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GUILLAUME FERRIERE

Univ. Lille, Inria, CNRS, UMR 8524 - Laboratoire Paul Painlevé, F-59000 Lille, France
guillaume.ferriere@inria.fr

Abstract

We consider an infinite ferromagnetic nanowire, with an energy functional E with easy-axis in the direction e_1 and a constant external magnetic field $E_{ext} = h_0 e_1$ along the same direction. The evolution of its magnetization is governed by the Landau-Lifshitz-Gilbert equation (LLG) associated to E. Under some assumptions on h_0, we prove the existence of stationary solutions with the same limits at infinity, their uniqueness up to the invariances of the equation and the instability of their orbits with respect to the flow. This property gives interesting new insights of the behavior of the solutions of (LLG), which are completed by some numerical simulations and discussed afterwards, in particular regarding the stability of 2-domain wall structures proven in [4] and more generally the interactions between domain walls.

1. INTRODUCTION

1.1. A model for a ferromagnetic nanowire

We are interested in a model of a ferromagnetic nanowire (of infinite length) as a straight line $\mathbb{R}e_1 \subset \mathbb{R}^3$ where

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

is the canonical basis of $\mathbb{R}^3$. The magnetization $m = (m_1, m_2, m_3) : \mathbb{R} \to \mathbb{S}^2$ of this nanowire takes its values into the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, and we introduce the energy functional

$$E(m) = \frac{1}{2} \int_{\mathbb{R}} |\partial_x m|^2 + (1 - m_1^2) \, \mathrm{d}x,$$

where x is the variable in direction e_1 of the nanowire, and $\cdot$ is the scalar product in $\mathbb{R}^3$. We refer to [5] where this model was derived from the full 3D system by Γ-convergence in a special regime. We want to study the evolution of the magnetization under the Landau-Lifshitz-Gilbert flow associated to E, that is the equation:

$$\partial_t m = m \wedge H(m) - \alpha m \wedge (m \wedge H(m)), \tag{LLG}$$

where now $m : I \times \mathbb{R} \to \mathbb{S}^2$ is the time dependent magnetization (I is an interval of time of $\mathbb{R}$). In this equation, $\wedge$ designates the cross product in $\mathbb{R}^3$, $\alpha > 0$ is the damping coefficient, and the magnetic field H is given by

$$H(m) = -\delta E(m) + h(t) e_1.$$

The last term $H_{ext}(t) \coloneqq h(t) e_1$ is the applied magnetic field, whereas $\delta E(m)$ in the first term is the variation of the energy, which writes (recall that $m_1^2 + m_2^2 + m_3^2 = 1$)

$$\delta E(m) = -\partial_{xx}^2 m + m_2 e_2 + m_3 e_3. \tag{1.1}$$

The function $h : I \to \mathbb{R}$ is the (given) intensity of an applied external field, oriented on the axis e_1. Classically, it solely depends on the time variable t. In this article, we will assume that it is constant: $h(t) = h_0$.

The (LLG) flow is equivariant under the following set of transformations:

- translations in space $\tau_y m(x) \coloneqq m(x-y)$ for $y \in \mathbb{R}$,
- rotations $R_\phi \coloneqq \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$ around the axis e_1 with angle $\phi \in \mathbb{R}$.

We define the group $G \coloneqq \mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}$ which naturally acts on functions $w : \mathbb{R} \to \mathbb{R}^3$ as follows: if $g = (y, \phi) \in G$, $g.w \coloneqq R_\phi \tau_y w = \tau_y R_\phi w$. The action of G preserves $\mathbb{S}^2$ valued functions, and so acts on magnetizations; it also extends naturally to functions of space and time, for which it preserves solutions to (LLG). Also, we endow G with the natural quotient distance over $\mathbb{R}^2$:

$$\forall g = (y, \phi) \in G, \quad |g| \coloneqq |y| + \inf\left\{|\phi + 2k\pi|, k \in \mathbb{Z}\right\}.$$

1.2. Functional spaces and Cauchy theory

We define for $s \geq 1$ the spaces

$$\mathcal{H}^s := \left\{m = (m_1, m_2, m_3) \in \mathscr{C}(\mathbb{R}, \mathbb{S}^2) \mid \|m\|_{\mathcal{H}^s} < +\infty\right\}$$
$$\text{with} \quad \|m\|_{\mathcal{H}^s} := \|m_2\|_{L^2} + \|m_3\|_{L^2} + \|m\|_{\dot{H}^s}.$$

The $\mathcal{H}^s$ spaces are modelled on the usual Sobolev spaces H^s (see Section 1.6.1 for general notations), but adapted to the geometry of the target manifold $\mathbb{S}^2$ and to the energy functional E: the main point is that $|m_1| \to 1$ at $\pm\infty$ so that $m_1 \notin L^2$. The space $\mathcal{H}^1$ corresponds to the set of finite energy configurations $E(m) < +\infty$, in which case the energy gradient $\delta E(m) \in H^{-1}$.

The Cauchy theory of (LLG) has been a challenging question, prompting numerous studies. Notably, Alouges and Soyeur [1] investigated weak solutions, Lakshmanan and Nakamura [7] employed the stereographic projection to transform (LLG) into a quasilinear dissipative Schrödinger equation, and Gutiérrez and de Laire [6] delved into the Cauchy problem in the BMO space. The following well-posedness result, quoted from [5], may not be the most optimal solution for initial data in $\mathcal{H}^s$ with $s \geq 1$, as discussed in [5, Section 4]. Despite this, it suffices for our purpose and, notably, yields an energy dissipation identity of significant interest.

Theorem 1.1 (Local well-posedness in $\mathcal{H}^s$)**.** *Let $\alpha > 0$ and $h \in L^\infty((0, +\infty), \mathbb{R})$. Assume $s \geq 1$ and $m_0 \in \mathcal{H}^s$. Then there exist a maximal time $T_+ = T_+(m_0) \in (0, +\infty]$ and a unique solution $m \in \mathscr{C}([0, T_+), \mathcal{H}^s)$ to* (LLG) *with initial data m_0. Moreover,*

1. *if $T_+ < +\infty$, then $\|m(t)\|_{\mathcal{H}^1} \to +\infty$ as $t \uparrow T_+$;*
2. *for $T < T_+$ (with T_+ finite or infinite), the map $\tilde{m}_0 \in \mathcal{H}^s \to \tilde{m} \in \mathscr{C}([0, T], \mathcal{H}^s)$ is continuous in a small $\mathcal{H}^s$ neighbourhood of m_0 (for every initial data $\tilde{m}_0$ in that neighborhood, the maximal time of the corresponding solution $\tilde{m}$ satisfies $T_+(\tilde{m}_0) > T$);*
3. *if $s \geq 2$, one has the energy dissipation identity : $t \mapsto E(m(t))$ is a locally Lipschitz function in $[0, T_+)$ (even $\mathscr{C}^1$ provided h is continuous) and for all $t \in [0, T_+)$,*

$$\frac{d}{dt}E(m) = -\alpha \int \left(|\delta E(m)|^2 - |m \cdot \delta E(m)|^2\right) \mathrm{d}x + \alpha h(t) \int (m \wedge e_1) \cdot (m \wedge \delta E(m)) \, \mathrm{d}x.$$

1.3. Main result: existence, uniqueness and instability of stationary solutions

Following the establishment of well-posedness, a natural question arises regarding the dynamics of (LLG). In particular, the large-time behavior of solutions becomes an important subject, extensively explored in the context of many partial differential equations, as well as the study of stationary solutions, solitons and progressive waves as a first step. It is indeed expected that generic solutions of this equation decompose into a superposition of independent structures called *domain walls*. This behavior aligns with the *Soliton Resolution Conjecture*, commonly invoked in the context of dispersive PDEs. This conjecture (vaguely formulated) suggests that any global solution of a nonlinear

dispersive PDE will eventually decompose at large time into a combination of non-scattering structures and a radiative term.

In this context, Côte and Ignat [5] studied these domain walls. These authors proved that domain walls are stationary solutions of (LLG) when $h \equiv 0$, they are precessing explicitly with respect to the external magnetic field h, and they exhibit asymptotic stability under a smallness assumption on h. Moreover, they established an exponential convergence rate for both the solutions and the associated parameters (translations and rotations). Building on this groundwork, Côte and the author of this paper [4] delved into the study of *2-domain walls*. These structures consist of two independent domain walls that move away from each other. While not being exact solutions to (LLG), these authors proved that these structures are still asymptotically stable under a suitable external magnetic field. These findings contribute to our understanding of the intricate behavior of solutions to (LLG).

This paper is focused on the study of the stationary solutions of (LLG) in the presence of a constant external magnetic field $h(t) = h_0 \neq 0$. As previously explained, the inclusion of stationary solutions is imperative for any comprehensive understanding of the (generic) large-time behavior. However, to the best of the author's knowledge, such stationary solutions have not yet undergone dedicated numerical or analytic study, as they present different properties compared to domain walls. The first result of this paper asserts the existence of these stationary solutions and, notably, establishes their uniqueness up to the invariances of the equation.

Theorem 1.2.

- *Let $h_0 > 0$. There exists a unique non-constant stationary solution w_{h_0} to* (LLG) *in $\mathcal{H}^1$ such that $\lim_{\pm\infty} w_{h_0} = e_1$, up to the invariances. Such a solution can be taken with values in $\mathbb{S}^1 \times \{0\}$ and symmetric: $w_{h_0} = (\cos\theta_{h_0}, \sin\theta_{h_0}, 0)$ with θ_{h_0} smooth and satisfying*
  - $\theta_{h_0}(0) = \pi$ *and* $\partial_x\theta_{h_0}(0) = \sqrt{4h_0}$
  - $\lim_{-\infty}\theta_{h_0} = 0$ *and* $\lim_{+\infty}\theta_{h_0} = 2\pi$,
  - $\theta_{h_0} - \pi$ *is odd,*
  - θ_{h_0} *is a solution of the following ODEs:*

$$-\partial_{xx}\theta + \sin\theta\cos\theta + h_0\sin\theta = 0. \tag{1.2}$$

$$\partial_x\theta = \sqrt{\sin^2\theta + 2h_0(1-\cos\theta)}, \tag{1.3}$$

  - $\sin\theta_{h_0} \in L^2$, $1 - \cos\theta_{h_0} \in L^1$, $\partial_x\theta_{h_0} \in H^k$ *for any* $k \geq 1$.
  - *Last, there holds*

$$H(w_{h_0}) = \Lambda(x)\, w_{h_0}, \tag{1.4}$$

  where

$$\Lambda := -2\sin^2\theta_{h_0} + 3h_0\cos\theta_{h_0} - 2h_0. \tag{1.5}$$

- *Let $h_0 \in (-1, 0)$. There exists a unique non-constant stationary solution w_{h_0} to* (LLG) *in $\mathcal{H}^1$ such that $\lim_{\pm\infty} w_{h_0} = e_1$, up to the invariances. Such a solution can be taken with values in $\mathbb{S}^1 \times \{0\}$ and symmetric: $w_{h_0} = (\cos\theta_{h_0}, \sin\theta_{h_0}, 0)$ with θ_{h_0} smooth and satisfying*
  - $\theta_{h_0}(0) = \arccos(-1 - 2h_0)$ *and* $\partial_x\theta_{h_0}(0) = 0$
  - $\lim_{\pm\infty}\theta_{h_0} = 0$,
  - θ_{h_0} *is even,*
  - θ_{h_0} *satisfies* (1.2) *and:*

$$\partial_x\theta = \operatorname{sgn}(x)\sqrt{\sin^2\theta + 2h_0(1-\cos\theta)}, \tag{1.6}$$

  - $\sin\theta_{h_0} \in L^2$, $1 - \cos\theta_{h_0} \in L^1$, $\partial_x\theta_{h_0} \in H^k$ *for any* $k \geq 1$.
  - *Last, there holds* (1.4) *with* (1.5).

The proof of the existence and uniqueness of such solutions is similar to the proof of [5, Theorem 1.1]. We adapted it to the case with an external magnetic field, which in turn implies different limits at infinity for m_{h_0} and θ_{h_0}.

Remark 1.3. Similar statements hold for $\lim_{\pm\infty} w_{h_0} = -e_1$ and $h_0 < 0$ or $h_0 \in (0,1)$ throughout this paper. This can be easily seen by the symmetry of (LLG), which is preserved by transforming a solution $m = (m_1, m_2, m_3)$ with external magnetic field $h(t)\, e_1$ into another solution $(-m_1, m_2, -m_3)$ with external magnetic field $-h(t)\, e_1$. Therefore, we will focus on the case $\lim_{\pm\infty} m = e_1$ in this paper.

With the existence and uniqueness of stationary solutions of (LLG) established, the natural progression of inquiry leads to the question of their stability. In other words, we now turn our attention to the discerning whether the stationary structures identified in the previous result are stable configurations, maintaining their form over time, or if they are sensitive to perturbations and evolve into different states. This quest for stability, whose definition is stated hereafter in the nonlinear setting, is crucial for characterizing the large-time behavior and practical implications of the identified stationary solutions.

Definition 1.4. *1. The stationary solution w_{h_0} is (nonlinearly) orbitally stable if for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for any initial data $m_0 \in \mathcal{H}^2$ satisfying*

$$\|m_0 - w_{h_0}\|_{H^1} < \delta,$$

the solution m to (LLG) *satisfies, for any $t > 0$,*

$$\inf_{g \in G} \|m(t) - g.w_{h_0}\|_{H^1} < \varepsilon.$$

2. The stationary solution w_{h_0} is (nonlinearly) orbitally unstable if there exists $\varepsilon > 0$ such that for any $\delta > 0$, there exists some initial data $m_0 \in \mathcal{H}^2$ satisfying

$$\|m_0 - w_{h_0}\|_{H^1} < \delta,$$

and the solution m to (LLG) *satisfies, for some $t > 0$,*

$$\inf_{g \in G} \|m(t) - g.w_{h_0}\|_{H^1} \geq \varepsilon.$$

The second main result of this paper is the nonlinear instability of the stationary solution w_{h_0}.

Theorem 1.5. *For any $h_0 > 0$ or $h_0 \in (-1, 0)$, w_{h_0} is (nonlinearly) orbitally unstable.*

The instability of these stationary structures implies that, to some extent, an initial data which is a perturbation of this stationary structure yields a solution which will not remain near the orbit of this structure (constructed by translation and rotation around e_1). This explains why such solutions has not been observed in numerical simulations for very general initial data.

1.4. A more precise statement

Actually, the previous statement can be precised thanks to a modified energy. For any $m \in \mathcal{H}^1$ satisfying $\lim_{\pm\infty} = e_1$ and using the fact that $m_2^2 + m_3^2 = 1 - m_1^2 = (1 - m_1)(1 + m_1)$, there holds $m_1 - 1 \in L^1$. Thus, we can define a modified energy E_{h_0} which takes into account the external magnetic field:

$$E_{h_0}(m) \coloneqq E(m) - h_0 \int (m_1 - 1)\, \mathrm{d}x. \tag{1.7}$$

Its variation is related not only to the variation δE of the initial energy E, but also to the effective magnetic field H:

$$\delta E_{h_0}(m) = \delta E(m) - h_0 e_1 = -H(m). \tag{1.8}$$

Remark 1.6. Any stationary solution w of (LLG) satisfies $w \wedge H(w) = 0$. Therefore, w_{h_0} is also the unique non-constant critical point of E_{h_0} in $\mathcal{H}^1$, up to the invariances.

This modified energy plays an important role in the evolution of the solution. In particular, the statement of Theorem 1.5 can refined by an estimate: the time during which the solution stays close to the orbit of the stationary solution w_{h_0} is bounded by above by an explicit value. This estimate is valid as long as the modified energy of the initial data is smaller than the modified energy of the stationary solution. We also prove that such initial data can be found as close to w_{h_0} as one wants.

Theorem 1.7. *There exists $\lambda, \varepsilon > 0$ such that for any $0 < \delta < \varepsilon$, the following property holds. Define*

$$V_\delta = \left\{ m_0 \in \mathcal{H}^2 \,|\, \|m_0 - w_{h_0}\|_{H^1} < \delta, E_{h_0}(m_0) < E_{h_0}(w_{h_0}) \right\}.$$

Then $V_\delta \neq \emptyset$ and any initial data $m_0 \in V_\delta$ satisfies the properties of part 2 of Definition 1.4 for some $t > 0$ such that

$$t \leq \frac{1}{\lambda^2} \ln \left(\frac{\varepsilon}{\lambda |E_{h_0}(m_0) - E_{h_0}(w_{h_0})|} \right). \tag{1.9}$$

This modified energy is actually related to the effective magnetic field E_{ext}, and is decreasing through the flow of (LLG). Although w_{h_0} is a critical point with respect to the modified energy E_{h_0}, it is only a saddle point, which explains its instability. The proof of this theorem is, once again, inspired by [5, Theorem 1.4]. First, a moving frame, adapted to the geometry of the equation, is introduced. It allows to reduce the dimension of the problem, from 3d (with a constraint) to 2d, thanks to a smallness assumption. Then, expansions in the two remaining unknowns are performed on the modified energy E_{h_0} and its dissipation through the flow of the equation (Proposition 5.1). Two Schrödinger operators appear at the first order, from which some estimates are deduced. Such estimates allow to get a global estimate on the evolution of the energy, from which a Gronwall lemma allows to conclude.

Remark 1.8. The proof of Theorem 1.7 shows an even stronger property: as long as

$$\inf_{g \in G} \|m(t) - g.w_{h_0}\|_{H^1} < \varepsilon$$

for all $t \in [0, \tau]$ for some $\tau > 0$, then this infimum also satisfies the following lower bound on this interval if $m_0 \in V_\delta$:

$$\inf_{g \in G} \|m(t) - g.w_{h_0}\|_{H^1} \geq \lambda |E_{h_0}(m_0) - E_{h_0}(w_{h_0})| e^{\lambda^2 t}.$$

Remark 1.9. For the instability definition and result, the initial data m_0 are taken in $\mathcal{H}^2$, whereas the assumption regarding the closeness of this initial data to the stationary solution only involves the H^1 norm. This simplification is employed here in order to use the energy dissipation identity in Theorem 1.1 and so that $m \wedge H(m)$ is in L^2 for all time $t \geq 0$. It is purely technical: a limiting argument can be used to restrain this assumption to $m_0 \in \mathcal{H}^1$, in the same way as in [5, 4].

Remark 1.10. Similarly as in [5], the analysis of this paper can also be performed with the addition of the Dzyaloshinskii-Moriya interaction. Like the domain walls, the stationary solutions would be modified by a dilatation and a rotation around e_1 with an angle linearly dependent of the space variable x, both transformations depending on the intensity of the aforementioned interaction. Apart from this change, we believe that our arguments still apply in this case.

1.5. Outline of the paper

This paper is organized as follows. Existence and uniqueness of the stationary solutions are proven in Section 2, completed by exponential decay estimates at infinity. The proof of the instability of the stationary solution is inspired from the [5]. In particular, we use a special space-dependent basis similar to the one used for the stability of the domain wall in the latter. This basis is introduced in Section 3, along with two Schrödinger operators and their properties. These two operators appear in the expansion at the main order of the modified energy and its dissipation, performed in Section 4. The proof of Theorem 1.7 is then performed in Section 5. We complete our study with numerical simulations, displayed and discussed in Section 6. This paper ends with a discussion about the conclusions, the open problems and some conjectures related to this study.

1.6. Notations

1.6.1. *General notations.*

- $(L^p)^k$ is usually used as short notation for the Lebesgue space $(L^p(\mathbb{R},\mathbb{R}))^k$ (or equivalently $L^p(\mathbb{R},\mathbb{R}^k)$) with $p\in[1,\infty]$. Similarly, $(\dot H^s)^k$ is classically used for the homogeneous Sobolev space $(\dot H^s(\mathbb{R},\mathbb{R}))^k$ (or equivalently $\dot H^s(\mathbb{R},\mathbb{R}^k)$) with $s\geq 0$ and $k\in\mathbb{N}$ whose seminorm is given through the Fourier transform:

$$\|f\|^2_{(\dot H^s)^k} \coloneqq \frac{1}{2\pi}\int_{\mathbb{R}} |\hat f(\xi)|^2|\xi|^{2s}\,\mathrm{d}\xi, \quad \text{where} \quad \hat f(\xi)=\int_{\mathbb{R}} e^{-ix\xi}f(x)\,\mathrm{d}x.$$

  In particular, $\|f\|_{(\dot H^n)^k}=\|\partial_x^n f\|_{(L^2)^k}$ for any $n\in\mathbb{N}$. Last, the Sobolev spaces $(H^s(\mathbb{R},\mathbb{R}))^k$ (with $s\geq 0$ and $k\in\mathbb{N}$) are also denoted $(H^s)^k$. Though, most of the time, we will omit the k exponent and simply denote these spaces L^p, $\dot H^s$ and H^s when there is no possible confusion.

- We denote $\mathbb{S}^2\subset\mathbb{R}^3$ the 2-dimensional sphere: $\mathbb{S}^2=\big\{X\in\mathbb{R}^3\,|\,|X|=1\big\}$.

- We define for $s\geq 1$ the spaces

$$\begin{aligned}\mathcal{H}^s &:= \big\{m=(m_1,m_2,m_3)\in\mathscr{C}(\mathbb{R},\mathbb{S}^2)\,|\,\|m\|_{\mathcal{H}^s}<+\infty\big\}\\ &\text{with}\quad \|m\|_{\mathcal{H}^s}:=\|m_2\|_{L^2}+\|m_3\|_{L^2}+\|m\|_{\dot H^s}.\end{aligned}$$

  For any interval $I\subset\mathbb{R}$, the space $\mathscr{C}(I,\mathcal{H}^s)$ is the set of maps m

$$\begin{aligned} m: I &\to \mathcal{H}^s\\ t &\mapsto m(t)\end{aligned}$$

  continuous with respect to the H^s norm.

1.6.2. *Multilinear estimates in Sobolev spaces.* We will use the same notation $O^\ell_k(f)$ as in [5] to express pointwise bounds that turn into Sobolev bounds with linear dependence in the highest term.

Definition 1.11. *For $k\geq 0$ and $\ell\geq 1$, and given a (possibly vector valued) function $f=(f_j)_{1\leq j\leq J}$, we use the notation*

$$g=O^\ell_k(f)$$

for a (possibly vector valued) function g if (each component of) g is an homogeneous polynomial of degree ℓ in the components of f and their derivatives such that the total number of derivatives in each term is at most k, and whose coefficients are $\mathscr{C}^\infty_b(\mathbb{R})$ functions. g is then the sum of terms of the form

$$\alpha\prod_{j=1}^{J}\prod_{\kappa=0}^{k}(\partial_x^\kappa f_j)^{\ell_{j,\kappa}},\qquad \text{where}\quad \sum_{j,\kappa}\ell_{j,\kappa}=\ell,\qquad \sum_{j,\kappa}l_{j,\kappa}\kappa\leq k,\quad\text{and}\quad \alpha\in\mathscr{C}^\infty_b.$$

Lemma 1.12. *1. If $k'\geq k$, then $O^\ell_k(f)=O^\ell_{k'}(f)$.*

2. If $\alpha\in\mathscr{C}^\infty_b$, then $\alpha O^\ell_k(f)=O^\ell_k(f)$.

3. $O^\ell_k(f_1)O^{\ell'}_{k'}(f_2)=O^{\ell+\ell'}_{k+k'}(f_1,f_2)$.

4. $\partial_x O^\ell_k(f)=O^\ell_{k+1}(f)$,

5. $O^\ell_k(f_1+f_2)=O^\ell_k(f_1,f_2)$.

Lemma 1.13. *Assume that $u=O^\ell_k(f)$ with $\ell\geq 1$.*

- *If $k\geq 2$ and $f\in H^k$, then there holds*

$$\|u\|_{L^2}\lesssim \|f\|^{\ell-1}_{H^{k-1}}\|f\|_{H^k}.$$

- *If $k \in \{0,1\}$ and $f \in H^1$, then*
$$\|u\|_{L^2} \lesssim \|f\|_{H^1}^{\ell}.$$
- *If $\ell \geq 2$, $k = 2$ and $f \in H^1$, then*
$$\left| \int u \, \mathrm{d}x \right| \lesssim \|f\|_{H^1}^{\ell}.$$
- *If $\ell \geq 2$, $k \in \{3,4\}$ and $f \in H^2$, then*
$$\left| \int u \, \mathrm{d}x \right| \lesssim \|f\|_{H^1}^{\ell-k+2} \|f\|_{H^2}^{k-2}.$$

The proof of such properties and bounds can be found in Claims 4.7 to 4.9 of [5].

2. Existence, uniqueness and properties of the stationary solution

This section is devoted to the general properties of the stationary solution.

2.1. Existence and uniqueness

We begin by proving Theorem 1.2, which asserts the existence and uniqueness of the stationary solution.

Proof of Theorem 1.2. The proof is quite similar to the proof of the existence of static domain walls in [5, Theorem 1.1], and we use the same tactic.

Step 1: Equation on m.
Let $m \in \mathcal{H}^1$ be a critical point to E_{h_0} in $\mathcal{H}^1$ such that $\lim_{\pm\infty} m = e_1$. It satisfies $m \wedge H(m) = 0$ and $|m| = 1$. The first equation can be re-written as $H(m) = \Lambda(x)m$, which is

$$\partial_{xx} m - m_2 e_2 - m_3 e_3 + h_0 e_1 = \Lambda(x) m. \tag{2.1}$$

The scalar function Λ, defined on $\mathbb{R}$, can be explicited thanks to the second equation:

$$\Lambda(x) = H(m) \cdot m = -(|\partial_x m|^2 + m_2^2 + m_3^2) + h_0 m_1. \tag{2.2}$$

Since $m \in \mathcal{H}^1$, by a bootstrap argument, there holds $m \in \mathscr{C}^\infty(\mathbb{R})$.

Step 2: m is planar.
First, we show that

$$m_2 \partial_x m_3 - m_3 \partial_x m_2 = 0 \quad \text{on } \mathbb{R}. \tag{2.3}$$

Indeed, using (2.1), there holds

$$\begin{aligned} \partial_x \Big(m_2 \partial_x m_3 - m_3 \partial_x m_2 \Big) &= m_2 \partial_{xx} m_3 - m_3 \partial_{xx} m_2 \\ &= m_2(\partial_{xx} m_3 - m_3) - m_3(\partial_{xx} m_2 - m_2) \\ &= m_2(\Lambda(x) m_3) - m_3(\Lambda(x) m_2) \\ &= 0, \end{aligned}$$

which shows that $m_2 \partial_x m_3 - m_3 \partial_x m_2$ is constant on $\mathbb{R}$. On the other hand, since $m \in \mathcal{H}^1$, we have $m_2, m_3 \in H^1$, and thus $m_2 \partial_x m_3 - m_3 \partial_x m_2 \in L^1$. This proves (2.3), which implies that $(m_2(x), \partial_x m_2(x))$ and $(m_3(x), \partial_x m_3(x))$ are collinear in $\mathbb{R}^2$ for every x in $\mathbb{R}$. On the other hand, by (2.1), these two vector fields solve the same first order linear ODE system in (u, v):

$$\partial_x u = v, \quad \partial_x v = (1 + \Lambda(x)) u.$$

By uniqueness in the Cauchy-Lipschitz theorem, the collinearity factor is constant in x, and thus there exists $\beta \in \mathbb{R}$ such that $m_2 = \beta m_3$ (or $m_3 = \beta m_2$, respectively) in $\mathbb{R}$. Therefore, choosing ϕ such that $\cot \phi = \beta$ (or $\tan \phi = \beta$,

respectively), we conclude that $R_{-\phi}m$ satisfies the same equation and regularity as m, with the additional property that its 3rd component vanishes.

Step 3: Lifting.

Since m can be taken as a smooth function with values in $\mathbb{S}^1 \times \{0\}$, there exists a smooth lifting $\theta : \mathbb{R} \to \mathbb{R}$ such that $m = (\cos\theta, \sin\theta, 0)$. For such a lifting, there holds

$$\begin{aligned} |\partial_x \theta|^2 = |\partial_x m|^2 \in L^1, \\ \sin^2(\theta) = m_2^2 \in L^1, \\ \cos\theta - 1 = m_1 - 1 \in L^1. \end{aligned}$$

Therefore, we get

$$\begin{aligned} E(m) &= \frac{1}{2}\int \Big[(\partial_x\theta)^2 + \sin^2\theta\Big]\,\mathrm{d}x < \infty, \\ E_{h_0}(m) &= \frac{1}{2}\int \Big[(\partial_x\theta)^2 + \sin^2\theta - 2h_0(\cos\theta - 1)\Big]\,\mathrm{d}x < \infty, \end{aligned}$$

and the equation (2.1) can be re-written in terms of θ as (1.2). Moreover, since $\lim_{\pm\infty} m = e_1$, there also holds $\lim_{\pm\infty}\theta = 2k_\pm\pi$ for some $k_\pm \in \mathbb{N}$. We also point out that, conversely, if θ satisfies (1.2) and $\lim_{\pm\infty}\theta = 2k_\pm\pi$, then $m = (\cos\theta, \sin\theta, 0)$ satisfies (2.1) and $\lim_{\pm\infty} m = e_1$.

Since θ is smooth, we can multiply and integrate (1.2) by θ, which yields

$$-(\partial_x\theta)^2 + \sin^2\theta - 2h_0(\cos\theta - 1) = cst.$$

All these terms are integrable on $\mathbb{R}$, so this constant is zero and we get

$$-(\partial_x\theta)^2 + \sin^2\theta + 2h_0(1 - \cos\theta) = 0. \tag{2.4}$$

Step 4: Trajectories of the lifting for $h_0 > 0$.

Let $X \coloneqq (\theta, \partial_x\theta)$ and set the Hamiltonian

$$\mathrm{Ham}(X_1, X_2) = \frac{1}{2}\Big(X_2^2 - \sin^2 X_1 - 2h_0(1 - \cos X_1)\Big), \qquad X = (X_1, X_2) \in \mathbb{R}^2.$$

From the previous equations, X satisfies the dynamical system

$$\partial_x X = \Big(\frac{\partial\,\mathrm{Ham}}{\partial X_2}(X), -\frac{\partial\,\mathrm{Ham}}{\partial X_1}(X)\Big),$$

and the trajectory $\{X(x)\}_{x\in\mathbb{R}}$ is included in the zero set of Ham. We emphasize that $1 - \cos\theta \geq 0$, and vanishes only on $2\pi\mathbb{Z}$. For $h_0 > 0$, the zero set of Ham can be denoted as $Z^- \cup Z^0 \cup Z^+$ where

$$Z^\pm \coloneqq \left\{(X_1, X_2) \in \mathbb{R}^2 \mid \pm X_2 > 0, \mathrm{Ham}(X_1, X_2) = 0\right\},$$

and

$$Z^0 \coloneqq \{(X_1, 0) \mid \mathrm{Ham}(X_1, 0) = 0\} = 2\pi\mathbb{Z} \times \{0\}.$$

In particular, any connected component of Z^+ and Z^- ends at two consecutive points of Z^0, which is actually the set of constant solutions of the previous dynamical system. Therefore, by uniqueness in the Cauchy-Lipschitz theorem, the trajectory $\{X(x)\}_{x\in\mathbb{R}}$ either:

- is included in Z^0: the trajectory is stationary, i.e. $\theta \equiv 2k\pi$ for some $k \in \mathbb{N}$, corresponding to a constant solution of (2.1),
- begins and ends at two consecutive points of Z^0. In particular, the sign of $\partial_x\theta$ does not change, and the total rotation of a critical point θ is given by $\int_{\mathbb{R}} \partial_x\theta\,\mathrm{d}x = \pm 2\pi$. This also means that $|k_+ - k_-| = 1$.

Step 5: Conclusion on existence and uniqueness for $h_0 > 0$.

Assuming that m is not constant, we are in the second case of the previous disjunction. Since θ is defined up to a $2\pi\mathbb{Z}$ additive constant, we can assume that $\theta(x_0) = \pi$ at some $x_0 \in \mathbb{R}$, and that $x_0 = 0$ up to a translation τ_{x_0}. Moreover, if $\lim_{-\infty} \theta = 2\pi$ and $\lim_{+\infty} \theta = 0$, then $\tilde{\theta} \coloneqq 2\pi - \theta$ satisfies (1.2), $\lim_{-\infty} \tilde{\theta} = 0$ and $\lim_{+\infty} \tilde{\theta} = 2\pi$, and corresponds to a lifting of $R_\pi m$. Therefore, we can assume that $\lim_{-\infty} \theta = 0$ and $\lim_{+\infty} \theta = 2\pi$. Since the sign of $\partial_x \theta$ does not change, there holds $\partial_x \theta > 0$. Thus, with this property along with (2.4), θ satisfies

$$\begin{cases} \partial_x \theta = \sqrt{\sin^2 \theta + 2h_0(1 - \cos\theta)}, \\ \theta(0) = \pi. \end{cases} \tag{2.5}$$

The function $f : y \mapsto \sqrt{\sin^2 y + 2h_0(1 - \cos y)}$ is smooth on $(0, 2\pi)$, and we know that $\theta(x) \in (0, 2\pi)$ for all $x \in \mathbb{R}$. Therefore, by the uniqueness in the Cauchy-Lipschitz theorem, the previous properties uniquely defines θ.

For the existence, let θ_{h_0} satisfying (2.5), given by the existence part of Cauchy-Lipschitz theorem. Then θ also satisfies (1.2). Moreover, one can compute an expansion of f at 0^+ and $2\pi^-$. For instance, at 0, we have $\sin y = y + O(y^3)$, $\sin^2 y = y^2 + O(y^4)$ and $1 - \cos y = \frac{y^2}{2} + O(y^4)$. Therefore, $f(y) = \sqrt{1 + h_0} y + O(y^3)$. Thus, it is easy to prove that not only θ_{h_0} does not reach 0, but θ_{h_0} and $\partial_x \theta_{h_0}$ also converge exponentially to 0 at $-\infty$ by Gronwall lemma, and so are $\partial_x^k \theta_{h_0}$ for any $k \geq 1$ thanks to (1.2). A similar conclusion can be achieved for $\theta_{h_0} - 2\pi$ at $+\infty$, therefore θ is globally defined and $\theta(x) \in (0, 2\pi)$ for all $x \in \mathbb{R}$. This shows that $w_{h_0} = (\cos\theta_{h_0}, \sin\theta_{h_0}, 0) \in \mathcal{H}^1$ is a non-constant critical point to E_{h_0} such that $\lim_{\pm\infty} w_{h_0} = e_1$. Last, we emphasize that (2.4) along with (2.2) yields (1.4).

Step 6: Trajectories of the lifting for $h_0 \in (-1, 0)$.

In this case, the trajectories are different. Indeed, $\mathrm{Ham}(\pi, X_2) = \frac{1}{2} X_2^2 - 2h_0 > 0$ for all $X_2 \in \mathbb{R}$, which shows that θ cannot cross π. However, defining $g(y) \coloneqq \sin^2 y + 2h_0(1 - \cos y) = 1 - \cos^2(y) + 2h_0(1 - \cos y)$, there holds $g(y) = g_2(\cos y)$ where $g_2(z) \coloneqq 1 - z^2 + 2h_0(1 - z)$ is a polynomial function of degree 2 which has two roots: 1 and $-1 - 2h_0$. By a further analysis, it is easy to check that $g > 0$ on $(0, \theta_c)$ and $g < 0$ on (θ_c, π) where $\theta_c = \arccos(-1 - 2h_0)$. Thus, θ cannot reach (θ_c, π), neither $(-\pi, -\theta_c)$ since g is even. Similarly as before, θ is defined up to a 2π additive constant, so that we can assume that $\theta(0) \in (-\pi, \pi)$, and thus $-\theta_c \leq \theta(x) \leq \theta_c$ for all $x \in \mathbb{R}$. Moreover, similarly as before, if $\theta(x_0) = 0$ for some x_0, then $\theta \equiv 0$ is constant.

Assuming now that θ is not constant, this means that the sign of θ cannot change. We can assume that $\theta > 0$ (up to a rotation of angle π about the axis e_1 for w_{h_0}). Then, θ reaches θ_c at some point x_0. Indeed, if not, then $\partial_x \theta = g(\theta)$ does not vanish, which means that the sign of $\partial_x \theta$ does not change, that θ is strictly monotone and has two different limits at infinity, limits which are in $[0, \theta_c]$ and must be constant solutions of (1.2) and (1.3). Thus, these limits must be 0 and θ_c. However, if θ_c is a constant solution to (1.2), then $\cos\theta_c = -h_0$, whereas $\cos\theta_c = -1 - 2h_0$ from the expression of θ_c, leading to a contradiction as $h_0 > -1$.

Up to a translation in space, we can assume that $\theta(0) = \theta_c$. Then, $\partial_x \theta(0) = 0$ and $\partial_{xx} \theta(0) = \sin(\theta_c)(h_0 + \cos\theta_c) < 0$. Since θ cannot cross 0 and $\partial_x \theta$ cannot change sign unless $\theta = \theta_c$, it is easy to prove that $\partial_x \theta > 0$ on $(-\infty, 0)$ and $\partial_x \theta < 0$ on $(0, \infty)$. Therefore, by (2.4), we obtain (1.6).

Step 7: Conclusion on the existence and uniqueness for $h_0 \in (-1, 0)$.

The equation (1.2) along with $\theta(0) = \theta_c$ and $\partial_x \theta(0) = 0$ ensures that θ is global and unique by Cauchy-Lipschitz theorem.

On the other hand, if we take θ_{h_0} satisfying (1.2) along with $\theta_{h_0}(0) = \theta_c$ and $\partial_x \theta_{h_0}(0) = 0$, which exists by Cauchy-Lipschitz theorem, then θ_{h_0} satisfies all the above properties. Moreover, it is easy to compute that $f = \sqrt{g}$ satisfies $f(y) = \sqrt{1 + h_0}\, y + O(y^3)$ for $y \to 0$, which proves (thanks to (1.6) and the fact that $\lim_{\pm\infty} \theta = 0$) that θ_{h_0} and $\partial_x \theta_{h_0}$ decays exponentially, so that $\theta_{h_0} \in H^1$. Last, the same arguments show that $w_{h_0} = (\cos\theta_{h_0}, \sin\theta_{h_0}, 0)$ satisfies all the above equations and is thus a critical point of E_{h_0} in $\mathcal{H}^1$. □

2.2. Exponential convergence at infinity

Next, we explore additional properties of the stationary solution outlined in Theorem 1.2. Specifically, we focus on deriving decay estimates for the solution and its derivatives as $|x|$ tends to $\pm\infty$.

Lemma 2.1. *For any $k \in \mathbb{N}$, there exists $C_k > 0$ such that, for any $x \in \mathbb{R}$,*

- *if* $h_0 > 0$,

$$\left|\partial_x^k \theta_{h_0}\right| \leq C_k e^{-\sqrt{1+h_0}|x|} \qquad \text{for } x < 0,$$
$$\left|\partial_x^k (\theta_{h_0} - 2\pi)\right| \leq C_k e^{-\sqrt{1+h_0}|x|} \qquad \text{for } x > 0,$$

- *if* $h_0 \in (-1, 0)$,

$$\left|\partial_x^k \theta_{h_0}\right| \leq C_k e^{-\sqrt{1+h_0}|x|}.$$

- *in both cases,*

$$\left|\partial_x^k (w_{h_0} - e_1)\right| \leq C_k e^{-\sqrt{1+h_0}|x|}.$$

Proof. Let f defined by

$$f : (0, 2\pi) \to \mathbb{R}$$
$$y \mapsto \sqrt{\sin^2 y + 2h_0(1 - \cos y)}.$$

Then $\partial_x \theta_{h_0} = f(\theta_{h_0})$. Moreover, by the previous expansion provided in the proof of Theorem 1.2, we know that $f(y) = \sqrt{1+h_0}\, y + O(y^3)$ near 0^+. If $h_0 > 0$, since $\lim_{-\infty} \theta_{h_0} = 0$, this shows that there exists $C > 0$ and $A < 0$ such that, for every $x < A$,

$$(\sqrt{1+h_0} - C\theta_{h_0}(x)^2)\, \theta_{h_0}(x) \leq \partial_x \theta_{h_0}(x) \leq (\sqrt{1+h_0} + C\theta_{h_0}(x)^2)\, \theta_{h_0}(x). \tag{2.6}$$

By a backward Gronwall lemma, we can estimate that, for all $x < A$,

$$\theta_{h_0}(x) \leq \theta_{h_0}(A) \exp\Big(-\sqrt{1+h_0}(A - x) + C \int_x^A \theta_{h_0}(y)^2 \, \mathrm{d}y\Big).$$

We obtain the announced estimate for θ_{h_0} by using the fact that $\theta_{h_0} \in L^2(-\infty, 0)$, since $\sin \theta_{h_0} \in L^2$ and $\lim_{-\infty} \theta_{h_0} = 0$. Then, using again (2.6), we obtain the conclusion for $\partial_x \theta_{h_0}$. As for $\partial_{xx} \theta_{h_0}$, we use (1.2), and more generally the estimate for $\partial_x^k \theta_{h_0}$ (for $k \geq 2$) can be reached by induction, by differentiating (1.2) $k - 2$ times and applying the estimates of the lower derivatives.

A similar conclusion can be reached for $2\pi - \theta_{h_0}$ at $+\infty$, for θ_{h_0} at $\pm\infty$ if $h_0 \in (-1, \infty)$, and finally the estimate for w_{h_0} in both cases comes from these estimates on θ_{h_0} and the fact that $w_{h_0} = (\cos \theta_{h_0}, \sin \theta_{h_0}, 0)$. □

3. Associated basis and Schrödinger operators

In this section, we introduce an associated space-dependent basis and two Schrödinger operators related to our problem. This specialized basis is carefully selected to capture the essential features of the system dynamics near the stationary solution w_{h_0}, providing a valuable framework for the analysis of the evolution and for the computation of the expansions of some quantities. But first, we introduce and analyze two significant Schrödinger operators. They play a fundamental role in our analysis, as they appear both in the operator of the linearized equation and in the nonlinear analysis through the modified energy.

3.1. Two important Schrödinger operators

3.1.1. *Definition and first properties.* We begin by defining the following operators, denoted L_1 and L_2:

$$L_1 \coloneqq -\partial_{xx} + 1 - 2\sin^2 \theta_{h_0} + h_0 \cos \theta_{h_0},$$
$$L_2 \coloneqq L_1 - 2h_0(1 - \cos \theta_{h_0}) = -\partial_{xx} + 1 - 2\sin^2 \theta_{h_0} + 3h_0 \cos \theta_{h_0} - 2h_0.$$

As it is classical for Schrödinger operators, they are unbounded self-adjoint operators in $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R})$, and their quadratic form can be extended to $H^1(\mathbb{R})$. Our primary interest lies in the spectrum of these operators, particularly the discrete spectrum contained in $(-\infty, 0]$. As an initial observation, we establish the presence of certain functions in their kernel.

Proposition 3.1. *There holds* $L_1\partial_x\theta_{h_0} = L_2 \sin\theta_{h_0} = 0$.

Proof. Coming back to (1.2) and differentiating with respect to x since θ_{h_0} is smooth, we get

$$0 = -\partial_{xx}(\partial_x\theta_{h_0}) + (\cos^2\theta_{h_0} - \sin^2\theta_{h_0})\partial_x\theta_{h_0} + h_0\,\partial_x\theta_{h_0}\,\cos\theta_{h_0} = L_1\partial_x\theta_{h_0}.$$

On the other hand, as $w_{h_0} = (\cos\theta_{h_0}, \sin\theta_{h_0}, 0)$ satisfies (1.4), taking the second component of this ODE leads to

$$\partial_{xx}(\sin\theta_{h_0}) - \sin\theta_{h_0} = \Lambda(x)\sin\theta_{h_0},$$

and therefore $L_2\sin\theta_{h_0} = 0$ thanks to (1.5). □

Remark 3.2. We point out that the functions $\partial_x\theta_{h_0}$ and $\sin\theta_{h_0}$ are actually related to the invariances of the equation and of the energies E and E_{h_0}. Indeed, $\partial_x\theta_{h_0}$ is related to the invariance by translation as there holds

$$w_{h_0}(\cdot + \varepsilon) = w_{h_0} + \varepsilon\partial_x\theta_{h_0}n_{h_0} + O(\varepsilon^2),$$

and $\sin\theta_{h_0}$ is related to the invariance by rotation along e_1:

$$R_\varepsilon w_{h_0} = \begin{pmatrix} \cos\theta_{h_0} \\ \sin\theta_{h_0}\cos\varepsilon \\ \sin\theta_{h_0}\sin\varepsilon \end{pmatrix} = w_{h_0} + \varepsilon\sin\theta_{h_0}e_3 + O(\varepsilon^2).$$

3.1.2. *Abstract approach.* The functions identified in the kernel provide deeper insights than initially apparent. Given that these operators are in dimension $d = 1$, the Sturm-Liouville theory offers a more precise understanding of the nature of the eigenvalues of the operators L_1 and L_2. To this end, we introduce two abstract lemmas in the following subsection. These lemmas not only address the existence (or absence) of eigenvalues in the interval $(-\infty, 0)$ but also provide precise estimates regarding the lack of coercivity of these operators.

Lemma 3.3. *Let* $L = -\partial_{xx} + V$ *where* $V \in L^\infty(\mathbb{R})$ *such that* $\lim_{\pm\infty} V > 0$. *Assume there exists* $\phi \in H^2$ *such that* $L\phi = 0$ *and* $\phi > 0$ *in* $\mathbb{R}$. *Then* $\ker L = \phi$ *and there exists* $\lambda > 0$ *such that for all* $v \in H^1$,

$$0 \le \langle Lv, v\rangle \le \frac{1}{\lambda}\|v\|_{H^1}^2,$$
$$\langle Lv, v\rangle \ge 4\lambda\|v\|_{H^1}^2 - \frac{1}{\lambda}\langle v, \phi\rangle^2,$$

and, for all $v \in H^2$,

$$\|Lv\|_{L^2}^2 \ge 4\lambda\|v\|_{H^2}^2 - \frac{1}{\lambda}\langle v, \phi\rangle^2.$$

Proof. See [5, Lemma C.1]. □

Lemma 3.4. *Let* $L = -\partial_{xx} + V$ *where* $V \in L^\infty(\mathbb{R})$ *such that* $\ell := \lim_{\pm\infty} V > 0$. *Assume there exists* $\phi \in H^2$ *such that* $L\phi = 0$ *and that* ϕ *vanishes only once in* $\mathbb{R}$. *Then it has a unique negative eigenvalue* $\gamma < 0$, *which is simple, and we denote* $\psi \in H^2$ *a normalized eigenfunction related to this eigenfunction:* $\|\psi\|_{L^2} = 1$ *and* $L\psi = \gamma\psi$. *Its second eigenvalue is* 0, *which is also simple:* $\ker L = \phi$. *As a consequence, there exists* $\lambda > 0$ *such that for all* $v \in H^1$, *there holds*

$$|\langle Lv, v\rangle| \le \frac{1}{\lambda}\|v\|_{H^1}^2, \tag{3.1}$$
$$\langle Lv, v\rangle \le \frac{1}{\lambda}\|v - \langle v, \psi\rangle\psi\|_{H^1}^2 - \lambda\langle v, \psi\rangle^2, \tag{3.2}$$

and, for all $v \in H^2$,

$$\|Lv\|_{L^2}^2 \ge 4\lambda\|v\|_{H^2}^2 - \frac{1}{\lambda}\langle v, \phi\rangle^2.$$

Proof. Using Sturm-Liouville theory on $\mathbb{R}$ (see [2, Chapter B.5], especially the proof of Theorem B.61 which can be easily adapted to this case), the fact that ϕ vanishes once shows that 0 is the second eigenvalue for L, there is a unique negative eigenvalue $\gamma < 0$ so that $L - \gamma$ is a positive operator, and both are simple. It also means that there exists $\delta > 0$ such that every $\lambda_0 \in \sigma(L_2) \setminus \{\gamma, 0\}$ (where $\sigma(L)$ is the spectrum of L) satisfies $\lambda_0 \geq \delta$.

Denoting ψ a normalized eigenfunction related to γ, we can easily derive the first two estimates. From classic estimates, there holds $\psi \in H^2$. First, (3.1) is an easy consequence of the fact that L is a Schrödinger operator with an L^∞ potential. Therefore, for any $v \in H^1$, denoting $\tilde{v} = v - \langle v, \psi\rangle\psi$, we get:

$$\begin{aligned}\frac{1}{\lambda}\|\tilde{v}\|^2 \geq \langle L\tilde{v}, \tilde{v}\rangle &= \langle Lv, v\rangle - \langle v, \psi\rangle\langle L\psi, v\rangle - \langle v, \psi\rangle\langle \psi, Lv\rangle + \langle v, \psi\rangle^2\langle L\psi, \psi\rangle \\ &= \langle Lv, v\rangle - 2\langle v, \psi\rangle\langle L\psi, v\rangle + \langle v, \psi\rangle^2\langle L\psi, \psi\rangle \\ &= \langle Lv, v\rangle - 2\gamma\langle v, \psi\rangle^2 + \gamma\langle v, \psi\rangle^2 \\ &= \langle Lv, v\rangle - \gamma\langle \psi, v\rangle^2,\end{aligned}$$

which gives (3.2) by the fact that $\gamma < 0$, up to decreasing λ. Last, let

$$a \coloneqq \inf\left\{\langle Lv, v\rangle \,|\, v \in H^1, \|v\|_{L^2} = 1, \langle v, \phi\rangle = \langle v, \psi\rangle = 0\right\}.$$

We aim to prove that $a > 0$. This quantity is actually related to the third Rayleigh quotient (see Lemma A.1):

$$a = \mu_3(L) \coloneqq \sup_{\psi_1, \psi_2} \inf_{u \in \mathrm{span}(\psi_1, \psi_2)^\perp \cap H^1 \setminus \{0\}} \frac{\langle Lu, u\rangle}{\|u\|_{L^2}^2}.$$

Moreover, we know that, either $\mu_3(L)$ is the third eigenvalue (counted with multiplicity) of L or the bottom of its essential spectrum. By the assumption $\lim_{\pm\infty} V = \ell > 0$, it is known that the essential spectrum of L is $[\ell, \infty)$. On the other hand, 0 is the second eigenvalue and it is simple. Thus, we conclude that $a > 0$. Therefore, if $v \in H^2$ with $\langle v, \psi\rangle = \langle v, \sin\theta_{h_0}\rangle = 0$, there holds

$$\|Lv\|_{L^2}\|v\|_{L^2} \geq \langle Lv, v\rangle \geq a\|v\|_{L^2}^2,$$

which yields $\|Lv\|_{L^2} \geq a\|v\|_{L^2}$. Let $b \coloneqq \frac{a}{1+2a+2\|V\|_{L^\infty}} \in (0, \frac{1}{2})$. We thus get

$$\begin{aligned}\|Lv\|_{L^2}^2 &= 2b\|\Delta v\|_{L^2}^2 + (1-2b)\langle Lv, Vv\rangle + 2b\langle \Delta v, Vv\rangle + 2b\|Vv\|_{L^2}^2 \\ &\geq 2b\|\Delta v\|_{L^2}^2 + a(1-2b)\|v\|_{L^2}^2 - 4b\|\Delta v\|_{L^2}\|Vv\|_{L^2} + 2b\|Vv\|_{L^2}^2 \\ &\geq 2b\|\Delta v\|_{L^2}^2 + a(1-2b)\|v\|_{L^2}^2 - b\|\Delta v\|_{L^2}^2 - 4b\|Vv\|_{L^2}^2 + 2b\|Vv\|_{L^2}^2 \\ &\geq b\|\Delta v\|_{L^2}^2 + a(1-2b)\|v\|_{L^2}^2 - 2b\|Vv\|_{L^2}^2 \\ &\geq b\|\Delta v\|_{L^2}^2 + \underbrace{(a(1-2b) - 2b\|V\|_{L^\infty}^2)}_{=b}\|v\|_{L^2}^2 = b(\|\Delta v\|_{L^2}^2 + \|v\|_{L^2}^2).\end{aligned}$$

As $\|\Delta v\|_{L^2} + \|v\|_{L^2}$ controls the H^2 norm, we can conclude that, for some $b' > 0$,

$$\|Lv\|_{L^2}^2 \geq b'\|v\|_{H^2}^2.$$

Let us emphasize that we can reduce b', therefore we can assume that b' is as small as we want it to be. Now, if we take a general $v \in H^2$, we can apply the previous estimate to $v - \langle v, \psi\rangle\psi - \frac{1}{\|\phi\|_{L^2}^2}\langle v, \phi\rangle\phi$. Moreover,

$$\begin{aligned}\left\|L\Big(v - \langle v, \psi\rangle\psi - \frac{1}{\|\phi\|_{L^2}^2}\langle v, \phi\rangle\phi\Big)\right\|_{L^2}^2 &= \|Lv - \gamma\langle v, \psi\rangle\psi\|_{L^2}^2 \\ &= \|Lv\|_{L^2}^2 - 2\gamma\langle v, \psi\rangle \underbrace{\langle Lv, \psi\rangle}_{=\langle v, L\psi\rangle = \gamma\langle v, \psi\rangle} + \gamma^2\langle v, \psi\rangle^2 \\ &= \|Lv\|_{L^2}^2 - \gamma^2\langle v, \psi\rangle^2.\end{aligned}$$

On the other hand,

$$\left\| v - \langle v, \psi\rangle\psi - \frac{1}{\|\phi\|_{L^2}^2}\langle v, \phi\rangle\phi \right\|_{H^2}^2 \geq \frac{1}{2}\|v\|_{H^2}^2 - 2\langle v, \psi\rangle^2\|\psi\|_{H^2}^2 - \frac{2}{\|\phi\|_{L^2}^4}\langle v, \phi\rangle^2\|\phi\|_{H^2}^2,$$

where we have used $(a-b)^2 \geq \frac{1}{2}a^2 - b^2$ and $(b+c)^2 \leq 2b^2 + 2c^2$, so that $(a-b-c)^2 \geq \frac{1}{2}a^2 - 2b^2 - 2c^2$. Therefore, we obtain

$$\begin{aligned}\|Lv\|_{L^2}^2 &\geq \frac{b'}{2}\|v\|_{H^2}^2 + \Big(\gamma^2 - 2b'\|\psi\|_{H^2}^2\Big)\langle v, \psi\rangle^2 - \frac{2b'}{\|\phi\|_{L^2}^4}\langle v, \phi\rangle^2\|\phi\|_{H^2}^2 \\ &\geq \frac{b'}{2}\|v\|_{H^2}^2 - \frac{2b'}{\|\phi\|_{L^2}^4}\langle v, \phi\rangle^2\|\phi\|_{H^2}^2,\end{aligned}$$

by assuming $b' \leq \frac{\gamma^2}{2\|\psi\|_{H^2}^2}$. □

3.1.3. *Application to L_1 and L_2.* The previous lemmas, Lemma 3.3 and Lemma 3.4, when applied to the two operators, offer valuable insights into the spectral properties of L_1 and L_2. However, the behaviour of the two functions in the kernel ($\partial_x\theta_{h_0}$ and $\sin\theta_{h_0}$) depends on the sign of h_0 (see Theorem 1.2). Therefore, we have to distinguish the two cases.

For $h_0 > 0$. In this case, $\partial_x\theta_{h_0}$ remains positive on $\mathbb{R}$, whereas $\sin\theta_{h_0}$ vanishes at $x = 0$.

Lemma 3.5. *Assume that $h_0 > 0$.*

- *L_1 is a self-adjoint positive operator on L^2 with dense domain H^2, and has 0 as first simple eigenvalue with eigenfunction $\partial_x\theta_{h_0} > 0$:*

$$L_1\partial_x\theta_{h_0} = 0. \tag{3.3}$$

As a consequence, there exists $\lambda_1 > 0$ such that for all $v \in H^1$,

$$0 \leq \langle L_1 v, v\rangle \leq \frac{1}{\lambda_1}\|v\|_{H^1}^2,$$
$$\langle L_1 v, v\rangle \geq 4\lambda_1\|v\|_{H^1}^2 - \frac{1}{\lambda_1}\langle v, \partial_x\theta_{h_0}\rangle^2,$$

and for all $v \in H^2$

$$\|L_1 v\|_{L^2}^2 \geq 4\lambda_1\|v\|_{H^2}^2 - \frac{1}{\lambda_1}\langle v, \partial_x\theta_{h_0}\rangle^2.$$

- *L_2 is a self-adjoint operator on L^2 with dense domain H^2. It has a unique negative eigenvalue $\gamma_2 < 0$, which is simple, and we denote $\psi_{h_0} \in H^2$ a normalized eigenfunction related to this eigenvalue: $\|\psi_{h_0}\|_{L^2} = 1$ and*

$$L_2\psi_{h_0} = \gamma_2\psi_{h_0}.$$

Its second eigenvalue is 0, which is also simple, with eigenfunction $\sin\theta_{h_0}$:

$$L_2\sin\theta_{h_0} = 0. \tag{3.4}$$

As a consequence, there exists $\lambda_2 > 0$ such that for all $v \in H^1$, there holds

$$\langle L_2 v, v\rangle \leq \frac{1}{\lambda_2}\|v - \langle v, \psi_{h_0}\rangle\psi_{h_0}\|_{H^1}^2 - \lambda_2\langle v, \psi_{h_0}\rangle^2,$$
$$|\langle L_2 v, v\rangle| \leq \frac{1}{\lambda_2}\|v\|_{H^1}^2,$$

and for all $v \in H^2$,

$$\|L_2 v\|_{L^2}^2 \geq 4\lambda_2\|v\|_{H^2}^2 - \frac{1}{\lambda_2}\langle v, \sin\theta_{h_0}\rangle^2.$$

Proof. First of all, we emphasize that L_1 and L_2 are two self-adjoint Schrödinger operators, whose potentials have finite limits at infinity. Indeed, since $\lim_{-\infty} \theta_{h_0} = 0$ and $\lim_{+\infty} \theta_{h_0} = 2\pi$, we have

$$\lim_{\pm\infty} 1 - 2\sin^2\theta_{h_0} + h_0\cos\theta_{h_0} = \lim_{\pm\infty} 1 - 2\sin^2\theta_{h_0} + 3h_0\cos\theta_{h_0} - 2h_0 = 1 + h_0.$$

We also have (3.3) by Proposition 3.1. Since $\partial_x\theta_{h_0} \in H^2$ and $\partial_x\theta_{h_0} > 0$ (see (1.3)), we can apply Lemma 3.3.

On the other hand, we have (3.4) by Proposition 3.1. However, $\sin\theta_{h_0}$ vanishes at $x = 0$ and its sign changes at this point. Nonetheless, as θ_{h_0} is increasing with values in $(0, 2\pi)$, we know that it is the only point where $\sin\theta_{h_0}$ vanishes. Thus, we can apply Lemma 3.4, leading to the conclusion. □

Remark 3.6. Let us point out that, due to (3.3) and since $2h_0(1 - \cos\theta_{h_0}) > 0$ on $\mathbb{R}$, we obtain that

$$\langle L_2\partial_x\theta_{h_0}, \partial_x\theta_{h_0}\rangle = -2h_0\int (1 - \cos\theta_{h_0})|\partial_x\theta_{h_0}|^2\,\mathrm{d}x < 0.$$

Moreover, since L_1 has no negative eigenvalue, it is positive, so that for any $f \in H^2$,

$$\langle L_2 f, f\rangle = \langle L_1 f, f\rangle - 2h_0\int (1 - \cos\theta_{h_0})|f|^2\,\mathrm{d}x \geq -2h_0\int (1 - \cos\theta_{h_0})|f|^2\,\mathrm{d}x \geq -4h_0\|f\|_{L^2}^2.$$

With its link with the first Rayleigh quotient (see Appendix A), we can probably estimate that the first eigenvalue has the same order as $-h_0$, at least when h_0 is small.

For $h_0 \in (-1, 0)$. If $h_0 \in (-1, 0)$, conversely, $\sin\theta_{h_0}$ remains positive on $\mathbb{R}$, whereas $\partial_x\theta_{h_0}$ vanishes at $x = 0$.

Lemma 3.7. *Assume that $h_0 \in (-1, 0)$.*

- *L_1 is a self-adjoint operator on L^2 with dense domain H^2. It has a unique negative eigenvalue $\gamma_1 < 0$, which is simple, and we denote $\psi_{h_0} \in H^2$ a normalized eigenfunction related to this eigenvalue: $\|\psi_{h_0}\|_{L^2} = 1$ and*

$$L_1\psi_{h_0} = \gamma_1\psi_{h_0}.$$

Its second eigenvalue is 0, which is also simple, with eigenfunction $\partial_x\theta_{h_0}$:

$$L_1\partial_x\theta_{h_0} = 0. \tag{3.5}$$

As a consequence, there exists $\lambda_1 > 0$ such that for all $v \in H^1$,

$$\langle L_1 v, v\rangle \leq \frac{1}{\lambda_1}\|v - \langle v, \psi_{h_0}\rangle\psi_{h_0}\|_{H^1} - \lambda_1\langle v, \psi_{h_0}\rangle^2,$$
$$|\langle L_1 v, v\rangle| \leq \frac{1}{\lambda_1}\|v\|_{H^1}^2,$$

and for all $v \in H^2$,

$$\|L_1 v\|_{L^2}^2 \geq 4\lambda_1\|v\|_{H^2}^2 - \frac{1}{\lambda_1}\langle v, \sin\theta_{h_0}\rangle^2.$$

- *L_2 is a self-adjoint positive operator on L^2 with dense domain H^2, and has 0 as first simple eigenvalue with eigenfunction $\sin\theta_{h_0} > 0$:*

$$L_2\sin\theta_{h_0} = 0. \tag{3.6}$$

As a consequence, there exists $\lambda_2 > 0$ such that for all $v \in H^1$, there holds

$$0 \leq \langle L_2 v, v\rangle \leq \frac{1}{\lambda_2}\|v\|_{H^1}^2,$$
$$\langle L_2 v, v\rangle \geq 4\lambda_2\|v\|_{H^1}^2 - \frac{1}{\lambda_2}\langle v, \partial_x\theta_{h_0}\rangle^2,$$

and for all $v \in H^2$

$$\|L_2 v\|_{L^2}^2 \geq 4\lambda_2\|v\|_{H^2}^2 - \frac{1}{\lambda_2}\langle v, \partial_x\theta_{h_0}\rangle^2.$$

Proof. Once again, L_1 and L_2 are two self-adjoint Schrödinger operators, whose potentials have finite limits at infinity. Indeed, since $\lim_{\pm\infty} \theta_{h_0} = 0$, we have

$$\lim_{\pm\infty} 1 - 2\sin^2\theta_{h_0} + h_0\cos\theta_{h_0} = \lim_{\pm\infty} 1 - 2\sin^2\theta_{h_0} + 3h_0\cos\theta_{h_0} - 2h_0 = 1 + h_0.$$

We also have (3.5) and (3.6) by Proposition 3.1. Since $\sin\theta_{h_0} \in H^2$ and $\sin\theta_{h_0} > 0$ (see (1.3)), we can apply Lemma 3.3 to L_2. On the other hand, $\partial_x\theta_{h_0}$ vanishes at $x = 0$. Nonetheless, we also know that $\partial_x\theta_{h_0} > 0$ on $(-\infty, 0)$ and $\partial_x\theta_{h_0} < 0$ on $(0, \infty)$. Thus, we can apply Lemma 3.4 to L_1. □

Remark 3.8. Let us point out that, due to (3.4) and since $2h_0(1 - \cos\theta_{h_0}) < 0$ on $\mathbb{R}$, we obtain that

$$\langle L_1 \sin\theta_{h_0}, \sin\theta_{h_0}\rangle = 2h_0 \int (1 - \cos\theta_{h_0})|\sin\theta_{h_0}|^2 \,\mathrm{d}x < 0.$$

Moreover, since L_2 has no negative eigenvalue, it is positive, so that for any $f \in H^2$,

$$\langle L_1 f, f\rangle = 2h_0 \int (1 - \cos\theta_{h_0})|f|^2 \,\mathrm{d}x + \langle L_2 f, f\rangle \geq 2h_0 \int (1 - \cos\theta_{h_0})|f|^2 \,\mathrm{d}x \geq -4h_0\|f\|_{L^2}^2.$$

With its link with the first Rayleigh quotient (see Appendix A), we can probably estimate that the first eigenvalue is of order of h_0, at least when h_0 is small.

3.2. The associated basis

3.2.1. *Definition and equivalence.* Let us introduce the following vector, depending on x:

$$n_{h_0} = \begin{pmatrix} -\sin\theta_{h_0} \\ \cos\theta_{h_0} \\ 0 \end{pmatrix}.$$

Similarly to [5], such a vector allows us to introduce the frame $(w_{h_0}(x), n_{h_0}(x), e_3)$. This frame is a direct orthogonal basis for any $x \in \mathbb{R}$, and it is better adapted to a $\mathbb{S}^2$-valued magnetisation m close to a stationary solution w_{h_0}. Indeed, if $m = w_{h_0} + \eta \in \mathbb{S}^2$ with η small (in some sense), we can decompose η in this frame: $\eta = \mu w_{h_0} + \nu n_{h_0} + \rho e_3$. Then μ is quadratic in η, whose norm is thus equivalent to that of ν and ρ. The precise statement is as follows.

Lemma 3.9. *There exists $\delta_3 > 0$ and $C_2 > 0$ such that the following holds. Let $m = w_{h_0} + \eta : \mathbb{R} \to \mathbb{S}^2$ be such that*

$$\|\eta\|_{H^1} < \delta_3.$$

We decompose η in the (w_{h_0}, n_{h_0}, e_3) basis pointwise in x:

$$\eta = \mu w_{h_0} + \nu n_{h_0} + \rho e_3 \quad \textit{where} \quad \mu \coloneqq \eta \cdot w_{h_0}, \quad \nu = \eta \cdot n_{h_0}, \quad \rho = \eta \cdot e_3.$$

Then $\mu, \nu, \rho \in H^1$, with

$$\|\mu\|_{H^1} \leq C_2\|\eta\|_{H^1}^2, \qquad \frac{1}{C_2}\|\eta\|_{H^1} \leq \|(\nu, \rho)\|_{H^1} \leq C_2\|\eta\|_{H^1}. \tag{3.7}$$

In particular, $\mu = -\frac{1}{2}|\eta|^2 = -\frac{1}{2}(\nu^2 + \rho^2) + O_0^4(\eta)$, and $\mu \in L^1$. We also have

$$\eta \cdot e_1 = \cos\theta_{h_0}\,\mu - \sin\theta_{h_0}\,\nu \in L^1. \tag{3.8}$$

If furthermore $\eta \in H^2$, then $\mu, \nu, \rho \in H^2$ and

$$\|(\nu, \rho)\|_{H^2} \leq C_2\|\eta\|_{H^2}$$

Last, there also hold

$$\rho\sin\theta_{h_0} = \eta \cdot (e_1 \wedge w_{h_0}), \qquad \nu\,\partial_x\theta_{h_0} = \eta \cdot \partial_x w_{h_0}.$$

Proof. The proof is similar to the first step of the proof of [5, Proposition 4.16]. First, the relations between μ, ν, ρ and η along with Lemma 1.13 and the regularity and integrability result on θ_{h_0} in Theorem 1.2 give

$$\|\mu\|_{H^k} + \|\nu\|_{H^k} + \|\rho\|_{H^k} \lesssim \|\eta\|_{H^k}.$$

On the other side, $\eta = \mu w_{h_0} + \nu n_{h_0} + \rho e_3$ and therefore

$$\|\eta\|_{H^k} \lesssim \|\mu\|_{H^k} + \|\nu\|_{H^k} + \|\rho\|_{H^k}.$$

The equality $\mu = \frac{1}{2}|\eta|^2$ comes from the expansion of $|w_* + \eta|^2 = 1$, which gives the first inequality of (3.7) with Lemma 1.13. It also gives the fact that $\mu \in L^1$, and then (3.8) is deduced by the relation

$$e_1 = \cos\theta_{h_0} w_{h_0} - \sin\theta_{h_0} n_{h_0}.$$

We conclude that $\eta \cdot e_1 \in L^1$ thanks to $\mu \in L^1$, $\sin\theta_{h_0} \in L^2$ and $\nu \in L^2$. As soon as $\|\eta\|_{H^1}$ is small enough the second inequality in (3.7) is then straightforward. Eventually, the last equality comes from the formulas

$$\partial_x w_{h_0} = \partial_x \theta_{h_0}\, n_{h_0}, \qquad e_1 \wedge w_{h_0} = \sin\theta_{h_0}\, e_3. \qquad \square$$

3.2.2. *Expansions of some expressions.* In a similar vein to the approach taken in [5], we now turn our attention to a lemma providing expansions in η (or equivalently, in ν and ρ) for several crucial expressions.

Lemma 3.10. *Let $m \in \mathcal{H}^2$ satisfying the assumptions of Lemma 3.9, and let η, μ, η, ρ as defined in the same Lemma. Then there holds*

$$\begin{aligned}\delta E(\eta) = O_2^2(\eta) &+ (\partial_{xx}\theta_{h_0}\nu + 2\partial_x\theta_{h_0}\partial_x\nu + \cos\theta_{h_0}\sin\theta_{h_0}\nu)\, w_{h_0} \\ &+ (-\partial_{xx}\nu + (\partial_x\theta_{h_0})^2\nu + \cos^2\theta_{h_0}\nu)\, n_{h_0} \\ &+ (-\partial_{xx}\rho + \rho)\, e_3,\end{aligned} \tag{3.9}$$

$$\begin{aligned}\delta E(\eta)\cdot\eta = O_2^3(\eta) &+ (-\partial_{xx}\nu + (1 + 2h_0(1-\cos\theta_{h_0}))\nu)\,\nu \\ &+ (-\partial_{xx}\rho + \rho)\,\rho.\end{aligned} \tag{3.10}$$

Proof. First, from the definition of w_{h_0} and n_{h_0}, there holds $\partial_x w_{h_0} = \partial_x\theta_{h_0} n_{h_0}$ and $\partial_x n_{h_0} = -\partial_x\theta_{h_0} w_{h_0}$. Therefore, we can compute the derivatives of $\eta = \mu\, w_{h_0} + \nu\, n_{h_0} + \rho\, e_3$:

$$\begin{aligned}\partial_x\eta &= (\partial_x\mu - \partial_x\theta_{h_0}\nu)\, w_{h_0} + (\partial_x\nu + \partial_x\theta_{h_0}\mu)\, n_{h_0} + \partial_x\rho\, e_3, \\ \partial_{xx}\eta &= (\partial_{xx}\,\mu - \partial_{xx}\theta_{h_0}\,\nu - 2\partial_x\theta_{h_0}\partial_x\nu - (\partial_x\theta_{h_0})^2\mu) w_{h_0} \\ &\quad + (\partial_{xx}\nu + \partial_{xx}\theta_{h_0}\,\mu + \partial_x\theta_{h_0}\partial_x\mu - (\partial_x\theta_{h_0})^2\nu)\, n_{h_0} \\ &\quad + \partial_{xx}\rho\, e_3.\end{aligned} \tag{3.11}$$

Moreover, we see that $e_2 = \sin\theta_{h_0}\, w_{h_0} + \cos\theta_{h_0}\, n_{h_0}$. Thus, $\eta_2 = \eta\cdot e_2 = \sin\theta_{h_0}\,\mu + \cos\theta_{h_0}\,\nu$ and therefore

$$\eta_2 e_2 = (\sin\theta_{h_0}\,\mu + \cos\theta_{h_0}\,\nu)\sin\theta_{h_0}\, w_{h_0} + (\sin\theta_{h_0}\,\mu + \cos\theta_{h_0}\,\nu)\cos\theta_{h_0}\, n_{h_0}.$$

Last, $\eta_3 e_3 = \rho e_3$. Thus, using the fact that $\mu = -\frac{1}{2}|\eta|^2 = O_0^2(\eta)$ (see Lemma 3.9) and Lemma 1.13, all the terms involving μ in (3.11) are $O_2^2(\eta)$. Then, we obtain (3.9) by the expression of δE in (1.1). As for (3.10), it is obtained by expanding $\delta E(\eta)$ with (3.9), by neglecting once again all the terms involving μ and by using (1.3). $\square$

3.2.3. *Modulation and decomposition of the magnetisation.* In our framework, decomposing directly m with the aforementioned associated basis is not the best option. Indeed, in order to prove Theorem 1.7, we need to prove not only that w_{h_0} is (nonlinearly) unstable, but also its orbit. Therefore, formally, this decomposition should be used on $(-g_m).w_{h_0}$ where g_m minimizes the following minimization problem:

$$\min_{g\in G}\|m - g.w_{h_0}\|_{H^1}.$$

However, such a g_m is hard to track down, and might not be smooth enough in time. Therefore, we will more generally use this decomposition on $(-g).m$ for some g well chosen. To be more specific, we would like g to be near g_m, to be smooth enough and to make vanish the bad negative terms in the estimations of Lemma 3.5 or Lemma 3.7. This is the goal of the following result of modulation and decomposition of a magnetization m near w_{h_0}.

Lemma 3.11. *There exists $\delta_1 > 0$ and $C_1 \geq 1$ such that the following holds. For all $w \in \mathcal{M}$ where $\mathcal{M} \coloneqq (e_1 + H^1) \cap \mathscr{C}(\mathbb{R}, \mathbb{S}^2)$ such that*

$$\|w - w_{h_0}\|_{H^1} \leq \delta_1, \tag{3.12}$$

there exists a gauge $g = (y, \phi) \in G$ such that the following orthogonality constraints hold:

$$\int \eta \cdot \partial_x w_{h_0} \,\mathrm{d}x = \int \eta \cdot (e_1 \wedge w_{h_0}) \,\mathrm{d}x = 0,$$

where $\eta \in H^1$ is defined by

$$\eta \coloneqq (-g).w - w_{h_0},$$

so that $w = g.(w_{h_0} + \eta)$. Furthermore,

$$|g| + \|\eta\|_{H^1} \leq C_1 \|w - w_{h_0}\|_{H^1}.$$

The gauge g is unique with the above properties and the map $w \in \mathcal{M} \mapsto g \in G$ is of class $\mathscr{C}^\infty$ in a neighborhood of w_{h_0}.

Of course, such a lemma can be applied to $(-g_m).m$, which leads to the requested properties. This lemma is proven in Appendix B.

4. The modified energy: dispersion and expansion

We first recall the definition of the modified energy E_{h_0} given in (1.7)

$$E_{h_0}(m) \coloneqq E(m) - h_0 \int (m_1 - 1) \,\mathrm{d}x.$$

We also recall that its variation is related to the variation δE of the initial energy E and to the effective magnetic field H:

$$\delta E_{h_0}(m) = \delta E(m) - h_0 e_1 = -H(m).$$

In particular, $\delta E_{h_0}(m)$ and $H(m)$ are in $L^2 + L^\infty$ if $m \in \mathcal{H}^2$, and in $H^{-1} + L^\infty$ if m is only $\mathcal{H}^1$. Another property of E_{h_0} can be deduced from (1.8): this modified energy is decreasing through the flow of (LLG).

Proposition 4.1. *With the assumption of Theorem 1.1, assuming furthermore that $h(t) = h_0 > 0$ is constant, $m_0 \in \mathcal{H}^s$ for some $s \geq 2$ and $\lim_{\pm\infty} m_0 = e_1$, then for all $t \in [0, T_+(m))$,*

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{h_0}(m(t)) = -\alpha \int |m \wedge H(m)|^2 \,\mathrm{d}x.$$

Proof. From (LLG), there holds

$$\begin{aligned}
\partial_t m_1 &= (m \wedge H(m)) \cdot e_1 - \alpha \Big(m \wedge (m \wedge H(m)) \Big) \cdot e_1, \\
\partial_t (m_1 - 1) &= -H(m) \cdot (m \wedge e_1) + \alpha (m \wedge e_1) \cdot (m \wedge H(m)) \\
&= -\partial_{xx} m \cdot (m \wedge e_1) - \alpha (m \wedge e_1) \cdot (m \wedge \delta E(m)) + h_0 \alpha (m \wedge e_1) \cdot (m \wedge e_1).
\end{aligned}$$

As we know that $m \in \mathcal{H}^2$ and thus $\delta E(m) \in L^2$ on $[0, T_+(m))$, with uniform bounds on every compact subintervals, we can easily infer that all the terms in the right-hand side of the last equality are in L^1. Therefore, since $m_1(t) - 1 \in L^1$ for all $t \geq 0$, we get $m_1 - 1 \in C^1([0, T_+(m)), L^1)$ and

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int (m_1 - 1) \,\mathrm{d}x &= -\int \partial_{xx} m \cdot (m \wedge e_1) \,\mathrm{d}x - \alpha \int (m \wedge e_1) \cdot (m \wedge \delta E(m)) \,\mathrm{d}x + h_0 \alpha \int |m \wedge e_1|^2 \,\mathrm{d}x \\
&= \int \partial_x m \cdot (\partial_x m \wedge e_1) \,\mathrm{d}x - \alpha \int (m \wedge e_1) \cdot (m \wedge \delta E(m)) \,\mathrm{d}x + h_0 \alpha \int |m \wedge e_1|^2 \,\mathrm{d}x
\end{aligned}$$

$$= -\alpha \int (m \wedge e_1) \cdot (m \wedge \delta E(m)) \, \mathrm{d}x + h_0 \alpha \int |m \wedge e_1|^2 \, \mathrm{d}x.$$

The conclusion is reached by using this equality into the relation between $E(m)$ and $E_{h_0}(m)$, which yields

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{h_0}(m(t)) = \frac{\mathrm{d}}{\mathrm{d}t} E(m(t)) - h_0 \frac{\mathrm{d}}{\mathrm{d}t} \int (m_1 - 1) \, \mathrm{d}x,$$

and using the fact that $|\delta E(m)|^2 - |m \cdot \delta E(m)|^2 = |\delta E(m) \wedge m|^2$. □

This property shows that E_{h_0} is therefore better suited to describe the behavior of any solution near its critical point w_{h_0}, and more generally near the orbit of w_{h_0} since E_{h_0} is invariant by translation and rotation around e_1. The results in Theorem 1.2 and in Section 3.2 allow to expand the modified energy E_{h_0} and its dispersion through the flow of the equation (see Proposition 4.1) near w_{h_0} in both cases $h_0 > 0$ and $h_0 \in (-1, 0)$. First, we begin by the expansion of the modified energy.

Lemma 4.2. *Let $m \in \mathcal{H}^2$ satisfying the assumptions of Lemma 3.9, and let η, μ, η, ρ as defined in the same Lemma. Then there exists $C > 0$ such that*

$$\left| E_{h_0}(m) - E_{h_0}(w_{h_0}) - \frac{1}{2} \Big(\langle L_1 \nu, \nu \rangle + \langle L_2 \rho, \rho \rangle \Big) \right| \leq C \|\eta\|_{H^1}^3. \tag{4.1}$$

Proof. By the definition of E and δE, we know that $E(m) = \frac{1}{2} \int \delta E(m) \cdot m \, \mathrm{d}x$. Since $m = w_{h_0} + \eta$, there holds

$$\begin{aligned} E_{h_0}(m) = E_{h_0}(w_{h_0} + \eta) &= E(w_{h_0} + \eta) - h_0 \int (w_{h_0} \cdot e_1 - 1 + \eta \cdot e_1) \, \mathrm{d}x \\ &= \underbrace{\frac{1}{2} \int \Big(\delta(w_{h_0}) \cdot w_{h_0} - 2h_0 (w_{h_0} \cdot e_1 - 1) \Big) \, \mathrm{d}x}_{I_1} \\ &\qquad + \frac{1}{2} \Big(\underbrace{\int \delta E(w_{h_0}) \cdot \eta \, \mathrm{d}x + \int \delta E(\eta) \cdot w_{h_0} \, \mathrm{d}x - 2h_0 \int \eta \cdot e_1 \, \mathrm{d}x}_{I_2} \Big) \\ &\qquad + \frac{1}{2} \underbrace{\int \delta E(\eta) \cdot \eta \, \mathrm{d}x}_{I_3}. \end{aligned}$$

First, there holds $I_1 = E_{h_0}(w_{h_0})$. As for I_2, we emphasize that $\eta \cdot e_1 \in L^1$ by Lemma 3.9, thus the last integral is well-defined. By grouping all the integrals, using (1.4) and that δE is a linear operator which is self-adjoint with respect to the L^2 scalar product, we get

$$\begin{aligned} I_2 &= \int \Big(2\delta E(w_{h_0}) \cdot \eta - 2h_0 \eta \cdot e_1 \Big) \, \mathrm{d}x \\ &= -2 \int H(w_{h_0}) \cdot \eta \, \mathrm{d}x \\ &= -2 \int \Lambda(x) w_{h_0} \cdot \eta \, \mathrm{d}x \\ &= 2 \int (2 \sin^2 \theta_{h_0} - 3h_0 \cos \theta_{h_0} + 2h_0) \mu \, \mathrm{d}x \\ &= - \int (2 \sin^2 \theta_{h_0} - 3h_0 \cos \theta_{h_0} + 2h_0)(\nu^2 + \rho^2) \, \mathrm{d}x + \int O_0^4(\eta) \, \mathrm{d}x. \end{aligned}$$

Last, using (3.10), I_3 can be expanded as follows:

$$I_3 = \int (-\partial_{xx} \nu + (1 + 2h_0(1 - \cos \theta_{h_0})) \nu) \, \nu \, \mathrm{d}x + \int (-\partial_{xx} \rho + \rho) \, \rho \, \mathrm{d}x + \int O_2^3(\eta) \, \mathrm{d}x.$$

The last term for both I_2 and I_3 can be estimated thanks to Lemma 1.13. Summing the other terms lead to (4.1). □

Then, we continue with the expansion of the dispersion.

Lemma 4.3. *Let $m \in \mathcal{H}^2$ satisfying the assumptions of Lemma 3.9, and let η, μ, η, ρ as defined in the same Lemma. Then there exists $C > 0$ such that*

$$\left| \int |m \wedge H(m)|^2 \, \mathrm{d}x - (\|L_1 \nu\|_{L^2}^2 + \|L_2 \rho\|_{L^2}^2) \right| \leq C \|\eta\|_{H^1} \|\eta\|_{H^2}^2. \tag{4.2}$$

Proof. First, since $H(m) = -\delta E(m) + h_0 e_1$ with δE linear in $m = w_{h_0} + \eta$, then

$$m \wedge H(m) = w_{h_0} \wedge H(w_{h_0}) + \eta \wedge H(w_{h_0}) - w_{h_0} \wedge \delta E(\eta) - \eta \wedge \delta E(\eta).$$

Since $H(w_{h_0}) = \Lambda(x) w_{h_0}$, the first term of this expansion vanishes. Therefore, we can expand $|m \wedge H(m)|^2$ as follows:

$$\begin{aligned} |m \wedge H(m)|^2 &= |\eta \wedge H(w_{h_0}) - w_{h_0} \wedge \delta E(\eta) - \eta \wedge \delta E(\eta)|^2 \\ &= |\Lambda(x) \eta \wedge w_{h_0} - w_{h_0} \wedge \delta E(\eta)|^2 \\ &\quad + 2(\eta \wedge \delta E(\eta)) \cdot (w_{h_0} \wedge \delta E(\eta) - \Lambda(x) \eta \wedge w_{h_0}) \\ &\quad + |\eta \wedge \delta E(\eta)|^2. \end{aligned}$$

From (3.9), we have

$$\begin{aligned} w_{h_0} \wedge \delta E(\eta) = O_2^2(\eta) &- (-\partial_{xx} \rho + \rho) \, n_{h_0} \\ &+ (-\partial_{xx} \nu + (\partial_x \theta_{h_0})^2 \nu + \cos^2 \theta_{h_0} \nu) \, e_3. \end{aligned}$$

On the other hand, $\eta \wedge w_{h_0} = \rho n_{h_0} - \nu e_3$, so that, using (1.3),

$$\begin{aligned} \Lambda(x) \eta \wedge w_{h_0} - w_{h_0} \wedge \delta E(\eta) = O_2^2(\eta) &+ \Big(-\partial_{xx} \rho + (1 + \Lambda(x)) \rho \Big) \, n_{h_0} \\ &- \Big(-\partial_{xx} \nu + \Big((\partial_x \theta_{h_0})^2 + \cos^2 \theta_{h_0} + \Lambda(x) \Big) \nu \Big) \, e_3. \end{aligned}$$

Using (1.5) in Theorem 1.2, we get

$$\begin{aligned} 1 + \Lambda(x) &= 1 - 2 \sin^2 \theta_{h_0} + 3 h_0 \cos \theta_{h_0} - 2 h_0, \\ (\partial_x \theta_{h_0})^2 + \cos^2 \theta_{h_0} + \Lambda(x) &= 1 - 2 \sin^2 \theta_{h_0} + h_0 \cos \theta_{h_0}. \end{aligned}$$

Therefore, we get

$$\Lambda(x) \eta \wedge w_{h_0} - w_{h_0} \wedge \delta E(\eta) = O_2^2(\eta) - L_2 \rho \, n_{h_0} + L_1 \nu \, e_3.$$

This leads to

$$|\Lambda(x) \eta \wedge w_{h_0} - w_{h_0} \wedge \delta E(\eta)|^2 = |L_2 \rho|^2 + |L_1 \nu|^2 + O_4^3(\eta) + O_4^4(\eta).$$

Last, we also have $(\eta \wedge H(\eta)) \cdot (w_{h_0} \wedge H(\eta) + \Lambda(x) \eta \wedge w_{h_0}) = O_4^3(\eta)$ and $|\eta \wedge \delta E(\eta)|^2 = O_4^4(\eta)$. (4.2) is then obtained by combining and integrating all these equalities and by using Lemma 1.13 for any $\int O_4^\ell(\eta) \, \mathrm{d}x$. □

5. Proof of the instability

We now delve into the proof of Theorem 1.7. In brief, the idea of the proof consists in tracking down the evolution of the modified energy. We know it is decreasing thanks to Proposition 4.1 (which is recalled in Section 5.1.1), but the evolution can be better described thanks to the expansions of the energy (Lemma 4.2) and of the dispersion (Lemma 4.3). As soon as this energy is smaller than the energy of the stationary solution, it decreases exponentially in time and this estimate can be pushed back to the distance between the solution and $g.w_{h_0}$ for any $g \in G$.

5.1. Summary of the main properties

5.1.1. *Evolution of the modified energy.* First, we recall here the evolution of the modified energy, *i.e.* Proposition 4.1.

Proposition 5.1. *With the assumption of Theorem 1.1, assuming furthermore that $h(t) = h_0 > 0$ is constant, $m_0 \in \mathcal{H}^s$ for some $s \geq 2$ and $\lim_{\pm\infty} m_0 = e_1$, then for all $t \in [0, T_+(m))$,*

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{h_0}(m(t)) = -\alpha \int |m \wedge H(m)|^2 \, \mathrm{d}x.$$

5.1.2. *Expansion of the energy for $h_0 > 0$.* Now, we recall the expansion of the energy in terms of ν and ρ, starting with the case $h_0 > 0$. From now on, let $\delta_3 > 0$ as defined in Lemma 3.9.

Lemma 5.2. *Assume that $h_0 > 0$. There exists $C, \lambda > 0$ such that, for any $m = w_{h_0} + \eta : \mathbb{R} \to \mathbb{S}^2$ satisfying*

$$\|\eta\|_{H^1} < \delta_3,$$

there holds

$$E_{h_0}(m) - E_{h_0}(w_{h_0}) \leq C\Big(\|\nu\|_{H^1}^2 + \|\rho - \langle \rho, \psi_{h_0}\rangle \psi_{h_0}\|_{H^1}^2 + \|\eta\|_{H^1}^3\Big) - \lambda \langle \rho, \psi_{h_0}\rangle^2,$$
$$|E_{h_0}(m) - E_{h_0}(w_{h_0})| \leq \frac{1}{\lambda} \|\eta\|_{H^1}^2,$$

where ν, ρ are defined in Lemma 3.9 and ψ_{h_0} in Lemma 3.5.

This lemma is straightforwardly obtained by Lemma 4.2 and Lemma 3.5.

5.1.3. *For $h_0 \in (-1, 0)$.* Now, we recall the same result for the case $h_0 \in (-1, 0)$.

Lemma 5.3. *Assume that $h_0 \in (-1, 0)$. There exists $C, \lambda > 0$ such that, for any $m = w_{h_0} + \eta : \mathbb{R} \to \mathbb{S}^2$ satisfying*

$$\|\eta\|_{H^1} < \delta_3,$$

there holds

$$E_{h_0}(m) - E_{h_0}(w_{h_0}) \leq C\Big(\|\rho\|_{H^1}^2 + \|\nu - \langle \nu, \psi_{h_0}\rangle \psi_{h_0}\|_{H^1}^2 + \|\eta\|_{H^1}^3\Big) - \lambda \langle \nu, \psi_{h_0}\rangle^2,$$
$$|E_{h_0}(m) - E_{h_0}(w_{h_0})| \leq \frac{1}{\lambda} \|\eta\|_{H^1}^2,$$

where ν, ρ are defined in Lemma 3.9 and ψ_{h_0} in Lemma 3.7.

Once again, this lemma is easily obtained by Lemma 4.2 and Lemma 3.7.

5.1.4. *Expansion of the dispersion.* Last, we summarize the expansion of the dispersion in terms of ν and ρ.

Lemma 5.4. *Assume that $h_0 > 0$ or $h_0 \in (-1, 0)$. There exists $\delta_3 > 0$ and $C, \lambda > 0$ such that, for any $m = w_{h_0} + \eta : \mathbb{R} \to \mathbb{S}^2$ satisfying*

$$\|\eta\|_{H^1} < \delta_3,$$

there holds

$$-\int |m \wedge H(m)|^2 \, \mathrm{d}x \leq -\lambda \|\eta\|_{H^1}^2 + \frac{1}{\lambda}(\langle \nu, \partial_x \theta_{h_0}\rangle^2 + \langle \rho, \sin \theta_{h_0}\rangle^2)$$

where ν, ρ are defined in Lemma 3.9 and ψ_{h_0} in Lemma 3.5.

This last result is obtained thanks to Lemma 4.3 along with Lemma 3.5 (when $h_0 > 0$) and Lemma 3.7 (when $h_0 \in (-1, 0)$).

5.2. V_δ is non-empty

We are now in position to prove Theorem 1.7. We begin by the first part, which is the existence of initial data arbitrarily close to m_{h_0} with a strictly smaller modified energy. We recall this here.

Proposition 5.5. *Let $h_0 \in (-1,0)\cup(0,\infty)$. For any $0<\delta$, the following set*

$$V_\delta \coloneqq \left\{m_0 \in \mathcal{H}^2 \,|\, \|m_0 - w_{h_0}\|_{H^1} < \delta, E_{h_0}(m_0) < E_{h_0}(w_{h_0})\right\}$$

is non-empty.

The proof is constructivist, although it can be easily shown that an infinite number of functions are in the set V_δ, which takes the form of a cone near m_{h_0}.

Proof. 1st case: $h_0 > 0$

First, we point out that $\psi_{h_0} \in H^1 \subset L^\infty$. Let $\varepsilon_0 \neq 0$ small enough to be fixed later, with $\varepsilon_0 < \frac{1}{2\|\psi\|_{L^\infty}}$. Let $\eta_0 \coloneqq \varepsilon_0(\psi_{h_0}e_3 + \Phi w_{h_0})$ where Φ is defined by

$$\Phi \coloneqq \frac{-1+\sqrt{1-2\varepsilon_0^2|\psi_{h_0}|^2}}{\varepsilon_0} = -\frac{2\varepsilon_0|\psi_{h_0}|^2}{1+\sqrt{1-2\varepsilon_0^2|\psi_{h_0}|^2}}.$$

By assumption on ε_0, we know that

$$\frac{1}{2} < 1 - 2\varepsilon_0^2|\psi_{h_0}|^2 < 1.$$

Moreover, since $\psi_{h_0} \in H^k$ for all $k \in \mathbb{N}$, then so is $|\psi_{h_0}|^2$. This proves that Φ is well defined and it is of the same regularity as ψ_{h_0}: $\Phi \in H^k$ for all $k \in \mathbb{N}$, with

$$\|\Phi\|_{H^k} \leq C_k\varepsilon_0,$$

for some $C_k > 0$ depending on k and on $\|\psi_{h_0}\|_{H^k}$. By the definition of Φ, we have

$$\Phi = -\frac{\varepsilon_0}{2}(|\psi_{h_0}|^2 + |\Phi|^2),$$

and

$$|w_{h_0}+\eta_0|^2 = 1 + 2w_{h_0}\cdot\eta_0 + |\eta_0|^2 = 1 + 2\varepsilon_0\Phi + \varepsilon_0^2(|\psi|^2+|\Phi|^2) = 1.$$

Thus, there holds

$$m_0 \coloneqq w_{h_0} + \eta_0 \in \mathcal{H}^k$$

and

$$\|m_0 - w_{h_0}\|_{H^k} \leq C_k\varepsilon_0$$

for all $k \geq 1$. Assuming $C_1\varepsilon_0 \leq \delta$ leads to $\|m_0 - w_{h_0}\|_{H^1} \leq \delta$. Last, using Lemma 5.2, we get, for some $C > 0$,

$$E_{h_0}(m_0) - E_{h_0}(w_{h_0}) \leq -\lambda_2\varepsilon_0^2 + C\varepsilon_0^3.$$

By taking $|\varepsilon_0|$ small enough, we get

$$E_{h_0}(m_0) - E_{h_0}(w_{h_0}) \leq -\frac{1}{2}\lambda_2\varepsilon_0^2 < 0.$$

Therefore, $m_0 \in V_\delta$.

2nd case: $h_0 \in (-1,0)$

Once again, $\psi_{h_0} \in H^1 \subset L^\infty$, and for ε_0 small enough with $\varepsilon_0 < \frac{1}{2\|\psi\|_{L^\infty}}$, we define $\eta_0 \coloneqq \varepsilon_0(\psi_{h_0}n_{h_0} + \Phi w_{h_0})$ where Φ is defined like previously, by

$$\Phi \coloneqq \frac{-1+\sqrt{1-2\varepsilon_0^2|\psi_{h_0}|^2}}{\varepsilon_0} = -\frac{2\varepsilon_0|\psi_{h_0}|^2}{1+\sqrt{1-2\varepsilon_0^2|\psi_{h_0}|^2}}.$$

The same arguments as for $h_0 > 0$ hold here, and we get $m_0 \in V_\delta$ for any ε_0 small enough □

Remark 5.6. One can also substitute ψ_{h_0} by $\partial_x\theta_{h_0}$ in the case $h_0 > 0$ and by $\sin\theta_{h_0}$ in the case $h_0 \in (-1, 0)$. Indeed, Remarks 3.6 and 3.8 show that, in their respective case, $\langle L_2\partial_x\theta_{h_0}, \partial_x\theta_{h_0}\rangle < 0$ and $\langle L_1 \sin\theta_{h_0}, \sin\theta_{h_0}\rangle < 0$, and the same arguments still apply. This remark will be important for numerical simulations, as ψ_{h_0} is not explicit whereas $\partial_x\theta_{h_0}$ and $\sin\theta_{h_0}$ can be easily computed once we have (numerically) computed θ_{h_0}.

5.3. Nonlinear instability of the orbit of w_{h_0}

Now, we are in position to finish the proof of Theorem 1.7.

Proof of Theorem 1.7. Let $\varepsilon \coloneqq \min(\frac{\delta_1}{2}, \frac{\delta_3}{1+2C_1})$ and $0 < \delta < \varepsilon$, where δ_1 and C_1 are defined in Lemma 3.11 and δ_3 is defined in Lemma 3.9. Let $m_0 \in V_\delta$. By Proposition 5.1, there obviously holds that $E_{h_0}(m(t))$ is non-increasing, and thus

$$E_{h_0}(m(t)) \leq E_{h_0}(m_0) < E_{h_0}(w_{h_0}) = E_{h_0}(g.w_{h_0}),$$

for any $g \in G$ and for all $t \geq 0$. In particular, $\mathcal{E}(t) \coloneqq E_{h_0}(m(t)) - E_{h_0}(w_{h_0}) < 0$. Define

$$T \coloneqq \inf\left\{\tau \geq 0 \,|\, \forall t \in [0,\tau], \inf_{g\in G}\|m(t) - g.w_{h_0}\|_{H^1} \leq \varepsilon\right\}.$$

Thanks to the continuity of the flow in $\mathcal{H}^1$ (see part 2 of Theorem 1.1) and with the assumptions on the initial data, there holds $T > 0$. Let $0 < \tau < T$. By definition of T, we have $\inf_{g\in G}\|m(t) - g.w_{h_0}\|_{H^1} \leq \varepsilon \leq \frac{1}{2}\delta_1$ for all $t \in [0,\tau]$. Therefore, for any $t \in [0,T]$, we can find some $\tilde{g}(t) \in G$ such that $\|m(t) - \tilde{g}(t).w_{h_0}\|_{H^1} = \|(-\tilde{g}(t)).m(t) - w_{h_0}\|_{H^1} < 2\varepsilon \leq \delta_1$ Then, from Lemma 3.11 applied to $(-\tilde{g}(t)).m(t)$ for any $t \in [0,\tau]$, we get some $\hat{g}(t)$ such that, by defining $g \coloneqq \tilde{g} + \hat{g}$ and $\eta = (-g(t)).m(t) - w_{h_0}$

$$\|m(t) - g(t).w_{h_0}\|_{H^1} = \|\eta\|_{H^1} \leq C_1\|(-\tilde{g}(t)).m(t) - w_{h_0}\|_{H^1} < 2C_1\varepsilon < \delta_3,$$

with the orthogonality conditions

$$\int \eta(t)\cdot\partial_x w_{h_0}\,\mathrm{d}x = \int \eta(t)\cdot(e_1 \wedge w_{h_0})\,\mathrm{d}x = 0, \qquad \text{for all } t \in [0,\tau].$$

Let us define $\mu(t), \nu(t), \rho(t)$ from Lemma 3.9 applied to $(-g(t)).m(t)$, and apply Lemma 5.2 if $h_0 > 0$ and Lemma 5.3 if $h_0 \in (-1, 0)$, so that there holds for all $t \in [0,\tau]$

$$\|\eta(t)\|_{H^1}^2 \geq \lambda|E_{h_0}((-g(t)).m(t)) - E_{h_0}(w_{h_0})| = -\lambda\mathcal{E}(t), \tag{5.1}$$

since $\mathcal{E}(t) < 0$. On the other hand, by Lemma 5.4,

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{h_0}(m(t)) \leq -\lambda\|\eta(t)\|_{H^1}^2,$$

thus there also holds for all $t \in [0,\tau]$

$$\mathcal{E}(t) - \mathcal{E}(0) \leq -\lambda\int_0^t \|\eta(s)\|_{H^1}^2\,\mathrm{d}s.$$

Using (5.1), we obtain

$$\mathcal{E}(t) \leq \mathcal{E}(0) + \lambda^2\int_0^t \mathcal{E}(s)\,\mathrm{d}s, \tag{5.2}$$

for all $t \in [0,\tau]$. This proves that $g(t) \coloneqq e^{-\lambda^2 t}\int_0^t \mathcal{E}(s)\,\mathrm{d}s$ is decreasing, and more exactly

$$g'(t) \leq \mathcal{E}(0)e^{-\lambda^2 t}.$$

By integration and since $g(0) = 0$, it yields

$$g(t) \leq \frac{1 - e^{-\lambda^2 t}}{\lambda^2}\mathcal{E}(0),$$

which leads to

$$\int_0^t \mathcal{E}(s)\,\mathrm{d}s \leq \frac{e^{\lambda^2 t}-1}{\lambda^2}\mathcal{E}(0).$$

Putting this estimate into (5.2), we finally obtain

$$\mathcal{E}(t) \leq e^{\lambda^2 t}\mathcal{E}(0). \tag{5.3}$$

Now, we show that such an estimate proves that, for all $t \in [0,\tau]$,

$$\inf_{g\in G}\|m(t) - g.w_{h_0}\|_{H^1} \geq \lambda|\mathcal{E}(0)|e^{\lambda^2 t}. \tag{5.4}$$

Indeed, let $t \in [0,\tau]$ and $g_0 \in G$. Since we already know that $\inf_{g\in G}\|m(t) - g.w_{h_0}\|_{H^1} \leq \varepsilon < \delta_3$, then we can assume that $\|m(t) - g_0.w_{h_0}\|_{H^1} < \delta_3$. Therefore, we can apply once again Lemma 5.2 to $(-g_0).m(t)$, which gives

$$\begin{aligned}\|m(t) - g_0.w_{h_0}\|_{H^1}^2 &= \|(-g_0).m(t) - w_{h_0}\|_{H^1}^2 \\ &\geq \lambda|E_{h_0}((-g_0(t)).m(t)) - E_{h_0}(w_{h_0})| = \lambda|E_{h_0}(m(t)) - E_{h_0}(w_{h_0})| = -\lambda\mathcal{E}(t).\end{aligned}$$

Thus, using (5.3) along with the fact that $\mathcal{E}(0) < 0$, we get $\|m(t) - (g_0).w_{h_0}\|_{H^1} \geq \lambda|\mathcal{E}(0)|e^{\lambda^2 t}$. This estimate is true for general $g_0 \in G$, which shows (5.4).

Now, applying this estimate at $t = \tau$, and using the fact that $\inf_{g\in G}\|m(\tau) - g.w_{h_0}\|_{H^1} \leq \varepsilon$ since $\tau < T$ and by definition of T, we get

$$\lambda|\mathcal{E}(0)|e^{\lambda^2\tau} \leq \varepsilon,$$

which yields

$$\tau \leq \frac{1}{\lambda^2}\ln\Big(\frac{\varepsilon}{\lambda|\mathcal{E}(0)|}\Big).$$

As this is true for general $\tau < T$, we obtain that T satisfies the same upper bound, which obviously leads to (1.9) by continuity of $t \mapsto \inf_{g\in G}\|m(t) - g.w_{h_0}\|_{H^1}$. □

6. Numerical simulations

In this section, we delve into the numerical approximations of stationary solutions and explore the evolution of solutions to (LLG) through numerical simulations, all conducted using Python. We begin by examining stationary solutions and proceed to investigate the behavior of solutions under small perturbations from these stationary states. These numerical investigations are supplemented with critical observations, related remarks, and comprehensive discussions regarding the general dynamics of (LLG).

6.1. Numerical approximations of the stationary solutions

The stationary solutions are explicitly related to θ_{h_0} obtained as the solution of an ODE with explicit initial data given by Theorem 1.2. This solution are approximated numerically using the *ode45* function from the *matplotlib* library and displayed in Figures 1 and 2 for several h_0.

It is important to point out that direct use of (1.6) is impractical in the case $h_0 \in (-1,0)$. This ODE involves a function that lack $\mathscr{C}^1$ regularity, particularly at $\theta = \arccos(-1-2h_0)$. Even more problematic, the constant function $\arccos(-1-2h_0)$ is a solution to (1.6), and it is probably the solution that a numerical approximation would find in this setting. Thus, the 2nd order ODE (1.2) will be used when $h_0 \in (-1,0)$, whereas this problem does not arise in the case $h_0 > 0$, giving the ability to use (1.3).

For small h_0 (either positive or negative), the stationary solution w_{h_0} looks like a 2-domain wall, as described in [4], separated from a distance of order roughly $\ln|h_0|$. The distinction between positive and negative h_0 lies in the direction of the transition between $-e_1$ and $+e_1$ of the "domain wall" in $(0,+\infty)$: a rotation around e_1 of angle π separates the two cases. In the case $h_0 \in (-1,0)$, this transition takes the same values as the first transition. Conversely, for $h_0 > 0$, it mirrors the first transition with respect to a rotation around e_1 of angle π.

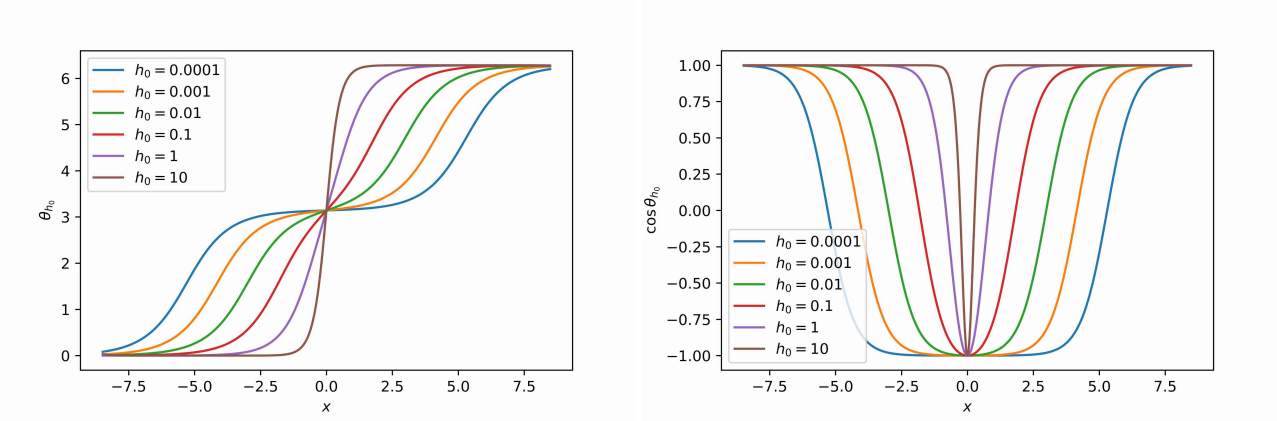

Figure 1: Plot of θ_{h_0} (on the left) and of $\cos\theta_{h_0}$, which is the first component of m_{h_0} (on the right), for several positive h_0.

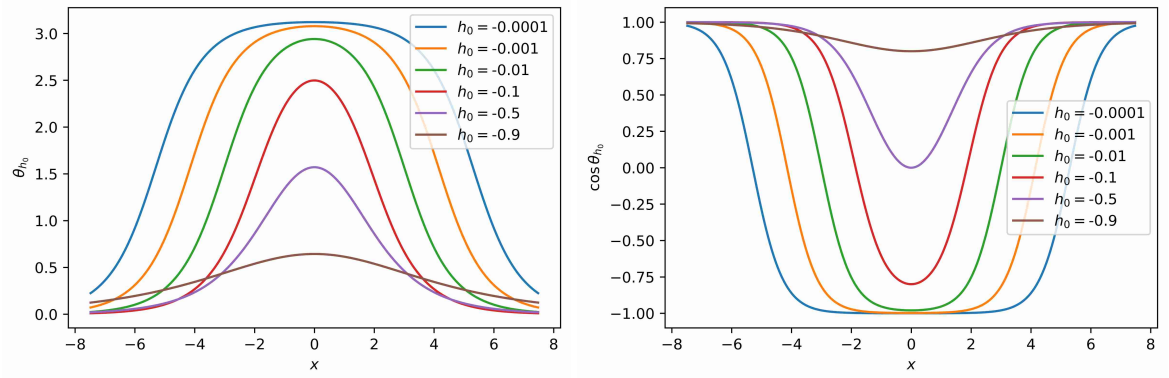

Figure 2: Plot of θ_{h_0} (on the left) and of $\cos\theta_{h_0}$, which is the first component of m_{h_0} (on the right), for several negative h_0.

On the contrary, when $h_0 > 0$ is large, w_{h_0} still performs a transition between e_1 and $-e_1$ back and forth, but the larger h_0, the more abrupt the transition and the shorter the interval of w_{h_0} remaining close to $-e_1$. It is rather intriguing that a solution with such a fast transition with h_0 large can be a stationary solution to (LLG).

There is no stationary solution when $h_0 \leq -1$. As h_0 approaches -1^+, the stationary solution is close to be the constant solution e_1. This is most probably related to the fact that the constant solution e_1 becomes unstable when $h_0 < 0$ becomes large, with the threshold seemingly occurring at $h_0 = -1$.

6.2. Numerical simulation

6.2.1. *Numerical scheme.* The numerical scheme used for the computation of the evolution of (LLG) is a simple explicit finite-difference scheme, on a finite interval $(-L, L)$ with Neumann boundary conditions. Moreover, between each time step, the solution is renormalized so that $|m| = 1$ anytime and anywhere. The simulations we will present in the following sections are performed with $L = 15$, space step $dx = 0.2$ and time step $dt = 5.10^{-5}$. All the plots displayed in the following sections concern m_1, as it is the most important component of the magnetization in our problem.

6.2.2. *Instability of the stationary solution.* In Figure 3, a numerical simulation of (LLG) with the aforementioned scheme has been performed for $h_0 = 0.1$. The initial data displayed here is the function m_0 constructed in Section 5.2 with Remark 5.6, perturbation of the stationary solution, with $\varepsilon_0 = 0.1$. Theorem 1.7 establishes that for, for sufficiently small ε_0, such initial data subsequently evolves into a distinct state over time. The numerical simulation validates this statement: we see that the solution evolves, deviates from the stationary solution and, in this case, collapses into the constant solution e_1.

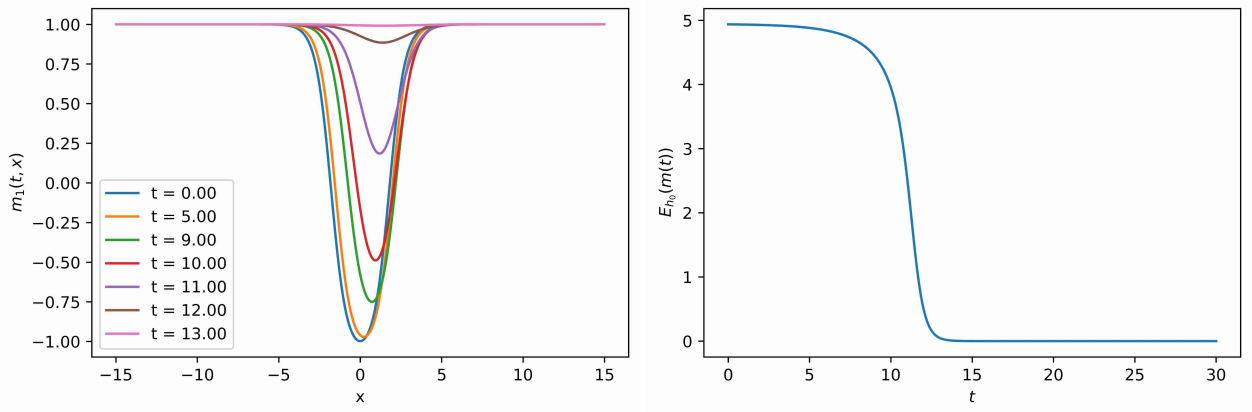

Figure 3: Left: Plot of $m_1(t)$ for several time between 0 and 13. Right: Evolution of the modified energy $E_{h_0}(m(t))$. The initial data is a perturbation of the stationary solution w_{h_0} with $h_0 = 0.1$, as depicted in Section 5.2.

We want to point out that this collapse is not symmetric, because the initial data is not symmetric anymore. While collapsing, the transition of the left-hand side moves to the right. If we take $\varepsilon_0 = -0.1$, the evolution is symmetric with respect to the case $\varepsilon_0 = 0.1$: there is still collapse, but now the transition of the right-hand side moves to the left. Last, the (discrete) modified energy is indeed decreasing, as expected. More precisely, at first, E_{h_0} is exponentially decreasing. After some time, this decrease starts being smaller and smaller, and converges exponentially to 0 when $t \to \infty$. This corresponds to the fact that $E_{h_0}(m) \geq 0$ in the case $h_0 > 0$ and the minimum 0 is reached at $m \equiv e_1$, which is indeed the limit of the magnetization as depicted in Figure 3.

In the case $h_0 = -0.1$, the situation is more involved and even more interesting at this point. The initial data is again the function m_0 constructed in Section 5.2 with Remark 5.6, with $\varepsilon_0 = \pm 0.1$. However, the behavior of the solution is quite different depending on the sign of ε. When $\varepsilon = -0.1$, the situation is rather similar to the previous case, with a collapse onto the constant solution e_1, although it is more symmetric now (Figure 4).

On the other hand, when $\varepsilon = 0.1$, the evolution is rather different. Indeed, instead of collapsing, the structure evolves into a 2-domain wall structure, delimiting a magnetic domain of magnetization $-e_1$. Moreover, the domain

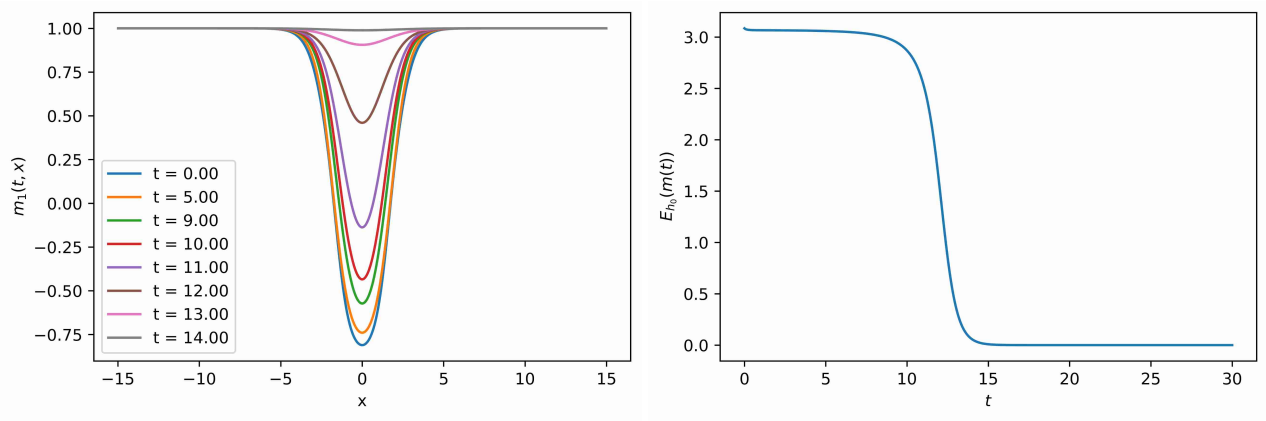

Figure 4: Left: Plot of $m_1(t)$ for several time between 0 and 13. Right: Evolution of the modified energy $E_{h_0}(m(t))$. The initial data is a perturbation of the stationary solution w_{h_0} with $h_0 = -0.1$, as depicted in Section 5.2 with $\varepsilon = -0.1$.

walls seem to go away to infinity at a linear rate. This is also confirmed by the linear decreasing rate of the modified energy, which shows that the area of magnetization $-e_1$ is growing linearly.

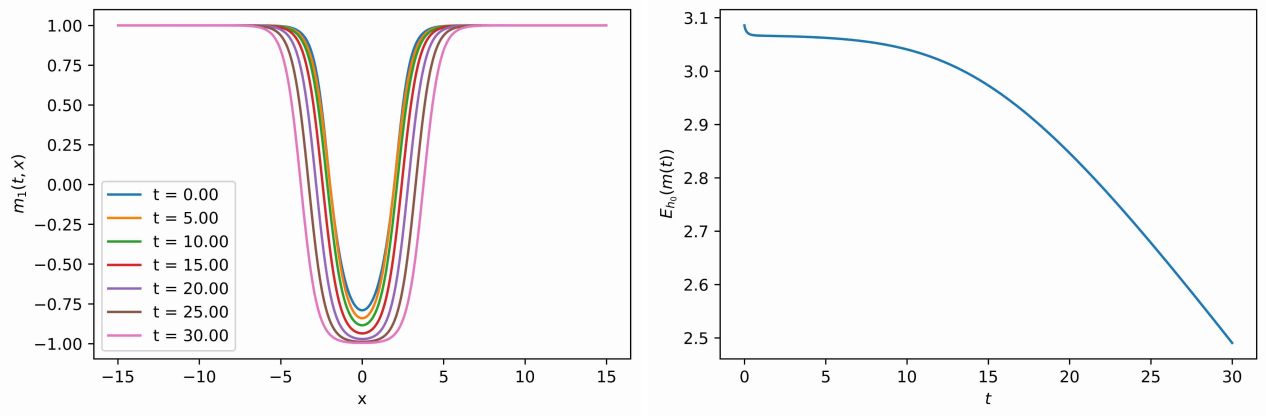

Figure 5: Left: Plot of $m_1(t)$ for several time between 0 and 13. Right: Evolution of the modified energy $E_{h_0}(m(t))$. The initial data is a perturbation of the stationary solution w_{h_0} with $h_0 = -0.1$, as depicted in Section 5.2 with $\varepsilon = 0.1$.

This linear evolution of a 2-domain wall confirms the result of [4, Theorem 1.2]. It also give some information about the assumptions regarding the initial distance between the two domain walls. The minimal distance seems to be related to this stationary solutions, as the sign of ε gives a different behavior of the solution (evolution to a 2-domain wall or collapsing). This minimal distance thus seem to be of order $\ln|h_0|$ when $h_0 \to 0^-$. This conjecture might give some insights on the internal interactions between domain walls, especially their intensity, but these details are behind the scope of this paper.

6.2.3. *Evolution without external magnetic field.* We also look at the evolution of the solution whose initial data is a stationary solution w_{h_0} for some $h_0 > 0$ or $h_0 \in (-1, 0)$, but with no external magnetic field in (LLG) (i.e. $H_{ext} = 0$).

In the case $h_0 = -0.1$ (figure 6), the structure collapses into the constant solution e_1. This behavior was expected: the external magnetic field $h_0 e_1$, maintaining the structure when triggered in (LLG), is "pushing" it towards the magnetization $-e_1$ since $h_0 < 0$, countering the internal forces of the structure which push it towards e_1. But when this external magnetic field is cut off, there is nothing which counters these internal interactions anymore. This was mostly expected when h_0 is close to -1^+, as the stationary solution was already close to e_1. Other simulations with smaller h_0 shows the same behavior, although the collapse appear at larger time on which we cannot not assure the

validity of our simple numerical scheme.

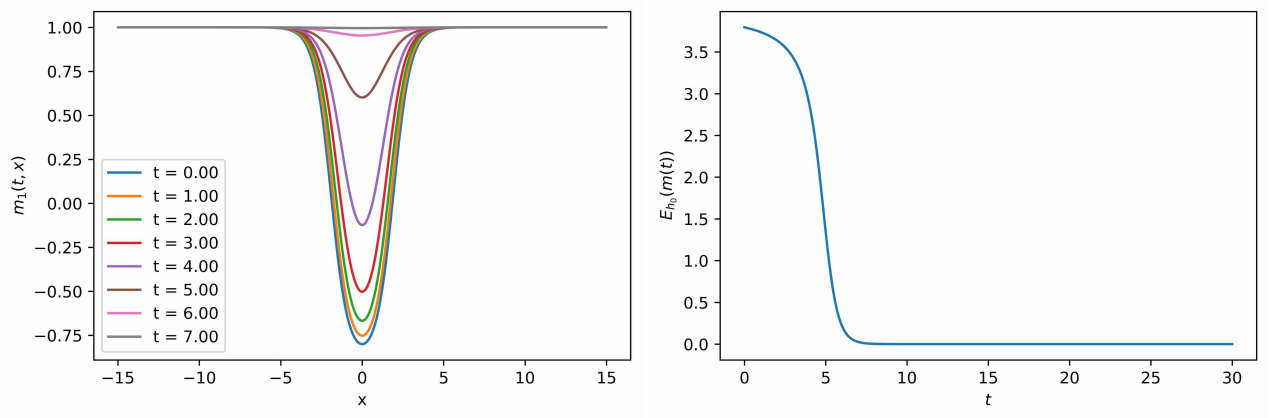

Figure 6: Left: Plot of the first component $m_1(t)$ of $m(t)$, solution to (LLG) with $H_{ext} = 0$, for several time between 0 and 7. Right: Evolution of the energy $E(m(t))$. The initial data is the stationary solution w_{h_0} with $h_0 = -0.1$.

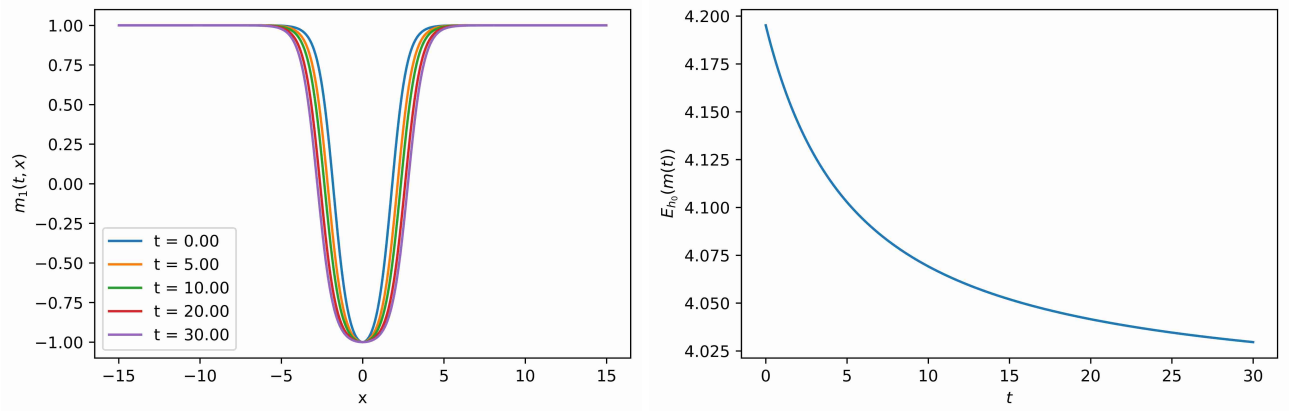

Figure 7: Left: Plot of the first component $m_1(t)$ of $m(t)$, solution to (LLG) with $H_{ext} = 0$, for several time between 0 and 30. Right: Evolution of the energy $E(m(t))$. The initial data is the stationary solution w_{h_0} with $h_0 = 0.1$.

The case $h_0 > 0$ is more interesting. The external magnetic field H_{ext} is now pushing towards e_1, and therefore one gets nontrivial dynamics when $t \to \infty$ if we cut off this external magnetic field. When h_0 is small, so that m_{h_0} looks like a 2-domain wall, the two domain walls might get closer one from each other. This is what happens if $h_0 < 0$, as we saw previously. On the contrary, if $h_0 > 0$, it looks like the internal interactions of this structure push them away from each other (see Figure 7).

When h_0 is large, the dynamics is also very interesting. The initial data has a rapid transition from e_1 to $-e_1$ and back near $x = 0$, but is almost $-e_1$ everywhere else. Contrarily to what one might expect, the structure does not collapse like previously. It even grows, so that a real magnetic domain of magnetization $-e_1$ appear, delimited by two domain walls which move away from each other (see Figure 8 for the simulation with $h_0 = 10$). The evolution of the energy seems to converge to 4, which is twice the energy of a domain wall, and is therefore in agreement with the previous discussion. A small random regular perturbation of this initial data still leads to the same behavior, which proves that this phenomenon is not caused by symmetry.

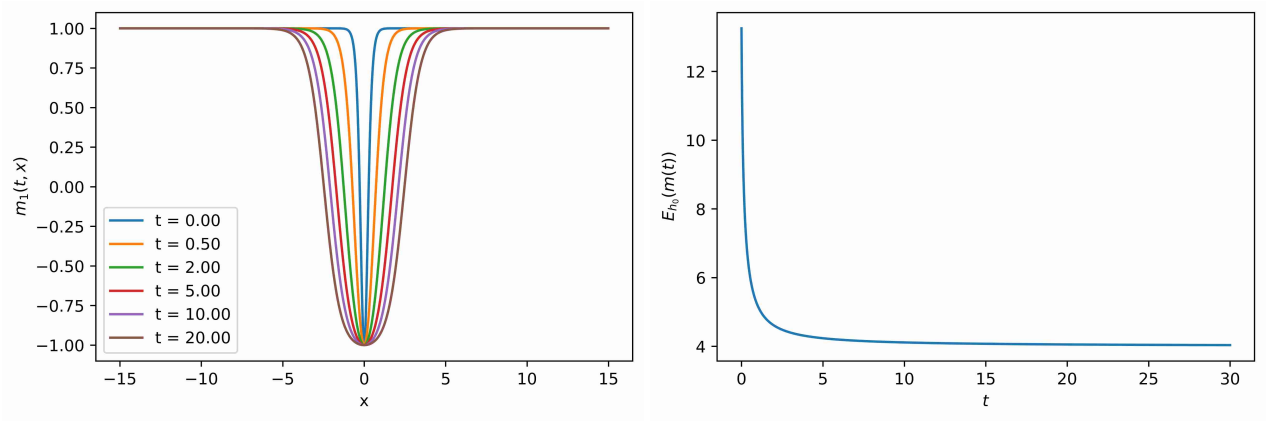

Figure 8: Left: Plot of $m_1(t)$, solution to (LLG) with $H_{ext} = 0$, for several time between 0 and 30. Right: Evolution of the energy $E(m(t))$. The initial data is the stationary solution w_{h_0} with $h_0 = 10$.

7. Discussion and open problems

In this final section, we engage in a brief discussion regarding two intriguing aspects arising from our numerical investigation into the evolution of solutions of the Landau-Lifshitz-Gilbert equation. First, we explore the nature of interactions between domain walls. Secondly, we discuss about the spectrum of the operator derived from a direct linearization of the equation, which could suggest further insights of the stability properties of this stationary structure. These open problems offer avenues for future research, which might enrich our understanding of magnetization dynamics in nanoscale systems.

7.1. Attractive and repulsive domain walls ?

The main question arising from this work concern the interactions between domain walls. The result of [4] is valid only when the two initial domain walls are far away enough and with a non-negligible external magnetic field which pushes them away from each other. But it does not say anything about the interactions between these two structures, except that they are negligible enough compared to the external magnetic field in this framework.

On the other hand, the stationary solutions m_{h_0} looks like 2-domain walls when h_0 is small enough. When h_0 is negative, one would expect the two domain walls to move away from each other in view of [4, Theorem 1.2]. Since this structure does not move through the flow of (LLG), this would mean that the forces exerted by the external magnetic field are countered by internal interactions which tend to make them get closer. This is also shown by the simulation in Figure 5. On the other hand, when h_0 is positive, the exact opposite happens: one would expect the external magnetic field to make the two domain walls get closer, but it is probably countered by internal interactions which make them move away from each other.

In conclusion, a domain wall $w_*(x)$ and its opposite $w_*(-x)$ must have *attractive* interactions, whereas the interactions with the latter being turned by a rotation R_π around e_1 of angle π must be *repulsive*. Yet, this interpretation needs to be attenuated. Such considerations are mostly on the positions of the domain walls, but it does not say anything on the evolution of the rotation between the two interfering domain walls. In particular, for $h_0 > 0$, the analysis performed in this paper shows that the instability of the stationary solution comes from a term $L_2\rho$, where ρ is strongly related to rotations around e_1 (ν is related to translations, see also Remark 3.2). Thus, we cannot rule out a possible mechanism which would make the two domain walls to rotate at the same time as they move away from each other at first, which may lead to a new situation where their interactions become attractive.

In the end, these nonlinear dynamics considering the interactions between domain walls remain an open question, which would require a detailed analysis, going further in the direction of [4].

7.2. Spectrum of the linearized operator

A direct linearization of (LLG) around a stationary solution w_{h_0} is easily available. By decomposing the solution as in Lemma 3.9 and using the computations in Lemma 4.2, we obtain that (η, ρ) must satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\eta\\ \rho\end{pmatrix} = -L\begin{pmatrix}\eta\\ \rho\end{pmatrix} + O_2^2(\eta),$$

where the operator L is given by

$$L = \begin{pmatrix}\alpha L_1 & -L_2\\ L_1 & \alpha L_2\end{pmatrix}$$

This operator is not self-adjoint. Yet, it is a relatively compact perturbation of the operators

$$\begin{pmatrix}\alpha L_1 & -L_1\\ L_1 & \alpha L_1\end{pmatrix} = L_1 M \qquad \text{and} \qquad \begin{pmatrix}\alpha L_2 & -L_2\\ L_2 & \alpha L_2\end{pmatrix} = L_2 M,$$

where M is a matrix with constant coefficients

$$M \coloneqq \begin{pmatrix}\alpha & -1\\ 1 & \alpha\end{pmatrix}.$$

This matrix has two complex eigenvalues:

- $\alpha + i$, with related eigenvector $(1, i)$,
- $\alpha - i$, with related eigenvector $(1, -i)$.

We know that the potential in both L_1 and L_2 has a limit at $\pm\infty$, which is $1 + h_0$. Thus, the essential spectrum of L_1 and L_2 is given by $\sigma_{ess}(L_1) = \sigma_{ess}(L_2) = [1 + h_0, \infty)$, and we obtain straightforwardly the essential spectrum of $L_1 M$ and $L_2 M$, which in turn is also the essential spectrum of L: $(\alpha + i).[1 + h_0, \infty) \cup (\alpha - i).[1 + h_0, \infty)$. A similar analysis leads to the same result for the adjoint of this operator.

In particular, this essential spectrum is completely embedded in $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$. Thus, in order to study the linear stability, one should study the remaining eigenvalues of L, or rather of $L^\top$, and in particular the eigenvalues embedded in $\{z \in \mathbb{C} \mid \operatorname{Re} z \leq 0\}$. The kernel of $L^\top$ can be easily computed thanks to the kernel of L_1 and L_2 given in Proposition 3.1:

$$\ker L^\top = \mathbb{C}\begin{pmatrix}\alpha\\ 1\end{pmatrix}\partial_x\theta_{h_0} + \mathbb{C}\begin{pmatrix}-1\\ \alpha\end{pmatrix}\sin\theta_{h_0}.$$

These two functions are related to the invariance by translation and rotation around e_1, as already explained previously, and can therefore be put aside by some modulation. In view of the results found above on the dynamics in the nonlinear setting, one would expect at least another eigenvalue with negative real part, and more probably two (L is real, so any eigenvalue $\lambda \in \mathbb{C} \setminus \mathbb{R}$ would give rise to a second eigenvalue $\overline{\lambda}$). However, since the operator L is not self-adjoint, this analysis is much more difficult and it is still an open problem to the best of the author's knowledge. A positive answer to this question would still give a more precise hint of the linear and nonlinear dynamics, as it would roughly say that most solutions to (LLG) close to a stationary solution would evolve into a different state, except maybe on a manifold of dimension 2. This would be better than the result in Theorem 1.7, which states this behaviour only on a subset in form of a cone near the stationary solution.

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A. Rayleigh quotients and eigenfunctions

In this appendix, we recall a particular case of the Rayleigh quotients, where the supremum over linear subspace of codimension fixed can be explicited thanks to the eigenfunctions of the previous eigenvalues.

Lemma A.1. *Let $\lambda_1 < \lambda_2$ be the first two eigenvalues of a self-adjoint bounded-from-below operator $(\mathrm{Dom}(L), L)$ on an Hilbert space H, associated with the quadratic form $Q(u) \coloneqq \langle Lu, u\rangle_H$. Assume that these eigenvalues are simple with respecting normalized eigenvectors $\psi_1, \psi_2 \in \mathrm{Dom}(L) \subset \mathrm{Dom}(Q)$ and that the essential spectrum is included in $[\lambda_\infty, \infty)$ where $\lambda_\infty > \lambda_2$. Then*

$$\inf_{u\in\mathrm{span}(\psi_1,\psi_2)^\perp\cap\mathrm{Dom}(Q)\setminus\{0\}} \frac{\langle Lu,u\rangle_H}{\|u\|_H^2} =: a = \mu_3(L) \coloneqq \sup_{\phi_1,\phi_2} \inf_{u\in\mathrm{span}(\phi_1,\phi_2)^\perp\cap\mathrm{Dom}(Q)\setminus\{0\}} \frac{\langle Lu,u\rangle_H}{\|u\|_H^2},$$

where $\mu_3(L)$ is the third Rayleigh quotient of L.

Proof. By definition of $\mu_3(L)$, we obviously have $a \le \mu_3(L)$. By contradiction, assume that $a < \mu_3(L)$. By definition of a, there exists a non-trivial vector $\psi_3 \in \mathrm{Dom}(Q)\setminus\{0\}$ such that $\langle\psi_3,\psi_1\rangle_H = \langle\psi_3,\psi_2\rangle_H = 0$ and $Q(\psi_3) < \lambda\|\psi_3\|_H^2$ for any $\lambda \in (a, \mu_3(L))$ to be fixed later. We can assume that $\|\psi_3\|_H = 1$. Let $\mu_2(L)$ be the second Rayleigh quotient

$$\mu_2(L) \coloneqq \sup_{\phi_1} \inf_{u\in\mathrm{span}(\phi_1)^\perp\cap\mathrm{Dom}(Q)\setminus\{0\}} \frac{\langle Lu,u\rangle_H}{\|u\|_H^2}.$$

It is known that $\mu_2(L) = \lambda_2$ (see [3]). Moreover, since λ_2 is simple and since the bottom of the essential spectrum is larger than $\lambda_\infty > \lambda_2$, we also know that $\mu_2(L) < \mu_3(L)$. Then, we can assume furthermore $\lambda \ge \mu_2(L) = \lambda_2$.

Now, let $\phi_1, \phi_2 \in H$. We know that $\mathrm{span}(\psi_1,\psi_2,\psi_3)$ is a vector space with dimension 3 which is included in $\mathrm{Dom}(Q) \subset H$, whereas $\mathrm{span}(\phi_1,\phi_2)^\perp$ is a vector space with codimension (at least) 2 in H. Therefore, there exists a non-trivial vector $u \in \mathrm{span}(\psi_1,\psi_2,\psi_3) \cap \mathrm{span}(\phi_1,\phi_2)^\perp$, and this u is obviously in $\mathrm{Dom}(Q)$. We can decompose u in the basis (ψ_1,ψ_2,ψ_3):

$$u = \langle u,\psi_1\rangle_H\psi_1 + \langle u,\psi_2\rangle_H\psi_2 + \langle u,\psi_3\rangle_H\psi_3.$$

In particular, we can compute $Q(u)$ with this expansion. Indeed, since ψ_1 and ψ_2 are eigenvectors, and as (ψ_1,ψ_2,ψ_3) are orthogonal to each other in H, all the cross terms vanish:

$$\begin{aligned} Q(u) &= \langle u,\psi_1\rangle_H^2 Q(\psi_1) + \langle u,\psi_2\rangle_H^2 Q(\psi_2) + \langle u,\psi_3\rangle_H^2 Q(\psi_3) \\ &= \underbrace{\lambda_1}_{<\lambda_2\le\lambda}\langle u,\psi_1\rangle_H^2 + \underbrace{\lambda_2}_{\le\lambda}\langle u,\psi_2\rangle_H^2 + \langle u,\psi_3\rangle_H^2\underbrace{Q(\psi_3)}_{<\lambda} \\ &\le \lambda\Big(\langle u,\psi_1\rangle_H^2 + \langle u,\psi_2\rangle_H^2 + \langle u,\psi_3\rangle_H^2\Big) = \lambda\|u\|_H^2. \end{aligned}$$

Therefore, $\inf_{u\in\mathrm{span}(\phi_1,\phi_2)^\perp\cap\mathrm{Dom}(Q)\setminus\{0\}} \frac{\langle Lu,u\rangle_H}{\|u\|_H^2} \le \lambda$. Since this is true for any $\phi_1,\phi_2 \in H$, we get $\mu_3(L) \le \lambda < \mu_3(L)$, and thus the contradiction. □

B. Proof of the modulation lemma

In this section, we prove Lemma 3.11 which decomposes the magnetization with a nice gauge satisfying two orthogonal equalities. In brief, this result is a consequence of the implicit function theorem. We start with defining and studying the appropriate function: let

$$\begin{aligned} \mathfrak{F} : (L^\infty + L^2)\times G &\to \mathbb{R}^2 \\ (m,g) &\mapsto \begin{pmatrix} \int m\cdot g.\partial_x w_{h_0}\,\mathrm{d}x \\ \int m\cdot(e_1\wedge g.w_{h_0})\,\mathrm{d}x \end{pmatrix}. \end{aligned}$$

This integrals are well-defined since both $\partial_x w_{h_0}$ and $e_1\wedge w_{h_0}$ converge exponentially to 0 at $\pm\infty$ (they are also of class $W^{k,1}$ for all $k\in\mathbb{N}$). We begin by some obvious properties on this function.

Lemma B.1. *The function$\mathfrak{F}$ is linear in the first variable and of class $\mathscr{C}^\infty$. Moreover, there also holds*

$$\mathfrak{F}(m,g) = \begin{pmatrix} \int (-g).m \cdot \partial_x w_{h_0} \,\mathrm{d}x \\ \int (-g).m \cdot (e_1 \wedge w_{h_0}) \,\mathrm{d}x \end{pmatrix} = \begin{pmatrix} \int ((-g).m - w_{h_0}) \cdot \partial_x w_{h_0} \,\mathrm{d}x \\ \int ((-g).m - w_{h_0}) \cdot (e_1 \wedge w_{h_0}) \,\mathrm{d}x \end{pmatrix}.$$

Then, we show that $\partial_g \mathfrak{F}(w_{h_0},(0,0))$ is invertible.

Lemma B.2. *Writing $g=(y,\phi)$, there holds $\mathfrak{F}(w_{h_0},(0,0))=0$ and*

$$\partial_y \mathfrak{F}(w_{h_0},(0,0)) = \begin{pmatrix} -\int |\partial_x w_{h_0}|^2 \,\mathrm{d}x \\ 0 \end{pmatrix}, \qquad \partial_\phi \mathfrak{F}(w_{h_0},(0,0)) = \begin{pmatrix} 0 \\ -\int |e_1 \wedge w_{h_0}|^2 \,\mathrm{d}x \end{pmatrix}.$$

Therefore, $\partial_g \mathfrak{F}(w_{h_0},(0,0))$ is invertible.

Proof. Since $\partial_y((-g).m) = (-g).\partial_x m$ and $\partial_\phi((-g).m) = (-g).(e_1 \wedge m)$, we get

$$\partial_y \mathfrak{F}(w_{h_0},(0,0)) = \begin{pmatrix} -\int |\partial_x w_{h_0}|^2 \,\mathrm{d}x \\ -\int \partial_x w_{h_0} \cdot (e_1 \wedge w_{h_0}) \end{pmatrix}, \qquad \partial_\phi \mathfrak{F}(w_{h_0},(0,0)) = \begin{pmatrix} -\int \partial_x w_{h_0} \cdot (e_1 \wedge w_{h_0}) \\ -\int |e_1 \wedge w_{h_0}|^2 \,\mathrm{d}x \end{pmatrix}.$$

On the other hand, we know that $\partial_x w_{h_0} = \partial_x \theta_{h_0}\, n_{h_0}$ and $e_1 \wedge w_{h_0} = \sin\theta_{h_0}\, e_3$, so that $\partial_x w_{h_0} \cdot (e_1 \wedge w_{h_0}) \equiv 0$ on $\mathbb{R}$. Last, we know that $\int |\partial_x w_{h_0}|^2 \,\mathrm{d}x > 0$ and $\int |e_1 \wedge w_{h_0}|^2 \,\mathrm{d}x > 0$, which leads to the conclusion. □

Last before proving Lemma 3.11, we show that $g.f$ is close to f when g is close to $(0,0)$.

Lemma B.3. *There exists $C>0$ such that, for any $g \in G$, there holds*

$$\|g.w_{h_0} - w_{h_0}\|_{H^1} \leq C|g|.$$

Proof. First, we know that, for any $f \in \mathcal{H}^2$ and $y \in \mathbb{R}$, there holds

$$\|\tau_y.f - f\|_{H^1} \leq C|y| \|\partial_x f\|_{H^1}.$$

Moreover, using the expression of w_{h_0} in terms of θ_{h_0}, there holds

$$R_\phi.w_{h_0} - w_{h_0} = \sin\theta_{h_0} \begin{pmatrix} 0 \\ \cos\phi - 1 \\ \sin\phi \end{pmatrix},$$

so that

$$\|R_\phi.w_{h_0} - w_{h_0}\|_{H^1} \leq C|\phi|.$$

Therefore,

$$\begin{aligned} \|g.w_{h_0} - w_{h_0}\|_{H^1} &\leq \|\tau_y R_\phi.w_{h_0} - R_\phi.w_{h_0}\|_{H^1} + \|R_\phi.w_{h_0} - w_{h_0}\|_{H^1} \\ &\leq C|y| \|w_{h_0}\|_{H^1} + C|\phi| \leq C|g|. \end{aligned}$$

□

Proof of Lemma 3.11. By Lemma B.2, the implicit function theorem gives some $\tilde{\delta}_1 >$ such that, for all $w \in L^\infty$ satisfying

$$\|m - w_{h_0}\|_{L^\infty} < \tilde{\delta}_1,$$

there exists a unique $g \in G$ satisfying $|g| < C\|m - w_{h_0}\|_{L^\infty}$ for some $C>0$ such that

$$\int m \cdot g.\partial_x w_{h_0} \,\mathrm{d}x = \int m \cdot (e_1 \wedge g.w_{h_0}) \,\mathrm{d}x = 0.$$

Then, Lemma B.1 implies that $\eta \coloneqq (-g).m - w_{h_0}$ satisfies

$$\int \eta \cdot \partial_x w_{h_0} \,\mathrm{d}x = \int \eta \cdot (e_1 \wedge w_{h_0}) \,\mathrm{d}x = 0.$$

Now, we use the embedding of H^1 into L^∞, that is the existence of $C_S > 0$ such that for all $f \in H^1$,

$$\|f\|_{L^\infty} < C_S \|f\|_{H^1}.$$

Therefore, a similar conclusion can be reached if $m - w_{h_0} \in H^1$ satisfies (3.12) for $\delta_1 \coloneqq \frac{\tilde{\delta}_1}{C_S}$. Last, in such a case, we can estimate $\|\eta\|_{H^1}$ as follows:

$$\begin{aligned}
\|\eta\|_{H^1} &= \|m - g.w_{h_0}\|_{H^1} \\
&\leq \|m - w_{h_0}\|_{H^1} + \|g.w_{h_0} - w_{h_0}\|_{H^1} \\
&\leq \|m - w_{h_0}\|_{H^1} + C|g| \\
&\leq \|m - w_{h_0}\|_{H^1} + C\|m - w_{h_0}\|_{L^\infty} \\
&\leq (1 + CC_S)\|m - w_{h_0}\|_{H^1},
\end{aligned}$$

where we used Lemma B.3 from the second to the third line. □

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