



**HAL**  
open science

# Spanning isogeny classes of principally polarized abelian surfaces with RM

Jean Kieffer

► **To cite this version:**

Jean Kieffer. Spanning isogeny classes of principally polarized abelian surfaces with RM. 2024. hal-04592523

**HAL Id: hal-04592523**

**<https://hal.science/hal-04592523v1>**

Preprint submitted on 29 May 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# SPANNING ISOGENY CLASSES OF PRINCIPALLY POLARIZED ABELIAN SURFACES WITH RM

JEAN KIEFFER

**ABSTRACT.** We describe how isogenies between principally polarized abelian surfaces (PPAS) with real multiplication (RM) can be decomposed into elementary isogeny types. This leads to a strategy for enumerating the isogeny class of a given PPAS with RM. This note is part of an ongoing work with R. van Bommel, S. Chidambaram, and E. Costa and is not intended for separate publication.

## 1. ISOGENIES AND REAL ENDOMORPHISMS

**1.1. Notation.** We keep the same setting as [1, §2].  $k$  is a field of characteristic zero (e.g. a number field). We identify group schemes over  $k$  with the set of their  $\bar{k}$ -points endowed with an action of  $\text{Gal}(\bar{k}/k)$ .

We only consider abelian varieties, polarizations, isogenies and endomorphisms that are defined over  $k$ . Two isogenies are declared isomorphic if they have the same domain and differ by an isomorphism on their targets.

If  $(A, \lambda_A)$  is a principally polarized abelian variety (PPAV) over  $k$ , then the *Rosati involution* on  $\text{End}_k(A)$  is  $\dagger : \varphi \mapsto \lambda_A^{-1} \varphi^\vee \lambda_A$ , where  $\varphi^\vee$  denotes the dual of  $\varphi$ . Note that the Rosati involution depends on the choice of  $\lambda_A$ .

An endomorphism  $\beta$  of  $A$  is called *symmetric* if  $\beta^\dagger = \beta$ . The roots of its characteristic polynomial (a polynomial of degree  $\dim A$ ) are then real numbers. We say that  $\beta$  is *totally positive* if these roots are positive. The set of symmetric (resp. symmetric and totally positive) endomorphisms is denoted by  $\text{End}_k(A)^\dagger$  (resp.  $\text{End}_k(A)_{>0}^\dagger$ ). If  $\beta \in \text{End}_k(A)_{>0}^\dagger$ , then the  $\beta$ -torsion subgroup  $A[\beta]$  carries a non-degenerate Weil pairing.

**1.2. Correspondence between isogenies and endomorphisms.** We recall [1, Lemma 2.1], which summarizes results of Mumford [4].

**Proposition 1.1.** *Let  $(A, \lambda_A)$  be a PPAV over any field  $k$ . Then there is a one-to-one correspondence between*

---

*Date:* May 29, 2024.

- isomorphism classes of isogenies  $A \rightarrow A'$  where  $A'$  is also endowed with a principal polarization  $\lambda_{A'}$ , and
- pairs  $(\beta, G)$  where  $\beta \in \text{End}(A)$  is symmetric and totally positive, and  $G$  is a subgroup of  $A[\beta]$  which is maximal isotropic with respect to the Weil pairing.

The correspondence is as follows. Given an isogeny  $\varphi : A \rightarrow A'$ , we

$$(1) \quad \beta = \lambda_A^{-1} \circ \varphi^\vee \circ \lambda_{A'} \circ \varphi$$

and  $G = \ker \varphi$ . Given a pair  $(\beta, G)$ , we let  $\varphi : A \rightarrow A/G$  be the quotient isogeny, and endow  $A' = A/G$  with the unique principal polarization  $\lambda_{A'}$  such that (1) holds.

We have  $\deg(\beta) = \deg(\varphi)^2$ . We say that  $\varphi$  is a  $\beta$ -isogeny. Note that with this terminology, a “2-step  $\ell$ -isogeny” as in [1] is an  $\ell^2$ -isogeny.

**1.3. Isogeny classes.** Before using Proposition 1.1 to enumerate isogeny classes, we must specify exactly what classes we are looking at.

**Definition 1.2.** Let  $(A, \lambda_A)$  be a PPAV over  $k$ . We define

$$\text{Isog}_{\text{pp}}(A) = \{(A', \lambda_{A'}) \text{ PPAV over } k : A' \text{ and } A \text{ are isogenous}\} / \sim .$$

In other words,  $\text{Isog}_{\text{pp}}(A)$  consists of all isomorphism classes of *abelian varieties endowed with a principal polarization* that are isogenous to  $A$ , without any a priori compatibility between isogenies and polarizations.

**Example 1.3.** Assume that the only abelian variety over  $k$  that is isogenous to  $A$  (as a nonpolarized abelian variety) is  $A$  itself. Then determining  $\text{Isog}_{\text{pp}}(A)$  means determining the set of possible principal polarizations on  $A$  up to isomorphism.

We would like to compute  $\text{Isog}_{\text{pp}}(A)$  when  $k$  is a number field and  $A$  is a given abelian surface with real multiplication. (It is a finite set by Faltings [2].) The purpose of this note is to present the first step in such a computation: we explain how any isogeny between such abelian surfaces can be decomposed into isogenies of “elementary” types, using Proposition 1.1 as our main tool.

**1.4. Assumptions in the rest of this note.**  $F = \text{End}_k(A)^\dagger \otimes \mathbb{Q}$  is either  $\mathbb{Q} \times \mathbb{Q}$  or a real quadratic field. (In the former case,  $A$  is isogenous over  $k$  to a product of non-isogenous elliptic curves.)  $\mathbb{Z}_F$  is the maximal order of  $F$  (where  $\mathbb{Z}_F = \mathbb{Z} \times \mathbb{Z}$  if  $F = \mathbb{Q} \times \mathbb{Q}$ ) and  $\text{Cl}(F)$  (resp.  $\text{Cl}^+(F)$ ) is the class group (resp. narrow class group) of  $\mathbb{Z}_F$  (both trivial if  $F = \mathbb{Q} \times \mathbb{Q}$ ).

The identification  $F = \text{End}_k(A)^\dagger \otimes \mathbb{Q}$  propagates through isogenies: if  $\varphi : A \rightarrow A'$  then there exists a unique bijection  $F \rightarrow \text{End}_k(A')^\dagger \otimes \mathbb{Q}$

such that  $\varphi$  is compatible with these RM structures. This allows us to identify  $F = \text{End}_k(A')^\dagger \otimes \mathbb{Q}$  as well.

**Remark 1.4.** Actually, the content of this note may also be useful if  $\text{End}_k(A)^\dagger$  is more complicated than a quadratic field, i.e. if the endomorphism algebra of  $A$  is QM,  $M_2(\mathbb{Q})$ , or  $M_2(\text{CM})$ : even in such a case, for every  $\alpha \in \text{End}_k(A)^\dagger_{>0}$ ,  $\mathbb{Q}(\alpha)$  is either  $\mathbb{Q}$ ,  $\mathbb{Q} \times \mathbb{Q}$  or a real quadratic fields, and the methods of this note apply to decompose an  $\alpha$ -isogeny into more elementary ones.

1.5. **Acknowledgements.** I thank Raymond van Bommel for his comments on an earlier version of this note.

## 2. REDUCTION TO MAXIMAL RM

If  $A$  has RM by a non-maximal order, then it is isogenous to another abelian surface with real multiplication by  $\mathbb{Z}_F$ .

**Proposition 2.1.** *Assume that  $A$  has RM by  $\mathcal{O} \subset \mathbb{Z}_F$ , and let  $c$  be the conductor of  $\mathcal{O}$  in  $\mathbb{Z}_F$ . Let  $c = \prod_{i=1}^r \ell_i$  be a decomposition of  $c$  as a product of prime numbers. Then there exists an isogeny  $\varphi : A \rightarrow B$  defined over  $k$  such that  $B$  has RM by  $\mathbb{Z}_F$ , and  $\varphi$  decomposes as  $\varphi = \varphi_1 \circ \dots \circ \varphi_r$ , where  $\varphi_i$  is an  $\ell_i$ -isogeny for every  $1 \leq i \leq r$ .*

*Proof.* This is “well known”. We can always find  $x \in \mathbb{Z}_F$  such that  $\mathbb{Z}_F = \mathbb{Z}[x]$  and  $\mathcal{O} = \mathbb{Z}[cx]$ . One can take  $G = A[c] \cap A[cx]$  as the kernel of  $\varphi$ . Since  $G \simeq (\mathbb{Z}/c\mathbb{Z})^2$  as an abstract group and  $G$  is maximal isotropic in  $A[c]$ , we can indeed decompose  $\varphi$  as claimed.  $\square$

**Corollary 2.2.** *We can enumerate  $\text{Isog}_{\text{pp}}(A)$  as follows:*

- (1) Find  $(B, \lambda_B) \in \text{Isog}_{\text{pp}}(A)$  such that  $B$  has RM by  $\mathbb{Z}_F$  as in Proposition 2.1;
- (2) Starting from  $B$ , enumerate all  $(B', \lambda_{B'}) \in \text{Isog}_{\text{pp}}(B)$  such that  $B'$  also has RM by  $\mathbb{Z}_F$ ;
- (3) From each  $B'$  found in step (2), look for sequences of  $\ell$ -isogenies where  $\ell \in \mathbb{Z}$  are primes (possibly distinct ones), and add the isogenous p.p. abelian surfaces we find to  $\text{Isog}_{\text{pp}}(A)$ .

We know how  $\ell$ -isogenies for  $\ell \in \mathbb{Z}_{\geq 1}$  can be classified and computed, see [1]. (Only Dieulefait’s criteria have to be modified to account for the larger endomorphism algebra.) In the rest of this note, we concentrate on step (2) in this procedure, and we add to the assumptions in §1.4 that  $A$  has RM by  $\mathbb{Z}_F$ .

## 3. HURWITZ-MAASS ISOGENIES

If  $A$  is a p.p. abelian surface with RM by  $\mathbb{Z}_F$ , then  $A$  always admits rational isogenies of the following form. Let  $\mathfrak{p}$  be an ideal in  $\mathbb{Z}_F$  such that  $\mathfrak{p}^2$  is trivial in  $\text{Cl}^+(F)$ , and let  $\alpha \in \mathbb{Z}_F$  be a totally positive generator of  $\mathfrak{p}^2$ . (This includes the case:  $\mathfrak{p} = \mathbb{Z}_F$  and  $\alpha$  is a totally positive unit.) Then the subgroup  $A[\mathfrak{p}]$  is  $k$ -rational, and is maximal isotropic in  $A[\alpha]$ . By Proposition 1.1, the quotient abelian surface  $B = A/A[\mathfrak{p}]$  thus admits a unique principal polarization  $\lambda_B$  for which  $\varphi : (A, \lambda_A) \rightarrow (B, \lambda_B)$  is an  $\alpha$ -isogeny.

We refer to  $\varphi$  as the *Hurwitz–Maass isogeny* attached to  $(\mathfrak{p}, \alpha)$ . In the moduli interpretation, they correspond to the action of the Hurwitz–Maass extension of the Hilbert modular group on the Hilbert surface that classifies p.p. abelian surfaces with RM by  $\mathbb{Z}_F$  [3, §I.4]. Because  $A[\mathfrak{p}]$  is stable under  $\mathbb{Z}_F$ ,  $B$  still has RM by  $\mathbb{Z}_F$ .

If  $\beta \in \mathbb{Z}_F \setminus \{0\}$ , then replacing  $\mathfrak{p}$  by  $\beta\mathfrak{p}$  and  $\alpha$  by  $\beta^2\alpha$  yields an isomorphic PPAS. Moreover, composing Hurwitz–Maass isogenies corresponds to multiplying ideals and their generators. Thus, the Hurwitz–Maass isogeny class of  $A$  can be exhausted by considering only:

- (1) Fixed representatives  $\mathfrak{p}$  of ideal classes in  $\text{Cl}^+(F)$  that are squares of elements in  $\text{Cl}(F)$ , and arbitrary generators  $\alpha_{\mathfrak{p}}$  of  $\mathfrak{p}^2$  fixed once and for all; and
- (2) Fixed representatives  $\varepsilon$  of totally positive units in  $\mathbb{Z}_F$  modulo squares (for  $\mathfrak{p} = \mathbb{Z}_F$ ).

**Remark 3.1.** By genus theory, we in fact find an action of a finite abelian 2-group on the set of PPAS’s with RM by  $\mathbb{Z}_F$  by means of rational isogenies. If  $\#\text{Cl}^+(F) = 1$  and the fundamental unit of  $\mathbb{Z}_F$  has negative norm, then this group is trivial and we don’t have to worry about Hurwitz–Maass isogenies. This includes the cases  $F = \mathbb{Q}(\sqrt{\Delta})$  for all prime fundamental discriminants  $\Delta < 220$  and  $\Delta = 8$ , as well as  $F = \mathbb{Q} \times \mathbb{Q}$ .

Not all isogenies are Hurwitz–Maass: in particular,  $\beta$ -isogenies when  $\beta \in \mathbb{Z}_F$  is a split prime are not, so the classification is not yet complete.

## 4. THE FULL CLASSIFICATION

Let  $f : A \rightarrow A'$  be an  $\alpha$ -isogeny between PPAS with RM by  $\mathbb{Z}_F$ , where  $\alpha$  is a totally positive element of  $\mathbb{Z}_F$ . Consider the factorization

$$\alpha = \prod \mathfrak{p}^{e(\mathfrak{p})}$$

into prime ideals of  $\mathbb{Z}_F$ . By assumption,  $\ker(f)$  is stable under  $\mathbb{Z}_F$ , so by the Chinese remainder theorem, we have a decomposition

$$\ker(f) = \bigoplus_{\mathfrak{p}} K_{\mathfrak{p}}, \quad \text{where } K_{\mathfrak{p}} = \ker(f) \cap A[\mathfrak{p}^{e(\mathfrak{p})}].$$

Each subgroup  $K_{\mathfrak{p}}$  is maximal isotropic for the alternating pairing induced on  $A[\mathfrak{p}^{e(\mathfrak{p})}]$  from  $A[\alpha]$ .

Fix an ideal  $\mathfrak{p}$  appearing in this decomposition that is split in  $\mathbb{Z}_F$  (i.e.  $N_{F/\mathbb{Q}}(\mathfrak{p})$  is a prime  $p$ ). After choosing a convenient basis, we can identify  $A[\mathfrak{p}^{e(\mathfrak{p})}]$  with  $(\mathbb{Z}/p^{e(\mathfrak{p})}\mathbb{Z})^2$ , where  $p = N_{F/\mathbb{Q}}(\mathfrak{p})$  with its standard alternating pairing

$$((a, b), (c, d)) \mapsto ad - bc \pmod{p^{e(\mathfrak{p})}}.$$

For each  $0 \leq j \leq e(\mathfrak{p}) - 1$ , let  $n_j \in \{0, 1, 2\}$  be the integer such that  $\#(p^j K_{\mathfrak{p}} \cap A[\mathfrak{p}]) = p^{n_j}$ . Since  $K_{\mathfrak{p}}$  is maximal isotropic, we have the following constraints:

- (1)  $n_j + n_{e(\mathfrak{p})-1-j} \leq 2$  for every  $j$ ,
- (2)  $\sum_{j=0}^{e(\mathfrak{p})-1} n_j = e(\mathfrak{p})$ ,
- (3) The sequence  $n_0, \dots, n_{e(\mathfrak{p})-1}$  is nonincreasing.

Therefore, there exists an integer  $0 \leq m(\mathfrak{p}) \leq (e(\mathfrak{p}) - 1)/2$  such that

$$n_j = \begin{cases} 2 & \text{if } j < m(\mathfrak{p}), \\ 1 & \text{if } m(\mathfrak{p}) \leq j < e(\mathfrak{p}) - m(\mathfrak{p}), \\ 0 & \text{if } j \geq e(\mathfrak{p}) - m(\mathfrak{p}). \end{cases}$$

**Definition 4.1.** With the above notation, we say that

- the *cyclic type* of  $f$  is  $\text{cyc}(f) = \prod_{\mathfrak{p} \text{ split}} \mathfrak{p}^{e(\mathfrak{p})-2m(\mathfrak{p})}$ ,
- the *torsion type* of  $f$  is  $\text{tor}(f) = \prod_{\mathfrak{p} \text{ split}} \mathfrak{p}^{m(\mathfrak{p})}$ ,
- the *inert type* of  $f$  is  $\text{inert}(f) = \prod_{\mathfrak{p} \text{ inert}} \mathfrak{p}^{e(\mathfrak{p})}$ .

We call  $\bigoplus_{\mathfrak{p} \text{ split}} K_{\mathfrak{p}}$  the *split subgroup* of  $\ker(f)$ , denoted by  $\ker(f)_{\text{split}}$ .

Note that  $A[\text{tor}(f)] \subset \ker(f)_{\text{split}}$ , while  $\ker(f)_{\text{split}}/A[\text{tor}(f)]$  is a cyclic subgroup of  $A[\text{cyc}(f) \text{tor}(f)]/A[\text{tor}(f)]$ , which explains the name *torsion type* and *cyclic type* respectively. Note also that both  $\text{inert}(f)$  and  $\text{tor}(f)^2 \text{cyc}(f)$  are trivial in  $\text{Cl}^+(F)$ .

The inert type of an isogeny always admits a totally positive generator (in fact, an element of  $\mathbb{Z}$ ). Thus, we can always decompose an isogeny to separate the inert type.

**Proposition 4.2.** *Let  $f : A \rightarrow A'$  be an  $\alpha$ -isogeny as above, and let  $n \in \mathbb{Z}_{\geq 1}$  be a generator of  $\text{inert}(f)$ . Then we can decompose  $f$  over  $k$*

as

$$A \xrightarrow{g} B \xrightarrow{h} A',$$

where  $B$  is a PPAS with RM by  $\mathbb{Z}_F$ ;  $g$  is an  $\alpha/n$ -isogeny;  $h$  is an  $n$ -isogeny; and  $f = h \circ g$ . Thus  $\text{inert}(g)$  is the trivial ideal,  $\text{inert}(h) = (n)$ , and both  $\text{tor}(h)$  and  $\text{cyc}(h)$  are trivial.

*Proof.*  $\ker(f)_{\text{split}}$  is maximal isotropic in  $A[\alpha/n]$  and is stable under  $\mathbb{Z}_F$ , so the quotient isogeny  $g : A \rightarrow B = A/\ker(f)_{\text{split}}$  fits in a diagram as claimed. (Note that by considering the inert part of  $\ker(f)$ , we could also write  $f = g \circ h$  where  $h$  is an  $n$ -isogeny and  $g$  is an  $\alpha/n$ -isogeny.)  $\square$

In general,  $\text{Cl}^+(F)$  is not trivial, so separating the cyclic and torsion types is not so straightforward. Nevertheless, we have:

**Proposition 4.3.** *Let  $f : A \rightarrow A'$  be an  $\alpha$ -isogeny with trivial inert type. Let  $\mathfrak{q}$  and  $\mathfrak{r}$  be divisors of  $\text{cyc}(f)$  and  $\text{tor}(f)$  respectively such that  $\mathfrak{q}\mathfrak{r}^2$  is trivial in  $\text{Cl}^+(F)$ , and let  $\beta$  be a totally positive generator of that ideal. Then there exists a diagram defined over  $k$ :*

$$A \xrightarrow{g} B \xrightarrow{h} A'$$

where  $B$  is a PPAS with RM by  $\mathbb{Z}_F$ ;  $g$  is a  $\beta$ -isogeny with cyclic and torsion types  $\mathfrak{q}$  and  $\mathfrak{r}$ ;  $h$  is an  $\alpha/\beta$ -isogeny with cyclic and torsion types  $\text{cyc}(f)/\mathfrak{q}$  and  $\text{tor}(f)/\mathfrak{r}$ ; and  $f = h \circ g$ .

*Proof.* For a split prime  $\mathfrak{p}$  appearing in the decomposition of  $\alpha$ , let  $p$ ,  $e(\mathfrak{p})$  and  $m(\mathfrak{p})$  be as above. Let  $e'$  and  $e''$  be the exponents of  $\mathfrak{p}$  in  $\mathfrak{q}\mathfrak{r}$  and  $\mathfrak{q}$  respectively. We then write

$$K'_{\mathfrak{p}} = A[\mathfrak{p}^{e''}] + p^{e(\mathfrak{p})-m(\mathfrak{p})-e'} K_{\mathfrak{p}}.$$

The subgroup  $K' = \bigoplus_{\mathfrak{p}} K'_{\mathfrak{p}}$  is maximal isotropic in  $A[\beta]$  and  $k$ -rational. Let  $g$  be the quotient isogeny  $A \rightarrow A/K'$ : it has the correct torsion and cyclic types. There exists a unique isogeny  $h$  such that  $f = h \circ g$ , and we can also check that  $h$  has the correct torsion and cyclic types.  $\square$

**Corollary 4.4.** *Let  $f : A \rightarrow A'$  be an  $\alpha$ -isogeny with trivial inert type. Let  $\mathfrak{q}$  be a divisor of  $\text{cyc}(f)$  which is a square in  $\text{Cl}^+(F)$ , and let  $\mathfrak{c}$  be any ideal such that  $\mathfrak{c}^2\mathfrak{q}$  is trivial in  $\text{Cl}^+(F)$ . Choose a totally positive generator  $\beta$  of  $\mathfrak{q}\mathfrak{c}^2$ . Let  $n = N_{F/\mathbb{Q}}(\mathfrak{c})$ . Then there exists a commutative diagram*

$$\begin{array}{ccccc} A & \xrightarrow{g} & B & \xrightarrow{h} & A' \\ \downarrow [n] & & & \nearrow f & \\ A & & & & \end{array}$$

where  $B$  is a PPAS with RM by  $\mathbb{Z}_F$ ,  $g$  is a  $\beta$ -isogeny with torsion and cyclic types  $\mathfrak{c}$  and  $\mathfrak{q}$ , and  $h$  is an  $\alpha/\beta$ -isogeny with torsion and cyclic types  $\text{tor}(f)\bar{\mathfrak{c}}$  and  $\text{cyc}(f)/\mathfrak{q}$ .

*Proof.* We can assume that  $\mathfrak{c}$  is a split ideal. Then  $f \circ [n]$  has torsion type  $\text{tor}(f)\bar{\mathfrak{c}}$  and cyclic type  $\text{cyc}(f)$ . Now we apply Proposition 4.3 using that  $\mathfrak{c}^2\mathfrak{q}$  is trivial in  $\text{Cl}^+(F)$ .  $\square$

This leads us to the following way of enumerating  $\text{Isog}_{\text{pp}}(A)$ .

**Algorithm 4.5. Input:** A PPAS  $A/k$  with real multiplication by  $\mathbb{Z}_F$ .  
**Output:** The set  $\text{Isog}_{\text{pp}}(A)$ .

- (1) Let  $S$  be the Hurwitz–Maass isogeny class of  $A$ .
- (2) In this step, we consider a fixed element  $B \in S$ . (At first,  $B = A$ .) Let  $Q$  be a set of nontrivial ideals  $\mathfrak{q}$  of  $\mathbb{Z}_F$  such that:
  - (a)  $\mathfrak{q}$  is a square in  $\text{Cl}^+(F)$ , and  $\mathfrak{q}$  is a product of split ideals;
  - (b)  $A[\mathfrak{q}]$  admits a maximal cyclic subgroup that is  $k$ -rational;
  - (c) Every nontrivial ideal  $\mathfrak{q}'$  satisfying (a) and (b) is divisible by some  $\mathfrak{q} \in Q$ .

For every  $\mathfrak{q} \in Q$ ,

- (a) Choose an ideal  $\mathfrak{c}$ , prime to  $\mathfrak{q}$ , so that  $\mathfrak{c}^2\mathfrak{q}$  is trivial in  $\text{Cl}^+(F)$ , and let  $\beta$  be a totally positive generator of that ideal.
  - (b) For every maximal cyclic  $k$ -rational subgroup  $G \subset A[\mathfrak{q}]$ , the subgroup  $G' = A[\mathfrak{c}] \oplus G$  is maximal isotropic in  $A[\beta]$ . Let  $B$  be the PPAS  $A/G'$ . If  $B \in S$ , then do nothing, otherwise add the Hurwitz–Maass isogeny class of  $B$  to  $S$  and apply step (2) to this new  $B$ .
- (3) Apply step (4) to every  $B \in S$ .
  - (4) In this step, we consider a fixed element  $B \in S$ . Let  $L_1$  (resp.  $L_2$ ) be the set of primes  $\ell \in \mathbb{Z}_{\geq 1}$  such that  $A[\ell]$  (resp.  $A[\ell^2]$ ) admits a maximal isotropic  $k$ -rational subgroup isomorphic to  $(\mathbb{Z}/\ell\mathbb{Z})^2$  (resp.  $(\mathbb{Z}/\ell\mathbb{Z})^2 \times (\mathbb{Z}/\ell^2\mathbb{Z})$ ).
    - (a) For every  $\ell \in L_1$ , add all the PPAS's  $B'$  that are 1-step  $\ell$ -isogenous to  $B$  to  $S$ .
    - (b) For every  $\ell \in L_2$ , add all the PPAS's  $B'$  that are 2-step  $\ell$ -isogenous to  $B$  to  $S$ .
    - (c) Apply step (4) to every element of  $S$  we added in (a) or (b).
  - (5) Output  $S = \text{Isog}_{\text{pp}}(A)$ .

For the notion of 1-step and 2-step  $\ell$ -isogenies where  $\ell$  is a prime number, see [1].

**Theorem 4.6.** *Algorithm 4.5 is correct.*



*Proof.* First, the algorithm terminates, as  $\text{Isog}_{\text{pp}}(A)$  is finite and step (2) is applied at most once on each PPAS in this set.

Let  $A' \in \text{Isog}_{\text{pp}}(A)$ , and fix an isogeny  $f : A \rightarrow A'$ . By Corollary 2.2, Proposition 4.2 and step (4), we can assume that  $A'$  has RM by  $\mathbb{Z}_F$  and  $f$  has trivial inert type. If  $\text{cyc}(f)$  is nontrivial, then when we apply step (2) to  $A$ , we find  $\mathfrak{q} \in Q$  dividing  $\text{cyc}(f)$ . Choosing  $\mathfrak{c}$  as in step (2), by Corollary 4.4, we find a decomposition  $f \circ [n] = g \circ h$  where  $h$  has cyclic type  $\text{cyc}(f)/\mathfrak{q}$ , and moreover step (2) has been applied to the domain of  $h$ . After a finite number of such steps, we reduce to the case where the cyclic type of  $f$  is trivial. Then  $f$  is a Hurwitz–Maass isogeny. Since step (2) adds the Hurwitz–Maass isogeny class of any PPAS it finds,  $A'$  will indeed be part of  $S$ .  $\square$

In order to actually implement Algorithm 4.5, we have to do more. Computing  $Q$  in step (2), and  $L_1, L_2$  in step (4), will rely on Dieulefait’s criteria. Computing isogenies will rely on analytic methods as in the typical case [1].

#### REFERENCES

- [1] R. van Bommel, S. Chidambaram, E. Costa, J. Kieffer. Computing isogeny classes of typical principally polarized abelian surfaces over the rationals. Proceedings of the conference *LMFDB, Computation, and Number Theory (LuCaNT)*, 2024.
- [2] G. Faltings. Endlichkeitssätze für abelsche Varietäten über Zahlkörper. *Invent. Math.* 73 (3), 349–366, 1983.
- [3] G. van der Geer. *Hilbert modular surfaces*. Springer, 1988.
- [4] D. Mumford. *Abelian varieties*. Oxford University Press, 1970.