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# Input redundancy of switched linear systems via polynomial parameter-dependent systems

Valessa V. Viana, Jérémie Kreiss, and Marc Jungers. \*

## Abstract

In this paper, we introduce a definition of input redundancy for switched systems, particularly focusing on the continuous input. We establish a criterion for assessing whether a switched linear system is input redundant with respect to the continuous input. Our approach initially involves transforming the switched linear system into a linear system depending polynomially on a univariate parameter through Lagrange polynomial interpolation. This allows us to leverage recent results in geometric control theory and input redundancy for parameter-dependent systems, and adapt them to the context of switched systems in order to obtain the desired conditions. To illustrate the application of the proposed strategy, we provide two numerical examples.

## 1 Introduction

Input redundancy is a property enjoyed by a system where a desired output trajectory does not uniquely determine the input one [1]. It is usually the case for over-actuated systems for which more actuators than strictly necessary are involved, for example, to improve the performances, to get away from physical constraints, or to be more resilient to failure. Input redundant (IR) systems benefit from degrees of freedom in the way the control input is chosen, leading to an interesting and original control design, referred to as control allocation methods [2], in order to meet secondary objectives that improve the closed-loop behavior.

The characterization of input redundancy is the cornerstone of the development of control allocation methods. Yet, even for linear time-invariant (LTI) systems, this notion was puzzled out only recently. In 2009, [3] pointed out a new category of input redundant LTI systems, and the property was revised and fully characterized (see [1, 4]) by using the tools from the geometric control theory [5], such as the controlled invariant and output invisible subspaces, bringing up to date this old-fashioned theory. These recent results lead the way to promising research by considering IR to extended classes of dynamical systems, as evidenced by the extension for linear systems depending polynomially on a parameter [6, 7] and also by the preprint [8]. It is noteworthy to mention that for parameter-dependent systems, the property of input redundancy may depend on whether the parameter is unknown (robust case) or known (adaptive case). If the early result [6]

only deals with the robust case, in the preprint [8], both cases are handled. These two distinct paradigms rely on the fundamental concept of *generalized controlled invariance and output invisibility*, which can be adaptive if the control input making a subspace invariant may depend on the parameter.

In this work, we aim to dive the property of IR into the class of switched systems, motivated by the fact that many power electronic devices exhibit both switching and IR properties [9, 10]. As far as the authors are concerned, no results are available in the literature about IR of switched systems and many directions are open, even in the way to define the IR property. Indeed, it could be viewed concerning either the continuous input, the switched sequence, or both of these signals. As a first step, we decided here to focus on the case where IR refers to the continuous input, which is already relevant and challenging. To tackle this open question, our strategy is to imitate the well-known result of [11] which highlights the link between a switched linear system and a linear parameter-varying (LPV) one. In [11], the relation is made between the number of modes of a switched linear system and the number of parameters of the associated LPV system. For us, since the results about IR of parameter-dependent systems [8] are only valid for a unique parameter, we use a slightly different link: the number of modes is related to the degree of the polynomial dependence on a *single* parameter of the system. This relation, based on the Lagrange interpolating polynomials, is also used in [12] in the context of stability analysis of switched non-linear systems. Here, it allows us to provide insights about IR for switched linear systems.

Once this relation is stated, we use the results of [7] to determine the largest generalized adaptively controlled invariant and output invisible subspace. Note that this extension allows us to retrieve a concept of controlled invariance and output invisibility for switched systems which goes even further than the *simultaneous controlled invariant and output invisible subspace* of the literature [13]. Finally, we can obtain constructive elements about the IR property of the parameter-dependent system by using [8] and derive it for the initial switched system.

Therefore, in this paper, after presenting the system description and problem formulation where the property of input redundancy for a switched linear system is established in Section 2, we show the relation between the initial switched linear system and a parameter-dependent linear system in Section 3. On the latter system, we explain in Section 4 how to use the results of [7, 8] in order to obtain the weakly unobservable subspace when looking at IR, and finally determine if the parameter-dependent system is IR or not. In

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\*V. V. Viana, J. Kreiss and M. Jungers are with Université de Lorraine, CNRS, CRAN, F-54000 Nancy, France. [valessa.valentim-viana;jeremie.kreiss;marc.jungers]@univ-lorraine.fr

Section 5, we exhibit how to conclude about the redundancy of the initial switched linear system. Numerical illustrations are widespread in the paper and simulations are provided in Section 6 before the conclusion.

**Notation.** Let  $\mathbb{R}$ ,  $\mathbb{R}^+$ ,  $\mathbb{Z}$ ,  $\mathbb{N}$ , and  $\mathbb{N}^*$  be, respectively, the set of (i) real numbers, (ii) real positive numbers including zero, (iii) relative numbers, (iv) natural numbers including zero, and (v) natural numbers without zero. For any  $a, b \in \mathbb{Z}$  satisfying  $a \leq b$ , the set  $\llbracket a, b \rrbracket$  refers to  $[a, b] \cap \mathbb{Z}$ . For a matrix  $M$ ,  $M^\top$  means its transpose.  $0_{n \times m}$  stands for the zero matrix of  $n$  rows and  $m$  columns,  $0_n := 0_{n \times n}$ , and  $I_n$  stands for the identity matrix of dimension  $n$ . The notation  $\text{diag}_{i=0}^N M_i$  denotes the block diagonal matrix where  $M_i$  is the  $i$ -th block.  $\text{Im}\{V\}$  is the vector space spanned by the columns of matrix  $V$ . The inverse map of set  $\mathcal{V}$  by an application  $B$  is given by  $B^{-1}\mathcal{V} := \{u \in \mathbb{R}^m; Bu \in \mathcal{V}\}$ , where  $B$  is not necessarily invertible. For time functions, the notation  $u_a \neq u_b$  means that  $\int \|u_a(\tau, \theta(\tau)) - u_b(\tau, \theta(\tau))\| d\tau > 0$ .

## 2 System description and problem formulation

Consider a switched linear system:

$$\begin{aligned} \dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \\ y(t) &= C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t), \end{aligned} \quad (1)$$

where, for all  $t \in \mathbb{R}^+$ ,  $x(t) \in \mathbb{R}^n$  is the state with initial condition  $x(0) := x_0 \in \mathbb{R}^n$ ,  $u(t) \in \mathbb{R}^m$  is the input,  $y(t) \in \mathbb{R}^p$  is the output of the system. For  $i \in \llbracket 1, N \rrbracket$ , matrices  $A_i$ ,  $B_i$ ,  $C_i$ ,  $D_i$  of appropriate dimension describe an LTI subsystem which is also called a mode of the switched system. The notation  $\sigma$  denotes a signal responsible for the switching between the subsystems, i.e., the modes of system (1). We consider that  $\sigma \in \mathcal{S}$ , where  $\mathcal{S}$  is the set of piecewise constant right-continuous functions from the time domain  $\mathbb{R}^+$  to the set of modes  $\llbracket 1, N \rrbracket$  with a finite number of discontinuities in a finite length interval<sup>1</sup>. The time-dependence of  $\sigma(t)$  may be omitted, in the sequel, to lighten the notation. Note that in this paper we suppose that  $\sigma$  is an external signal on which we cannot act. In order to improve the readability of the paper, let us define  $\forall i \in \llbracket 1, N \rrbracket$

$$\mathbb{R}^{(n+p) \times (n+m)} \ni M_i := \left[ \begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array} \right] \quad (2)$$

as the collection of system matrices of the  $i$ -th mode.

In the context of LTI systems, input redundancy is well-defined and characterized in the literature [1]. Nevertheless, the existing results do not readily apply for switched linear systems. Therefore, in this paper, we aim to investigate the input redundancy of switched linear systems. First, we propose the following definition for input redundancy of switched linear systems.

**Definition 1.** A switched linear system (1) is said input redundant w.r.t.  $u$  if, for all  $\sigma \in \mathcal{S}$ , there exists an output  $y$

<sup>1</sup>For an input  $u$ , considering a switching rule  $\sigma$  in  $\mathcal{S}$  allows the well-posedness and unicity of the state trajectory, without Zeno phenomenon.

that can be produced by (at least) two different input functions for some  $x_0 \in \mathbb{R}^n$ , i.e.,  $\forall \sigma \in \mathcal{S}$ ,  $\exists x_0 \in \mathbb{R}^n$ ,  $u_a \neq u_b$  such that<sup>2</sup>  $y_{u_a}(t, \sigma, x_0) = y_{u_b}(t, \sigma, x_0)$ ,  $\forall t \geq 0$ .

Our goal is to provide a tractable characterization of input redundancy for switched systems.

**Remark 1.** Note that in the definition above, we do not specify the switching signal dependence of the different inputs  $u_a$  and  $u_b$  and we will consider here that they possibly depend on  $\sigma$ , even if the case where they are not allowed to depend on it can be dealt by suitable choices in the development of the paper.

## 3 From switched linear systems to polynomial parameter-dependent systems

The main objective of this section is to provide a systematic process to transform a switched linear system into a univariate polynomial parameter-dependent system.

### 3.1 Lagrange interpolating polynomial

Let us define a parameter  $\theta \in [1, N]$ . The proposed transformation is based on the Lagrange interpolating polynomial [14]. We are going to construct the unique polynomial of degree at most  $N - 1$  over  $\theta$ , interpolating the  $N$  nodes  $(i, M_i)$ ,  $i \in \llbracket 1, N \rrbracket$ . To this end, let us consider the following  $N$  polynomials  $P_i : [1, N] \rightarrow \mathbb{R}$ ,  $i \in \llbracket 1, N \rrbracket$  such that

$$P_i : \theta \mapsto P_i(\theta) = \prod_{j=1, j \neq i}^N \frac{(\theta - j)}{(i - j)}. \quad (3)$$

We can easily see that  $P_i$  is of degree  $N - 1$  and is such that  $\forall j \in \llbracket 1, N \rrbracket \setminus \{i\}$ ,  $P_i(j) = 0$  and  $P_i(i) = 1$ . We have just constructed  $P_i$  as the Lagrange interpolating polynomial of the nodes  $\{(1, 0), \dots, (i - 1, 0), (i, 1), (i + 1, 0), \dots, (N, 0)\}$ .

**Lemma 1.** Let us define

$$M : \begin{cases} [1, N] & \rightarrow \mathbb{R}^{(n+p) \times (n+m)} \\ \theta & \mapsto M(\theta) = \sum_{j=1}^N P_j(\theta) M_j. \end{cases} \quad (4)$$

$M$  is the unique polynomial of degree at most  $N - 1$  such that  $\forall i \in \llbracket 1, N \rrbracket$ ,  $M(i) = M_i$  where  $M_i$  is given by (2).

*Proof.* By definition,  $P_j$ ,  $j \in \llbracket 1, N \rrbracket$  is a polynomial of degree  $N - 1$ . Therefore, the degree of  $M$  is at most  $N - 1$ . Similar to [12, Theorem 1], using the mentioned properties of  $P_j$ , we get that  $\forall i \in \llbracket 1, N \rrbracket$ ,  $M(i) = \sum_{j=1}^N P_j(i) M_j = P_i(i) M_i = M_i$ . We just shown that  $M$  is a polynomial of degree at most  $N - 1$  interpolating the  $N$  following nodes  $(i, M_i)$ ,  $i \in \llbracket 1, N \rrbracket$ . As a consequence,  $M$  is the Lagrange interpolating polynomial and is unique [14, Sec. 2.5].  $\square$

<sup>2</sup>The term  $y_u(t, \sigma, x_0)$  means standardly the value of the output at time  $t$  generated by the dynamics (1) for a given initial input  $x_0$ , switched rule  $\sigma$  and input  $u$ .

It is of foremost importance to mention that (i) we built a unique transformation of the  $N$  modes  $M_i$  of the switched linear system to the polynomial  $M$  and (ii) this transformation can be reversed to obtain the  $N$  modes  $M_i$  from the polynomial  $M$ . Thus, the representation (1) is equivalent to

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = M(\sigma(t)) \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad (5)$$

where  $\sigma \in \mathcal{S}$ .

### 3.2 Convexification of $\llbracket 1, N \rrbracket$ into $[1, N]$

It is also important to note that even if the two representations (1) and (5) are equivalent, we will consider in the following the polynomial  $M(\theta)$  where  $\theta$  is not specifically linked to  $\sigma$  but can be freely selected in the continuous interval  $[1, N]$ . Therefore, a convexification of the discontinuous interval for  $\sigma$  is made to obtain the continuous one for  $\theta$  and consequently, the equivalence does not hold anymore. However, all the following developments on the polynomial representation apply to the switched representation (1). Hence, the switched linear system (1) is included in the representation of a univariate polynomial parameter-dependent system:

$$\begin{aligned} \dot{x}(t) &= A(\theta(t))x(t) + B(\theta(t))u(t), \\ y(t) &= C(\theta(t))x(t) + D(\theta(t))u(t), \end{aligned} \quad (6)$$

with  $\theta(t) \in [1, N], \forall t \geq 0$ , where  $\left[ \begin{array}{c|c} A(\theta) & B(\theta) \\ \hline C(\theta) & D(\theta) \end{array} \right] := M(\theta)$  of appropriate dimension.

The following numerical example illustrates this transformation of a switched linear system to a polynomial parameter-dependent one.

**Example 1.** *Let us consider a switched linear system of three modes defined by the following matrices:*

$$\begin{aligned} A_1 &= \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 0 \\ 4 & -1 \end{bmatrix}, A_3 = \begin{bmatrix} -1 & 0 \\ 9 & -1 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, B_3 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \\ C_1 &= C_2 = C_3 = [0 \quad 1], D_1 = D_2 = D_3 = [0 \quad 0]. \end{aligned} \quad (7)$$

with  $N = 3$ . To obtain the corresponding parameter-dependent matrices, we have to compute  $P_i(\theta)$ , given in (3), for  $i \in \llbracket 1, 3 \rrbracket$ :

$$\begin{aligned} P_1(\theta) &= \frac{1}{2}(\theta - 2)(\theta - 3), \quad P_2(\theta) = -(\theta - 1)(\theta - 3), \\ P_3(\theta) &= \frac{1}{2}(\theta - 1)(\theta - 2), \end{aligned}$$

such that  $M(\theta) = \sum_{i=1}^3 P_i(\theta)M_i$ . By developing this expression, we find  $M(\theta) = \tilde{M}_0 + \theta\tilde{M}_1 + \theta^2\tilde{M}_2$  where  $\tilde{M}_0 = (3M_1 - 3M_2 + M_3)$ ,  $\tilde{M}_1 = (-2.5M_1 + 4M_2 - 1.5M_3)$ ,  $\tilde{M}_2 = (0.5M_1 - M_2 + \frac{1}{2}M_3)$ . Therefore, the switched system can be represented by a polynomial parameter-dependent system:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -1 & 0 \\ \theta^2(t) & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ \theta(t) & 1 \end{bmatrix} u(t), \\ y(t) &= [0 \quad 1] x(t), \end{aligned} \quad (8)$$

with  $\theta(t) \in [1, 3], \forall t \geq 0$ .

**Remark 1.** *Throughout the paper,  $\theta$  can represent a function  $t \mapsto \theta(t)$ , the value of this function at time  $t$ , or the indeterminate of a polynomial with  $\theta \in [1, N]$ . In section 3.1 and 4,  $\theta$  is mostly considered as the indeterminate of a polynomial.*

So far, we lack the means to characterize the input redundancy of the switched system (1). However, there exist techniques to verify the input redundancy of polynomial parameter-dependent systems, that were developed in [8]. In the next section, let us recapitulate the previous literature results.

## 4 Geometric control theory and input redundancy for parameter-dependent systems

### 4.1 A Geometric control tool: the weakly unobservable subspace $\mathcal{V}^*$

Geometric control theory has been extensively used, in the literature, for characterization of the concept of input redundancy [1, 4, 6, 8] which is strongly related to the controlled invariant and output invisible subspaces [5]. In the context of parameter-dependent systems as (6), a subspace  $\mathcal{V} \subseteq \mathbb{R}^n$  is called generalized adaptively controlled invariant and output invisible if there exists a matrix  $F : [1, N] \rightarrow \mathbb{R}^{m \times n}$  such that, for all  $\theta \in [1, N]$ ,

$$\begin{aligned} (A(\theta) + B(\theta)F(\theta))\mathcal{V} &\subseteq \mathcal{V}, \\ (C(\theta) + D(\theta)F(\theta))\mathcal{V} &= 0. \end{aligned} \quad (9)$$

Matrix  $F(\theta)$  is called a friend of  $\mathcal{V}$ . Moreover, the set of these subspaces admits a maximal element, called the generalized adaptively weakly unobservable subspace and denoted  $\mathcal{V}^*$ .

**Computation of the generalized adaptively weakly unobservable subspace  $\mathcal{V}^*$**  Recently, [7] proposed an algorithm to compute generalized adaptively weakly unobservable subspaces for parameter-dependent systems. The algorithm aims to find, after a finite number of steps, the largest subspace with the assumption that there is a polynomial friend of a given degree making the subspace invariant. To apply this algorithm, [7] assumes that  $\theta$  lies in a polytopic set, which is the case for (6), and then transforms the polynomial parameter-dependent system into an equivalent homogeneous polynomial parameter-dependent system depending on a parameter that lies in a unit simplex set. The inputs required for this algorithm are the degree of dependence of a polynomial friend  $F(\theta)$  and constant matrices derived from the weighting matrices of the homogeneous polynomial system. The steps of this algorithm are presented in detail in [7, Algorithm 1]. Additionally, a numerical example is provided to demonstrate the transformation into a homogeneous polynomial system [7, Example 1], and to detail the application of the algorithm [7, Example 1 (continued)].

To sum up, with this latter paper, we obtain  $\mathcal{V}^*$ , the largest subspace  $\mathcal{V}$  satisfying (9) under a polynomial friend

$F(\theta)$  of a given degree, for system (6) for all  $\theta \in [1, N]$ . Since  $\llbracket 1, N \rrbracket \subset [1, N]$ ,  $\mathcal{V}^*$  also satisfies (9) for system (5) and therefore for (1). Even if we can force  $F$  to be independent of  $\sigma$ , by setting the friend constant in the design of [7], and then recover the simultaneous controlled invariant and output invisible subspaces [13], allowing  $F$  to depend on  $\theta$  leads to a subspace  $\mathcal{V}^*$  that encompass this notion, as illustrated by the following example.

**Example 1 continued.** *On the switched system defined by (7), we can show that the simultaneous controlled invariant and output invisible subspaces [13] is  $\mathcal{V}^* = \{0\}$ . However, looking at (8) and applying the methodology of [7] (detailed in Example 1 of [7]) with a friend that can depend linearly on  $\theta$  leads to  $\mathcal{V}^* = \text{Im} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^\top \right\}$ , where  $F(\theta) = \begin{bmatrix} -\theta & 0 \\ 0 & 0 \end{bmatrix}$  is a friend of  $\mathcal{V}^*$ .*

Obviously, in the context of switched systems, the friend of  $\mathcal{V}^*$  can be represented by a matrix  $F_{\sigma(t)}$ , where

$$F_i := F(\theta = i), \quad \forall i \in \llbracket 1, N \rrbracket. \quad (10)$$

## 4.2 Characterization of adaptive input redundancy

Now that we have done with the generalized adaptively weakly unobservable subspace, we can use [8] to verify if the parameter-dependent (6) is adaptive input redundant. The definition of adaptive input redundancy of a parameter-dependent system is recalled from the paper [8] hereafter.

**Definition 2.** *The parameter-dependent system (6) is called adaptive input redundant if there exists an output  $y$  that can be produced by, at least, two distinct parameter-dependent inputs, for some function  $\theta$  and some initial condition, i.e., there exist  $\theta : t \mapsto \theta(t)$ ,  $x_0 \in \mathbb{R}^n$ , and two distinct input functions  $u_a \neq u_b$ , where  $u_a : (t, \theta) \mapsto u_a(t, \theta)$  and  $u_b : (t, \theta) \mapsto u_b(t, \theta)$ , such that  $y_{u_a}(t, \theta, x_0) = y_{u_b}(t, \theta, x_0)$  for all  $t \geq 0$ .*

It has been shown, in [8], that if the subspace

$$B^{-1}(\theta)\mathcal{V}^* \cap \ker \{D(\theta)\}, \quad (11)$$

with  $B^{-1}(\theta)\mathcal{V}^* := \{u \in \mathbb{R}^m; B(\theta)u \in \mathcal{V}^*\}$ , is not trivial, then the system (6) is adaptive input redundant. Actually, the subspace (11) contains input directions that, if  $\mathcal{V}^*$  is made invariant, are invisible from the output. We understand that if these directions are non-zero, we can construct two different inputs leading to the same output and recover the input redundancy property of the parameter-dependent system. A procedure to systematically compute (11) was proposed in [8], using the definition that  $\ker \{W(\theta)\} := B^{-1}(\theta)\mathcal{V}^* \cap \ker \{D(\theta)\}$ , where  $W(\theta)$  depends polynomially on  $\theta$ , and the Smith normal form of  $W(\theta)$  (see [15]). The Smith normal form is an approach tailored for univariate polynomial matrices. Henceforth, we will be interested in employing the transformation outlined in Section 3, which converts a switched system into a univariate polynomial parameter-dependent system.

The subspace (11) exhibits three different kinds of dependencies with respect to  $\theta$ : (i) some directions are independent of  $\theta$ , leading to the minimum dimension (with respect to  $\theta$ ) of (11) being greater than or equal to one, (ii) other directions may depend on  $\theta$  in such a way that these directions can vanish for some isolated value of  $\theta$  and (iii) the last directions appear only for specific isolated values of  $\theta$ . All of these cases are taken into account in the parameter-dependent matrix  $L(\theta)$  provided by [8] which is such that  $\text{Im} \{L(\theta)\} = \ker \{W(\theta)\} = B^{-1}(\theta)\mathcal{V}^* \cap \ker \{D(\theta)\}$ , and reads  $L(\theta) = \begin{bmatrix} \Delta_{\delta_1} \hat{L}_1(\theta) & \cdots & \Delta_{\delta_r} \hat{L}_r(\theta) & \tilde{L}(\theta) \end{bmatrix}$ , with  $\Delta_{\delta_j} = \begin{cases} 1, & \text{if } \theta = \delta_j \\ 0, & \text{otherwise,} \end{cases}$  for  $j \in \llbracket 1, r \rrbracket$ , where  $r$  is the normal rank of  $W(\theta)$  in the Smith normal form (The normal rank of  $W(\theta)$  is the maximum rank of  $W(\theta)$  for almost all values of  $\theta$ ). The parameters  $\delta_j, j \in \llbracket 1, r \rrbracket$ , that may be complex, correspond to the isolated values of  $\theta$  for which a direction may appear in  $\ker \{W(\theta)\}$ . Since we have that  $\theta \in [1, N]$ , the values of  $\delta_j$  for which  $\Delta_{\delta_j}$  is not always zero must be real and inside the interval  $[1, N]$ . Matrices  $\hat{L}_j(\theta)$ , for  $j \in \llbracket 1, r \rrbracket$ , and  $\tilde{L}(\theta)$  depend polynomially on  $\theta$ , and may be empty. Note that all of the components of  $L(\theta)$  are given by the methodology provided by [8].

**Example 1 continued.** *Let us take again the parameter-dependent system (8). [8] allows us to conclude that  $\dim(B^{-1}(\theta)\mathcal{V}^* \cap \ker \{D(\theta)\}) = 1 > 0$  such that the system is adaptive input redundant. Furthermore, we can compute  $L(\theta) = \begin{bmatrix} 1 & -\theta \end{bmatrix}^\top$ . As a result,  $\forall x_0 \in \mathcal{V}^*$  and  $\forall t \geq 0$ , the input  $u(t) = F(\theta(t))x(t) + L(\theta(t))w(t)$ , for some function  $w$  that can be freely chosen, leads to  $y(t) = 0, \forall t \geq 0$ .*

## 5 Input redundancy for switched linear systems

We are finally ready to propose a sufficient condition for the input redundancy property for switched systems given in Definition 1.

Consider the switched linear system (1) and its polynomial representation (6) with  $\theta(t) \in [1, N], \forall t \geq 0$ . Let  $\mathcal{V}^* \subseteq \mathbb{R}^n$  be the generalized adaptively weakly unobservable subspace of system (6), and  $B^{-1}(\theta)\mathcal{V}^* \cap \ker \{D(\theta)\} \subseteq \mathbb{R}^m$  be the parameter-dependent subspace of invisible inputs. The following theorem holds.

**Theorem 1.** *The switched linear system (1) is input redundant w.r.t.  $u$  if for all  $\theta \in \llbracket 1, N \rrbracket$ ,*

$$\dim(B^{-1}(\theta)\mathcal{V}^* \cap \ker \{D(\theta)\}) > 0. \quad (12)$$

*Proof.* Consider the parameter-dependent system (6) with  $\theta \in [1, N]$ . We know from [8, Lemma 3] that for  $x_0 \in \mathcal{V}^*$  the input  $u(t) = F(\theta(t))x(t) + L(\theta(t))w(t)$  where  $F(\theta)$  is a friend of  $\mathcal{V}^*$ ,  $L(\theta)$  is such that  $\text{Im} \{L(\theta)\} = B^{-1}(\theta)\mathcal{V}^* \cap \ker \{D(\theta)\}$ , for some function  $w$ , leads to  $y(t) = 0, \forall t \geq 0$ .

Consider now two state trajectories of system (6),  $x_a$  and  $x_b$ , associated to two input trajectories  $u_a$  and  $u_b$ , respectively. Suppose that  $x_a$  and  $x_b$  start from the same initial

condition  $x_a(0) = x_b(0) = x_0$  and that,  $\forall t \geq 0$ ,

$$u_a(t) - u_b(t) = F(\theta(t))(x_a(t) - x_b(t)) + L(\theta(t))w(t),$$

for some  $w(t)$ . Therefore, since  $x_a(0) - x_b(0) = 0 \in \mathcal{V}^*$ , it comes from previous discussion that  $y_a(t) = y_b(t)$ ,  $\forall t \geq 0$ . Clearly, when (12) is valid,  $L(\theta)$  is non-empty and since we can freely select the function  $w$ , we can easily obtain  $u_a \neq u_b$ . Note that in the theorem, we only consider (12),  $\forall \theta \in \llbracket 1, N \rrbracket$ , and not  $\forall \theta \in [1, N]$ . Therefore, it is ensured that,  $\forall \theta \in \llbracket 1, N \rrbracket$ ,  $\text{rk}(L(\theta)) > 0$ .

As a result,  $\forall i \in \llbracket 1, N \rrbracket$ , we have that  $n_i := \text{rk}(L(i)) > 0$ . Then, there exists a function  $w$  of appropriate dimension, such that

$$u_a(t) - u_b(t) = F(\sigma(t))(x_a(t) - x_b(t)) + L(\sigma(t))w(t) \neq 0. \quad (13)$$

Consequently,  $\forall \sigma \in \mathcal{S}$ , applying these two different inputs  $u_a$  and  $u_b$  on system (1) leads to the same output. This proves that system (1) is input redundant with respect to  $u$ .  $\square$

**Remark 2. Computation of matrices  $L_i$ :** From the knowledge of  $L(\theta)$ , the matrices  $L(i)$  may present zero columns or linearly dependent ones. In these cases, we can determine matrices  $L_i$  avoiding these columns such that  $L_i \in \mathbb{R}^{m \times n_i}$ , of smallest dimension, verifying  $\text{Im}\{L_i\} = \text{Im}\{L(i)\}$ , for  $i \in \llbracket 1, N \rrbracket$ . This approach differs from the case of  $F_{\sigma(t)}$  defined in (10). Consequently, we can choose  $N$  signals  $w_i(t) \in \mathbb{R}^{n_i}$  such that relation (13) reads:

$$u_a(t) - u_b(t) = F_{\sigma(t)}(x_a(t) - x_b(t)) + L_{\sigma(t)}w_{\sigma(t)}(t) \neq 0.$$

Observe that by Theorem 1, we need that all of the modes are input redundant to obtain an IR switched system  $\forall \sigma \in \mathcal{S}$ . Indeed, suppose that one of the modes, let us say the mode  $i \in \llbracket 1, N \rrbracket$ , is not IR, i.e.  $\dim(B_i^{-1}\mathcal{V}_i^* \cap \ker\{D_i\}) = 0$  where  $\mathcal{V}_i^*$  is the weakly unobservable subspace of mode  $i$ . Picking the particular switching sequence  $\sigma(t) = i$ ,  $\forall t \geq 0$  brings us back to an LTI system of matrices  $(A_i, B_i, C_i, D_i)$  which is not input redundant. Thus, we have a switched system that is not IR with respect to this particular switching sequence  $\sigma$ . Since we consider that  $\sigma$  is an external signal on which we cannot act, it seems appropriate to consider IR property independent from  $\sigma$ . The converse case, i.e. when IR depends on  $\sigma$  becomes relevant when it is a control input. This case is left for future work.

It is also important to emphasize that the proposed result takes into account IR even if the origin of the redundancy is different for each mode, i.e. for some of the modes, the different inputs leading to the same output produce different state trajectories whereas for other modes, the state trajectories are identical. The following example illustrates this point.

**Example 2.** Let us consider the two-mode switched linear system given by matrices

$$A_1 = A_2 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix},$$

$C_1 = C_2 = [0 \ 1]$  and  $D_1 = D_2 = 0_{1 \times 2}$  which can be represented by the following parameter-dependent system

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} \theta(t) - 1 & 0 \\ \theta(t) & 1 \end{bmatrix} u(t), \\ y(t) &= [0 \ 1] x(t), \end{aligned}$$

with  $\theta(t) \in [1, 2]$ ,  $\forall t \geq 0$ . Using [7, 8], we obtain  $\mathcal{V}^* = \text{Im}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$  with  $F = 0_2$  as a friend, and  $L(\theta) = \begin{bmatrix} 1 \\ -\theta \end{bmatrix}$ . We finally get  $L_1 = L(1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$  and  $L_2 = L(2) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$  and we can see that for the first mode,  $L_1$  is such that  $\text{Im}\{L_1\} = \ker\{B_1\}$  whereas for mode 2, we have that  $\text{Im}\{B_2 L_2\} = \mathcal{V}^*$ , showing that the kind of redundancy is different for the two modes.

Finally, let us also highlight that it is possible to recover an IR property independent of  $\sigma$ , that does not require the knowledge of  $\sigma$  with our techniques. To this end, we need to specify a constant friend  $F$  of  $\mathcal{V}^*$ , i.e.,  $F(\theta = i) = F, \forall i \in \llbracket 1, N \rrbracket$ , for its computation in [7]. We will obtain the so-called largest generalized robust controlled-invariant and output invisible subspace. Then, by selecting a matrix  $L$  such that  $\text{Im}\{L\} = \cap_{i=1}^N \text{Im}\{L_i\}$ , if  $\dim(\text{Im}\{L\}) > 0$ , we are able to construct different inputs that do not depend on the switching law while leading to the same output trajectory. This scenario may be useful when the switching signal is unknown.

**Remark 3.** It is important to highlight that although Definition 1 is presented for all  $\sigma \in \mathcal{S}$  and Definition 2 is for some function  $\theta$ , this discrepancy is not an obstacle for using the results of [8] to obtain the outcome of Theorem 1. The results of [8] cover a wider variety of situations than those of switched systems.

**Remark 4.** Following the result of [1, Proposition 2.1], by linearity, we can say that if a switched linear system, composed of LTI systems, is input redundant w.r.t.  $u$ , then there is an infinite number of distinct inputs leading to a variety of output trajectories for all initial conditions.

## 6 Additional numerical example

Example 1 is divided into four steps to illustrate the different parts of the procedure to verify if a switched system is input redundant w.r.t.  $u$ . In this section, we present its final part, where the input redundancy of the original switched system and simulations are provided.

**Example 1 continued.** Consider the three-mode switched linear system given in (7) such that transformation to a polynomial parameter-dependent system is explained in Section 3. We have already seen that (8) is adaptive input re-

dundant with  $\mathcal{V}^* = \text{Im}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ ,  $F(\theta) = \begin{bmatrix} -\theta & 0 \\ 0 & 0 \end{bmatrix}$  and  $L(\theta) = [1 \ -\theta]^\top$ . From Theorem 1, the switched linear system (7) is input redundant w.r.t.  $u$  since  $\dim(B^{-1}(\theta)\mathcal{V}^* \cap \ker D(\theta)) > 0$  for all  $\theta \in \llbracket 1, 3 \rrbracket$ . Then, for each mode of the switched

system, the following matrices are obtained  $F_i = \begin{bmatrix} -i & 0 \\ 0 & 0 \end{bmatrix}$  and  $L_i = [1 \quad -i]^\top$ ,  $i \in \llbracket 1, 3 \rrbracket$ . In this case, the input redundancy depends on the mode of the system. To verify the input redundancy with simulations, we defined  $x_0 = [0 \quad -1]^\top$ , and inputs,  $\forall t \geq 0$ ,  $u_a(t) = F_{\sigma(t)}x_a(t) + v$ , and  $u_b(t) = F_{\sigma(t)}x_b(t) + L_{\sigma(t)}w_{\sigma(t)}(t) + v$ , where  $w_i(t) = \sin(t)$ ,  $i \in \llbracket 1, 3 \rrbracket$ ,  $v = [-1 \quad 0]^\top$  and

$$\sigma(t) = \begin{cases} 1, & t \in [0, 2) \cup [6, 8) \\ 2, & t \in [2, 4) \\ 3, & t \in [4, 6) \end{cases}$$

such that  $u_a(t) - u_b(t) = F_{\sigma(t)}(x_a(t) - x_b(t)) + L_{\sigma(t)}(-w_{\sigma(t)}(t))$ . The trajectories of inputs  $u_a$  and  $u_b$  are illustrated in Figure 1. These two inputs follow distinct paths. Subsequently, the output trajectories resulting from these distinct inputs are depicted in Figure 2. The system initially operates in mode one, where both outputs  $u_a$  and  $u_b$  follow the same trajectory. At  $t = 2s$ , the system switches to mode two, followed by another transition to mode three at  $t = 4s$ , and finally returns to mode one at  $t = 6s$ . At each transition, the output dynamics change, yet the equality between the two outputs persists. Consequently, despite a switching signal cycling through all modes, the output trajectories  $y_a$  and  $y_b$  remain identical, confirming the input redundancy of the switch system w.r.t. the continuous input  $u$ .

## 7 Conclusion

In this paper, a definition of input redundancy concerning the continuous input was proposed for switched linear systems. We account for the dependence of the system input on the switching signal and provide a sufficient tractable condition to determine if the switched system is input redundant w.r.t.  $u$ . Our approach builds upon existing results for input redundancy of parameter-dependent systems. To leverage these results, we developed a systematic process, based on the Lagrange interpolation polynomial, to transform a switched linear system into a univariate polynomial parameter-dependent system. Consequently, we were able to use the existing results within the context of switched systems. Future research should extend the investigation of this notion beyond the confines of parameter-dependent systems in order to obtain an idea regarding the conservatism linked to the current strategy and possibly derive less conservative results. Furthermore, the exploration of other notions of input redundancy for switched systems is also desired, such as an input redundancy w.r.t  $u$  for some  $\sigma \in \mathcal{S}$ , an input redundancy w.r.t. switching signal or both the continuous input and the switching signal.

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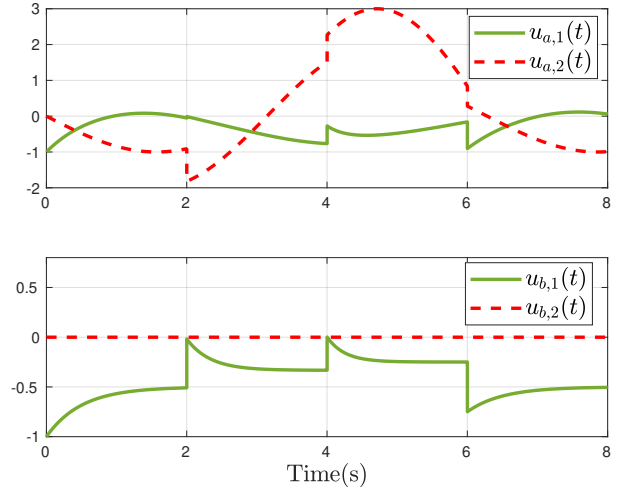


Figure 1: Ex. 1 - Input trajectories.

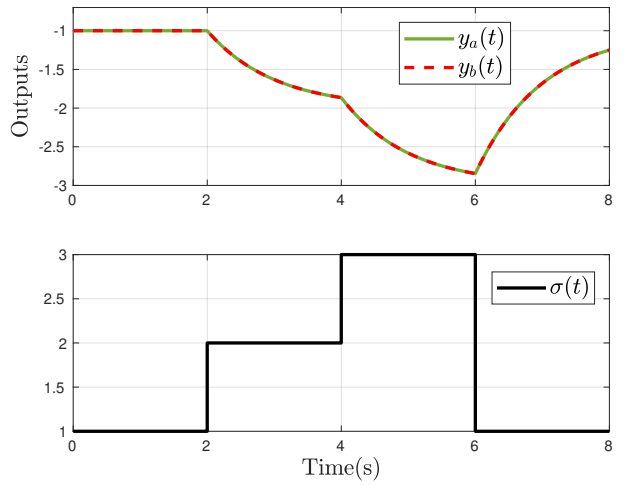


Figure 2: Ex. 1 - Output trajectories and switching signal.

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