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# Skew Cyclic and Skew Constacyclic Codes over a Mixed Alphabet

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**Abstract:** In this note, we study skew cyclic and skew constacyclic codes over the mixed alphabet  $\mathcal{R} = \mathbb{F}_q \mathcal{R}_1 \mathcal{R}_2$ , where  $q = p^m$ ,  $p$  is an odd prime with  $m$  odd and  $\mathcal{R}_1 = \mathbb{F}_q + u\mathbb{F}_q$  with  $u^2 = u$ , and  $\mathcal{R}_2 = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q$  with  $u^2 = u, v^2 = v, uv = vu = 0$ . Such codes consist of the juxtaposition of three codes of the same size over  $\mathbb{F}_q, \mathcal{R}_1$ , and  $\mathcal{R}_2$ , respectively. We investigate the generator polynomial for skew cyclic codes over  $\mathcal{R}$ . Furthermore, we discuss the structural properties of the skew cyclic and skew constacyclic codes over  $\mathcal{R}$ . We also study their  $q$ -ary images under suitable Gray maps.

**Keywords:** linear codes; skew-cyclic codes; Gray map; skew constacyclic codes

**MSC:** 94B05; 94B15; 94B35; 94B60



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## 1. Introduction

The most widely used family of linear codes consists of cyclic codes. Inspired by codes for the Lee metric [1], Berlekamp adapted them to constacyclic codes. Since then, as the following paragraphs demonstrate, they have happened in a number of circumstances.

Skew cyclic codes were first introduced as ideals in the skew polynomial ring  $\mathbb{F}[x; \theta]$  in 2007 by Boucher et al. [2], where  $\theta$  represents an automorphism of the finite field  $\mathbb{F}$ . The tables of the most well-known codes were enhanced by the numerous numerical samples that this technique created. The fact that the factorization of the polynomial  $x^n - 1$  is not unique gives skew polynomial rings an advantage over commutative polynomial rings. For a given length, these numerous factorizations produce a large number of additional codes. Boucher et al. further extended this technique to skew constacyclic codes in [3]. Siap et al. [4] investigated skew cyclic codes of any length in 2011 and produced maps using both classical and quasi-cyclic codes.

In 2012, Jitman et al. [5] studied skew constacyclic codes over finite chain rings and described the algebraic structure of Euclidean and Hermitian dual codes. Abualrub et al. [6] studied  $\theta$ -cyclic codes over the semilocal ring  $\mathbb{F}_2 + u\mathbb{F}_2, v^2 = v$  with respect to Euclidean and Hermitian inner products.

These codes over semilocal rings were further studied in many contexts. The rings  $\mathbb{F}_3 + v\mathbb{F}_3$  in [7],  $\mathbb{F}_q + v\mathbb{F}_q, v^2 = v$  in [8], and  $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q, u^2 = u, v^2 = v, uv = vu = 0$  in [9] were utilized as alphabets for skew cyclic codes, for example. Dertli and Cengellenmis [10] and Yao et al. [11] also examined these codes over  $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q, u^2 = u, v^2 = v, uv = vu$ . Skewed  $(-1 + 2v)$ -constacyclic codes were developed in 2017 by Gao et al. [12] after deriving the structure of skew constacyclic codes over the semilocal ring  $\mathbb{F}_q + v\mathbb{F}_q, v^2 = v$ .

In [13] and [14], respectively, Islam and Prakash established the algebraic structure of skew constacyclic codes over  $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q, u^2 = u, v^2 = v, uv = vu$ .

Using two non-trivial automorphisms, Bhardwaj and Raka [15] investigated the skew constacyclic codes over the ring  $\mathbb{F}_q[u, v] \langle f(u), g(v), uv - vu \rangle$  in 2019. Alternatively,  $\mathbb{Z}_2\mathbb{Z}_4$ -linear codes, or codes over the mixed alphabet  $\mathbb{Z}_2\mathbb{Z}_4$ , where a subset of coordinates is binary and the complement is quaternary, were introduced by Borges et al. [16]. They have calculated their generator matrices and described their dual codes. Fernandez-Cordoba et al. [17] obtained the rank and kernel of  $\mathbb{Z}_2\mathbb{Z}_4$ -linear codes in a follow-up experiment. Steganography is one field in which these codes have found industrial use [18].

In [19], additive codes over the mixed alphabet  $\mathbb{Z}_2\mathbb{Z}_{2^s}$  were examined. Next, Refs. [20–23] examined the mixed alphabet  $\mathbb{Z}_p\mathbb{Z}_{p^s}$  and, more broadly,  $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$ . Conversely, Abualrub et al. [24] defined  $\mathbb{Z}_2\mathbb{Z}_4$  in 2014, in line with the advancement of cyclic codes on mixed alphabets. The code for  $-$ additive cyclics is  $\mathbb{Z}_4[x]$ -submodule of  $\mathbb{Z}_2[x] / \langle x^r - 1 \rangle \times \mathbb{Z}_4[x] / \langle x^s - 1 \rangle$ , from which the smallest spanning set and unique set of generators for these codes, where  $s$  is an odd integer, were obtained.

Furthermore, generator polynomials and duals for  $\mathbb{Z}_2\mathbb{Z}_4$ -additive cyclic codes were discovered by Borges et al. [25]. In [26], Aydogdu et al. [27] introduced the novel mixed alphabets  $\mathbb{Z}_2\mathbb{Z}_2[u]$ -additive codes, where  $u^2 = 0$ . They also studied constacyclic codes over mixed alphabets by defining them as  $\mathbb{Z}_2[u][x] \times \mathbb{Z}_2[u][x] / \langle x^\beta - (1 + u) \rangle$ ,  $-$ submodules of  $\mathbb{Z}_2[x] / \langle x^\alpha - 1 \rangle$ .

As the Gray images of  $\mathbb{Z}_2\mathbb{Z}_2[u]$ -cyclic codes, they were able to derive several optimum binary linear codes. In the meanwhile,  $\mathbb{Z}_2\mathbb{Z}_2[u]$ -additive cyclic and constacyclic codes with the unit  $1 + u$ , respectively, were explored algebraically by [28]. Consequently, the predicted generalization in the continuation of these research should be  $\mathbb{Z}_{2^r}\mathbb{Z}_{2^s}[u]$ -additive cyclic codes,  $u^2 = 0$ , and constacyclic codes.

In this article, we examine a mixed alphabet  $\mathcal{R} = \mathbb{F}_q\mathcal{R}_1\mathcal{R}_2$ , where  $\mathcal{R}_1 = \mathbb{F}_q + u\mathbb{F}_q$  with  $u^2 = u$  and  $\mathcal{R}_2 = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q$  with  $u^2 = u, v^2 = v, uv = vu = 0$ . Moreover, we examine the cyclic codes  $\theta_t$  and  $(\theta_t, \alpha)$  over  $\mathcal{R}$ . The algebraic structure of these codes is fully determined. We examine their  $q$ -ary representations under Gray maps and provide a few brief numerical instances.

The contents are arranged as follows. The next section gathers some background information. Gray maps are examined in Section 3. Skewed cyclic codes are covered in Section 4, and skew constacyclic codes are covered in Section 5. The essay is concluded at Section 6.

## 2. Preliminaries

Let  $p$  be an odd prime, and let  $q = p^m$  with  $m$  being odd. Denote by  $\mathbb{F}_q$  the finite field of size  $q$ . The set  $\mathbb{F}_q^n$  of all ordered  $n$ -tuples over  $\mathbb{F}_q$  is equipped with the structure of an  $\mathbb{F}_q$  vector space by the usual addition and scalar multiplication of vectors.

A code of length  $n$  over  $\mathbb{F}_q$  is just any non-empty subset  $\mathcal{C}$  of  $\mathbb{F}_q^n$ . It is said to be linear if  $\mathcal{C}$  is an  $\mathbb{F}_q$  subspace of  $\mathbb{F}_q^n$ . From now on, we write  $\mathcal{R}_1 = \mathbb{F}_q + u\mathbb{F}_q$ , with  $u^2 = u$  and  $\mathcal{R}_2 = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q$ , with  $u^2 = u, v^2 = v, uv = vu = 0$ .

Note that  $\mathcal{R}_1$  and  $\mathcal{R}_2$  are finite non-chain rings. Let  $a + ub + vc$  be an element of  $\mathcal{R}_2$ . Then, we define two maps  $\eta$  and  $\delta$  as follows:

$$\begin{aligned} \eta : \mathcal{R}_2 &\rightarrow \mathbb{F}_q, & \delta : \mathcal{R}_2 &\rightarrow \mathcal{R}_1, \\ \eta(a + ub + vc) &= a, & \delta(a + ub + vc) &= a + ub, \end{aligned}$$

It is clear that  $\eta$  and  $\delta$  are ring homomorphisms. We consider the ring  $\mathcal{R}$ :

$$\mathcal{R} = \mathbb{F}_q\mathcal{R}_1\mathcal{R}_2 = \{(x, y, z) \mid x \in \mathbb{F}_q, y \in \mathcal{R}_1 \text{ and } z \in \mathcal{R}_2\}$$

We define a  $\mathcal{R}_2$ -multiplication in this ring as follows:

$$\begin{aligned} \star : \mathcal{R}_2 \times \mathcal{R} &\rightarrow \mathcal{R} \\ r \star (x, y, z) &= (\eta(r)x, \delta(r)y, rz) \end{aligned}$$

This is a well defined multiplication and it can be extended componentwise to  $\mathcal{R}_\circ = \mathbb{F}_q^{n_1} \times \mathcal{R}_1^{n_2} \times \mathcal{R}_2^{n_3}$  by:

$$\begin{aligned} \star : \mathcal{R}_2 \times \mathcal{R}_\circ &\rightarrow \mathcal{R}_\circ \\ r \star (x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2}, z_1, \dots, z_{n_3}) &= (\eta(r)x_1, \dots, \eta(r)x_{n_1}, \delta(r)y_1, \dots, \delta(r)y_{n_2}, rz_1, \dots, rz_{n_3}) \end{aligned}$$

where  $(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2}, z_1, \dots, z_{n_3}) \in \mathcal{R}_\gamma$ . Equipped with this multiplication,  $\mathcal{R}_\circ$  becomes an  $\mathcal{R}_2$  module. A non-empty subset  $\mathcal{C}$  of  $\mathcal{R}_\circ$  is said to be a  $\mathcal{R}$ -linear code of length  $(n_1, n_2, n_3)$  if  $\mathcal{C}$  is an  $\mathcal{R}_2$ -submodule of  $\mathcal{R}_\circ$ . Now we define the inner product by the formula:

$$\langle c, c' \rangle = \sum_1^{n_1} x_i x'_i + \sum_1^{n_2} y_j y'_j + \sum_1^{n_3} z_k z'_k,$$

where  $c = (x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2}, z_1, \dots, z_{n_3})$ ,  $c' = (x'_1, \dots, x'_{n_1}, y'_1, \dots, y'_{n_2}, z'_1, \dots, z'_{n_3})$  are in  $\mathcal{R}_\gamma$ . Let  $\mathcal{C}$  be an  $\mathcal{R}$ -linear code of length  $(n_1, n_2, n_3)$ . Then, the dual code of  $\mathcal{C}$  is defined as:

$$\mathcal{C}^\perp = \{c' \in \mathcal{R}_\gamma \mid \langle c, c' \rangle = 0 \forall c \in \mathcal{C}\}$$

### 3. Decomposition and Properties of Gray Maps

Recall that,  $\mathcal{R}_1 = \mathbb{F}_q + u\mathbb{F}_q$ , with  $u^2 = u$ . Consider the idempotent orthogonal elements  $e_1 = u$  and  $e_2 = 1 - u$ . Then, we have the decomposition:

$$\mathcal{R}_1 = e_1\mathcal{R}_1 \oplus e_2\mathcal{R}_1 \cong e_1\mathbb{F}_q \oplus e_2\mathbb{F}_q,$$

where  $e_1e_2 = 0, e_1^2 = e_1, e_1 + e_2 = 1$ . Hence,  $\mathcal{R}_1 = \{ae_1 + be_2 \mid a, b \text{ in } \mathbb{F}_q\}$ . We now define the Gray map:

$$\begin{aligned} \varphi_1 : \mathcal{R}_1 &\rightarrow \mathbb{F}_q^2 \\ \varphi_1(ae_1 + be_2) &= (a, b) \end{aligned}$$

It can be extended to the length  $n$  by:

$$\begin{aligned} \varphi_1 : \mathcal{R}_1^n &\rightarrow \mathbb{F}_q^{2n} \\ \varphi_1((a_1, \dots, a_n)e_1 + (b_1, \dots, b_n)e_2) &= (a_1, \dots, a_n, b_1, \dots, b_n) \end{aligned}$$

Note that it is a linear map. We define the Gray weight of a codeword in  $\mathcal{R}_1$  as:

$$wt_G(ae_1 + be_2) = wt_H(a, b)$$

where  $wt_H$  denotes the Hamming weight. If  $x, y$  lie in  $\mathcal{R}_1^n$ , then their mutual distance is given by:

$$d_G(x, y) = \sum_1^n wt_G(x_i - y_i) = \sum_1^{2n} wt_H(\varphi_1(x) - \varphi_1(y)) = d_H(\varphi_1(x), \varphi_1(y)).$$

Hence,  $\varphi_1$  is a weight preserving map. A non-empty subset  $\mathcal{C}$  of  $\mathcal{R}_i^n$  is said to be a linear code of length  $n$  if  $\mathcal{C}$  is  $\mathcal{R}_i$ -submodule of  $\mathcal{R}_i^n$ .

For  $i \in \{1, 2\}, A_i \subseteq \mathcal{R}_1$ :

$$A_1 \oplus A_2 = \{a_1 + a_2 \mid a_i \in A_i\} \text{ and } A_1 \otimes A_2 = \{(a_1, a_2) \mid a_i \in A_i\}.$$

Let  $\mathcal{C}_e$  be a linear code of length  $n$  over  $\mathcal{R}_1$ . Then, we define:

$$\mathcal{C}_{e_1} = \{y_1 \in \mathbb{F}_q^n \mid e_1y_1 + e_2y_2 \in \mathcal{C}_e, \text{ for some } y_2 \in \mathbb{F}_q^n\}$$

$$\mathcal{C}_{e_2} = \{y_2 \in \mathbb{F}_q^n \mid e_1y_1 + e_2y_2 \in \mathcal{C}_e, \text{ for some } y_1 \in \mathbb{F}_q^n\}$$

Therefore, any linear code  $\mathcal{C}_e$  over  $\mathcal{R}_1$  can be represented as  $\mathcal{C}_e = e_1\mathcal{C}_{e_1} \oplus e_2\mathcal{C}_{e_2}$  and  $\varphi_1(\mathcal{C}_e) = \mathcal{C}_{e_1} \otimes \mathcal{C}_{e_2}$ . Hence,  $\mathcal{C}_{e_1}$  and  $\mathcal{C}_{e_2}$  are  $\mathbb{F}_q$ -linear codes. Also note that  $\varphi_1(\mathcal{C}_e^\perp) = \varphi_1(\mathcal{C}_e)^\perp$ .

Recall that  $\mathcal{R}_2 = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q$ , with  $u^2 = u, v^2 = v, uv = vu = 0$ . Let  $o_1 = (1 - u - v), o_2 = u, o_3 = v$  be idempotent orthogonal elements in  $\mathcal{R}_2$ , then:

$$\mathcal{R}_2 = o_1\mathcal{R}_2 \oplus o_2\mathcal{R}_2 \oplus o_3\mathcal{R}_2 \cong o_1\mathbb{F}_q \oplus o_2\mathbb{F}_q \oplus o_3\mathbb{F}_q,$$

where  $o_i o_j = 0 (i \neq j), o_i^2 = o_i, o_1 + o_2 + o_3 = 1$ . Hence, any element in  $\mathcal{R}_2$  can be written as  $ao_1 + bo_2 + co_3$ . We now define a weight preserving linear Gray map  $\varphi_2$ :

$$\begin{aligned} \varphi_2 : \mathcal{R}_2 &\rightarrow \mathbb{F}_q^3 \\ \varphi_2(ao_1 + bo_2 + co_3) &= (a, b, c) \end{aligned}$$

It can be extended to length  $n$  by the formula:

$$\varphi_2((a_1, \dots, a_n)o_1 + (b_1, \dots, b_n)o_2 + (c_1, \dots, c_n)o_3) = (a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_n).$$

We define the Gray weight of a codeword in  $\mathcal{R}_2$  as:

$$wt_G(ao_1 + bo_2 + co_3) = wt_H(a, b, c)$$

where  $wt_H$  denotes the Hamming weight. If  $x, y$  are in  $\mathcal{R}_2^n$ , then their Gray distance is given by:

$$d_G(x, y) = \sum_1^n wt_G(x_i - y_i) = \sum_1^{3n} wt_H(\varphi_2(x) - \varphi_2(y)) = d_H(\varphi_2(x), \varphi_2(y)).$$

For  $i \in \{1, 2, 3\}, A_i \subseteq \mathcal{R}_2$ :

$$A_1 \oplus A_2 \oplus A_3 = \{a_1 + a_2 + a_3 \mid a_i \in A_i\} \text{ and } A_1 \otimes A_2 \otimes A_3 = \{(a_1, a_2, a_3) \mid a_i \in A_i\}.$$

Let  $\mathcal{C}_o$  be a linear code of length  $n$  over  $\mathcal{R}_2$ . We define the three codes:

$$\mathcal{C}_{o_1} = \{z_1 \in \mathbb{F}_q^n \mid o_1z_1 + o_2z_2 + o_3z_3 \in \mathcal{C}_o, \text{ for some } z_2, z_3 \in \mathbb{F}_q^n\},$$

$$\mathcal{C}_{o_2} = \{z_2 \in \mathbb{F}_q^n \mid o_1z_1 + o_2z_2 + o_3z_3 \in \mathcal{C}_o, \text{ for some } z_1, z_3 \in \mathbb{F}_q^n\},$$

$$\mathcal{C}_{o_3} = \{z_3 \in \mathbb{F}_q^n \mid o_1z_1 + o_2z_2 + o_3z_3 \in \mathcal{C}_o, \text{ for some } z_1, z_2 \in \mathbb{F}_q^n\}.$$

Then, any linear code  $\mathcal{C}_o$  over  $\mathcal{R}_2$  can be represented as  $\mathcal{C}_o = o_1\mathcal{C}_{o_1} \oplus o_2\mathcal{C}_{o_2} \oplus o_3\mathcal{C}_{o_3}$  and  $\varphi_2(\mathcal{C}_o) = \mathcal{C}_{o_1} \otimes \mathcal{C}_{o_2} \otimes \mathcal{C}_{o_3}$ , where  $\mathcal{C}_{o_1}, \mathcal{C}_{o_2}$ , and  $\mathcal{C}_{o_3}$  are  $\mathbb{F}_q$ -linear codes. Also note that  $\varphi_2(\mathcal{C}_o^\perp) = \varphi_2(\mathcal{C}_o)^\perp$ .

Henceforth, we define the Gray map  $\varphi$  on  $\mathcal{R}$  using the maps defined previously:

$$\begin{aligned} \varphi : \mathcal{R} &\rightarrow \mathbb{F}_q^6 \\ \varphi(x, y, z) &= (x, \varphi_1(y), \varphi_2(z)) \end{aligned}$$

now we can extend this map to  $\mathcal{R}_\gamma$ :

$$\varphi(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2}, z_1, \dots, z_{n_3}) = (x_1, \dots, x_{n_1}, \varphi_1(y_1), \dots, \varphi_1(y_{n_2}), \varphi_2(z_1), \dots, \varphi_2(z_{n_3}))$$

then, the Gray weight of an element in  $\mathcal{R}_\gamma$  can be denoted by  $wt_G(\alpha) = wt_H(\varphi(\alpha))$ . Any linear code  $\mathcal{C}$  of  $\mathcal{R}_\gamma$  can be represented by  $\mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_e \otimes \mathcal{C}_o$ , where  $\mathcal{C}_1, \mathcal{C}_e$ , and  $\mathcal{C}_o$  are linear code over  $\mathbb{F}_q, \mathcal{R}_1$ , and  $\mathcal{R}_2$ . Let  $G_{\mathbb{F}_q}$  be the generator matrix for linear code over  $\mathbb{F}_q$ . The generator matrix  $G_{\mathcal{R}_1}$  for a linear code over  $\mathcal{R}_1$  is denoted by:

$$G_{\mathcal{R}_1} = \begin{bmatrix} e_1 G_{e_1} \\ e_2 G_{e_2} \end{bmatrix}$$

where  $G_{e_i}$  is the generator matrix for the linear code  $\mathcal{C}_{e_i}$ , for  $i = \{1, 2\}$ . The generator matrix  $G_{\mathcal{R}_2}$  for the linear code over  $\mathcal{R}_2$  is:

$$G_{\mathcal{R}_2} = \begin{bmatrix} o_1 G_{o_1} \\ o_2 G_{o_2} \\ o_3 G_{o_3} \end{bmatrix}$$

where  $G_{o_i}$  is the generator matrix for the linear code  $\mathcal{C}_{o_i}$ , for  $i = \{1, 2, 3\}$ . Using the generator matrices above, we can say that the generator matrix  $G$  for the linear code over  $\mathcal{R}$  is:

$$G = \begin{bmatrix} G_{\mathbb{F}_q} & 0 & 0 \\ 0 & G_{\mathcal{R}_1} & 0 \\ 0 & 0 & G_{\mathcal{R}_2} \end{bmatrix}.$$

Note that the minimum distance of  $\mathcal{C}$  is  $\min\{d_H(\mathcal{C}_1), d_H(\varphi_1(\mathcal{C}_e)), d_H(\varphi_2(\mathcal{C}_o))\}$ . The following theorem provides the weight preserving nature of the Gray map.

**Theorem 1.** *The Gray map  $\varphi$  defined above is linear and weight preserving.*

**Proof.** Let  $x = (x_1, x_2, x_3), x' = (x'_1, x'_2, x'_3)$  be in  $\mathcal{R}_\gamma$  where  $x_1, x'_1 \in \mathbb{F}_q^{n_1}, x_2, x'_2 \in \mathcal{R}_1^{n_2}, x_3, x'_3 \in \mathcal{R}_2^{n_3}$ . We have:

$$\begin{aligned} \varphi(x + x') &= \varphi(x_1 + x'_1, x_2 + x'_2, x_3 + x'_3) \\ &= (x_1 + x'_1, \varphi_1(x_2 + x'_2), \varphi_2(x_3 + x'_3)) \\ &= (x_1, \varphi_1(x_2), \varphi_2(x_3)) + (x'_1, \varphi_1(x'_2), \varphi_2(x'_3)) (\because \varphi_1 \text{ and } \varphi_2 \text{ are linear}) \\ &= \varphi(x) + \varphi(x') \end{aligned}$$

Using the linear map  $\varphi$ ,

$$d_G(x, x') = wt_G(x - x') = wt_H(\varphi(x) - \varphi(x')) = d_H(\varphi(x), \varphi(x'))$$

Hence,  $\varphi$  is a weight preserving linear map.  $\square$

The following theorem gives the parameters of the Gray image of a linear code.

**Theorem 2.** *If  $\mathcal{C} \subseteq \mathcal{R}_\gamma$  is an  $(n_1 + n_2 + n_3, d_G)$  linear code then  $\varphi(\mathcal{C})$  is an  $(n_1 + 2(n_2) + 3(n_3), d_H)$  linear code over  $\mathbb{F}_q$ , where  $d_G = d_H$ .*

**Proof.** The proof can extended from the proof of Theorem 1.  $\square$

The following Theorem characterizes  $\varphi(\mathcal{C})$ :

**Theorem 3.** *If  $\mathcal{C} \subseteq \mathcal{R}_\gamma$  is linear, then  $\varphi(\mathcal{C}) = \mathcal{C}_1 \otimes_{i=1}^{i=2} \mathcal{C}_{e_i} \otimes_{j=1}^{j=3} \mathcal{C}_{o_j}, |\mathcal{C}| = |\mathcal{C}_1| \prod_{i=1}^{i=2} |\mathcal{C}_{e_i}| \prod_{j=1}^{j=3} |\mathcal{C}_{o_j}|$ .*

**Proof.** Let:  $\varphi(x, y, z) = (x, \varphi_1(y), \varphi_3(z)) = (a_1, \dots, a_6) \in \varphi(\mathcal{C}) \subseteq \mathbb{F}_q^6$ .

Note that  $\varphi$  is bijective and  $\mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_e \otimes \mathcal{C}_o$  is linear. Thus,  $a_1 = x \in \mathcal{C}_1$ . Also note that  $\varphi_1(y) = (a_2, a_3), \varphi_2(z) = (a_4, a_5, a_6)$ . Since  $\varphi_i$ 's are bijective,  $a_2e_1 + a_3e_2 \in e_1\mathcal{C}_{e_1} \oplus e_2\mathcal{C}_{e_2} = \mathcal{C}_e$ . Hence,  $(a_2, a_3) \in \mathcal{C}_{e_1} \otimes \mathcal{C}_{e_2}$  and similarly  $(a_4, a_5, a_6) \in \mathcal{C}_{o_1} \otimes \mathcal{C}_{o_2} \otimes \mathcal{C}_{o_3}$ . The converse holds in a similar way. The second part of the statement follows from the fact that  $\varphi$  is bijective.  $\square$

The following Theorem furnishes the decomposition of the dual of the linear code  $\mathcal{C}$ .

**Theorem 4.** *If  $\mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_e \otimes \mathcal{C}_o$  is a linear code over  $\mathcal{R}$  then  $\mathcal{C}^\perp = \mathcal{C}_1^\perp \otimes \mathcal{C}_e^\perp \otimes \mathcal{C}_o^\perp$ , where  $\mathcal{C}_1^\perp, \mathcal{C}_e^\perp$  and  $\mathcal{C}_o^\perp$  are duals for the respective linear codes.*

**Proof.** Let  $\mathcal{C}^\perp = \{c' \in \mathcal{R}_\gamma \mid \langle c, c' \rangle = 0 \text{ for all } c \in \mathcal{C}\} = \{(x', y', z') \in \mathcal{R}_\circ \mid x' \in \mathbb{F}_q^{n_1}, y' \in \mathcal{R}_1^{n_2}, z' \in \mathcal{R}_2^{n_3}\}$ . Let  $c = (x, y, z) \in \mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_e \otimes \mathcal{C}_o$ . Then:

$$\langle c, c' \rangle = xx' + yy' + zz' = 0.$$

Thus,  $x' \in \mathcal{C}_1^\perp, y' \in \mathcal{C}_e^\perp, z' \in \mathcal{C}_o^\perp$  and so  $\mathcal{C}^\perp \subseteq \mathcal{C}_1^\perp \otimes \mathcal{C}_e^\perp \otimes \mathcal{C}_o^\perp$ . Since  $|\mathcal{C}^\perp| = |\mathcal{C}_1^\perp| |\mathcal{C}_e^\perp| |\mathcal{C}_o^\perp|$ , the statement holds.

$\square$

The next result shows that the Gray maps is compatible with duality.

**Theorem 5.** *If  $\mathcal{C} \subseteq \mathcal{R}_\gamma$  is linear, then  $\varphi(\mathcal{C}^\perp) = \varphi(\mathcal{C})^\perp$ .*

**Proof.** Let  $(x, y, z) \in \mathcal{C}$  and  $(x', y', z') \in \mathcal{C}^\perp$ , where  $x \in \mathcal{C}_1, y \in \mathcal{C}_e, z \in \mathcal{C}_o$  and  $x' \in \mathcal{C}_1^\perp, y' \in \mathcal{C}_e^\perp, z' \in \mathcal{C}_o^\perp$ , then  $\langle (x|y|z, x'|y'|z') \rangle = 0$ . Using Theorem 4,  $\mathcal{C}_1^\perp, \mathcal{C}_e^\perp$  and  $\mathcal{C}_o^\perp$  are duals for  $\mathcal{C}_1, \mathcal{C}_e$ , and  $\mathcal{C}_o$ . Now, we have  $\varphi(x, y, z) = (x, \varphi_1(y), \varphi_2(z)), \varphi(x', y', z') = (x', \varphi_1(y'), \varphi_2(z'))$ , then the inner product is given by:

$$\begin{aligned} \langle \varphi(x, y, z), \varphi(x', y', z') \rangle &= \langle (x, \varphi_1(y), \varphi_2(z)), (x', \varphi_1(y'), \varphi_2(z')) \rangle \\ &= \langle (x, 0, 0), (x', 0, 0) \rangle + \langle (0, \varphi_1(y), 0), (0, \varphi_1(y'), 0) \rangle \\ &\quad + \langle (0, 0, \varphi_2(z)), (0, 0, \varphi_2(z')) \rangle \\ &= 0 \quad (\because \varphi_1(\mathcal{C}^\perp) = \varphi_1(\mathcal{C})^\perp, \varphi_2(\mathcal{C}^\perp) = \varphi_2(\mathcal{C})^\perp) \end{aligned}$$

Thus,  $\varphi(\mathcal{C}^\perp) \subseteq \varphi(\mathcal{C})^\perp$ . Since the cardinality is the same on both sides, the statement holds.  $\square$

The following result provides the self duality nature of the linear code and its Gray image.

**Corollary 1.** *If  $\mathcal{C}$  is a linear  $\mathcal{R}_\gamma$ -code, then  $\mathcal{C}$  is self-dual iff  $\varphi(\mathcal{C})$  is self-dual. Moreover,  $\varphi(\mathcal{C})$  is a self-orthogonal code over  $\mathbb{F}_q$  iff  $\mathcal{C}$  is self-orthogonal.*

**Proof.** Let  $\mathcal{C}$  be a self-dual linear code of length  $n$  over  $\mathcal{R}$ . Thus,  $\mathcal{C} = \mathcal{C}^\perp$ . Then,  $\varphi(\mathcal{C}) = \varphi(\mathcal{C}^\perp)$ , and hence, by Theorem 5, we have  $\varphi(\mathcal{C}) = (\varphi(\mathcal{C}))^\perp$ . Thus,  $\varphi(\mathcal{C})$  is a self-dual linear code of length  $n_1 + 2n_2 + 3n_3$  over  $\mathbb{F}_q$ . Conversely, let  $\varphi(\mathcal{C})$  be a self-dual linear code of length  $n_1 + 2n_2 + 3n_3$  over  $\mathbb{F}_q$ . Then,  $\varphi(\mathcal{C}) = (\varphi(\mathcal{C}))^\perp$ , and hence, by Theorem 5, we have  $\varphi(\mathcal{C}) = \varphi(\mathcal{C}^\perp)$ . Since  $\varphi$  is bijection,  $\mathcal{C} = \mathcal{C}^\perp$ . Therefore,  $\mathcal{C}$  is a self-dual linear code over  $\mathcal{R}_\gamma$ . Similarly, the self orthogonal case holds.  $\square$

#### 4. Skew Cyclic $\mathcal{R}$ -Codes

Let  $\theta_t$  be a non-trivial Frobenius automorphism defined by:

$$\theta_t : \mathbb{F}_q \rightarrow \mathbb{F}_q, \theta_t(a) = a^{p^t},$$

where  $t$  divides  $m$ . It can be extended to  $\mathcal{R}_1$  and  $\mathcal{R}_2$  by:

$$\theta_t(a + ub) = \theta_t(a) + u\theta_t(b), \theta_t(a + ub + vc) = \theta_t(a) + u\theta_t(b) + v\theta_t(c).$$

Since  $t|m$ , the order of automorphism  $\theta_t$  is  $\frac{m}{t}$ . We define a polynomial ring  $\mathcal{R}_i[x, \theta_t]$  ( $1 \leq i \leq 2$ ) as follows:

$$\mathcal{R}_i[x, \theta_t] = \{a_1 + \dots + a_n x^n \mid a_j \in \mathcal{R}_i, 1 \leq j \leq n\}$$

Clearly,  $\mathcal{R}_i[x, \theta_t]$  is a ring with respect to usual addition and the multiplication defined by:

$$ax^m bx^n = a\theta_t^m(b)x^{m+n}$$

Note that it is a non-commutative ring unless  $\theta_t$  is an identity map. A non-empty set  $\mathcal{C}$  is said to be a linear code of length  $n_i$  over  $\mathcal{R}_i$  if it is a  $\mathcal{R}_i$  submodule of  $\mathcal{R}_i^{n_i}$ . Using the above polynomial rings above, we extend the polynomial ring to  $\mathcal{R}$  by:

$$\mathcal{R}[x, \theta_t] = \{(a(x), b(x), c(x)) : a(x) \in \mathbb{F}_q[x], b(x) \in \mathcal{R}_1[x], c(x) \in \mathcal{R}_2[x]\}.$$

It can be seen that  $\mathcal{R}[x, \theta_t]$  is a  $\mathcal{R}_2[x; \theta_t]$  submodule with respect to usual addition and multiplication defined by:

$$\begin{aligned} \star : \mathcal{R}_2[x] \times \mathcal{R}[x, \theta_t] &\rightarrow \mathcal{R}[x, \theta_t] \\ (ax^s) \star (b_1 x^i, b_2 x^j, b_3 x^k) &= (\eta(a)x^s b_1 x^i, \delta(a)x^s b_2 x^j, ax^s b_3 x^k) \\ &= (\eta(a)\theta_t^s(b_1)x^{s+i}, \delta(a)\theta_t^s(b_2)x^{s+j}, a\theta_t^s(b_3)x^{s+k}) \end{aligned}$$

However, under associative and distributive laws, the multiplication can be extended to  $\mathcal{R}_\gamma[x; \theta_t] = \frac{\mathbb{F}_q[x; \theta_t]}{\langle x^{n_1} - 1 \rangle} \times \frac{\mathcal{R}_1[x; \theta_t]}{\langle x^{n_2} - 1 \rangle} \times \frac{\mathcal{R}_2[x; \theta_t]}{\langle x^{n_3} - 1 \rangle}$  as follows:

$$\star : \mathcal{R}_2[x; \theta_t] \times \mathcal{R}_\gamma[x; \theta_t] \rightarrow \mathcal{R}_\gamma[x; \theta_t]$$

$$r(x) \star (f_1(x) + \langle x^{n_1} - 1 \rangle, f_2(x) + \langle x^{n_2} - 1 \rangle, f_3(x) + \langle x^{n_3} - 1 \rangle) = (\eta(r(x))f_1(x) + \langle x^{n_1} - 1 \rangle, \delta(r(x))f_2(x) + \langle x^{n_2} - 1 \rangle, r(x)f_3(x) + \langle x^{n_3} - 1 \rangle).$$

**Definition 1 ([2]).** We say that an  $\mathcal{R}$ -submodule  $\mathcal{C}$  of  $\mathcal{R}^n$  is a  $\theta_t$ -cyclic code if for any  $c = (c_0, c_1, \dots, c_{n-1}) \in \mathcal{C}$ ,  $\sigma_1(c) = (\theta_t(c_{n-1}), \theta_t(c_0), \dots, \theta_t(c_{n-2})) \in \mathcal{C}$ . The operator  $\sigma_1$  is then said to be a  $\theta_t$ -cyclic shift operator on  $\mathcal{R}^n$ .

**Definition 2.** A non-trivial  $\mathcal{R}_2$ -submodule  $\mathcal{C}$  of  $\mathcal{R}_\gamma$  is called a  $\theta_t$ -cyclic code if for any  $c = (c_{0,1}, c_{1,1}, \dots, c_{n_1-1,1}, c_{0,e}, c_{1,e}, \dots, c_{n_2-1,e}, c_{0,o}, c_{1,o}, \dots, c_{n_3-1,o}) \in \mathcal{C}$ ,  $\sigma(c) = (\theta(c_{n_1-1,1}), \theta(c_{0,1}), \dots, \theta(c_{n_1-2,1}), \theta(c_{n_2-1,e}), \theta(c_{0,e}), \dots, \theta(c_{n_2-2,e}), \theta(c_{n_3-1,o}), \theta(c_{0,o}), \dots, \theta(c_{n_3-2,o})) \in \mathcal{C}$ . The operator  $\sigma$  is called a  $\theta_t$ -cyclic shift operator on  $\mathcal{R}^n$ .

The following result yields the relationship between the  $\theta_t$ -cyclic codes over  $\mathcal{R}$  and  $\mathbb{F}_q$ .

**Theorem 6.** Let  $\mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_e \otimes \mathcal{C}_o \subseteq \mathcal{R}_\gamma$  be linear. Then  $\mathcal{C}$  is a  $\theta_t$ -cyclic code if and only if  $\mathcal{C}_1, \mathcal{C}_e$  and  $\mathcal{C}_o$  are  $\theta_t$ -cyclic codes of length  $n_1, n_2$  and  $n_3$  over  $\mathbb{F}_q, \mathcal{R}_1$  and  $\mathcal{R}_2$  respectively.

**Proof.** Let  $\mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_e \otimes \mathcal{C}_o$  be a  $\theta_t$ -cyclic code over  $\mathcal{R}$ . Let  $z = (z_1, z_e, z_o) \in \mathcal{C}$ , that is:

$$z = (z_{0,1}, z_{1,1}, \dots, z_{n_1-1,1}, z_{0,e}, z_{1,e}, \dots, z_{n_2-1,e}, z_{0,o}, z_{1,o}, \dots, z_{n_3-1,o}) \in \mathcal{C}.$$

Then,  $\sigma(z) = (\theta(z_{n_1-1,1}), \theta(z_{0,1}), \dots, \theta(z_{n_1-2,1}), \theta(z_{n_2-1,e}), \theta(z_{0,e}), \dots, \theta(z_{n_2-2,e}), \theta(z_{n_3-1,o}), \theta(z_{0,o}), \dots, \theta(z_{n_3-2,o})) = (\sigma(z_1), \sigma(z_e), \sigma(z_o)) \in \mathcal{C}$ . From this, we can conclude that:

$$\sigma(z_1) \in \mathcal{C}_1, \sigma(z_e) \in \mathcal{C}_e \text{ and } \sigma(z_o) \in \mathcal{C}_o$$



Hence,  $C_1, C_e,$  and  $C_o$  are  $\theta_t$ -cyclic code of length  $n_i$ . The converse holds in a similar way.

□

We recall the following Theorem from [9].

**Theorem 7 ([9]).** Let  $C_o = o_1C_{o_1} \oplus o_2C_{o_2} \oplus o_3C_{o_3}$  be a linear code over  $\mathcal{R}_2$  of length  $n_3$ , then  $C_o$  is  $\theta_t$ -cyclic code iff  $C_{o_i}(1 \leq i \leq 3)$  is a  $\theta_t$ -cyclic code of length  $n_3$  over  $\mathbb{F}_q$ .

The analogue of this result in our setting is as follows.

**Theorem 8 ([8]).** Let  $C_e = e_1C_{e_1} \oplus e_2C_{e_2}$  be a linear code over  $\mathcal{R}_1$  of length  $n_2$  then  $C_e$  is  $\theta_t$ -cyclic code iff  $C_{e_i}(1 \leq i \leq 2)$  is a  $\theta_t$ -cyclic code of length  $n_2$  over  $\mathbb{F}_q$ .

**Theorem 9.** If  $C = C_1 \otimes C_e \otimes C_o$  is a linear code of length  $\gamma = n_1 + n_2 + n_3$ , then  $C$  is  $\theta_t$ -cyclic iff  $C_1, C_{e_i}, C_{o_j}(1 \leq i \leq 2, 1 \leq j \leq 3)$  are  $\theta_t$ -cyclic code of length  $n_1, n_2, n_3$  over  $\mathbb{F}_q$  respectively.

**Proof.** We obtain the proof on combining proofs of Theorems 6–8. □

These notions are well-behaved with respect to duality as the next result shows.

**Theorem 10.** If  $C$  is a  $\theta_t$ -cyclic code of length  $n$ , then its dual  $C^\perp$  is also a  $\theta_t$ -cyclic code.

**Proof.** From Theorem 9,  $C_1, C_{e_i}, C_{o_j}(1 \leq i \leq 2, 1 \leq j \leq 3)$  are  $\theta_t$ -cyclic codes over  $\mathbb{F}_q$ . Then,  $C_1^\perp, C_{e_i}^\perp, C_{o_j}^\perp(1 \leq i \leq 2, 1 \leq j \leq 3)$  are  $\theta_t$ -cyclic codes over  $\mathbb{F}_q$  from [29] and once again by using Theorem 9,  $C^\perp$  becomes a  $\theta_t$ -cyclic code. □

Recall the following result from [4].

**Lemma 1 ([4]).** Let  $C$  be a  $\theta_t$ -cyclic code of length  $n$  over  $\mathbb{F}_q$ . Then, there exists a polynomial  $f(x) \in \mathbb{F}_q[x; \theta_t]$  such that  $C = \langle f(x) \rangle$  and  $x^n - 1 = g(x)f(x)$  in  $\mathbb{F}_q[x; \theta_t]$ .

By assuming  $o(\theta_t) | n$ , the counterpart follows.

**Theorem 11.** Let  $C = C_1 \otimes C_e \otimes C_o$  be a  $\theta_t$ -cyclic code of length  $n$  over  $\mathcal{R}$  and assume that the order of  $\theta_t$  divides  $n$ . Then,  $C = \langle B_1, B_e, B_o \rangle$ , where  $B_1 = \langle (f_1(x), 0, 0) \rangle, B_e = \langle (0, f_e(x), 0) \rangle$ , and  $B_o = \langle (b_1(x), b_e(x), f_o(x)) \rangle$ , such that  $C_1 = \langle f_1(x) \rangle, C_e = \langle f_e(x) \rangle, C_o = \langle f_o(x) \rangle, b_1(x) \in C_1$  and  $b_2(x) \in C_e$ .

**Proof.** Let  $C = C_1 \otimes C_e \otimes C_o$  be a  $\theta_t$ -cyclic code of length  $\gamma = n_1 + n_2 + n_3$  over  $\mathcal{R}$ . Then, by Theorem 6,  $C_1, C_e, C_o$  are  $\theta_t$ -cyclic codes of length  $n_i$  over  $\mathbb{F}_q, \mathcal{R}_1$  and  $\mathcal{R}_2$ . Define a homomorphism from  $C$  to  $\mathcal{R}$  as follows:

$$\begin{aligned} \psi : C &\mapsto \mathcal{R} \\ \psi(c_1(x), c_e(x), c_o(x)) &= (0, 0, c_o(x)) \end{aligned}$$

Define:

$$\ker(\psi) = \{(c_1(x), c_e(x), 0) : c_1(x) \in C_1, c_e(x) \in C_e\}$$

$$\mathcal{I} = \{(c_1(x), c_e(x)) \in \mathbb{F}_q[x; \theta_t] \times \mathcal{R}_1[x; \theta_t] : (c_1(x), c_e(x), 0) \in \ker(\psi)\}.$$

Clearly,  $\mathcal{I} = \mathcal{I}_1 \times \mathcal{I}_e$  forms a submodule of  $\mathbb{F}_q[x; \theta_t] \times \mathcal{R}_1[x; \theta_t]$ . Therefore, there exist a polynomial  $f_1(x)$  and  $f_e(x)$  in  $\mathbb{F}_q[x; \theta_t]$  and  $\mathcal{R}_1[x; \theta_t]$ , respectively, generating  $\mathcal{I}_1$  and  $\mathcal{I}_e$  with  $f_1(x)|x^{n_1} - 1$  and  $f_e(x)|x^{n_e} - 1$ . Thus,  $\mathcal{I} = \langle (f_1(x), 0), (0, f_e(x)) \rangle$ , then for any  $(c_1(x), c_e(x), 0) \in \ker(\psi), (c_1(x), c_e(x)) = v(x) \star (f_1(x), 0), (0, f_e(x))$  for some  $v(x) \in$

$\mathcal{R}_1[x; \theta_t]$ . Finally, it leads to  $\ker(\psi) = \langle (f_1(x), 0, 0), (0, f_e(x), 0) \rangle$ . The fact that  $\mathcal{C}$  is a submodule implies that  $\psi(\mathcal{C})$  is a submodule. By using the first isomorphism theorem:

$$\mathcal{C} / \ker(\psi) \cong \psi(\mathcal{C}).$$

Let  $(b_1(x), b_e(x), f_o(x)) \in \mathcal{C}$ , then  $\psi(b_1(x), b_e(x), f_o(x)) = (0, 0, f_o(x))$ . From this, any  $\theta_t$ -cyclic code of length  $n$  can be represented by  $\mathcal{C} = \langle (f_1(x), 0, 0), (0, f_e(x), 0), (b_1(x), b_e(x), f_o(x)) \rangle$ , where  $f_1(x)|(x^{n_1} - 1)$ ,  $f_e(x)|(x^{n_2} - 1)$  and  $f_o(x)|(x^{n_3} - 1)$ .  $\square$

Furthermore, we have  $\mathcal{C}$  is  $\theta_t$ -cyclic, then  $\mathcal{C}_k$ , where  $k \in \{1, e_1, e_2, o_1, o_2, o_3\}$  is skew  $\theta_t$ -cyclic code over  $\mathbb{F}_q$  with respective lengths. From Theorem 3,  $|\mathcal{C}| = |\mathcal{C}_1| \prod_{i=1}^{i=2} |\mathcal{C}_{e_i}| \prod_{j=1}^{j=3} |\mathcal{C}_{o_j}|$ , since each  $\mathcal{C}_k$  is  $\theta_t$ -cyclic it is generated by a polynomial  $f_k(x)$ , and thus,  $|\mathcal{C}| = q^{\gamma - \sum_{i=1}^6 \epsilon_k}$ , where  $\gamma = n_1 + 2(n_2) + 3(n_3)$ . The following Theorem provides the generator polynomials for  $\theta_t$ -cyclic codes over  $\mathbb{F}_q$ .

**Theorem 12.** Let  $\mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_e \otimes \mathcal{C}_o$  be a skew cyclic code over  $\mathcal{R}$  of length  $\gamma = n_1 + n_2 + n_3$ . Then, there exists a polynomial:

- (i)  $f_1(x) \in \mathbb{F}_q[x; \theta_t]$  such that  $\mathcal{C}_1 = \langle f_1(x) \rangle$  and  $x^{n_1} - 1 = g_1(x)f_1(x)$ .
- (ii)  $f_e(x) \in \mathcal{R}_1[x; \theta_t]$  such that  $\mathcal{C}_e = \langle f_e(x) \rangle$  and  $x^{n_2} - 1 = g_e(x)f_e(x)$  where  $f_e(x) = \sum_{i=1}^2 e_i f_{e_i}(x)$ .
- (iii)  $f_o(x) \in \mathcal{R}_2[x; \theta_t]$  such that  $\mathcal{C}_o = \langle f_o(x) \rangle$  and  $x^{n_3} - 1 = g_o(x)f_o(x)$  where  $f_o(x) = \sum_{i=1}^3 o_i f_{o_i}(x)$ .

**Proof.** Let  $\mathcal{C}$  be a  $\theta_t$ -cyclic code of length  $\gamma = n_1 + n_2 + n_3$ . From Theorem 6, we have that  $\mathcal{C}_1, \mathcal{C}_e$ , and  $\mathcal{C}_o$  are  $\theta_t$ -cyclic codes. Using Lemma 1, (i) follows.

Then, the proof of (ii) is as follows. Let  $\mathcal{C}_e = e_1 \mathcal{C}_{e_1} \oplus e_2 \mathcal{C}_{e_2}$  be a  $\theta_t$ -cyclic code of length  $n_2$  over  $\mathcal{R}_1$ . Theorem 7 says that,  $\mathcal{C}_{e_1}$  and  $\mathcal{C}_{e_2}$  are  $\theta_t$ -cyclic codes of length  $n_2$  over  $\mathbb{F}_q$ . Lemma 1 says that we have  $\mathcal{C}_i = \langle f_{e_i}(x) \rangle$  and  $x^{n_2} - 1 = g_{e_i}(x)f_{e_i}(x)$  in  $\mathbb{F}_q[x; \theta_t]$  for  $i \in \{1, 2\}$ . Then,  $e_i f_{e_i}(x) \in \mathcal{C}$  for  $i \in \{1, 2\}$ . Also, for any  $f_e(x) \in \mathcal{C}$ , we have  $f_e(x) = \sum_{i=1}^2 e_i h_{e_i}(x) f_{e_i}(x)$ , where  $h_{e_i}(x) \in \mathbb{F}_q[x; \theta_t]$  for  $i \in \{1, 2\}$ . Thus,  $f_e(x) \in \langle e_1 f_{e_1}(x), e_2 f_{e_2}(x) \rangle$ . Therefore,  $\mathcal{C} = \langle e_1 f_{e_1}(x), e_2 f_{e_2}(x) \rangle$ . As  $x^{n_2} - 1 = g_{e_i}(x)f_{e_i}(x)$  in  $\mathbb{F}_q[x; \theta_t]$  for  $i \in \{1, 2\}$ . Let  $f_e(x) = e_1 f_{e_1}(x) + e_2 f_{e_2}(x) \in \mathcal{R}_1[x; \theta_t]$ . Then,  $f_e(x) \in \mathcal{C}$ . On the other hand  $e_i f_{e_i}(x) = e_i f_e(x) \in \langle f_e(x) \rangle$  for  $i = 1, 2$ . Consequently,  $\mathcal{C} = \langle f_e(x) \rangle$ . Furthermore,  $[\sum_{i=1}^2 e_i g_{e_i}(x)] f_e(x) = \sum_{i=1}^2 e_i g_{e_i}(x) f_{e_i}(x) = \sum_{i=1}^2 e_i (x^{n_2} - 1) = x^{n_2} - 1$ . Then,  $x^{n_2} - 1 = g_e(x)f_e(x)$  in  $\mathcal{R}_1[x; \theta_t]$ , where  $g_e(x) = \sum_{i=1}^2 e_i g_{e_i}(x)$ . Thus, (ii) follows. (iii) is similar to the proof of (ii).  $\square$

### 5. Skew Constacyclic Code over $\mathcal{R}$

In this section, we study skew  $\theta_t$ -constacyclic codes over  $\mathcal{R}$ . We choose a unit element  $\alpha \in \mathcal{R}_2^*$  such that  $\alpha$  satisfies the condition  $\alpha^2 = 1, (\alpha = 1, -1, \dots)$ .

**Definition 3.** Let  $\alpha_i \in \mathbb{F}_{p^t} \setminus \{0\}$ . A linear code  $\mathcal{C} \subseteq \mathcal{R}_\gamma[x, \theta]$  is called skew  $\alpha = \alpha_1 + u\alpha_2 + v\alpha_3$ -constacyclic code if it is invariant under the cyclic shift operator  $\lambda_\alpha$ , which is whenever:

$$\begin{aligned} c &= (x_0, x_1, \dots, x_{n_1-1}, y_0, y_1, \dots, y_{n_2-1}, z_0, z_1, \dots, z_{n_3-1}) \in \mathcal{C} \\ \lambda_\alpha(c) &= (\alpha_1 \theta_t(x_{n_1-1}), \theta_t(x_0), \dots, \theta_t(x_{n_1}), (\alpha_1 + u\alpha_2) \theta_t(y_{n_1-1}), \theta_t(y_1), \dots, \theta_t(y_{n_2-2}), \\ &\quad (\alpha_1 + u\alpha_2 + v\alpha_3) \theta_t(z_{n_3-1}), \theta_t(z_0), \dots, \theta_t(z_{n_3-2})) \in \mathcal{C} \end{aligned}$$

The following two results translate symmetry conditions into algebraic constraints. We give the first result without proof.

**Theorem 13.** Let  $R_{n,\lambda} = R[x; \theta_t] / \langle x^n - \lambda \rangle$ . A linear code  $\mathcal{C}$  of length  $n$  over  $R$  is  $(\theta_t, \lambda)$ -cyclic code if and only if  $\mathcal{C}$  is a left  $R[x; \theta_t]$ -submodule of  $R_{n,\lambda}$ .

The second result is less immediate.

**Theorem 14.** A code  $\mathcal{C}$  is skew  $\alpha$ -cyclic code over  $\mathcal{R}_\gamma = \frac{\mathbb{F}_q[x, \theta_1]}{x^{n_1-\alpha}} \times \frac{\mathcal{R}_1[x, \theta_1]}{x^{n_2-\alpha}} \times \frac{\mathcal{R}_2[x, \theta_1]}{x^{n_3-\alpha}}$  iff  $\mathcal{C}$  is a left  $\mathcal{R}_2[x, \theta_1]$  module over  $\mathcal{R}_\gamma$ .

**Proof.** Let  $\mathcal{C}$  be a skew  $\alpha$ -cyclic code. Then, by definition  $x \star (f(x)|g(x)|h(x)) \in \mathcal{C}$ :

$$x \star (f(x)|g(x)|h(x)) = (\theta_t(f_0)x + \theta_t(f_1)x^2 + \dots + \alpha_1\theta_t(f_{n_1-1}), \theta_t(g_0)x + \theta_t(g_1)x^2 + \dots + (\alpha_1 + u\alpha_2)\theta_t(g_{n_2-1}), \\ \theta_t(h_0)x + \theta_t(h_1)x^2 + \dots + (\alpha_1 + u\alpha_2 + v\alpha_3)\theta_t(h_{n_3-1})) \in \mathcal{C}$$

Moreover, by using linearity of  $\mathcal{C}$ :

$$r(x) \star (g_1(x)|g_2(x)|g_3(x)) \in \mathcal{C}$$

for some  $r(x) \in \mathcal{R}_2[x, \theta_1]$ . Hence,  $\mathcal{C}$  is an left  $\mathcal{R}_2[x, \theta_1]$  submodule over  $\mathcal{R}_\gamma$ . Conversely, assume that  $\mathcal{C}$  is an left  $\mathcal{R}_2[x, \theta_1]$  submodule over  $\mathcal{R}_\gamma$ , then we have  $x \star (f(x)|g(x)|h(x)) \in \mathcal{C}$  implies  $\mathcal{C}$  is skew  $\alpha$ -cyclic code.  $\square$

**Theorem 15.** The code  $\mathcal{C}_o \subseteq \mathcal{R}_2^n$  is skew  $\alpha = \alpha_1 + u\alpha_2 + v\alpha_3$ -cyclic of length  $n$  iff  $\mathcal{C}_{o_1}, \mathcal{C}_{o_2}$ , and  $\mathcal{C}_{o_3}$  are skew  $\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3$ -cyclic codes over  $\mathbb{F}_q$  of length  $n$ .

**Proof.** Let  $\mathcal{C}_o$  be a skew  $\alpha$ -cyclic code. Let  $a = x_0 + y_0 + z_0 \in \mathcal{C}_o$ , where  $x = (x_0, x_1, \dots, x_{n-1}) \in \mathcal{C}_{o_1}, y = (y_0, y_1, \dots, y_{n-1}) \in \mathcal{C}_{o_2}$  and  $z = (z_0, z_1, \dots, z_{n-1}) \in \mathcal{C}_{o_3}$ . Then, we have by definition,  $\lambda_\alpha(x) \in \mathcal{C}_o$ :

$$\begin{aligned} \lambda_\alpha(o_1(x_0, x_1, \dots, x_{n-1}) + o_2(y_0, y_1, \dots, y_{n-1}) \\ + o_3(z_0, z_1, \dots, z_{n-1})) &= ((\alpha_1 + u\alpha_2 + v\alpha_3) \star o_1(\theta_t(x_{n-1}), \theta_t(x_0), \dots, \theta_t(x_{n-2})) + \\ & (\alpha_1 + u\alpha_2 + v\alpha_3) \star o_2(\theta_t(y_{n-1}), \theta_t(y_0), \dots, \theta_t(y_{n-2})) + \\ & (\alpha_1 + u\alpha_2 + v\alpha_3) \star o_3(\theta_t(z_{n-1}), \theta_t(z_0), \dots, \theta_t(z_{n-1}))) \\ \implies & \lambda_{\alpha_1}(x) + \lambda_{\alpha_1+\alpha_2}(y) + \lambda_{\alpha_1+\alpha_3}(z) \in \mathcal{C}_o \\ \implies & \lambda_{\alpha_1}(x) \in \mathcal{C}_{o_1}, \lambda_{\alpha_1+\alpha_2}(y) \in \mathcal{C}_{o_2}, \lambda_{\alpha_1+\alpha_3}(z) \in \mathcal{C}_{o_3} \end{aligned}$$

Hence,  $\mathcal{C}_{o_1}, \mathcal{C}_{o_2}$ , and  $\mathcal{C}_{o_3}$  are skew  $\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3$ -cyclic codes over  $\mathbb{F}_q$  of length  $n$ .

Conversely, assume that  $\mathcal{C}_{o_1}, \mathcal{C}_{o_2}$  and  $\mathcal{C}_{o_3}$  are skew  $\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3$ -cyclic codes over  $\mathbb{F}_q$  of length  $n$ . Let  $m_0, m_1, \dots, m_{n-1}$  be an element in  $\mathcal{C}_o$ , where  $m_i = o_1x_i + o_2y_i + o_3z_i$  such that  $x = (x_0, x_2, \dots, x_{n-1}) \in \mathcal{C}_{o_1}, y = (y_0, y_2, \dots, y_{n-1}) \in \mathcal{C}_{o_2}$  and  $z = (z_0, z_2, \dots, z_{n-1}) \in \mathcal{C}_{o_3}$ . Then we have  $\lambda_{\alpha_1}(x) \in \mathcal{C}_{o_1}, \lambda_{\alpha_1+\alpha_2}(y) \in \mathcal{C}_{o_2}$  and  $\lambda_{\alpha_1+\alpha_3}(z) \in \mathcal{C}_{o_3}$ . So we get,

$$\begin{aligned} o_1\lambda_{\alpha_1}(x) + o_2\lambda_{\alpha_1+\alpha_2}(y) + o_3\lambda_{\alpha_1+\alpha_3}(z) &= o_1\lambda_{\alpha_1}(x_0, x_1, \dots, x_{n-1}) + o_2\lambda_{\alpha_1+\alpha_2}(y_0, y_1, \dots, y_{n-1}) \\ & + o_3\lambda_{\alpha_1+\alpha_3}(z_0, z_1, \dots, z_{n-1}) \in \mathcal{C} \\ &= \lambda_\alpha(m_0, m_1, \dots, m_{n-1}) \in \mathcal{C} \end{aligned}$$

Hence,  $\mathcal{C}$  is skew  $\alpha$ -cyclic code over  $\mathcal{R}_2^n$ .  $\square$

**Theorem 16.**  $\mathcal{C}_e$  be a skew  $\alpha = \alpha_1 + u\alpha_2$ -cyclic code over  $\mathcal{R}_1$  iff  $\mathcal{C}_{e_1}$  and  $\mathcal{C}_{e_2}$  are skew  $\alpha_1 + \alpha_2$  and  $\alpha_1$ -cyclic codes over  $\mathbb{F}_q$ .

**Proof.** The proof is similar to Theorem 15 taking mod  $v$  to the above condition.  $\square$

**Theorem 17.**  $\mathcal{C}$  be a skew  $\alpha = \alpha_1 + u\alpha_2 + v\alpha_3$ -cyclic code over  $\mathcal{R}$  of length  $\gamma = n_1 + n_2 + n_3$  iff  $\mathcal{C}_1, \mathcal{C}_e$  and  $\mathcal{C}_o$  are  $\alpha_1, \alpha_1 + u\alpha_2$ , and  $\alpha_1 + u\alpha_2 + v\alpha_3$ -cyclic codes over  $\mathbb{F}_q, \mathcal{R}_1$  and  $\mathcal{R}_2$ , respectively.

**Proof.**  $C_1, C_e$  and  $C_o$  be  $\alpha_1, \alpha_1 + u\alpha_2$ , and  $\alpha_1 + u\alpha_2 + v\alpha_3$ -cyclic. Consider  $x = (x_0, x_1, \dots, x_{n_1-1})$ ,  $y = (y_0, y_1, \dots, y_{n_2-1})$  and  $z = (z_0, z_1, \dots, z_{n_3-1})$ . Consider  $\alpha_1 + u\alpha_2 = \beta$ . Then, we have:

$$(x, y, z) \in \mathcal{C} \implies (\lambda_{\alpha_1}(x), \lambda_{\beta}(y), \lambda_{\alpha}(z)) \in \mathcal{C}$$

Hence,  $\mathcal{C}$  is skew  $\alpha$ -cyclic. The converse part holds similarly.  $\square$

**Theorem 18.**  $\mathcal{C}$  be a skew  $\alpha$ -cyclic code of length  $\gamma = n_1 + n_2 + n_3$  iff  $C_1$  is skew  $\alpha_1$ -cyclic code of length  $n_1$ ,  $C_{e_1}$ , and  $C_{e_2}$  are  $\alpha_1 + \alpha_2, \alpha_1$ -cyclic codes of length  $n_2$  and  $C_{o_1}, C_{o_2}$ , and  $C_{o_3}$  are skew  $\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3$ -cyclic codes over  $\mathbb{F}_q$  of length  $n_3$ .

**Proof.** Using Theorems 15–17 the result follows.  $\square$

**Theorem 19.**  $\mathcal{C}$  be a skew  $\alpha = \alpha_1 + u\alpha_2 + v\alpha_3$ -cyclic code over  $\mathcal{R}$  of length  $\gamma = n_1 + n_2 + n_3$  iff  $C_1^\perp, C_e^\perp$ , and  $C_o^\perp$  are  $(\alpha_1)^{-1}, (\alpha_1 + u\alpha_2)^{-1}$ , and  $(\alpha_1 + u\alpha_2 + v\alpha_3)^{-1}$ -cyclic.

**Proof.** Let  $\mathcal{C}$  be a skew  $\alpha$ -cyclic code, Lemma 3.1 [5] says that  $\mathcal{C}^\perp$  is skew  $(\alpha_1 + u\alpha_2 + v\alpha_3)^{-1}$ -cyclic code. From Theorem 17, we have  $C_1^\perp, C_e^\perp$  and  $C_o^\perp$  are skew  $(\alpha_1)^{-1}, (\alpha_1 + u\alpha_2)^{-1}$  and  $(\alpha_1 + u\alpha_2 + v\alpha_3)^{-1}$ -cyclic.  $\square$

**Corollary 2.** Let  $\mathcal{C} = C_1 \otimes C_e \otimes C_o$ . be a skew  $\alpha = \alpha_1 + u\alpha_2 + v\alpha_3$ -cyclic code over  $\mathcal{R}$  of length  $\gamma = n_1 + n_2 + n_3$ . Then, there exist polynomials:

- (i)  $f_1(x) \in \mathbb{F}_q[x; \theta_t]$  such that  $C_1 = \langle f_1(x) \rangle$  and  $x^{n_1} - \alpha_1 = g_1(x)f_1(x)$ .
- (ii)  $f_e(x) \in \mathcal{R}_1[x; \theta_t]$  such that  $C_e = \langle f_e(x) \rangle$  and  $x^{n_2} - (\alpha_1 + u\alpha_2) = g_e(x)f_e(x)$ .
- (iii)  $f_o(x) \in \mathcal{R}_2[x; \theta_t]$  such that  $C_o = \langle f_o(x) \rangle$  and  $x^{n_3} - (\alpha_1 + u\alpha_2 + v\alpha_3) = g_o(x)f_o(x)$ .

**Proof.** The proof is similar to the proof of Theorem 12.  $\square$

**Theorem 20.** Let  $\mathcal{C} = C_1 \otimes C_e \otimes C_o$  be a  $\theta_t$ -constacyclic code of length  $\gamma$  over  $\mathcal{R}$ . Then  $\mathcal{C} = \langle \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3 \rangle$ , where  $\mathcal{B}_1 = \langle (f_1(x), 0, 0) \rangle, \mathcal{B}_2 = \langle (0, f_e(x), 0) \rangle$ , and  $\mathcal{B}_3 = \langle (b_1(x), b_e(x), f_o(x)) \rangle$ .

**Proof.** The proof is similar to the proof of Theorem 11.  $\square$

**Example 1.** Let  $q = 9$  and  $\mathbb{F}_9 = \mathbb{F}_3[z]$  with  $z^2 + 1 = 0$ . Consider the ring  $R_\gamma = \frac{\mathbb{F}_9[x, \theta_3]}{x^4 - 1} \times \frac{\mathcal{R}_1[x, \theta_3]}{x^5 - 1} \times \frac{\mathcal{R}_2[x, \theta_3]}{x^5 - 1}$ , where  $\theta_3$  is the Frobenius automorphism defined by  $\theta_3(a) = a^3$  for any  $a \in \mathbb{F}_9$ . Write:

$$\begin{aligned} x^4 - 1 &= (x + 1)(x + 2)(x + z)(x + 2z) \in \mathbb{F}_9[x, \theta_3] \\ x^5 - 1 &= (x + 2)(x^4 + x^3 + x^2 + x + 1) \in \mathbb{F}_9[x, \theta_3] \\ f_1(x) &= \langle (x + 1) \rangle, f_e(x) = \langle e_1(x + 2) + e_2(x + 2) \rangle, f_o(x) = \langle o_1(x + 2) + o_2(x + 2) + o_3(x + 2) \rangle \end{aligned}$$

By Theorem 12, we have that  $f_i$  divides  $x^{n_i} - 1$  for  $(i = 1, e, o)$ , yielding a code with parameter  $[29, 18, 2]$  over  $\mathbb{F}_9$ .

**Example 2.** Let  $q = 25$  and  $\mathbb{F}_{25} = \mathbb{F}_5[z]$  with  $z^2 + z + 1 = 0$ . Consider the ring  $R_\gamma = \frac{\mathbb{F}_{25}[x, \theta_5]}{x^4 - 1} \times \frac{\mathcal{R}_1[x, \theta_5]}{x^6 - 1} \times \frac{\mathcal{R}_2[x, \theta_5]}{x^4 - 1}$ , where  $\theta_5$  is the Frobenius automorphism defined by  $\theta_5(a) = a^5$  for any  $a \in \mathbb{F}_{25}$ . Write

$$\begin{aligned} x^4 - 1 &= (x + 2)(x + 3)(x + z)(x + z + 1) \in \mathbb{F}_{25}[x, \theta_5] \\ x^6 - 1 &= (x^2 - 1)(x^2 + x + 1)(x^2 - x + 1) \in \mathbb{F}_{25}[x, \theta_5] \\ f_1(x) &= \langle (x + 2) \rangle, f_e(x) = \langle e_1(x^2 - 1) + e_2(x^2 - 1) \rangle, f_o(x) = \langle o_1(x + z + 1) + o_2(x + z + 1) + o_3(x + z + 1) \rangle \end{aligned}$$

By Theorem 12, we have that  $f_i$  divides  $x^{n_i} - 1$  for  $(i = 1, e, o)$  yielding a code with parameter  $[28, 20, 2]$  over  $\mathbb{F}_{25}$ .

## 6. Conclusions and Open Problems

In this note, we have studied the algebraic and metric structure of skew cyclic and skew constacyclic codes over a special mixed alphabet. Thus, our codes have a structure of

module over the largest of the three alphabets  $\mathcal{R}_2$ . Codes over the product ring  $\mathbb{F}_q \times \mathcal{R}_1 \times \mathcal{R}_2$  would be modules over that larger ring. The two algebraic structures are different and should not be confused.

The present work leads itself to two paths of generalization: consider different mixed alphabets or replace the concepts of cyclicity by that of quasi-cyclicity. The former path seems easier than the latter, in view of the many examples of rings that have been used as alphabets of cyclic codes in recent years. On the other hand, the structure of quasi-cyclic codes is always more subtle than that of cyclic codes.

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