications to modes detection, we are also interested in extension to the density model, which will be the object of forthcoming works.

A practical limitation of the proposed method is that it requires some knowledge on the parameters  $\mu$  and  $\alpha$  (at least lower bounds). The dependence on  $\alpha$  can be handled (while preserving the convergence rates), as in Section 2.4 of Henneuse (2024), via the Lepski’s method (Lepskii, 1991, 1992). The dependence on  $\mu$  is more problematic but rather common in homology/homotopy inference. Exploring methodologies to develop adaptive procedures requires further dedicated research.

Also, from a computational perspective (and in order to perform a numerical evaluation of this method), an important step forward would be to provide an efficient algorithm to compute the image persistence for general filtrations. This would permit to compare this method to plug-in approach, specifically to the histogram estimator. While Bauer and Schmahl (2022) offers an efficient such algorithm, it is tailored for Rips complexes, complicating its direct application to our context.

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## A Proof for $q$ -tameness

This section is devoted to prove the claim that the persistence diagrams we consider and estimated persistence diagrams we propose are well-defined, by proving that the underlying persistence modules are  $q$ -tame.

**Lemma 3.** *Let  $f \in S_d(L, \alpha, \mu, R_\mu)$ .  $\forall s \in \mathbb{N}, \forall h < \frac{R_\mu}{2}$ , there exist a morphism  $\phi$  such that,  $\forall \lambda \in \mathbb{R}$ ,*

$$\begin{array}{ccc}
 H_s(\mathcal{F}_\lambda) & \xrightarrow{\quad\quad\quad} & H_s(\mathcal{F}_{\lambda+L(4(1+2/\mu))^{\alpha}h^\alpha}) \\
 & \searrow & \nearrow \phi_\lambda \\
 & H_s(\mathcal{F}_\lambda^h) &
 \end{array} \tag{9}$$

*is a commutative diagram (unspecified map come from set inclusions).*

*Proof.* Let  $\tilde{\phi}_\lambda : H_s(\mathcal{K}_{\lambda,h}) \rightarrow H_s(\mathcal{G}_{\lambda,h})$  the morphism associated to the deformation retract from Proposition 2. We also denote  $i_{1,\lambda} : H_s(\mathcal{F}_\lambda^h) \rightarrow H_s(\mathcal{K}_{\lambda,h})$  the morphism induced by the inclusion  $\mathcal{F}_\lambda^h \subset \mathcal{K}_{\lambda,h}$  and  $i_{2,\lambda} : H_s(\mathcal{G}_{\lambda,h}) \rightarrow H_s(\mathcal{F}_{\lambda+L(4(1+2/\mu))^{\alpha}h^\alpha})$  the morphism induced by the inclusion  $\mathcal{G}_{\lambda,h} \subset \mathcal{F}_{\lambda+L(4(1+2/\mu))^{\alpha}h^\alpha}$ , also provided by Proposition 2. We take  $\phi_\lambda = i_{2,\lambda} \circ \tilde{\phi}_\lambda \circ i_{1,\lambda}$ . Diagram 9 then is (unspecified maps are the one induced by set inclusion),

$$\begin{array}{ccccc}
 H_s(\mathcal{F}_\lambda) & \xrightarrow{\quad\quad\quad} & H_s(\mathcal{F}_{\lambda+L(4(1+2/\mu))^{\alpha}h^\alpha}) & & \\
 \downarrow (F1) & \searrow (F2) & \searrow (F3) & & \uparrow i_{2,\lambda} \\
 H_s(\mathcal{F}_\lambda^h) & \xrightarrow{i_{1,\lambda}} & H_s(\mathcal{K}_{\lambda,h}) & \xleftarrow{\tilde{\phi}_\lambda} & H_s(\mathcal{G}_{\lambda,h})
 \end{array} \tag{10}$$

Faces (F1) and (F3) simply commute by inclusion. Face (F2) commutes as  $\tilde{\phi}_\lambda$  is an isomorphism. All faces of diagram 10 are commutative, hence diagram 10 (and equivalently diagram 9) is commutative.  $\square$

**Proposition 3.** *Let  $f \in S_d(L, \alpha, \mu, R_\mu)$  then  $f$  is  $q$ -tame.*

*Proof.* Let  $s \in \mathbb{N}$  and  $\mathbb{V}_{s,f}$  the persistence module (for the  $s$ -th homology) associated to the sublevel filtration,  $\mathcal{F}$  and for fixed levels  $\lambda < \lambda'$  let denote  $v_\lambda^{\lambda'}$  the associated map. Let  $\lambda \in \mathbb{R}$  and  $h < \frac{R_\mu}{2}$ . By Lemma 3,  $v_\lambda^{\lambda+L(4(1+2/\mu))^{\alpha}h^\alpha} = \phi_\lambda \circ \tilde{i}_\lambda$ , with  $\tilde{i}_\lambda : H_s(\mathcal{F}_\lambda) \rightarrow H_s(\mathcal{F}_\lambda^h)$ . And, due to sublevel thickening by  $h$ ,  $\overline{\mathcal{F}}_\lambda \subset \mathcal{F}_\lambda^h$ , and consequently  $\tilde{i}_\lambda$  is of finite rank. Thus,  $v_\lambda^{\lambda+L(4(1+2/\mu))^{\alpha}h^\alpha}$  is of finite rank for all  $0 < h < \frac{R_\mu}{2}$ . As for any  $\lambda < \lambda' < \lambda''$ ,  $v_\lambda^{\lambda''} = v_\lambda^{\lambda'} \circ v_\lambda^{\lambda'}$  we then have that  $v_\lambda^{\lambda'}$  is of finite rank for all  $\lambda < \lambda'$ . Hence,  $f$  is  $q$ -tame.  $\square$

**Proposition 4.** *Let  $f \in S_d(R, L, \alpha)$  and  $h > 0$  then, for all  $s \in \mathbb{N}$ ,  $\widehat{\mathbb{V}}_{s,f}^h$  is  $q$ -tame.*

*Proof.* Let  $h > 0$  and  $\lambda \in \mathbb{R}$ . Again, thanks to the  $\sqrt{dh}/\mu$ -thickening,  $\rho_{\lambda,h}$  is of finite rank. Hence,  $\text{Im}(\rho_{\lambda,h})$  is finite dimensional. Thus  $\widehat{\mathbb{V}}_{s,f}^h$  is  $q$ -tame by Theorem 1.1 of [Crawley-Boevey \(2012\)](#).  $\square$

## B Proof of the continuity of $F_{\lambda,h}$

This section is devoted to the proof of the continuity of the deformation retract  $F_{\lambda,h}$ , introduced in the proof of Proposition 2.

**Lemma 4.** *Let  $h > 0$  and  $\lambda \in \mathbb{R}$ ,  $F_{\lambda,h}$  is continuous.*

*Proof.* Let  $\delta, \delta' > 0$ ,  $x, y \in \mathcal{K}_{\lambda,h}$  such that  $\|x - y\|_2 \leq \delta$  and  $t, s \in [0, 1]$  such that  $|t - s| \leq \delta'$ . Let's look at the different cases.

First let's tackle the cases where  $x \in \overline{M}_i$  and  $y \in \overline{M}_j$ ,  $i \neq j$ .

- If  $x$  verifies 3 and 4. Supposing  $\frac{8h}{\mu} - d_2(C(x, s_x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) > 0$ , for sufficiently small  $\delta$ , we would have,

$$d_C(x, C(x, s_x)) \leq 2\delta/\mu < \left( \frac{4h}{\mu} - d_2(C(x, s_x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) \right)_+$$

which contradicts 4. Hence, we can suppose  $\left( \frac{4h}{\mu} - d_2(C(x, s_x), M_i \cap \mathcal{F}_{\lambda+Ld^\alpha/2h^\alpha}) \right)_+ = 0$ . Then,  $\tilde{s}_x = s_x$  and  $F_{\lambda,h}(x, t) = C(x, ts_x)$ , thus,

$$\|F_{\lambda,h}(x, t) - x\|_2 = \|x - C(x, ts_x)\|_2 \leq \|x - C(x, s_x)\|_2 \leq 2\delta/\mu.$$

- If  $x$  does not verify 3 or 4,  $F_{\lambda,h}(x, t) = x$ , and directly,

$$\|F_{\lambda,h}(x, t) - x\|_2 = 0.$$

following the same reasoning we also have, in both cases for  $y$ ,  $\|F_{\lambda,h}(y, s) - y\|_2 \leq 2\delta/\mu$ . Consequently, in any cases,

$$\|F_{\lambda,h}(x, t) - F_{\lambda,h}(y, s)\|_2 \leq \|F_{\lambda,h}(x, t) - x\|_2 + \|F_{\lambda,h}(y, s) - y\|_2 + \|x - y\|_2 \leq \delta(1 + 4/\mu).$$

This concludes for the cases where  $x \in \overline{M}_i$  and  $y \in \overline{M}_j$ ,  $i \neq j$ . From now, we suppose that  $x, y \in M_i$ .

- If  $x$  and  $y$  do not verify 3 or 4, then directly,

$$\|F_{\lambda,h}(x, t) - F_{\lambda,h}(y, s)\|_2 = \|x - y\|_2 \leq \delta.$$

- If  $x$  verifies 3 and 4 and  $y$  does not verify 3. Then,  $y \in \mathcal{F}_{\lambda+Lh^\alpha}$ . Thus,

$$d_2(C(x, s_x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) \leq d_C(x, C(x, s_x)) + \|x - y\|_2 \leq 2h/\mu + \delta$$

and,

$$4h/\mu - d_2(C(x, s_x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) \geq 2h/\mu - \delta.$$

Consequently,  $F_{\lambda,h}(x, t) \in \{C(x, t), t \in [0, \delta]\}$ . As,

$$\|F_{\lambda,h}(x, t) - F_{\lambda,h}(y, s)\|_2 = \|F_{\lambda,h}(x, t) - y\|_2 \leq \|F_{\lambda,h}(x, t) - x\|_2 + \delta$$

we can conclude in this case by continuity of  $C$ .

- If  $x$  verifies 3 and 4 and  $y$  verifies 3 but not 4, then,

$$\begin{aligned}
4h/\mu - d_2(C(x, s_x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) &= 4h/\mu - d_2(C(y, s_y), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) \\
&\quad + d_2(C(y, s_y), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) \\
&\quad - d_2(C(x, s_x), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) \\
&\geq 4h/\mu - d_2(C(y, s_y), M_i \cap \mathcal{F}_{\lambda+Lh^\alpha}) - \|C(x, s_x) - C(y, s_y)\|_2 \\
&\geq d_C(y, C(y, s_y)) - \|C(x, s_x) - C(y, s_y)\|_2.
\end{aligned}$$

Hence, by definition of  $\tilde{s}_x$ ,  $F_{\lambda,h}(x, t) \in \{C(x, t), x \in [0, |d_C(y, C(y, s_y)) - d_C(x, C(x, s_x))| + \|C(x, s_x) - C(y, s_y)\|_2]\}$ . Suppose without loss of generality that,  $s_x \geq s_y$ , then we can write,

$$C(x, s_x) = C(C(x, s_y), s_{C(x, s_y)}) = C(x, s_y + s_{C(x, s_y)}).$$

By 2,  $s_{C(x, s_y)} \leq 2\|C(x, s_y) - C(y, s_y)\|_2/\mu$ . And by continuity of  $C$ ,  $\lim_{\delta \rightarrow 0} \|C(x, s_y) - C(y, s_y)\|_2 = 0$ , thus,  $\lim_{\delta \rightarrow 0} |s_x - s_y| = 0$ . Hence, by continuity of  $C$  again,  $\lim_{\delta \rightarrow 0} \|F_{\lambda,h}(x, t) - x\|_2 = 0$ . As,

$$\|F_{\lambda,h}(x, t) - F_{\lambda,h}(y, s)\|_2 = \|F_{\lambda,h}(x, t) - y\|_2 \leq \|F_{\lambda,h}(x, t) - x\|_2 + \|x - y\|_2 \xrightarrow{\delta \rightarrow 0} 0$$

we conclude in this case.

- Finally, If both  $x$  and  $y$  verifies 3 and 4, then,

$$\|F_{\lambda,h}(x, t) - F_{\lambda,h}(y, s)\|_2 = \|C(x, t\tilde{s}_x) - C(y, s\tilde{s}_y)\|_2$$

As in the previous case, it follows again in this case by continuity of  $C$ , that  $\lim_{\delta \rightarrow 0} |s_x - s_y| = 0$ . Consequently, by continuity of  $C$  and definition of  $\tilde{s}_x$  and  $\tilde{s}_y$ ,  $\lim_{\delta \rightarrow 0} |\tilde{s}_x - \tilde{s}_y| = 0$ . And using again the continuity of  $C$ , we then have,

$$\lim_{\delta \rightarrow 0} \|C(x, t\tilde{s}_x) - C(y, s\tilde{s}_y)\|_2 = 0$$

which gives the conclusion in this case.

Combining all the previous cases, we proved that  $F_{\lambda,h}$  is continuous.  $\square$

## C Simplification for $H_0$ -homology

The trick of looking at the persistence module induced by the homology groups  $(\text{Im}(\rho_{\lambda,h,\theta,\alpha}))_{\lambda \in \mathbb{R}}$  instead of directly look at the one induced by  $(H_s(\widehat{\mathcal{F}}_{\lambda,h,\theta,\alpha}))$  permits to remove false topological features due to cubical approximation, thickening and noise. And particularly, the one lying around the discontinuity set that can potentially have an arbitrarily long lifetime. But, for inferring  $H_0$  persistence diagrams, we show that this is unnecessary. Informally, this holds because for any set  $A$  and  $h > 0$ , the number of connected components of  $A^h$  is less or equal than the number of connected components of  $A$  (which is not true for higher order homology, cycles can be created by the thickening).

Here,  $\widehat{\mathcal{V}}_{f,0}^{h\theta,\alpha}$  is defined as the persistence module corresponding to  $(H_0(\widehat{\mathcal{F}}_{\lambda,h,\theta,\alpha}))$  equipped with inclusion induced maps.

**Theorem 1 bis.** *There exists  $\tilde{C}_0$  and  $\tilde{C}_1$  such that, for all  $A > 0$ ,*

$$\mathbb{P} \left( \sup_{f \in \mathcal{S}_d(L, \alpha, \mu, R_\mu)} d_b \left( \text{dgm} \left( \hat{\mathbb{V}}_{f,0}^{h_{\theta,\alpha}} \right), \text{dgm} \left( \mathbb{V}_{f,0} \right) \right) \geq t \left( \theta^2 \log \left( \frac{1}{\theta} \right) \right)^{\frac{\alpha}{2\alpha+d}} \right) \leq \tilde{C}_0 \exp \left( -\tilde{C}_1 t^2 \right).$$

*Proof.* The proof follows essentially as the proof of Theorem 1. The only notable difference is contained in the two following remarks.

Let  $\lambda \in \mathbb{R}$ , Lemma 2 imply that any connected component  $A$  of  $\hat{\mathcal{F}}_{\lambda, h_{\theta,\alpha}}$  intersects a connected component  $B$  of  $\mathcal{F}_{\lambda + \|W\|_{\text{cube}} h_{\theta,\alpha}^\alpha}$ . Now suppose that  $A$  intersects 2 such components  $B_1$  and  $B_2$ , then, by Proposition 1,  $B_1$  and  $B_2$  are connected in  $\mathcal{F}_{\lambda + \|W\|_{\text{cube}, h_{\theta,\alpha}} h_{\theta,\alpha}^\alpha}^{\sqrt{d}(1+1/\mu)h_{\theta,\alpha}}$ . Thus, as a consequence of Lemma 3,  $B_1$  and  $B_2$  are connected in  $\mathcal{F}_{\lambda + (\|W\|_{\text{cube}} + L(4(\sqrt{d}(1+2/\mu)(1+1/\mu)))^\alpha) h_{\theta,\alpha}^\alpha}$ . We can then define properly the applications,

$$\begin{cases} \bar{\phi}_\lambda : H_0 \left( \hat{\mathcal{F}}_{\lambda, h_{\theta,\alpha}} \right) & \longrightarrow H_0 \left( \mathcal{F}_{\lambda + (\|W\|_{\text{cube}} + L(4(\sqrt{d}(1+2/\mu)(1+1/\mu)))^\alpha) h_{\theta,\alpha}^\alpha} \right) \\ [A] & \longmapsto [B] \end{cases}$$

with  $B$  of any connected components of  $\mathcal{F}_{\lambda + \|W\|_{\text{cube}, h_{\theta,\alpha}} h_{\theta,\alpha}^\alpha}$  intersecting  $A$ , the previous remark ensuring that  $[B]$  is independent of the choice of  $B$ .

Conversely, let  $A$  a connected component of  $\mathcal{F}_\lambda$ . By Proposition 1,  $A$  is contained in a connected component  $B$  of  $\hat{\mathcal{F}}_{\lambda + (\|W\|_{\text{cube}, h_{\theta,\alpha}} + Ld^{\alpha/2}(1/\mu+1)^\alpha) h_{\theta,\alpha}^\alpha}$ . We can then define properly the applications,

$$\begin{cases} \bar{\psi}_\lambda : H_0 \left( \mathcal{F}_\lambda \right) & \longrightarrow H_0 \left( \hat{\mathcal{F}}_{\lambda + (\|W\|_{\text{cube}, h_{\theta,\alpha}} + Ld^{\alpha/2}(1/\mu+1)^\alpha) h_{\theta,\alpha}^\alpha} \right) \\ [A] & \longmapsto [B] \end{cases}$$

with  $B$  the connected component of  $\hat{\mathcal{F}}_{\lambda + (\|W\|_{\text{cube}, h_{\theta,\alpha}} + Ld^{\alpha/2}(1/\mu+1)^\alpha) h_{\theta,\alpha}^\alpha}$  containing  $A$ .

Denote  $\tilde{K}_1 = \|W\|_{\text{cube}, h_{\theta,\alpha}} + Ld^{\alpha/2}(1+1/\mu)^\alpha$  and  $\tilde{K}_2 = \|W\|_{\text{cube}} + Ld^{\alpha/2}(4(1+2/\mu)(1+1/\mu))^\alpha$ . As in the proof of Theorem 1, one can check that the following diagrams commute (unspecified maps are the one induced by set inclusion),

$$\begin{array}{ccc} H_0 \left( \mathcal{F}_\lambda \right) & \longrightarrow & H_0 \left( \mathcal{F}_{\lambda'} \right) \\ \downarrow \bar{\psi}_\lambda & & \downarrow \bar{\psi}_{\lambda'} \\ H_0 \left( \hat{\mathcal{F}}_{\lambda + \tilde{K}_1 h_{\theta,\alpha}^\alpha, h_{\theta,\alpha}} \right) & \longrightarrow & H_0 \left( \hat{\mathcal{F}}_{\lambda' + \tilde{K}_1 h_{\theta,\alpha}^\alpha, h_{\theta,\alpha}} \right) \\ \\ H_0 \left( \hat{\mathcal{F}}_{\lambda, h_{\theta,\alpha}} \right) & \longrightarrow & H_0 \left( \hat{\mathcal{F}}_{\lambda', h_{\theta,\alpha}} \right) \\ \downarrow \bar{\phi}_\lambda & & \downarrow \bar{\phi}_{\lambda'} \\ H_0 \left( \mathcal{F}_{\lambda + \tilde{K}_2 h_{\theta,\alpha}^\alpha} \right) & \longrightarrow & H_0 \left( \mathcal{F}_{\lambda' + \tilde{K}_2 h_{\theta,\alpha}^\alpha} \right) \end{array}$$

$$\begin{array}{ccc}
H_0(\mathcal{F}_\lambda) & \xrightarrow{\quad\quad\quad} & H_0\left(\mathcal{F}_{\lambda+(\tilde{K}_1+\tilde{K}_2)h_{\theta,\alpha}^\alpha}\right) \\
\searrow \bar{\psi}_\lambda & & \nearrow \bar{\phi}_{\lambda+\tilde{K}_1 h_{\theta,\alpha}^\alpha} \\
& & H_0\left(\widehat{\mathcal{F}}_{\lambda+\tilde{K}_1 h_{\theta,\alpha}^\alpha, h_{\theta,\alpha}}\right) \\
\\
H_0\left(\widehat{\mathcal{F}}_{\lambda, h_{\theta,\alpha}}\right) & \xrightarrow{\quad\quad\quad} & H_0\left(\widehat{\mathcal{F}}_{\lambda+(\tilde{K}_1+\tilde{K}_2)h_{\theta,\alpha}^\alpha, h_{\theta,\alpha}}\right) \\
\searrow \bar{\phi}_\lambda & & \nearrow \bar{\psi}_{\lambda+\tilde{K}_2 h_{\theta,\alpha}^\alpha} \\
& & H_0\left(\mathcal{F}_{\lambda+\tilde{K}_2 h_{\theta,\alpha}^\alpha}\right)
\end{array}$$

Hence  $\widehat{\mathbb{V}}_{f,0}^{h_{\theta,\alpha}}$  and  $\mathbb{V}_{f,0}$  are  $(\tilde{K}_1 + \tilde{K}_2)h_{\theta,\alpha}^\alpha$ -interleaved, and we conclude as in the proof of Theorem 1.  $\square$

Note that the thickening of the first step is somehow necessary. A classical histogram estimator can fail to capture the right  $H_0$  persistence diagram, even in the noiseless setting, as illustrated in Figure 6.

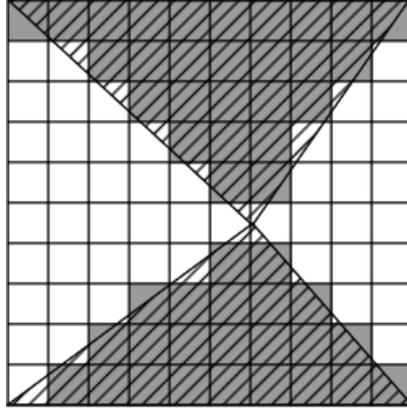


Figure 6:  $\lambda$ -sublevel cubical approximation for  $f$  the function defined as 0 on the hatched area and  $K$  outside (for arbitrarily large  $K$ ). The histogram approximation fails to identify the connectivity of the two triangle.

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