



HAL
open science

On the configurations of four spheres supporting the vertices of a tetrahedron

Marco Longinetti, Simone Naldi

► **To cite this version:**

Marco Longinetti, Simone Naldi. On the configurations of four spheres supporting the vertices of a tetrahedron. 2024. hal-04591613

HAL Id: hal-04591613

<https://hal.science/hal-04591613v1>

Preprint submitted on 29 May 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On the configurations of four spheres supporting the vertices of a tetrahedron

M. Longinetti¹

(marco.longinetti@unifi.it)

S. Naldi^{2,3}

(simone.naldi@unilim.fr)

May 29, 2024

Abstract

A reformulation of the three circles theorem of Johnson [10] with distance coordinates to the vertices of a triangle is explicitly represented in a polynomial system and solved by symbolic computation. A similar polynomial system in distance coordinates to the vertices of a tetrahedron $T \subset \mathbb{R}^3$ is introduced to represent the configurations of four spheres of radius R^* , which intersect in one point, each sphere containing three vertices of T but not the fourth one. This problem is related to that of computing the largest value r for which the set of vertices of T is an r -body [11]. For triangular pyramids we completely describe the set of geometric configurations with the required four balls of radius R^* . The solutions obtained by symbolic computation show that triangular pyramids are splitted into two different classes: in the first one R^* is unique, in the second one three values R^* there exist. The first class can be itself subdivided into two subclasses, one of which is related to the family of r -bodies.

Keywords. Tetrahedra, triangular pyramids, three circles theorem, R -bodies, distance geometry, Cayley-Menger determinants.

Mathematics Subject Classification (MSC2020). 68W30, 52A30, 51FXX.

1 Introduction

Let $V = \{v_0, v_1, v_2, v_3\} \subset \mathbb{R}^3$ be the set of vertices of a simplex $T = \text{co}(V)$ with circumradius R_T . Assume that there exist four distinct spheres $S_0, S_1, S_2, S_3 \subset \mathbb{R}^3$ of the same radius R^* such that

- (i) S_j contains the vertices v_i , for all $i \neq j$,
- (ii) S_j is the boundary of an open ball not containing v_j ,
- (iii) the intersection of the four spheres is one point $\{O^*\}$.

When T is the regular simplex, by a symmetry argument, it is not difficult to prove that O^* is necessarily the center of T and that $R^* = \frac{3}{2}R_T$ (Figure 1, on the right). It turns out that O^* does not belong to any open ball of radius greater than R^* and not containing the set V , see [11, Thm 5.6]. When $T \subset \mathbb{R}^n$ is a general simplex, $n \geq 3$, the question whether a configuration of $n + 1$ spheres satisfying the above properties exists, is open in its full generality.

This geometric configuration is a crucial example in [11] for some sentences and properties that hold in the class of R -bodies in \mathbb{R}^n . An R -body is the complement of the union of open balls with fixed radius R ; the smallest R -body containing a set $V \subset \mathbb{R}^n$ is called the R -hulloid of V and denoted $\text{co}_R(V)$. For a set V of vertices of a simplex T let us call R_T^* the supremum of all values

¹Dipartimento DIMAI, Università degli Studi di Firenze, Viale Morgagni 67/a, 50138, Firenze, Italy.

²Université de Limoges, CNRS, XLIM, UMR 7252, F-87000 Limoges, France.

³Sorbonne Université, CNRS, LIP6, F-75005 Paris, France.

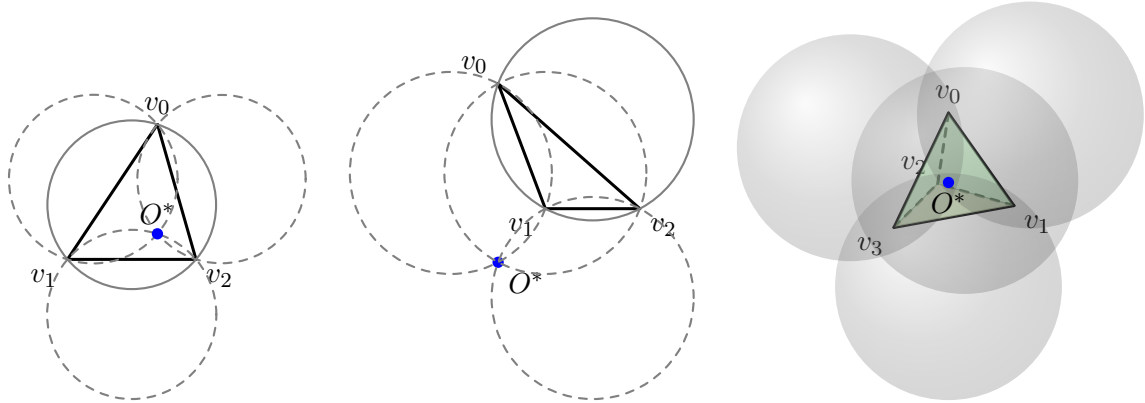


Figure 1: Configurations of three circles as in Theorem 1 for acute-angled (left) and obtuse-angled (center) plane triangles. Four spheres supporting the vertices of a simplex in \mathbb{R}^3 (right).

r such that V is an r -body; therefore for $r < R_T^*$ the complement of V is union of open balls of radius r .

Rephrasing in terms of R -bodies, this result implies that in case T is the regular simplex in \mathbb{R}^n , $n \geq 3$, the value R_T^* is the radius of n balls whose boundaries satisfy the conditions (i), (ii) and (iii): in particular $R_T^* = \frac{n}{2}R_T$ and its set of vertices V is an r -body if and only if $0 < r < \frac{n}{2}R_T$, whereas its R_T^* -hulloid is $\bar{V} \cup \{O^*\}$.

The planar case ($n = 2$), completely solved in [11, Thm. 4.2], is somehow special. Given a triangle $T = co(V) \subset \mathbb{R}^2$, it consists in the problem of determining three circles of equal radius, each containing two of the three vertices of T , but not the third one, and meeting in one point. In this special case one obtains the equality $R_T^* = R_T$, and that the set V is a r -body if and only if $r \leq R_T$. Indeed, this special property of dimension 2 follows directly from a classical result in Euclidean geometry of 1916, namely Johnson's Three Circles Theorem:

Theorem 1 (Johnson [10]) *Assume three distinct plane circles of the same radius R intersect in a point O^* and let $V = \{v_0, v_1, v_2\}$ be the other intersection points. Then the circle through V has radius R and O^* is the orthocenter of $T = co(V)$.*

The four points in $V \cup \{O^*\}$ in Theorem 1 form a rigid configuration in the following sense: every point is the orthocenter of the remaining three. The question whether a generalized version of Theorem 1 holds for $n > 2$ is one of the starting points of our work. It is open in its full generality, nevertheless, we give a partial answer for the three-dimensional case in this paper.

In particular, we are interested in checking the existence of special configurations of four distinct spheres satisfying properties (i) and (iii) and among these, understanding in which case a configuration exists satisfying the additional property (ii), which has a special meaning for the theory of R -bodies. Let us mention that the question of the existence of special arrangements of $n + 2$ spheres in \mathbb{R}^n , of equal radius, has been addressed in the work [14] for the general case of \mathbb{R}^n and not completely solved in \mathbb{R}^3 (see Remark 1).

Main contributions. For a general simplex of \mathbb{R}^3 with no special symmetries, the problem of computing the value R_T^* seems challenging. In Section 4 we present an algebraic formulation for the problem of finding, for a tetrahedron $T \subset \mathbb{R}^3$, the radius R^* of four spheres satisfying conditions (i) and (iii). This is based on the fact that conditions (i) and (iii) can be translated into equality constraints (parametric polynomial equations) through the so-called Cayley-Menger determinants, while condition (ii) is rather semialgebraic, that is, given by polynomial inequalities.

Whereas the algebraic approach seems challenging for the general class of tetrahedra, we consider in Section 5 the subclass of triangular pyramids (tetrahedra with a regular base and the fourth vertex equidistant from the base vertices, cf. [1]): in this special case the solutions of the polynomial system can be fully characterized. A complete description, depending on the shape of T , of all solutions R^* is given in Theorem 4; their corresponding geometric configurations are described next and contain in particular the critical value of R_T^* for the R -body property (ii). We show that for $n = 3$, the point O^* defined above is neither unique, nor coincides in general with a remarkable point of the tetrahedron T (except for special cases): in the planar case, by Theorem 1, O^* is the orthocenter of the triangle, while in \mathbb{R}^3 it is neither the orthocenter nor the Monge point of the tetrahedron T .

This symbolic approach is considered also in the planar case for a set $V = \{v_0, v_1, v_2\}$ of three non-collinear vertices of a triangle $T = \text{co}(V)$ with circumradius R_T . A polynomial system in distance coordinates is considered in Section 3.1 and its algebraic solutions described in Section 3.2 and connected with the corresponding geometric configurations in Section 3.3. This new approach carries out a new proof of a more general formulation of Johnson's Theorem introduced previously in Section 2.3.

Finally in Section 6 we consider any simplex $T \subset \mathbb{R}^3$ and we recall the assumption under which the critical value R_T^* is the value of the radius of four spheres satisfying conditions (i), (ii) and (iii). Such spheres are the boundary of open balls that are said to be R -supporting the set V , in the theory of R -bodies. In this paper we relax this definition saying that four spheres are R -supporting V if they satisfy properties (i) and (iii), and not necessarily (ii). We show that this assumption is satisfied in the case of triangular pyramids and we conclude with a characterization of R -hulloids for this class of tetrahedra.

Paper outline. In Section 2 the basics of computational and distance geometry are introduced, with focus on Cayley-Menger determinants. In Section 2.3 a reformulation of the original problem using equations representing double spheres is discussed. The main contributions of the paper are then organized starting from the planar case for a more clear description: Section 3 is completely devoted to the case $n = 2$. In Section 4 a polynomial system representing configurations of spheres supporting the vertices of a tetrahedron in \mathbb{R}^3 is presented, specialized to the case of regular tetrahedron for which a complete description of the configurations is given. Next a solution for the case $n = 3$ for the class of triangular pyramids is given in Section 5. In Section 6 the connections with the theory of R -bodies is presented.

2 Preliminaries and Notation

2.1 Basic algebraic geometry

Denote by $K[x_1, \dots, x_n]$ the ring of polynomials in variables x_1, \dots, x_n with coefficients over a field K . In this work, we address geometric questions that often depend on parameters a_1, \dots, a_n (that are considered free, unless otherwise stated), in which case $K = \mathbb{Q}(a_1, \dots, a_n)$; often parameters are specialized to algebraic numbers in which case the corresponding polynomials have coefficients in some algebraic extension of \mathbb{Q} .

We refer to [3] for a complete background on computational algebraic geometry and we only recall the main definitions needed for this work. The complex (resp. real) common vanishing locus of a family of polynomials $F \subset K[x_1, \dots, x_n]$ is denoted $\mathcal{Z}(F)$ (resp. $\mathcal{Z}_{\mathbb{R}}(F)$) and called a (real) algebraic variety. Every ideal $I \subset K[x_1, \dots, x_n]$ is generated by finitely many polynomials f_1, \dots, f_s , briefly $I = \langle f_1, \dots, f_s \rangle$ (if $s = 1$, I is called principal). The radical of I is the set $\sqrt{I} = \{f \in K[x_1, \dots, x_n] : f^m \in I, \exists m \in \mathbb{N}\}$ and of course $\mathcal{Z}(\sqrt{I}) = \mathcal{Z}(I) = \mathcal{Z}(f_1, \dots, f_s)$, for $I = \langle f_1, \dots, f_s \rangle$. To a set $Z \subset K^n$ one can associate its vanishing ideal $\mathcal{I}(Z) = \{f \in K[x_1, \dots, x_n] : f(z) = 0, \forall z \in Z\}$. For an algebraically closed field K , the operators \mathcal{Z} and \mathcal{I}

define a bijective correspondence between varieties and radical ideals: by Hilbert's Nullstellensatz [3, §4.1], one has indeed $\mathcal{S}(\mathcal{Z}(I)) = \sqrt{I}$.

A variety Z is called irreducible if it is not union of two algebraic varieties properly contained in Z , in which case its vanishing ideal $\mathcal{S}(Z)$ is prime. Every variety is the finite union of irreducible varieties, called its irreducible components. An ideal P is called primary if $fg \in P$ implies either $f \in P$ or $g \in \sqrt{P}$, in which case one also has that \sqrt{P} is a prime ideal and $\mathcal{Z}(P)$ irreducible. Every ideal I is decomposable as finite intersection of primary ideals $I = P_1 \cap P_2 \cap \dots \cap P_s$, called a primary decomposition of I . Such decomposition in particular yields the irreducible decomposition of the associated variety: $\sqrt{I} = \sqrt{P_1} \cap \sqrt{P_2} \cap \dots \cap \sqrt{P_s}$ and $\mathcal{Z}(P_i) = \mathcal{Z}(\sqrt{P_i})$, $i = 1, \dots, s$ are the irreducible components of $\mathcal{Z}(I)$. In this work we make use of the software Macaulay2 [6] for the computation of primary decompositions and radicals of ideals related to some special polynomial systems.

Finally we often make use of Sturm sequences and of Sturm Theorem, for which we refer to [2].

2.2 Distance geometry

For geometric-constrained problems, such as those treated in this work, it is often useful to use distance geometry as a coordinate model instead of the Cartesian coordinates, see for instance [8, 18]. We briefly recall the main aspects of distance geometry in \mathbb{R}^n and next focus on the case $n = 2$ and $n = 3$.

Given a set of $n + 1$ points $V = \{v_0, \dots, v_n\} \subset \mathbb{R}^n$, denote by $(X_0, \dots, X_n)_V$ the vector of squared Euclidean distances of $v \in \mathbb{R}^n$ to the elements of V , meaning that $\|v - v_i\|_2^2 = X_i$, $i = 0, \dots, n$ and call them the distance coordinates of v with respect to V . The convex hull of V is $\text{co}(V)$. The following classical rigidity property shows the good definition of distance coordinates to independent sets:

Proposition 1 *Let $V = \{v_0, \dots, v_n\} \subset \mathbb{A}^n$ be $n + 1$ affinely-independent vectors in the real affine space \mathbb{A}^n and let $P, Q \in \mathbb{A}^n$ with the same distance coordinates $(X_0, \dots, X_n)_V$ with respect to V . Then $P = Q$.*

Let now $d_{ij} = \|v_i - v_j\|_2^2$ be the squared Euclidean distance of v_i to v_j . We recall from [16] that the squared volume of $\text{co}(V)$ in \mathbb{R}^n is a constant multiple of the determinant of the Cayley-Menger matrix

$$\text{CM}_V = \begin{bmatrix} 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & d_{01} & d_{02} & \dots & d_{0n} \\ 1 & d_{01} & 0 & d_{12} & \dots & d_{1n} \\ 1 & d_{02} & d_{12} & 0 & & d_{2n} \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 1 & d_{0n} & d_{1n} & d_{2n} & \dots & 0 \end{bmatrix} \quad (1)$$

The following proposition, see [8] for more details, gives a relation satisfied by the distance coordinates of the point v .

Proposition 2 *Let d_{ij} , for $0 < i < j = 1, \dots, n$, be the squared relative distances of $n + 1$ affinely-independent points $V = \{v_0, \dots, v_n\}$ and let $D_0, \dots, D_n \geq 0$. Then there exists $v \in \text{span}(V)$ with distance coordinates $(D_0, \dots, D_n)_V$ if and only if the determinant of the augmented Cayley-Menger matrix $\text{CM}_{V \cup \{v\}}$ is zero:*

$$\det \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & D_0 & D_1 & D_2 & \cdots & D_n \\ 1 & D_0 & 0 & d_{01} & d_{02} & \cdots & d_{0n} \\ 1 & D_1 & d_{01} & 0 & d_{12} & \cdots & d_{1n} \\ 1 & D_2 & d_{02} & d_{12} & 0 & & d_{2n} \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \\ 1 & D_n & d_{0n} & d_{1n} & d_{2n} & \cdots & 0 \end{bmatrix} = 0 \quad (2)$$

Let us consider now the case $v \in \text{span}(V)$ lies on a sphere of radius r , with center $w \in \text{span}(V)$.

Proposition 3 *Let d_{ij} , for $0 < i < j = 1, \dots, n$, be the squared relative distances of $n+1$ affinely-independent points $V = \{v_0, \dots, v_n\}$ and let $D_0, \dots, D_n \geq 0$ and $\rho = r^2$. Let $w \in \text{span}(V)$ be the center of a sphere S in $\text{span}(V)$ of radius r containing V . Then there exists $v \in \text{span}(V) \cap S$ with distance coordinates $(D_0, \dots, D_n)_V$ if and only if the determinant of the augmented Cayley-Menger matrix $\text{CM}_{V \cup \{v\} \cup \{w\}}$ is zero:*

$$\det \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & \rho & \rho & \rho & \rho & \cdots & \rho \\ 1 & \rho & 0 & D_0 & D_1 & D_2 & \cdots & D_n \\ 1 & \rho & D_0 & 0 & d_{01} & d_{02} & \cdots & d_{0n} \\ 1 & \rho & D_1 & d_{01} & 0 & d_{12} & \cdots & d_{1n} \\ 1 & \rho & D_2 & d_{02} & d_{12} & 0 & & d_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & \\ 1 & \rho & D_n & d_{0n} & d_{1n} & d_{2n} & \cdots & 0 \end{bmatrix} = 0 \quad (3)$$

2.3 Double circles and spheres

Let $V = \{v_0, v_1, v_2\} \subset \Pi$ be a set of three affinely independent points of a real plane Π and let $r \geq \frac{1}{2}\|v_i - v_j\|_2$ for all $0 \leq i < j \leq 2$. Denote by $\mathcal{C}_0^r, \mathcal{C}_1^r, \mathcal{C}_2^r$ the union of the two circles through v_1, v_2 (resp. v_0, v_2 and v_0, v_1), contained in Π , of radius r and call it the double circle through v_1, v_2 (resp. v_0, v_2 and v_0, v_1) of radius r . For $r = \frac{1}{2}\|v_i - v_j\|_2$, the double circle \mathcal{C}_k^r , $k \neq i, j$, reduces to one circle. Define the intersection of the three double circles

$$\mathcal{C}_V^r := \bigcap_{i=0}^2 \mathcal{C}_i^r. \quad (4)$$

The following shows that Cayley-Menger constraints (2) and (3) naturally define plane double circles and thus can be used to model the configuration in (4):

Theorem 2 *Let $V = \{v_0, v_1, v_2\} \subset \Pi$ be affinely-independent points spanning a plane Π and let $\mathcal{C}_0^r, \mathcal{C}_1^r, \mathcal{C}_2^r \subset \Pi$ be the double circles of radius r as in (4). Denoted for $k = 0, 1, 2$, $V_k = V \setminus \{v_k\}$ and by w_k any of the circumcenters of \mathcal{C}_k^r , then $P \in \mathcal{C}_V^r$ if and only if*

$$\det(\text{CM}_{V \cup \{P\}}) = \det(\text{CM}_{V_0 \cup \{w_0, P\}}) = \det(\text{CM}_{V_1 \cup \{w_1, P\}}) = \det(\text{CM}_{V_2 \cup \{w_2, P\}}) = 0. \quad (5)$$

Proof: We prove that $P \in \mathcal{C}_0^r$ if and only if $\det(\text{CM}_{V \cup \{P\}}) = \det(\text{CM}_{\{w_0, v_1, v_2, P\}}) = 0$: this implies the statement. First remark that $\text{CM}_{\{w_0, v_1, v_2, P\}}$ is well-defined as it only depends on the relative distances of w_0, v_1, v_2, P and the two circumcenters w_0 have all distance r from v_1, v_2, P for $P \in \mathcal{C}_0^r$. Further, $P \in \mathcal{C}_0^r$ if and only if $P \in \Pi = \text{span}(V)$ and P belongs to a circle of radius r containing v_1 and v_2 . By Proposition 2, the first condition is equivalent to $\det(\text{CM}_{V \cup \{P\}}) = 0$ and by Proposition 3, the second condition is equivalent to $\det(\text{CM}_{V_0 \cup \{P\}}) = 0$: indeed, it holds

if and only if $V_0 \cup \{w_0, P\} = \{w_0, v_1, v_2, P\}$ spans a plane and that w_0 is the circumcenter of $\text{co}(\{v_1, v_2, P\})$. \square

Using double circles in place of circles, the following proposition is a reformulation of Johnson's Theorem (Theorem 1):

Proposition 4 *Let $V = \{v_0, v_1, v_2\} \subset \Pi$ be not collinear points in a real affine plane Π , $T = \text{co}(V)$ the triangle with vertices in a set V and let R_T be the circumradius of T . Then $\mathcal{C}_V^r = \emptyset$ if and only if $r \neq R_T$. For $r = R_T$, then \mathcal{C}_V^r consists of the union of the circumcircle of T and of its orthocenter O^* .*

The statement of Proposition 4 might appear more advanced than that of Theorem 1, however its proof is straightforward from the one of Theorem 1 given in [9]. The advantage of this new formulation is twofold. First, it makes the following fact explicit: in order for the three circles to intersect in one point, it is necessary that their radius coincide with the circumradius of T . This is a special property of dimension two, indeed, this paper shows that it fails in dimension three. Secondly, with the help of Theorem 2, the constraint $P \in \mathcal{C}_V^r$ translates into the system of polynomial equations (5) and thus allows to use computer algebra in order to give an algebraic proof of Proposition 4 and at the same time of the classical Johnson's Theorem [10], which is what we do in Section 3.

In a three dimensional real space E , a similar formulation can be developed for the set of vertices of a tetrahedron and a configuration of four spheres, such as those introduced at the beginning of Section 1. Let $V = \{v_0, v_1, v_2, v_3\} \subset E$ be affinely independent. Let us denote by \mathcal{S}_i^r , $i = 0, 1, 2, 3$, the union of the two spheres containing $V_i := V \setminus \{v_i\}$, for $r \geq r_i$, where r_i is the circumradius of $\text{co}(V_i)$ in dimension two (that is in $\text{span}(V_i)$). For $r \geq \max_i \{r_i\}$, let

$$\mathcal{S}_V^r := \bigcap_{i=0}^3 \mathcal{S}_i^r. \quad (6)$$

The following is a direct generalization of Theorem 2 to the case $n = 3$:

Proposition 5 *Let $V = \{v_0, v_1, v_2, v_3\} \subset E$ be the set of vertices of a tetrahedron $T = \text{co}(V)$ in a three-dimensional real affine space E , with $\text{span}(V) = E$. Let w_i denote any of the two circumcenters of \mathcal{S}_i^r , for $i = 0, \dots, 3$. Then $P \in \mathcal{S}_V^r$ if and only if*

$$\begin{aligned} \det(\text{CM}_{V \cup \{P\}}) &= 0 \\ \det(\text{CM}_{V_i \cup \{w_i, P\}}) &= 0, \text{ for } i = 0, \dots, 3. \end{aligned} \quad (7)$$

Let us remark that the distance coordinates of the circumcenters w_i of \mathcal{C}_V^r in Theorem 2 (and of \mathcal{S}_V^r in Proposition 5) coincide with the unique value r , which is an unknown of the polynomial systems (5) and (7). Therefore the number of variables in such systems equals the number of their equations, indeed, the distances $\|w_i - v_i\|_2$, for $i = 0, \dots, 3$, are all equal to the unknown r .

3 Plane triangles

This section contains a proof of Proposition 4 and as a byproduct, an alternative proof of Johnson's Theorem [10] (Theorem 1) based on Cayley-Menger determinants and symbolic computation. The equations in (5) defining the intersection of three double circles (cf. (4)) are polynomials in the distance coordinates, with coefficients depending on the parameters of the problem.

3.1 A polynomial system

Let $V = \{v_0, v_1, v_2\} \subset \Pi$ be three non-collinear points and let A (resp. B and C) be the square of the distance of v_1 to v_2 (resp. v_0 to v_2 and v_0 to v_1). Denote by $V_i = V \setminus \{v_i\}, i = 0, 1, 2$ and by $T = \text{co}(V)$. Moreover let X, Y, Z be the squared distances with respect to v_0, v_1, v_2 of an unknown fourth point $P \in \Pi$, so that $P = (X, Y, Z)_V$ and let ρ be the square of the unknown radius of each of the three double circles in (4).

We recall from Theorem 2 that the condition $P \in C_V^r$ is equivalent to (4), that is to the parametric polynomial system:

$$\begin{aligned}
[\det(\text{CM}_{V \cup \{P\}})] & -2A^2X - 2ABC + 2ABX + 2ABY + 2ACX + 2ACZ - 2AX^2 + 2AXY + \\
& + 2AXZ - 2AYZ - 2B^2Y + 2BCY + 2BCZ + 2BXY - 2BXZ - \\
& - 2BY^2 + 2BYZ - 2C^2Z - 2CXY + 2CXZ + 2CYZ - 2CZ^2 = 0 \\
[\det(\text{CM}_{V_0 \cup \{w_0, P\}})] & \rho(2(AY + AZ + YZ) - A^2 - Y^2 - Z^2) - AYZ = 0 \\
[\det(\text{CM}_{V_1 \cup \{w_1, P\}})] & \rho(2(BX + BZ + XZ) - X^2 - B^2 - Z^2) - BXZ = 0 \\
[\det(\text{CM}_{V_2 \cup \{w_2, P\}})] & \rho(2(CX + CY + XY) - X^2 - Y^2 - C^2) - CXY = 0
\end{aligned} \tag{8}$$

System (8) consists of four equations depending on four variables X, Y, Z, ρ and whose coefficients are polynomial functions of three parameters A, B, C .

3.2 Description of the algebraic solutions

In this section we give a complete description of the solutions to system (8), recovering the classical result by Johnson in the form of Proposition 4. The proofs are computer-algebra assisted and all computations are done with the use of Macaulay2 [6].

First, as a direct application of Proposition 3, we recall the classical formula (in distance coordinates) for the circumradius of a triangle $T = \text{co}(V)$. Denote by

$$\theta := 2(AB + AC + BC) - (A^2 + B^2 + C^2) = 16 \text{area}(T)^2.$$

Proposition 6 $R_T^2 = \frac{ABC}{\theta}$.

Then we show that there is a unique non-zero value of ρ , as function of the parameters defining the triangle, for which (8) is solvable, as foreseen by Johnson's result (Theorem 1).

Lemma 1 *Let X, Y, Z, ρ satisfy system (8). Then either $\rho = 0$ or $\theta = 0$ or $\rho = \frac{ABC}{\theta}$.*

Proof: Let $I \subset \mathbb{Q}(A, B, C)[X, Y, Z, \rho]$ be the ideal defined by polynomials in (8). The elimination ideal $I \cap \mathbb{Q}(A, B, C)[\rho]$ is principal and defined by the polynomial

$$\rho(ABC + A^2\rho - 2AB\rho + B^2\rho - 2AC\rho - 2BC\rho + C^2\rho)^2$$

thus if (X, Y, Z, ρ) is a solution, then either $\rho = 0$ or $\theta\rho = ABC$. □

Next, we describe the algebraic solutions of system (8) in the following theorem.

Theorem 3 *Let $\theta = 2(AB + AC + BC) - (A^2 + B^2 + C^2)$. The solutions to system (8) are organized in the following components:*

1. $\rho = \frac{ABC}{\theta}$ and $(X, Y, Z)_V = \left(\frac{A(B+C-A)^2}{\theta}, \frac{B(A+C-B)^2}{\theta}, \frac{C(A+B-C)^2}{\theta} \right)$
2. $\rho = \frac{ABC}{\theta}$ and (A, B, C, X, Y, Z) belongs to a complex irreducible subvariety of \mathbb{C}^6 of codimension two and degree six

3. $\rho = 0$ or $A = 0$ or $B = 0$ or $C = 0$ or $\theta = 0$

Proof: From Lemma 1, one has either $\rho = 0$ or $\theta = 0$ (Item 3) or $\rho = \frac{ABC}{\theta}$. Let $I \subset \mathbb{Q}(A, B, C)[X, Y, Z, \rho]$ be the ideal generated by polynomials in (8). The elimination ideal $J = (I + \langle \rho\theta - ABC \rangle) \cap \mathbb{Q}(B, C)[X, Y, Z]$ is radical of codimension 2 and degree 6. Its primary decomposition has five prime components $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ of codimension 2, 3, 3, 3, 3 and degree 6, 1, 1, 1, 3, respectively. The polynomials generating these components are given in Appendix A.

The first component \mathcal{P}_0 is the one of Item 2. Eliminating two of the three variables X, Y, Z from the component \mathcal{P}_0 , one gets the univariate linear polynomials $\theta X - A(B + C - A)^2$, $\theta Y - B(A + C - B)^2$ and $\theta Z - C(A + B - C)^2$, whose zero set is the singleton in Item 1 (unless $\theta = 0$, Item 3). The three remaining components are linear spaces and satisfy either $A = 0$ or $B = 0$ or $C = 0$ (Item 3). \square

3.3 Geometric solutions

In this section we assume that A, B, C are the squared edge lengths of a plane triangle T (that is, their square roots satisfy the triangle inequality). We describe the acceptable geometric configurations among the solutions given in Theorem 3, allowing to make the expressions in Items 1 and 2 more explicit. We call O^* the point in \mathbb{R}^2 with distance coordinates $(X, Y, Z)_V$ from the vertices of T .

The proof of Corollary 1 is by standard analytic geometry.

Corollary 1 *Assume that A, B, C are the squared edge lengths of a plane triangle $T = co(V)$. Then the solution O^* of Item 1 in Theorem 3 is the orthocenter of T .*

Corollary 2 *Assume that A, B, C are the squared edge lengths of a plane triangle $T = co(V)$. Then the real solutions in Item 2 of Theorem 3 form the circumcircle of T .*

Proof: Let (x, y) be Cartesian coordinates in \mathbb{R}^2 . Assume by simplicity that $A = 1$ (the case with general $A > 0$ is equivalently obtained by scaling). Without loss of generality, we put (with $\theta = 2(B + C + BC) - (1 + B^2 + C^2)$)

$$v_0 = \begin{bmatrix} (1 - B + C)/2 \\ \sqrt{\theta}/2 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Let \mathcal{P}_0 be the component related to solutions of Item 2 in Theorem 3 (cf. Appendix A). Substituting to X, Y, Z , in the defining polynomials of \mathcal{P}_0 , the squared distances of (x, y) to v_0, v_1, v_2 , respectively, with $L = \sqrt{\theta}$ and adding the constraint $L^2 = \theta$, one gets a polynomial system $F \subset \mathbb{Q}(B, C, L)[x, y]$ whose real zero set $\mathcal{Z}_{\mathbb{R}}(F)$ is irreducible and defined by the single equation

$$\sqrt{\theta}x^2 + \sqrt{\theta}y^2 + y(1 - B - C) - \sqrt{\theta}x = 0,$$

that is, $\mathcal{Z}_{\mathbb{R}}(F)$ is the circle circumscribing T . \square

3.4 The equilateral case

We explain the algebraic solutions given in Corollaries 1 and 2 in the special case of the equilateral plane triangle (Figure 2). Let $I \subset \mathbb{Q}[X, Y, Z, \rho]$ be the ideal defined by system (8) with $A = B = C = 1$.

As in the proof of Theorem 3, eliminating the distance variables yields the univariate polynomial $\rho(3\rho - 1)^2$. The case $\rho = 0$ corresponds to the geometrically-meaningless situation when the circles have radius zero. The second case corresponds to the circumradius of the equilateral triangle with

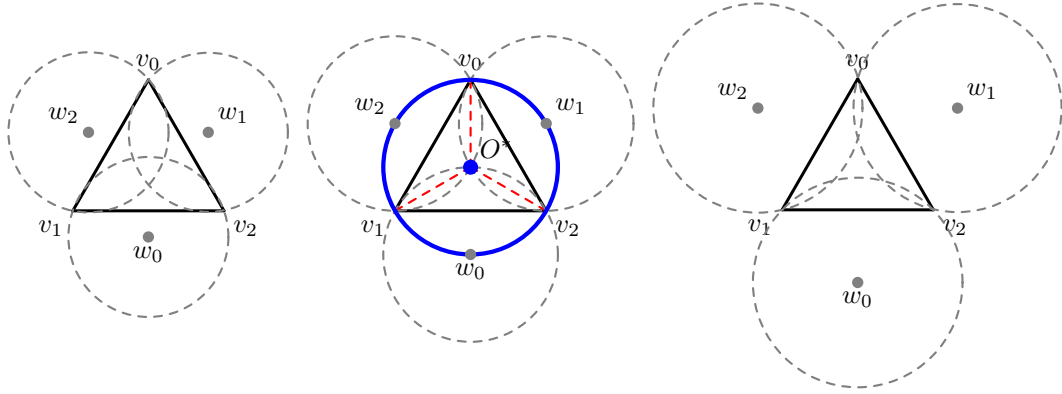


Figure 2: Johnson's Theorem for a plane equilateral triangle T .

side length 1, which is $\sqrt{1/3}$. This already shows the property mentioned in Proposition 4: in order for three circles of radius r , each containing two vertices of an equilateral triangle T , to intersect, it is necessary that $r = R_T$ (cf. Figure 2).

One verifies algebraically that this is also sufficient. Indeed, eliminating ρ from $I + \langle 3\rho - 1 \rangle$, one gets the ideal $J = I + \langle 3\rho - 1 \rangle \cap \mathbb{Q}[X, Y, Z]$, which is radical with two prime components: $\mathcal{P}_1 = \langle X + Y + Z - 2, X^2 + XY + Y^2 - 2X - 2Y + 1 \rangle$ and $\mathcal{P}_2 = \langle 3X - 1, 3Y - 1, 3Z - 1 \rangle$. The component \mathcal{P}_2 gives the distance coordinates $(X, Y, Z)_V = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})_V$ of the center of T (the blue dot in Figure 2).

Concerning \mathcal{P}_1 , first we remark that real solutions satisfying $X + Y + Z = 2$ correspond to the circumcircle of T . Indeed, let (x, y) be Cartesian coordinates in $\Pi = \text{span}(V)$. Since $\rho = \frac{1}{3}$ is the square of the circumradius, we can assume that the three vertices of the triangle are $v_0 = (0, 1/\sqrt{3})$, $v_1 = (-\frac{1}{2}, -\frac{1}{2\sqrt{3}})$ and $v_2 = (\frac{1}{2}, -\frac{1}{2\sqrt{3}})$. Thus the condition $X + Y + Z = 2$ translates into the condition $3x^2 + 3y^2 + 1 = 2$, that is $x^2 + y^2 = \frac{1}{3} = \rho$, the circumcircle. Finally, the second generator yields

$$X^2 + XY + Y^2 - 2X - 2Y + 1 = (1/3 - x^2 - y^2)(\sqrt{3}y - 3x^2 - 3y^2 - 3x),$$

that is, its zero set is the double circle of radius $\sqrt{1/3}$ through v_0, v_1 , namely $\mathcal{C}_2^{\sqrt{1/3}}$. In other words $\mathcal{Z}_{\mathbb{R}}(F)$, the real algebraic variety defined by system

$$F = \begin{cases} X + Y + Z = 2 \\ X^2 + XY + Y^2 - 2X - 2Y + 1 = 0 \end{cases}$$

is the circle circumscribing T (Figure 2, the blue circle).

4 Tetrahedra

Let $V = \{v_0, v_1, v_2, v_3\} \subset E$ be four affinely-independent points in a three dimensional real affine space E , forming the set of vertices of a tetrahedron $T = \text{co}(V)$. Let $d_{ij} = \|v_i - v_j\|_2^2$, for $0 \leq i < j \leq 3$ and let $P = (X, Y, Z, W)_V$ be a general point. Let $V_i = V \setminus \{v_i\}$, $i = 0, 1, 2, 3$.

Assume the point P satisfies conditions (i) and (iii) of the Introduction, for some given radius r : then necessarily P belongs to the set \mathcal{S}_V^r defined in (6) and by Proposition 5, the parameters

above satisfy the determinantal conditions (denoting $\rho = r^2$ and $|\cdot| := \det(\cdot)$):

$$\begin{aligned}
 & \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & \rho & Y & Z & W \\ 1 & \rho & 0 & \rho & \rho & \rho \\ 1 & Y & \rho & 0 & d_{12} & d_{13} \\ 1 & Z & \rho & d_{12} & 0 & d_{23} \\ 1 & W & \rho & d_{13} & d_{23} & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & X & \rho & Z & W \\ 1 & X & 0 & \rho & d_{02} & d_{03} \\ 1 & \rho & \rho & 0 & \rho & \rho \\ 1 & Z & d_{02} & \rho & 0 & d_{23} \\ 1 & W & d_{03} & \rho & d_{23} & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & X & Y & \rho & W \\ 1 & X & 0 & d_{01} & \rho & d_{03} \\ 1 & Y & d_{01} & 0 & \rho & d_{13} \\ 1 & \rho & \rho & \rho & 0 & \rho \\ 1 & W & d_{03} & d_{13} & \rho & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & X & Y & Z & \rho \\ 1 & X & 0 & d_{01} & d_{02} & \rho \\ 1 & Y & d_{01} & 0 & d_{12} & \rho \\ 1 & Z & d_{02} & d_{12} & 0 & \rho \\ 1 & \rho & \rho & \rho & \rho & 0 \end{vmatrix} = 0 \\
 & \hspace{15em} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & X & Y & Z & W \\ 1 & X & 0 & d_{01} & d_{02} & d_{03} \\ 1 & Y & d_{01} & 0 & d_{12} & d_{13} \\ 1 & Z & d_{02} & d_{12} & 0 & d_{23} \\ 1 & W & d_{03} & d_{13} & d_{23} & 0 \end{vmatrix} = 0
 \end{aligned} \tag{9}$$

Definition 1 A real solution $\rho, O^* = (X, Y, Z, W)_V$ to system (9) is called

- *geometrically-admissible*, if $X, Y, Z, W, \rho > 0$ and there is a configuration of four spheres meeting in O^* ;
- *trivial*, if it is geometrically-admissible but O^* lies on the circumscribed sphere of $T = \text{co}(V)$.

Giving a complete description of the solutions to this polynomial system with respect to the distance parameters of the tetrahedron is out of the scope of this paper. Nevertheless, for the regular tetrahedron it is possible to describe all solutions: this is done in the next example and completed later in Example 3.

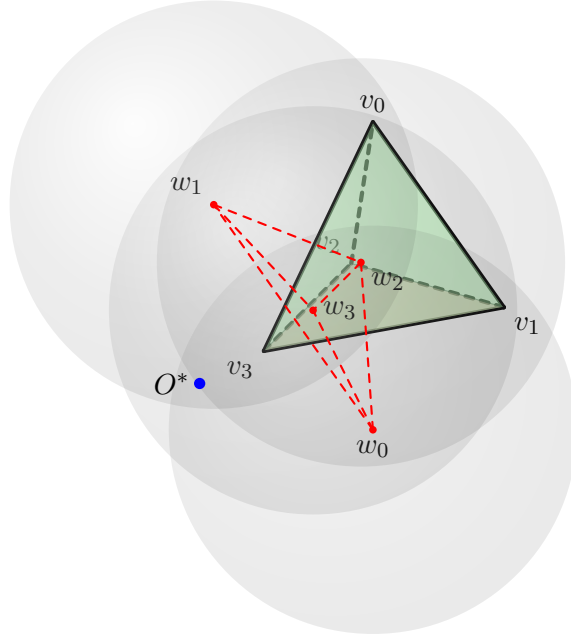


Figure 3: One of the six solutions with $\rho = \frac{5}{8}$ for the regular tetrahedron.

Example 1 (Regular tetrahedron) We assume $d_{01} = d_{02} = d_{03} = 1 = d_{12} = d_{13} = d_{23}$, so $T = \text{co}(V)$ is the regular tetrahedron with edge 1. Eliminating X, Y, Z, W from system (9) one gets the equation

$$(8\rho - 5)(8\rho - 3)^2(32\rho - 27)(64\rho^2 - 8\rho + 1) = 0.$$

The solutions $\rho = \frac{3}{8}$ and $\rho = \frac{27}{32}$ correspond to special configurations and will be discussed in Section 5 (Example 3). For $\rho = \frac{5}{8}$, each of the remaining variables X, Y, Z, W satisfies the condition $8\chi^2 - 20\chi + 9 = 0$ and the symmetric relation $X + Y + Z + W = 5$, thus there are exactly six solutions O^* for $\rho = \frac{5}{8}$, namely the ones obtained from

$$O^* = (X, Y, Z, W)_V = \left(\frac{5 + \sqrt{7}}{4}, \frac{5 + \sqrt{7}}{4}, \frac{5 - \sqrt{7}}{4}, \frac{5 - \sqrt{7}}{4} \right)_V \quad (10)$$

by applying a permutation of the variables.

Each solution yields the following configuration of spheres. Consider the four double spheres as in (6), of radius $r = \sqrt{\rho} = \sqrt{5/8}$ and supporting the set V of vertices of the regular tetrahedron T . These are boundaries of four double open balls of radius r : choose two open balls containing the fourth vertex and two open balls not containing it (there are $\binom{4}{2} = 6$ such choices). As an illustrative example (cf. Figure 3), the Cartesian coordinates of the point O^* and of the centers of the four spheres corresponding to the solution (10) are:

$$\begin{aligned} O^* &= \left(0, -\frac{\sqrt{21}}{6}, \frac{\sqrt{6} - \sqrt{42}}{12} \right) & w_0 &= \left(0, 0, -\frac{\sqrt{42}}{12} \right) & w_1 &= \left(0, -\frac{(1 + \sqrt{7})\sqrt{3}}{9}, \frac{(4 + \sqrt{7})\sqrt{6}}{36} \right) \\ w_2 &= \left(\frac{1 - \sqrt{7}}{6}, \frac{(1 - \sqrt{7})\sqrt{3}}{18}, \frac{(4 - \sqrt{7})\sqrt{6}}{36} \right) & w_3 &= \left(\frac{\sqrt{7} - 1}{6}, \frac{(1 - \sqrt{7})\sqrt{3}}{18}, \frac{(4 - \sqrt{7})\sqrt{6}}{36} \right) \end{aligned}$$

5 Triangular pyramids

In this section we make a twofold simplification to system (9): first we restrict it to a special class of tetrahedra (triangular pyramids) and further, we only look for solutions (X, Y, Z, W) satisfying $Y = Z = W$ (that will be called cyclic).

Consider the following geometric configuration: let $V = \{v_0, v_1, v_2, v_3\} \subset E$ be the set of four vertices of a triangular pyramid $T = \text{co}(V)$ in a three-dimensional real affine space E , that is, $T \subset E$ is a tetrahedron such that one of its faces, say $\text{co}(\{v_1, v_2, v_3\})$, is equilateral and the fourth vertex v_0 is equidistant to the base vertices.

We denote by λ the squared distance of v_0 to each of the base vertices and by η the squared edge length of the base face, for some $\lambda, \eta > 0$ (cf. Figure 4). Remark that the two parameters λ, η identify a solid pyramid if and only if $0 < \eta < 3\lambda$.

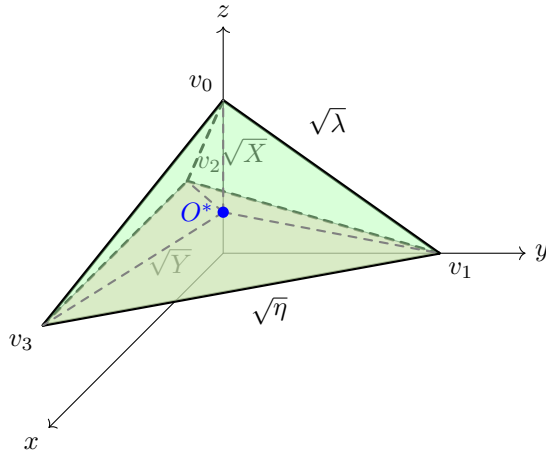


Figure 4: A triangular pyramid.

We denote by $O^* \in E$ a point satisfying properties (i) and (iii) given in Section 1, in other words, O^* is the unique intersection of four spheres $S_0, S_1, S_2, S_3 \subset \text{span}(V) = \mathbb{R}^3$, of the same radius, such that for every $0 \leq j \leq 3$, the sphere S_j contains v_i , for $i \neq j$ (Figure 1, right). In the context of R-bodies (cf. Section 6) we are particularly interested in configurations satisfying the additional property (ii), see Section 6 and particularly Definition 2.

Of course, as in the planar case, the trivial configuration where some of the four spheres coincide with the circumsphere is in principle included in the algebraic solutions of system (9), but not interesting for our purposes. By symmetry of T , the solution O^* of the polynomial system presented below in Section 5.1 is equidistant to the vertices of the base face, which leads to the trivial solutions consisting in the north pole and the south pole of the circumsphere. With the aim of generalizing Johnson's configuration to \mathbb{R}^3 , we would like to exclude these solutions and this is why we call them trivial in Definition 1.

5.1 A polynomial system

Denote by X (resp. Y) the squared distances of P to v_0 (resp. v_1, v_2 and v_3), so that the vector of distance coordinates of P with respect to V is $(X, Y, Y, Y)_V$. Let ρ be the squared radius of the four spheres. The center of the sphere S_k is denoted by w_k and $V_k := V \setminus \{v_k\}$, $k = 0, 1, 2, 3$. System (9) simplifies to the following parametric polynomial system:

$$\begin{aligned} [\det \text{CM}_{V \cup \{P\}}] & 3(X - Y + \lambda)^2 + 4\eta X - 12\lambda X = 0 \\ [\det \text{CM}_{V_0 \cup \{w_0, P\}}] & 3Y^2 - 4\rho(3Y - \eta) = 0 \\ [\det \text{CM}_{V_1 \cup \{w_1, P\}}] & 4\rho(4Y\lambda - (X - Y - \lambda)^2 - \eta X) - X(4\lambda Y - \eta X) = 0 \end{aligned} \quad (11)$$

Of course by symmetry of T the constraints $\det \text{CM}_{V_2 \cup \{w_2, P\}} = 0$ and $\det \text{CM}_{V_3 \cup \{w_3, P\}} = 0$ are equal to the third constraint in (11). Moreover up to scaling the tetrahedron, one can assume that $\lambda = 1$, which is what we do for most part of the remaining section.

Even if our model uses distance coordinates, the reader may assume that the three vertices v_1, v_2, v_3 lie in the plane $z = 0$ in \mathbb{R}^3 and bound an equilateral triangle of size $\sqrt{\eta}$ and the fourth vertex v_0 belongs to the z -axis at distance $\lambda = 1$ from v_1, v_2, v_3 , in other words that the Cartesian coordinates of V are:

$$v_0 = \begin{bmatrix} 0 \\ 0 \\ \sqrt{\frac{3-\eta}{3}} \end{bmatrix}, \quad v_1 = \begin{bmatrix} 0 \\ \sqrt{\frac{\eta}{3}} \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -\frac{\sqrt{\eta}}{2} \\ -\frac{1}{2}\sqrt{\frac{\eta}{3}} \\ 0 \end{bmatrix}, \quad \text{and} \quad v_3 = \begin{bmatrix} \frac{\sqrt{\eta}}{2} \\ -\frac{1}{2}\sqrt{\frac{\eta}{3}} \\ 0 \end{bmatrix}. \quad (12)$$

The circumcenter of the tetrahedron with vertices v_0, v_1, v_2, v_3 is then $w = (0, 0, \frac{3-2\eta}{2\sqrt{9-3\eta}})$. By symmetry, O^* is in the z -axis, thus $O^* = (0, 0, z)$, for z satisfying

$$X = (z - \sqrt{(3-\eta)/3})^2 \text{ and } Y = z^2 + \eta/3. \quad (13)$$

5.2 Description of the algebraic solutions

First we need a technical lemma.

Lemma 2 *Let*

$$\begin{aligned} g &= 1024(\eta - 3)\rho^3 + (-704\eta^2 + 1920\eta + 768)\rho^2 + \eta(196\eta^2 - 732\eta + 288)\rho + 27\eta^2 \in \mathbb{Q}(\eta)[\rho] \\ f &= 432(\eta - 3)t^3 + 108\eta(\eta - 2)t^2 - 9\eta^2(\eta + 1)t + \eta^4 \in \mathbb{Q}(\eta)[t] \end{aligned}$$

and let $\bar{\eta} = \frac{135}{98} + \frac{19}{98}\sqrt{57} \approx 2.841$ be the positive root of $49\eta^2 - 135\eta - 12$. The following holds:

1. For $\eta \in (0, \bar{\eta})$, f has exactly one real root and for $\eta \in (\bar{\eta}, 3)$, it has three distinct real roots

2. For $\eta \in (0, \bar{\eta}) \setminus \{\frac{12}{5}\}$, g has exactly one real root and for $\eta \in (\bar{\eta}, 3) \cup \{\frac{12}{5}\}$, it has three real roots counting multiplicities (a double one for $\eta = \frac{12}{5}$ and $\eta = \frac{20}{7}$)
3. For $\eta = \bar{\eta}$, both f and g have a simple real root and a double real root.

All the roots mentioned in Item 1, 2 and 3 are strictly positive.

Proof: The discriminant in t of f is $\Delta_t(f) = 314928\eta^7(3-\eta)(49\eta^2-135\eta-12)$ and the discriminant in ρ of g is $\Delta_\rho(g) = 196608\eta^3(3-\eta)(49\eta^2-135\eta-12)(5\eta-12)^2(7\eta-20)^2$. These discriminants vanish if and only if the corresponding polynomial has a double root (cf. [2, Prop. 4.3]) in which case, since they are cubics with real coefficients, they have one real root and one double real root (indeed the roots of a real polynomial come in pairs of conjugate complex numbers). This covers the cases $\eta = \frac{12}{5}, \frac{20}{7}$ and Item 3. Concerning f , one has $\Delta_t(f) < 0$ in $(0, \bar{\eta})$ and $\Delta_t(f) > 0$ in $(\bar{\eta}, 3)$, which proves Item 1, by applying [2, Prop. 4.5]. The same argument applied to $\Delta_\rho(g)$ yields Item 2.

Concerning the sign of the roots, for $\eta \in (0, 3)$ the leading coefficient of f and g is negative and the constant coefficient is always positive, thus they have at least one positive real root. In particular, when there is a unique real root (in $(0, \bar{\eta}) \setminus \{\frac{12}{5}\}$), it is positive. Now assume $\eta \in (\bar{\eta}, 3) \cup \{\frac{12}{5}\}$: in this interval, the coefficients of polynomials $t \mapsto f(-t)$ and $\rho \mapsto g(-\rho)$ are all positive, thus by Descartes' rule of signs [2, Thm. 2.33], $f(-t)$ and $g(-\rho)$ do not have positive roots. In other words, the roots of f and g in $(\bar{\eta}, 3) \cup \{\frac{12}{5}\}$ are positive and non-zero because $f(0) = \eta^4 > 0$ and $g(0) = 27\eta^2 > 0$ for $\eta > 0$. \square

Next we prove our main result. All the computations in the proof of Theorem 4 are done with Macaulay2 [6].

Theorem 4 *Let $\bar{\eta} = \frac{135}{98} + \frac{19}{98}\sqrt{57}$ be the positive root of $49\eta^2 - 135\eta - 12$. The solutions to the polynomial system (11) correspond to one of the following cases (unless otherwise stated, $\lambda = 1$ and η is free):*

1. $(X, Y, \rho) = (0, 1, \frac{3}{12-4\eta})$
2. $(X, Y, \rho) = (\frac{12}{12-4\eta}, \frac{4\eta}{12-4\eta}, \frac{3}{12-4\eta})$
3. for $\eta \in (0, \bar{\eta})$, except solutions of Item 1 and Item 2, there is a unique real and positive solution $\rho = \rho(\eta)$ and one corresponding point O^*
4. for $\eta \in (\bar{\eta}, 3)$, except solutions of Item 1 and Item 2, there are three real and positive solutions $\rho = \rho_1(\eta), \rho_2(\eta), \rho_3(\eta)$ and three corresponding points O_1^*, O_2^* and O_3^*
5. $\lambda = 0$ or $\eta \leq 0$ or $\eta \geq 3$
6. complex solutions

Proof: First, we exclude the case $\lambda = 0$ (Item 5), substitute $\lambda = 1$ in (11) and let $I \subset \mathbb{Q}(\eta)[X, Y, \rho]$ be the ideal generated by these polynomials. Eliminating X, Y from I , one gets a principal ideal $I \cap \mathbb{Q}(\eta)[\rho] = \langle \eta(4\eta\rho - 12\rho + 3)^2g \rangle$, where

$$g = 1024(\eta - 3)\rho^3 + (-704\eta^2 + 1920\eta + 768)\rho^2 + \eta(196\eta^2 - 732\eta + 288)\rho + 27\eta^2 \quad (14)$$

Necessarily $\eta(4\eta\rho - 12\rho + 3)^2g = 0$. The case $\eta = 0$ is covered by Item 5.

Next we consider the case $4\eta\rho - 12\rho + 3 = 0$, that is $\rho = \frac{3}{12-4\eta}$. Let $J = I + \langle 4\eta\rho - 12\rho + 3 \rangle \cap \mathbb{Q}[X, Y, \rho]$. The radical of J has three prime components:

$$\sqrt{J} = \langle Y - 1, X \rangle \cap \langle Y - 4\rho + 1, X - 4\rho \rangle \cap \langle Y - 1, 3X + 8\rho - 6, 8\rho^2 - 6\rho + 3 \rangle$$

The first component yields the solution $(X, Y, \rho) = (0, 1, \frac{3}{12-4\eta})$ of Item 1. The second one gives $(X, Y, \rho) = (\frac{12}{12-4\eta}, \frac{4\eta}{12-4\eta}, \frac{3}{12-4\eta})$ (Item 2). Finally, the third component yields complex solutions: $(\eta, X, Y, \rho) = ((12\rho - 3)/4\rho, (6 - 8\rho)/3, 1, \rho)$ with $\rho = (3 \pm i\sqrt{15})/8$ (Item 6).

Now we turn to the case when ρ and η satisfy $g(\eta, \rho) = 0$. We add this equation to I and we eliminate η and ρ , yielding $K = I + \langle g \rangle \cap \mathbb{Q}[X, Y]$. The ideal K is generated by the two polynomials $g_1 = (X - Y - 1)q$ and $g_2 = (Y - 4)(Y^2 - Y + 4)q$ where:

$$q = 3X^3 + 3X^2Y + XY^2 - 7Y^3 - 9X^2 - 2XY + 11Y^2 + 9X - Y - 3.$$

The system $g_1 = 0, g_2 = 0$ is satisfied in the following cases. Either $(X, Y) = (5, 4)$, which gives, by backward substitution, the unique solution $(\eta, X, Y, \rho) = (\frac{12}{5}, 5, 4, \frac{5}{4})$ and $O^* = (0, 0, -4/\sqrt{5})$, which is a special case of Item 2. Or Y satisfies $Y^2 - Y + 4 = 0$, leading to complex solutions (Item 6). Or $X = Y + 1$ and $q = 0$, in which case after substitution one gets that Y satisfies $Y^5 - 5Y^4 + 8Y^3 - 16Y^2 = Y^2(Y - 4)(Y^2 - Y + 4) = 0$, which gives $Y = 0$ (yielding the solution $(\eta, X, Y, \rho) = (0, 1, 0, \rho)$, Item 5) or $Y = 4$ (again Item 2) or complex solutions.

Finally, we are left with the case when the pair (X, Y) satisfies the cubic bivariate equation $q = 0$. Adding to $I + \langle g, q \rangle$ the polynomials $X - (z - s)^2$ and $3Y - z^2 - \eta$ and $3s^2 + \eta - 3$ from (13) and eliminating variables X, Y, ρ and s , we get an ideal whose radical has prime components: $\langle z - 1, \eta \rangle \cap \langle z + 1, \eta \rangle \cap \langle 432(\eta - 3)z^6 + 108\eta(\eta - 2)z^4 - 9\eta^2(\eta + 1)z^2 + \eta^4 \rangle$. The first two imply $\eta = 0$ (Item 5) and the last one is the following cubic evaluated in $t = z^2$:

$$f(t) = 432(\eta - 3)t^3 + 108\eta(\eta - 2)t^2 - 9\eta^2(\eta + 1)t + \eta^4 \in \mathbb{Q}(\eta)[t] \quad (15)$$

By Lemma 2, f and g (resp. in (15) and in (14)) have exactly one real and positive root for $\eta \in (0, \bar{\eta})$ and exactly three real and positive roots for $\eta \in (\bar{\eta}, 3)$. Thus for $\eta \in (0, \bar{\eta})$ there is exactly one real positive solution for ρ and for this solution at most two solutions for the coordinate z of O^* : one of these correspond to a positive s ($s = \sqrt{(3 - \eta)/3}$) and is acceptable, the other one corresponds to the situation where the tetrahedron is reflected with respect to the plane $z = 0$, which is excluded by our choice in (12). This is the solution in Item 3. Similarly, when $\eta \in (\bar{\eta}, 3)$, by Lemma 2 there are exactly three real positive solutions for ρ . For each of these solutions, the corresponding value of z can be computed by the second equation of system (11), combined with (13). This gives the solutions of Item 4. \square

5.3 Geometric solutions

We first prove that, if the parameter η defines a solid tetrahedron, then the solutions of the first two items of Theorem 4 are geometrically-admissible but trivial.

Corollary 3 *For $\eta \in (0, 3)$, $R_T^2 = \frac{3}{12-4\eta}$ and the solution of Item 1 (resp. of Item 2) of Theorem 4 is the northern (resp. southern) vertex of $T = co(V)$.*

Proof: First we prove that $\frac{3}{12-4\eta} = R_T^2$: this is straightforward from Proposition 2 with $n = 3$, $D_0 = D_1 = D_2 = D_3 = \rho$, $d_{01} = d_{02} = d_{03} = 1$ and $d_{12} = d_{13} = d_{23} = \eta$, indeed, the determinant of the matrix in (3) is $4\eta^3\rho - 12\eta^2\rho + 3\eta^2$. Of course $(X, Y, Y, Y)_V = (0, 1, 1, 1)_V$ are the distance coordinates of v_0 . The second solution $(X, Y) = (\frac{12}{12-4\eta}, \frac{4\eta}{12-4\eta})$, for $\eta \in (0, 3)$, corresponds to the distance coordinates of a point v whose Cartesian coordinates, applying (13), are $v = (0, 0, -\frac{\eta}{\sqrt{9-3\eta}})$: this point thus satisfies $\|v_0 - w\|^2 = \|w - v\|^2 = \frac{3}{12-4\eta} = \rho$, where w is the circumcenter of T . In other words, v is the opposite point of v_0 in the circumsphere of T . \square

The rest of the section investigates configurations of spheres supporting the vertices of a triangular pyramid T , with radius equal to R_T .

Theorem 5 A solution O^* to system (9) with $\rho = R_T^2$ is either equidistant to the base vertices ($Y = Z = W$) or coplanar with the base vertices (geometrically non-admissible) or lies on the sphere circumscribing T (trivial).

Proof: Recall from the proof of Corollary 3, that $R_T^2 = \frac{3}{12-4\eta}$ for $\eta \in (0, 3)$. Consider system (9) with $d_{01} = d_{02} = d_{03} = 1$ and $d_{12} = d_{13} = d_{23} = \eta$. Let I be the ideal generated by equations in (9). Adding $4\eta\rho - 12\rho + 3$ to I and eliminating η, ρ , one gets the ideal $(I + \langle 4\eta\rho - 12\rho + 3 \rangle) \cap \mathbb{Q}[X, Y, Z, W]$ generated by the reducible polynomial

$$(Y^2 - YZ + Z^2 - YW - ZW + W^2)(3X^2 + Y^2 - 2YZ + Z^2 - 2YW - 2ZW + W^2 - 6X + 3).$$

The zero locus of this polynomial defines the Zariski-closure of the set of solutions (X, Y, Z, W) to system (9) with $\rho = R_T^2$. Using (12), the first factor yields $3\eta(x^2 + y^2)$, with real solutions $x = y = 0$ (the z -axis). The second factor yields:

$$-8z \left(\frac{1}{2} \sqrt{9-3\eta} \left(x^2 + y^2 + z^2 - \frac{\eta}{3} \right) + \left(\eta - \frac{3}{2} \right) z \right)$$

whose real solutions are union of the plane $z = 0$ and the sphere circumscribing T . Thus a real solution O^* must be either equidistant to base vertices, or trivial or coplanar with v_1, v_2, v_3 : this last case is geometrically non-admissible since there is no sphere containing four coplanar vectors in \mathbb{R}^3 . \square

Example 2 Let I be the ideal given in Theorem 5. The eliminant $I \cap \mathbb{Q}(\eta)[\rho]$ is generated by $\eta^4(4\eta\rho - 12\rho + 3)^2gh$, where g is given in (14) and

$$\begin{aligned} h = & (16384\eta^3 - 81920\eta^2 + 114688\eta - 49152)\rho^5 + (-7168\eta^4 + 24576\eta^3 + 1024\eta^2 - \\ & - 43008\eta + 36864)\rho^4 + (576\eta^5 + 704\eta^4 - 8512\eta^3 + 7232\eta^2 - 7680\eta - 9216)\rho^3 + \\ & (-880\eta^4 + 2080\eta^3 + 208\eta^2 + 4736\eta + 768)\rho^2 + (140\eta^3 - 500\eta^2 - 480\eta)\rho + 75\eta^2. \end{aligned} \quad (16)$$

Excluding the case $\eta = 0$ and the case where $4\eta\rho - 12\rho + 3$ vanishes identically (treated in Theorem 5) one is left with $h = 0$ or $g = 0$. A direct computation yields

$$\begin{aligned} h \left(\frac{3}{12-4\eta} \right) &= \frac{-192\eta^3(\eta-2)(4\eta^2-18\eta+21)}{(\eta-3)^4} \\ g \left(\frac{3}{12-4\eta} \right) &= -\frac{12\eta(5\eta-12)(2\eta^2-9\eta+12)}{(\eta-3)^2}. \end{aligned}$$

Concerning h , the previous expression vanishes in $(0, 3)$ if and only if $\eta = 2$. Substituting $\eta = 2$ and $\rho = \frac{3}{4}$ in (9), one gets equations

$$X - Y - Z - W + 3 = Y^2 + YZ + Z^2 + YW + ZW + W^2 - 6Y - 6Z - 6W + 12 = 0.$$

Using (12), the previous constraints yield equations $f_1(x, y, z) = f_2(x, y, z) = 0$ with

$$\begin{aligned} f_1 &:= -2x^2 - 2y^2 - 2z^2 - \frac{2}{\sqrt{3}}z + \frac{4}{3} \\ f_2 &:= 12x^2y^2 + 12x^2z^2 + 12y^2z^2 + 6z^4 - 10z^2 - 8y^2 - 8x^2 + 6x^4 + 6y^4 + \frac{8}{3}. \end{aligned}$$

Now remark that $f_1 = 0$ is the equation of the sphere S containing V , namely the circumscribed sphere of T and further that $f_2 = (-3x^2 - 3y^2 - 3z^2 + 2 + z\sqrt{3})f_1$ is the equation of the double sphere of radius $\sqrt{3}/4$ containing v_1, v_2, v_3 . Thus $\langle f_1, f_2 \rangle = \langle f_1 \rangle$ and hence $\mathcal{Z}_{\mathbb{R}}(f_1, f_2) = \mathcal{Z}_{\mathbb{R}}(f_1) = S$. In

other words, a point $O^* = (X, Y, Z, W)_V$ is a solution with $\eta = 2$ and $\rho = \frac{3}{4} = R_T^2$ if and only if O^* is in the sphere circumscribing V .

The expression $g(\frac{3}{12-4\eta})$ vanishes for $\eta \in (0, 3)$ if and only if $\eta = \frac{12}{5}$. Substituting this value (and $\rho = \frac{5}{4}$) gives equations $9X - 5Y - 5Z - 5W + 15 = 13Y^2 - YZ + 13Z^2 - YW - ZW + 13W^2 - 60Y - 60Z - 60W + 144 = 0$ and similarly to the case $\eta = 2$, substituting (12) yields the circumsphere of V .

Remark 1 Maehara and Tokushige defined in [14] a sphere system in \mathbb{R}^n as a collection of $n + 2$ spheres of the same radius, with empty intersection, and such that each $n + 1$ spheres intersect in a unique point. Johnson's Theorem shows that for $n = 2$ circle systems exist where all radii are equal. In [14] it is proved that for every $2 \leq n \neq 3$ there exists a unit sphere system (all spheres with radius 1) and it is conjectured that this is not the case for $n = 3$.

As for the case of Johnson's Theorem, revisited in Section 3 fixing a priori the three mutual intersections of two circles out of three, in \mathbb{R}^3 our method fixes the set V of four vertices of a tetrahedron $T = \text{co}(V)$ and looks for a fifth point O^* defined as the intersection of four spheres satisfying conditions (i) and (iii) of the Introduction. When the radii of the four spheres are all equal to the circumradius of T , this situation determines five spheres of equal radius in \mathbb{R}^3 and hence could in principle lead to sphere systems validating the Maehara-Tokushige Conjecture, or to counterexamples.

The results in Theorem 5 and Example 2 show that there are no counterexamples to Maehara-Tokushige's conjecture such that four of the five intersections of the five spheres (taken four by four) are the vertices of a triangular pyramid T . Indeed, assume there is such a counterexample. In particular, O^* does not belong to the circumsphere of T . By Example 2 one can exclude the special cases $\eta = \frac{12}{5}$ and $\eta = 2$ with only trivial solutions with $\rho = R_T^2$. For a general η , Theorem 5 ensures that the solution O^* must be cyclic (that is with $Y = Z = W$). We deduce by Lemma 2 and Theorem 4 that the polynomial g must vanish, which is the case if and only if $\eta = \frac{12}{5}$, contradiction.

5.4 Examples

We discuss several examples of tetrahedra showing each of the solutions given in Theorem 4, Items 1 and 2 (which we call trivial) and Items 3 and 4 (non-trivial solutions). Indeed, solutions of Items 1-2 are expected as special points of the circumsphere of T (cf. Corollary 3) that correspond to the case when the four spheres coincide with the circumsphere.

We start illustrating Theorem 4 with the case of regular tetrahedron for which there is only one non-trivial solution O^* .

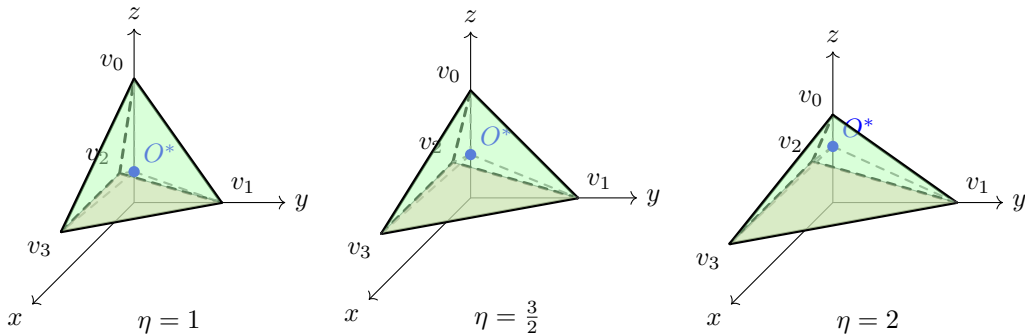


Figure 5: Regular tetrahedron of Example 3 (on the left); hemispherical pyramid of Example 4 (center); trirectangular pyramid of Example 5 (on the right).

Example 3 ($\eta = 1$, regular tetrahedron) Fix $\eta = 1$. The ideal $I \subset \mathbb{Q}[X, Y, \rho]$ generated by polynomials in (11) is zero-dimensional of degree 6. Eliminating X and Y from I , we get one equation for ρ :

$$\begin{aligned} 131072\rho^5 - 225280\rho^4 + 129536\rho^3 - 31488\rho^2 + 3528\rho - 243 &= \\ &= (8\rho - 3)^2(32\rho - 27)(64\rho^2 - 8\rho + 1) = 0 \end{aligned}$$

cf. also Example 1. There are two real solutions, $\rho = \frac{3}{8} = R_T^2$ and $\rho = \frac{27}{32}$. Replacing $\rho = \frac{3}{8}$, we get the equations $2Y^2 - 3Y + 1 = X + 3Y - 3 = 0$, thus either $(X, Y) = (0, 1)$, giving v_0 , or $(X, Y) = (\frac{3}{2}, \frac{1}{2})$, yielding the point $(0, 0, -1/\sqrt{6})$, the trivial solutions of Items 1-2. The second one $\rho = \frac{27}{32}$, the unique real root of g , yields $X = Y = \frac{3}{8}$, Cartesian coordinates $O^* = (0, 0, 1/\sqrt{24})$, the circumcenter of T (cf. Figure 5, left). ■

The next two examples show again pyramids with a unique real non-trivial solution O^* . The first one is the tetrahedron inscribed in a hemisphere.

Example 4 ($\eta = \frac{3}{2}$, hemispherical pyramid) For $\eta = \frac{3}{2}$, the circumcenter is coplanar with v_1, v_2, v_3 and the pyramid is inscribed in a hemisphere. The radius must satisfy $(2\rho - 1)^2(2048\rho^3 - 2752\rho^2 + 738\rho - 81) = 0$, thus either $\rho = R_T^2 = \frac{1}{2}$ (giving the trivial solutions $(X, Y) = (0, 1), (2, 1)$) or $\rho \approx 1.0316$, the unique real root of the cubic factor. Forcing $2048\rho^3 - 2752\rho^2 + 738\rho - 81 = 0$, that is adding this polynomial to the ideal, one gets equations $64X^3 - 120X^2 + 121X - 18 = 64Y^3 - 88Y^2 + 45Y - 9 = 0$, giving z satisfying $16z^3 + 4\sqrt{2}z^2 + 2z - \sqrt{2} = 0$, that is $z \approx 0.2865$ (Figure 5, center). ■

Recall that for a general tetrahedron, the orthocenter is not well-defined, contrary e.g. to circumcenter, incenter or baricenter. Indeed the four altitudes of a tetrahedron do not need to be concurrent and the orthocenter is usually replaced for tetrahedra with the so-called Monge point, see for instance [7]. A tetrahedron where the four altitudes are concurrent is called orthocentric and this is the case of triangular pyramids, where the orthocenter exists and coincides with its Monge point. As the next example shows, O^* differs in general from the orthocenter, and hence from the Monge point, of a triangular pyramid.

Example 5 ($\eta = 2$, trirectangular pyramid, continued) For $\eta = 2$, the edges of T containing v_0 are two-by-two orthogonal: the pyramid is called trirectangular. The Monge point (and orthocenter) of T is v_0 . The unique real root of $g = -1024\rho^3 + 1792\rho^2 - 784\rho + 108$ is $\rho \approx 1.1746$. The distance coordinates $(X, Y, Y, Y)_V$ of O^* must satisfy $36X^3 - 60X^2 + 73X - 3 = 0$ and $12Y^3 - 24Y^2 + 19Y - 6 = 0$, giving $O^* \approx (0, 0, 0.371)$. In particular O^* is not the Monge point of T . (See also Example 2 for a previous discussion on the trirectangular pyramid.)

The next three examples are special inasmuch as the parameter η is a root of either the discriminant of f or g (cf. Lemma 2). This represents some bifurcations of the roots of the two polynomials and, geometrically, to special configurations of spheres.

Example 6 ($\eta = \frac{12}{5}$, $\Delta_\rho(g) = 0$ and $\Delta_\rho(f) < 0$) The trivial solutions are $(X, Y) = (0, 1)$ and $(5, 4)$, with $\rho = R_T^2 = \frac{5}{4}$. The polynomial g in (14) factors

$$-\frac{3072}{5}\rho^3 + \frac{33024}{25}\rho^2 - \frac{101952}{125}\rho + \frac{3888}{25} = \frac{48}{125}(5 - 4\rho)(-9 + 20\rho)^2$$

The first factor gives again $\rho = R_T^2$ and the solutions $O^* = v_0 = (0, 1, 1, 1)_V$. However this solution corresponds to a non-trivial configuration of spheres: three spheres of radius equal to the circumradius $\sqrt{5}/4$, each one containing v_0 and two of the base vertices, but not the third one (in other words, the symmetric sphere of the circumsphere with respect to a lateral face); and the fourth sphere is the circumsphere. The four spheres intersect in $O^* = v_0$. The double root of g is $\rho = \frac{9}{20}$ yielding only complex solutions: indeed $\Delta_\rho(f) < 0$ for $\eta = \frac{12}{5}$ and by elimination (X, Y) must satisfy $25X^2 - 45X + 64 = 25Y^2 - 45Y + 36 = 0$, that have complex solutions. ■

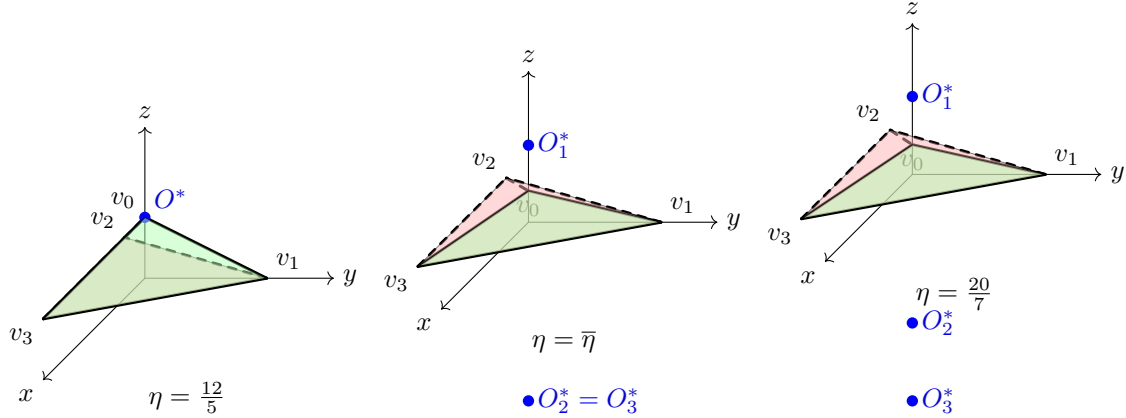


Figure 6: Tetrahedra of Example 6 (left), Example 8 (center) and Example 7 (right).

In the next example ($\eta = \frac{20}{7}$) the polynomial g has again a double root but yielding two distinct (real) configurations of spheres.

Example 7 ($\eta = \frac{20}{7}$, $\Delta_\rho(g) = 0$ and $\Delta_\rho(f) > 0$) *The circumradius is $\sqrt{21}/2$ and the trivial solutions are $v_0 = (0, 0, 1/\sqrt{21})$ (for $X = 0, Y = 1$) and the south pole $(0, 0, -20/\sqrt{21})$, (for $X = 21, Y = 20$). The polynomial g has a simple root $\rho_2 = \frac{27}{28}$, yielding $(X, Y) = (\frac{12}{7}, \frac{15}{7})$ and $O_2^* = (0, 0, -5/\sqrt{21})$; and a double root $\rho_1 = \rho_3 = \frac{5}{4}$, yielding the two solutions $O_3^* = (0, 0, \frac{-5\sqrt{21}-21\sqrt{5}}{42})$ (for $X = \frac{11+\sqrt{105}}{6}, Y = \frac{105+5\sqrt{105}}{42}$) and $O_1^* = (0, 0, \frac{-5\sqrt{21}+21\sqrt{5}}{42})$ (for $X = \frac{11-\sqrt{105}}{6}, Y = \frac{105-5\sqrt{105}}{42}$).* ■

The parameter η of the following tetrahedron (that we call discriminant tetrahedron) is a zero of both the discriminants $\Delta_\rho(f)$ and $\Delta_\rho(g)$, thus one of the configurations of spheres has multiplicity two.

Example 8 ($\eta = \frac{135}{98} + \frac{19}{98}\sqrt{57}$, **discriminant tetrahedron**) *For $\eta = \bar{\eta}$, $\Delta_\rho(f) = \Delta_\rho(g) = 0$ and f and g both have two real positive roots, one of which is double. The discriminant tetrahedron is pictured in Figure 6 (center). The roots of g are $\rho_1 = \frac{7911+1035\sqrt{57}}{12544} \approx 1.2536$, giving radius $\sqrt{\rho_1} \approx 1.1196$ and $\rho_2 = \rho_3 = \frac{9+\sqrt{57}}{16} \approx 1.0344$, giving radius $\sqrt{\rho_2} \approx 1.0170$. The circumradius is $R_T \approx 2.1739$. The non-trivial solutions to system (11) are given by $O_1^* \approx (0, 0, 0.5660)$ and $O_2^* = O_3^* \approx (0, 0, -1.3124)$.* ■

By Lemma 2, a tetrahedron with $\eta \in (\bar{\eta}, 3)$, admits three distinct configurations of spheres satisfying properties (i) and (iii). We end this section with one of such examples.

Example 9 ($\eta = \frac{29}{10}$) *For $\eta = \frac{29}{10}$, the squared radius ρ satisfies the quintic equation $\frac{29}{10}(3 - \frac{2}{5}\rho)^2 g = 0$, with $g = -\frac{512}{5}\rho^3 + \frac{10384}{25}\rho^2 - \frac{135169}{250}\rho + \frac{22707}{100}$. The factor $3 - \frac{2}{5}\rho$ gives the circumradius $\sqrt{\rho} = \sqrt{15}/2 \approx 2.7386$ and the two trivial solutions. The factor g has three distinct real roots, $\rho_1 \approx 1.2370$, $\rho_2 \approx 0.9687$ and $\rho_3 \approx 1.8506$. Adding g to the ideal, we are left with a zero-dimensional ideal of degree three, generated by equations $14400X^3 - 109320X^2 + 129649X - 18750 = 0$ and $4800Y^3 - 45240Y^2 + 106807Y - 73167 = 0$. This gives three real solutions in z , approximately 0.59227 (O_1^* , with radius $\sqrt{\rho_1}$), -0.93909 (O_2^* , with radius $\sqrt{\rho_2}$) and -2.3005 (O_3^* , with radius $\sqrt{\rho_3}$).* ■

6 Connections with R-bodies

This last section aims at making a connection of the results in Section 5 with the already mentioned theory of R -bodies. All along the section R denotes a fixed positive real number.

A general open ball of radius $r > 0$ is denoted by $B(r)$ and for the special case $r = R$ it is simply denoted by B . An R -body is the complement of a non empty union of open balls B of radius R in the Euclidean space \mathbb{R}^d , see [11]. These sets have been introduced and studied by Perkal [15], used in Walther [17] and in Cuevas, Fraiman, Pateiro-López [4]. For a closed set E , its R -hulloid $co_R(E)$, namely the minimal R -body containing E , was investigated recently in [12]. In case E is the set V of vertices of a plane triangle in \mathbb{R}^2 or the set of vertices of a tetrahedron T in \mathbb{R}^3 , a complete description of $co_r(V)$ is possible for all $r > 0$.

It is straightforward to see, for every simplex $T = co(V) \subset \mathbb{R}^n$ given by the convex hull of a set of $n + 1$ affinely independent vectors $V = \{v_0, v_1, \dots, v_n\} \subset \mathbb{R}^n$, that for $r \leq R_T$ and for every point $x \in int(T)$, there exists a ball $B(r)$ such that $x \in B(r)$ and $V \cap B(r) = \emptyset$ (such ball is included in the ball $B(R_T)$ circumscribing T). This fact implies that $co_r(V) = V$ for every $r \in (0, R_T]$ and hence the non-trivial case to be analyzed is $r > R_T$.

A preliminary result in the planar case $n = 2$ is in [11]:

Proposition 7 ([11, Thm. 4.2]) *Let $V = \{v_0, v_1, v_2\} \subset \mathbb{R}^2$ be the set of vertices of a triangle $T = co(V)$ with circumradius R_T . If $r > R_T$, then*

$$co_r(V) = V \cup \tilde{T}$$

where $\tilde{T} \subset T$ is the curvilinear triangle bordered by three arcs of circles of radius r , each one through two vertices of T . If T is right-angled or obtuse-angled then the vertex of the major angle of T is also a vertex of \tilde{T} .

The result in [11, Thm. 4.2] is a direct consequence of Johnson's Three Circles Theorem (Theorem 1 in the Introduction) and implies that $co_r(V)$ has non-empty interior for every $r > R_T$. Moreover one can show that for $r \rightarrow R_T^+$, the r -hulloid $co_r(V)$ tends to $V \cup \{O^*\}$, where O^* is the orthocenter of T , that is, $co_r(V)$ has a discontinuity for $r = R_T$ (indeed as mentioned one always has $co_{R_T}(V) = V$).

Determining the shape of $co_r(V)$ for a tetrahedron $T = co(V) \subset \mathbb{R}^3$ is a more subtle problem. A recent result given in [13] completes the description of r -huloids of the set of vertices of a general tetrahedron:

Proposition 8 ([13, Thm. 5.12, Cor. 5.13]) *Let $V = \{v_0, v_1, v_2, v_3\}$ be the set of vertices of a tetrahedron $T = co(V)$ in \mathbb{R}^3 with circumradius R_T . Under the assumption:*

$$\exists r > R_T : co_r(V) = V \tag{17}$$

there exists $R^* > R_T$ and four spheres $S_j := \partial B_j(R^*)$, $0 \leq j \leq 3$, such that

- (i) S_j contains the vertices v_i , for all $i \neq j$;
- (ii) $v_j \notin B_j(R^*)$;
- (iii) $\bigcap_{j=0}^3 S_j = \{O^*\}$;
- (iv) $O^* \in int(T)$.

Moreover

$$co_r(V) = V \quad \text{for } r < R^*; \tag{18}$$

$$co_r(V) = V \cup \{O^*\} \quad \text{for } r = R^*; \tag{19}$$

$$co_r(V) = V \cup \tilde{\Gamma} \quad \text{for } R^* < r, \tag{20}$$

where

$$\tilde{\Gamma} = \text{int}(T) \setminus \bigcup_{j=0}^3 B_j(r) \quad (21)$$

is a non-empty connected set, and $\partial\tilde{\Gamma}$ is the union of connected subsets of $\partial B_j(r)$, $0 \leq j \leq 3$, the sphere of radius r containing the vertices in $V \setminus \{v_j\}$.

In Proposition 8 the point O^* and the critical radius R^* are uniquely determined, and O^* is by construction the circumcenter of the tetrahedron W with vertices the centers of the spheres ∂B_j and circumradius $R_W = R^*$. Each ball $B_j(r)$ in the previous proposition is r -supporting V at each vertex in $V \setminus \{v_j\}$, according to [13, Def. 2.4] which we recall here: a ball $B(r)$ is r -supporting a closed set $E \subset \mathbb{R}^d$ at $x \in \partial E$ if $x \in \partial B(r)$ and $E \subset (B(r))^c$.

Definition 2 A sphere configuration (S_0, S_1, S_2, S_3) satisfying properties (i), (ii), (iii) and (iv) in Proposition 8 is called an R -body configuration for V . The spheres S_j are boundaries of open balls R -supporting V .

Assumption (17) guarantees the existence of R -body configurations for the set of vertices of any tetrahedron, but looks restrictive for such general class. The results of this paper allow to prove that this assumption holds for the subclass of triangular pyramids.

Lemma 3 For $0 < \eta < \frac{12}{5}$, then (17) is satisfied for the triangular pyramid T with parameter η .

Proof: Put $\rho = r^2$. We prove below that for $0 < \eta < \frac{12}{5}$, the polynomial g in (14) does not vanish in $\rho \in (0, R_T^2]$, where $R_T^2 = \frac{3}{12-4\eta}$ is the squared circumradius of T . Since the equality $co_r(V) = V$ is always satisfied for $r \in (0, R_T]$, we conclude that by continuity there must exist $r > R_T$ with $co_r(V) = V$. Remark that the cases $\rho = R_T^2$ and $\eta = \frac{12}{5}$ have already been treated in Example 2.

Consider $g \in \mathbb{Q}(\eta)[\rho]$ as a univariate polynomial in ρ with coefficients depending on η . Its Sturm sequence $s = (g_0, g_1, g_2, g_3) \in \mathbb{Q}(\eta)[\rho]^4$ is defined as

$$g_0 := g \quad g_1 := \frac{dg}{d\rho} \quad g_2 := -\text{rem}_\rho(g_0, g_1) \quad g_3 := -\text{rem}_\rho(g_1, g_2) \quad (22)$$

where $\text{rem}_\rho(a, b)$ is the remainder of the division of a by b as polynomials in ρ . Denote by $s[0] = (v_0, v_1, v_2, v_3) \in \mathbb{Q}(\eta)$ and by $s[\frac{3}{12-4\eta}] = (w_0, w_1, w_2, w_3) \in \mathbb{Q}(\eta)$ the evaluation of s at $\rho = 0$ and at $\rho = \frac{3}{12-4\eta}$, respectively ($s, s[0], s[\frac{3}{12-4\eta}]$ are given explicitly in Appendix B). Remark that $v_3 = w_3 = g_3 \in \mathbb{Q}(\eta)$ and that it is always positive in $\eta \in (0, \frac{12}{5})$. Moreover $v_0 = 27\eta^2$ and w_0 is a positive multiple of $\eta(12 - 5\eta)$, thus positive in $(0, \frac{12}{5})$.

Now define v_1^*, v_2^* the unique roots of v_1, v_2 in $(0, \frac{12}{5})$: one has $v_1 \geq 0$ in $(0, v_1^*)$, $v_1 \leq 0$ in $(v_1^*, \frac{12}{5})$, $v_2 \leq 0$ in $(0, v_2^*)$ and $v_2 \geq 0$ in $(v_2^*, \frac{12}{5})$. We deduce that for every $\eta \in (0, \frac{12}{5})$, there are exactly two sign variations in $s[0]$. Similarly, let $w_1^* < w_1^{**} \in (0, \frac{12}{5})$ be the two real roots of the numerator of w_1 . Then $w_1 \leq 0$ in $(0, w_1^*) \cup (w_1^{**}, \frac{12}{5})$ and $w_1 \geq 0$ in (w_1^*, w_1^{**}) , whereas the rational function w_2 is always negative in $(0, \frac{12}{5})$. Thus for every $\eta \in (0, \frac{12}{5})$, there are exactly two sign variations in $s[\frac{3}{12-4\eta}]$. We conclude by Sturm's Theorem [2, Thm. 2.62] that for every $\eta \in (0, \frac{12}{5})$, the polynomial g does not vanish in $(0, \frac{3}{12-4\eta})$, as claimed. \square

With a similar technique, one can prove the following result.

Lemma 4 For $\frac{12}{5} \leq \eta < 3$, then every solution O^* to system (11) for the triangular pyramid T with parameter η , is such that $O^* \notin \text{int}(T)$.

Proof: Recall that according to the Cartesian coordinates in (12), the north vertex of the tetrahedron has coordinates $v_0 = (0, 0, \sqrt{(3-\eta)/3})$, and the base vertices lie in the plane $z = 0$. Let $O^* = (0, 0, z^*)$ in the same Cartesian system. Then $t = (z^*)^2$ is one of the roots of f in (15). For $\eta = \frac{12}{5}$, the unique solution O^* coincides with v_0 , thus the statement is true. We show below that for $\eta \in (\frac{12}{5}, 3)$, the polynomial f does not vanish for $t \in (0, \frac{3-\eta}{3})$, in other words, that $z^* \notin (0, \sqrt{(3-\eta)/3})$, namely $O^* \notin \text{int}(T)$.

The Sturm sequence s of $f \in \mathbb{Q}(\eta)[t]$ with respect to t is defined as the one in (22), and is given in Appendix C together with its evaluations $s[0] = (v_0, v_1, v_2, v_3)$ and $s[\frac{3-\eta}{3}] = (w_0, w_1, w_2, w_3)$ at $t = 0$ and $t = \frac{3-\eta}{3}$, respectively. The sign pattern of $s[0]$ and $s[\frac{3-\eta}{3}]$ are respectively:

$$s[0] : \begin{cases} [+,-,-,+ & \text{for } \eta \in (\frac{12}{5}, v_2^*) \\ [+,-,+,+ & \text{for } \eta \in (v_2^*, \bar{\eta}) \\ [+,-,+,- & \text{for } \eta \in (\bar{\eta}, 3) \end{cases} \quad s\left[\frac{3-\eta}{3}\right] : \begin{cases} [+,-,-,+ & \text{for } \eta \in (\frac{12}{5}, w_2^*) \\ [+,-,+,+ & \text{for } \eta \in (w_2^*, \bar{\eta}) \\ [+,-,+,- & \text{for } \eta \in (\bar{\eta}, 3) \end{cases}$$

where $v_2^* \approx 2.74$ and $w_2^* \approx 2.71$ are the unique roots of v_2, w_2 in $[\frac{12}{5}, 3)$. Again applying Sturm's Theorem [2, Thm. 2.62], we conclude that for every $\eta \in (\frac{12}{5}, 3)$, the polynomial f does not have a root in $(0, \frac{3-\eta}{3})$. \square

We conclude with the the following result, direct consequence of Theorem 4 and Lemmas 3 and 4, summarizing the connection between Section 5 and the theory of R -bodies, for the class of triangular pyramids.

Theorem 6 *Let $T = \text{co}(V)$ be the triangular pyramid with vertices V and parameter η .*

1. *For $0 < \eta < \frac{12}{5}$, let ρ be the unique real root of the polynomial g in (14) and let $O^* = (X, Y, Y, Y)_V$ be the unique solution of system (11) related to ρ . Let $R^* = \sqrt{\rho}$. Then*

$$R^* > R_T \\ \text{co}_{R^*}(V) = V \cup \{O^*\}.$$

and $O^ \in \text{int}(T)$.*

2. *For $\frac{12}{5} \leq \eta < 3$, the solutions (R^*, O^*) are geometrically-admissible but do not yield R -body configurations.*

Proof: By Lemma 3, for $\eta \in (0, \frac{12}{5})$, the Assumption in (17) is satisfied. Thus by Proposition 8 there exists an R -body configuration for T for some O^* and for some $R^* > R_T$. Remark that O^* must belong to the symmetry axis of the base face, thus the unique solution $\rho = (R^*)^2, O^* = (X, Y, Y, Y)$ to system (11) (that exists and is unique by Theorem 4) satisfies the claim, with ρ the unique real root of g . Moreover O^* belongs to the segment joining the origin and the apex v_0 , in particular, $O^* \in \text{int}(T)$.

For $\frac{12}{5} \leq \eta < 3$, by Lemma 4, the solutions (R^*, O^*) given by Theorem 4 are such that $O^* \notin \text{int}(T)$, thus the corresponding configuration is not R -body in the sense of Definition 2. \square

Acknowledgments. This work has been partially supported by INDAM-GNAMPA (2022). The second author is supported by the ANR Project ANR-21-CE48-0006-01 ‘‘HYPERSPACE’’.

References

- [1] A.M. Balk, M.B. Balk. The enigma of the triangular pyramid. J. Geom. 62, 13–25 (1998). 3

- [2] S. Basu, R. Pollack, M.F. Roy. Algorithms in real algebraic geometry. Vol.10 of Algorithms and Computing Mathematics, Springer-Verlag II edition (2006). 4, 13, 20, 21
- [3] D. Cox, J. Little, D. O’Shea. Ideals, varieties and algorithms: an introduction to computational algebraic geometry and commutative algebra. Springer Science & Business Media, 2013. 3, 4
- [4] A. Cuevas, R. Fraiman, B. Pateiro-López. On statistical properties of sets fulfilling rolling-type conditions. Adv. App. Probab. (S.G.S.A.) 2012; 44: 311-329. 19
- [5] C. D’Andrea, M. Sombra. The Cayley-Menger determinant is irreducible for $n \geq 3$. Siberian Mathematical Journal 46.1 (2005): 71-76.
- [6] D.R. Grayson, M.E. Stillman. MACAULAY2, a software system for research in algebraic geometry. Available at <http://www2.macaulay2.com> 4, 7, 13
- [7] N. Hadjisavvas, J-B. Hiriart-Urruty, and P-J. Laurent. A Characterization by Optimization of the Monge Point of a Tetrahedron. Journal of Optimization Theory and Applications 171.3 (2016): 856-864. 17
- [8] T.F. Havel. Some examples of the use of distances as coordinates for Euclidean geometry. Journal of Symbolic Computation 11.5-6 (1991): 579-593. 4
- [9] R.A. Johnson. A Circle Theorem . Amer. Math. Monthly 23(1916), 161-162. 6
- [10] R.A. Johnson. Advanced Euclidean Geometry. Dover Publication (1960). 1, 2, 6
- [11] M. Longinetti, P. Manselli, A. Venturi. On the complements of union of open balls of fixed radius in the Euclidean space. Le Matematiche 1:3-22 (2023). 1, 2, 19
- [12] M. Longinetti, P. Manselli, A. Venturi. On the complements of union of open balls and related set classes. To appear. 19
- [13] M. Longinetti, S. Naldi, A. Venturi. R-hulloid of the vertices of a tetrahedron. To appear. 19, 20
- [14] H. Maehara, N. Tokushige. On a special arrangement of spheres. Ryukyu Math. J 19 (2006): 15-24. 2, 16
- [15] J. Perkal. Sur les ensembles ε -convexes. Colloq. Math 1956, Fasc 1, 1-10. 19
- [16] D.M.Y. Sommerville. An Introduction to the Geometry of n Dimensions. New York: Dover Publications (1958). 4
- [17] G. Walther. On a Generalization of Blaschke’s Rolling Theorem and the Smoothing of Surfaces. Math. Meth. Appl. Sci. 1999; 22: 301-316. 19
- [18] L. Yang, and Z. Zeng. An open problem on metric invariants of tetrahedra. Proceedings of the 2005 Intern. Symp. on Symbolic and Algebraic Computation (ISSAC), 2005. 4

A Prime decomposition for Theorem 3

The generators of \mathcal{P}_0 in the proof of Theorem 3 are

$$\begin{aligned}
q_1 &= BY^2 - BYZ - CYZ + CZ^2 - BY - XY - CZ - XZ + 2YZ + X \\
q_2 &= BXY - CXY - BXZ + CXZ + BC - X^2 - BY - CZ + YZ + X \\
q_3 &= BCY - CXY - C^2Z + BXZ - BYZ + CZ^2 - BC + CX - XZ + YZ \\
q_4 &= B^2Y - CXY - BCZ + BXZ - BYZ + CZ^2 + BC - BX - BY - CZ - XZ + YZ + X \\
q_5 &= B^2XZ - 2BCXZ + C^2XZ - B^2C + 2BCX - CX^2 + 2BCZ - 2BXZ - CZ^2 + XZ \\
q_6 &= CXY^2 - 2CXYZ + CXZ^2 - 2CXY - C^2Z - X^2Z + 2CYZ + 2XYZ - Y^2Z + CX \\
q_7 &= C^2XY - CX^2Y + BCXZ - 2C^2XZ + BX^2Z - CXYZ - BXZ^2 + 2CXZ^2 - BC^2 - \\
&\quad - BCX + 2CX^2 + CXY + BCZ + 2C^2Z - BXZ - 2X^2Z - CYZ + XYZ - 2CZ^2 + YZ^2 + \\
&\quad + BC - 2CX + 2XZ - YZ.
\end{aligned}$$

The further components are

$$\begin{aligned}
\mathcal{P}_1 &= \langle B - C + Y - Z, A - C + X - Z, C^3 - 2C^2X + CX^2 - 2C^2Y + 2CXY + \\
&\quad + CY^2 + C^2Z - 2CXZ + X^2Z - 2CYZ - 2XYZ + Y^2Z \rangle \\
\mathcal{P}_2 &= \langle Z, C, A - B + X - Y \rangle \\
\mathcal{P}_3 &= \langle Y, B, A - C + X - Z \rangle \\
\mathcal{P}_4 &= \langle X, A, B - C + Y - Z \rangle.
\end{aligned}$$

B Sturm sequence for Lemma 3

The Sturm sequence of the polynomial g in (14) is:

$$\begin{aligned}
g_0 &= g \\
g_1 &= (3072\eta - 9216)\rho^2 + (-1408\eta^2 + 3840\eta + 1536)\rho + 196\eta^3 - 732\eta^2 + 288\eta \\
g_2 &= \frac{(832\eta^4 - 10560\eta^3 + 39264\eta^2 - 43776\eta - 4608)\rho + 539\eta^5 - 3483\eta^4 + 6666\eta^3 - 2880\eta^2 - 864\eta}{3(3 - \eta)} \\
g_3 &= -\frac{27(5\eta - 12)^2(49\eta^2 - 135\eta - 12)(7\eta - 20)^2\eta^3(\eta - 3)^2}{(26\eta^4 - 330\eta^3 + 1227\eta^2 - 1368\eta - 144)^2}.
\end{aligned}$$

Its values at $\rho = 0$ and $\rho = \frac{3}{12-4\eta}$ are respectively

$$\begin{aligned}
s[0] &= \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 27\eta^2, \\ 4\eta(49\eta^2 - 183\eta + 72), \\ \frac{\eta(539\eta^4 - 3483\eta^3 + 6666\eta^2 - 2880\eta - 864)}{36(3-\eta)}, \\ g_3 \end{bmatrix} \\
s\left[\frac{3}{12-4\eta}\right] &= \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} \frac{12\eta(12-5\eta)(2\eta^2-9\eta+12)}{(\eta-3)^2}, \\ 4(49\eta^4 - 330\eta^3 + 885\eta^2 - 936\eta + 144), \\ \frac{\eta-3}{36(\eta-3)^2}, \\ -\frac{539\eta^6 - 5100\eta^5 + 16491\eta^4 - 14958\eta^3 - 21672\eta^2 + 35424\eta + 3456}{36(\eta-3)^2} \end{bmatrix}
\end{aligned}$$

C Sturm sequence for Lemma 4

The Sturm sequence of the polynomial f in (15) is:

$$\begin{aligned} f_0 &= f \\ f_1 &= (1296\eta - 3888)t^2 + (216\eta^2 - 432\eta)t - 9\eta^2(\eta + 1) \\ f_2 &= \frac{-\eta^2(-48\eta^2 + 144\eta - 24)t - \eta^2(5\eta^3 - 13\eta^2 - 2\eta)}{4\eta - 12} \\ f_3 &= \frac{-9(\eta - 3)^2(49\eta^2 - 135\eta - 12)\eta^3}{4(2\eta^2 - 6\eta + 1)^2}. \end{aligned}$$

Its values at $t = 0$ and $t = \frac{3-\eta}{3}$ are respectively

$$\begin{aligned} s[0] &= \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \eta^4 \\ -9\eta^2(\eta + 1) \\ \frac{\eta^3(5\eta^2 - 13\eta - 2)}{4(3-\eta)} \\ f_3 \end{bmatrix} \\ s\left[\frac{3-\eta}{3}\right] &= \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 9(5\eta - 12)(2\eta^2 - 9\eta + 12) \\ 63\eta^3 - 945\eta^2 + 3456\eta - 3888 \\ \frac{\eta^2(21\eta^3 - 109\eta^2 + 150\eta - 24)}{4(3-\eta)} \\ f_3 \end{bmatrix} \end{aligned}$$