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Article

Skew Cyclic and Skew Constacyclic Codes over a Mixed Alphabet

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Abstract: In this note, we study skew cyclic and skew constacyclic codes over the mixed alphabet $\mathcal{R} = \mathbb{F}_q \mathcal{R}_1 \mathcal{R}_2$, where $q = p^m$, p is an odd prime with m odd and $\mathcal{R}_1 = \mathbb{F}_q + u\mathbb{F}_q$ with $u^2 = u$, and $\mathcal{R}_2 = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q$ with $u^2 = u, v^2 = v, uv = vu = 0$. Such codes consist of the juxtaposition of three codes of the same size over $\mathbb{F}_q, \mathcal{R}_1$, and \mathcal{R}_2 , respectively. We investigate the generator polynomial for skew cyclic codes over \mathcal{R} . Furthermore, we discuss the structural properties of the skew cyclic and skew constacyclic codes over \mathcal{R} . We also study their q -ary images under suitable Gray maps.

Keywords: linear codes; skew-cyclic codes; Gray map; skew constacyclic codes

MSC: 94B05; 94B15; 94B35; 94B60



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1. Introduction

The most widely used family of linear codes consists of cyclic codes. Inspired by codes for the Lee metric [1], Berlekamp adapted them to constacyclic codes. Since then, as the following paragraphs demonstrate, they have happened in a number of circumstances.

Skew cyclic codes were first introduced as ideals in the skew polynomial ring $\mathbb{F}[x; \theta]$ in 2007 by Boucher et al. [2], where θ represents an automorphism of the finite field \mathbb{F} . The tables of the most well-known codes were enhanced by the numerous numerical samples that this technique created. The fact that the factorization of the polynomial $x^n - 1$ is not unique gives skew polynomial rings an advantage over commutative polynomial rings. For a given length, these numerous factorizations produce a large number of additional codes. Boucher et al. further extended this technique to skew constacyclic codes in [3]. Siap et al. [4] investigated skew cyclic codes of any length in 2011 and produced maps using both classical and quasi-cyclic codes.

In 2012, Jitman et al. [5] studied skew constacyclic codes over finite chain rings and described the algebraic structure of Euclidean and Hermitian dual codes. Abualrub et al. [6] studied θ -cyclic codes over the semilocal ring $\mathbb{F}_2 + u\mathbb{F}_2, v^2 = v$ with respect to Euclidean and Hermitian inner products.

These codes over semilocal rings were further studied in many contexts. The rings $\mathbb{F}_3 + v\mathbb{F}_3$ in [7], $\mathbb{F}_q + v\mathbb{F}_q, v^2 = v$ in [8], and $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q, u^2 = u, v^2 = v, uv = vu = 0$ in [9] were utilized as alphabets for skew cyclic codes, for example. Dertli and Cengellenmis [10] and Yao et al. [11] also examined these codes over $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q, u^2 = u, v^2 = v, uv = vu$. Skewed $(-1 + 2v)$ -constacyclic codes were developed in 2017 by Gao et al. [12] after deriving the structure of skew constacyclic codes over the semilocal ring $\mathbb{F}_q + v\mathbb{F}_q, v^2 = v$.

In [13] and [14], respectively, Islam and Prakash established the algebraic structure of skew constacyclic codes over $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$, $u^2 = u, v^2 = v, uv = vu$.

Using two non-trivial automorphisms, Bhardwaj and Raka [15] investigated the skew constacyclic codes over the ring $\mathbb{F}_q[u, v] \langle f(u), g(v), uv - vu \rangle$ in 2019. Alternatively, $\mathbb{Z}_2\mathbb{Z}_4$ -linear codes, or codes over the mixed alphabet $\mathbb{Z}_2\mathbb{Z}_4$, where a subset of coordinates is binary and the complement is quaternary, were introduced by Borges et al. [16]. They have calculated their generator matrices and described their dual codes. Fernandez-Cordoba et al. [17] obtained the rank and kernel of $\mathbb{Z}_2\mathbb{Z}_4$ -linear codes in a follow-up experiment. Steganography is one field in which these codes have found industrial use [18].

In [19], additive codes over the mixed alphabet $\mathbb{Z}_2\mathbb{Z}_{2^s}$ were examined. Next, Refs. [20–23] examined the mixed alphabet $\mathbb{Z}_p\mathbb{Z}_{p^s}$ and, more broadly, $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$. Conversely, Abualrub et al. [24] defined $\mathbb{Z}_2\mathbb{Z}_4$ in 2014, in line with the advancement of cyclic codes on mixed alphabets. The code for $-$ additive cyclics is $\mathbb{Z}_4[x]$ -submodule of $\mathbb{Z}_2[x] / \langle x^r - 1 \rangle \times \mathbb{Z}_4[x] / \langle x^s - 1 \rangle$, from which the smallest spanning set and unique set of generators for these codes, where s is an odd integer, were obtained.

Furthermore, generator polynomials and duals for $\mathbb{Z}_2\mathbb{Z}_4$ -additive cyclic codes were discovered by Borges et al. [25]. In [26], Aydogdu et al. [27] introduced the novel mixed alphabets $\mathbb{Z}_2\mathbb{Z}_2[u]$ -additive codes, where $u^2 = 0$. They also studied constacyclic codes over mixed alphabets by defining them as $\mathbb{Z}_2[u][x] \times \mathbb{Z}_2[u][x] / \langle x^\beta - (1 + u) \rangle$, $-$ submodules of $\mathbb{Z}_2[x] / \langle x^\alpha - 1 \rangle$.

As the Gray images of $\mathbb{Z}_2\mathbb{Z}_2[u]$ -cyclic codes, they were able to derive several optimum binary linear codes. In the meanwhile, $\mathbb{Z}_2\mathbb{Z}_2[u]$ -additive cyclic and constacyclic codes with the unit $1 + u$, respectively, were explored algebraically by [28]. Consequently, the predicted generalization in the continuation of these research should be $\mathbb{Z}_{2^r}\mathbb{Z}_{2^s}[u]$ -additive cyclic codes, $u^2 = 0$, and constacyclic codes.

In this article, we examine a mixed alphabet $\mathcal{R} = \mathbb{F}_q\mathcal{R}_1\mathcal{R}_2$, where $\mathcal{R}_1 = \mathbb{F}_q + u\mathbb{F}_q$ with $u^2 = u$ and $\mathcal{R}_2 = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q$ with $u^2 = u, v^2 = v, uv = vu = 0$. Moreover, we examine the cyclic codes θ_t and (θ_t, α) over \mathcal{R} . The algebraic structure of these codes is fully determined. We examine their q -ary representations under Gray maps and provide a few brief numerical instances.

The contents are arranged as follows. The next section gathers some background information. Gray maps are examined in Section 3. Skewed cyclic codes are covered in Section 4, and skew constacyclic codes are covered in Section 5. The essay is concluded at Section 6.

2. Preliminaries

Let p be an odd prime, and let $q = p^m$ with m being odd. Denote by \mathbb{F}_q the finite field of size q . The set \mathbb{F}_q^n of all ordered n -tuples over \mathbb{F}_q is equipped with the structure of an \mathbb{F}_q vector space by the usual addition and scalar multiplication of vectors.

A code of length n over \mathbb{F}_q is just any non-empty subset \mathcal{C} of \mathbb{F}_q^n . It is said to be linear if \mathcal{C} is an \mathbb{F}_q subspace of \mathbb{F}_q^n . From now on, we write $\mathcal{R}_1 = \mathbb{F}_q + u\mathbb{F}_q$, with $u^2 = u$ and $\mathcal{R}_2 = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q$, with $u^2 = u, v^2 = v, uv = vu = 0$.

Note that \mathcal{R}_1 and \mathcal{R}_2 are finite non-chain rings. Let $a + ub + vc$ be an element of \mathcal{R}_2 . Then, we define two maps η and δ as follows:

$$\begin{aligned} \eta : \mathcal{R}_2 &\rightarrow \mathbb{F}_q, & \delta : \mathcal{R}_2 &\rightarrow \mathcal{R}_1, \\ \eta(a + ub + vc) &= a, & \delta(a + ub + vc) &= a + ub, \end{aligned}$$

It is clear that η and δ are ring homomorphisms. We consider the ring \mathcal{R} :

$$\mathcal{R} = \mathbb{F}_q\mathcal{R}_1\mathcal{R}_2 = \{(x, y, z) \mid x \in \mathbb{F}_q, y \in \mathcal{R}_1 \text{ and } z \in \mathcal{R}_2\}$$

We define a \mathcal{R}_2 -multiplication in this ring as follows:

$$\begin{aligned} \star : \mathcal{R}_2 \times \mathcal{R} &\rightarrow \mathcal{R} \\ r \star (x, y, z) &= (\eta(r)x, \delta(r)y, rz) \end{aligned}$$

This is a well defined multiplication and it can be extended componentwise to $\mathcal{R}_\circ = \mathbb{F}_q^{n_1} \times \mathcal{R}_1^{n_2} \times \mathcal{R}_2^{n_3}$ by:

$$\begin{aligned} \star : \mathcal{R}_2 \times \mathcal{R}_\circ &\rightarrow \mathcal{R}_\circ \\ r \star (x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2}, z_1, \dots, z_{n_3}) &= (\eta(r)x_1, \dots, \eta(r)x_{n_1}, \delta(r)y_1, \dots, \delta(r)y_{n_2}, rz_1, \dots, rz_{n_3}) \end{aligned}$$

where $(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2}, z_1, \dots, z_{n_3}) \in \mathcal{R}_\gamma$. Equipped with this multiplication, \mathcal{R}_\circ becomes an \mathcal{R}_2 module. A non-empty subset \mathcal{C} of \mathcal{R}_\circ is said to be a \mathcal{R} -linear code of length (n_1, n_2, n_3) if \mathcal{C} is an \mathcal{R}_2 -submodule of \mathcal{R}_\circ . Now we define the inner product by the formula:

$$\langle c, c' \rangle = \sum_{i=1}^{n_1} x_i x'_i + \sum_{j=1}^{n_2} y_j y'_j + \sum_{k=1}^{n_3} z_k z'_k,$$

where $c = (x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2}, z_1, \dots, z_{n_3})$, $c' = (x'_1, \dots, x'_{n_1}, y'_1, \dots, y'_{n_2}, z'_1, \dots, z'_{n_3})$ are in \mathcal{R}_γ . Let \mathcal{C} be an \mathcal{R} -linear code of length (n_1, n_2, n_3) . Then, the dual code of \mathcal{C} is defined as:

$$\mathcal{C}^\perp = \{c' \in \mathcal{R}_\gamma \mid \langle c, c' \rangle = 0 \forall c \in \mathcal{C}\}$$

3. Decomposition and Properties of Gray Maps

Recall that, $\mathcal{R}_1 = \mathbb{F}_q + u\mathbb{F}_q$, with $u^2 = u$. Consider the idempotent orthogonal elements $e_1 = u$ and $e_2 = 1 - u$. Then, we have the decomposition:

$$\mathcal{R}_1 = e_1 \mathcal{R}_1 \oplus e_2 \mathcal{R}_1 \cong e_1 \mathbb{F}_q \oplus e_2 \mathbb{F}_q,$$

where $e_1 e_2 = 0, e_1^2 = e_1, e_1 + e_2 = 1$. Hence, $\mathcal{R}_1 = \{ae_1 + be_2 \mid a, b \text{ in } \mathbb{F}_q\}$. We now define the Gray map:

$$\begin{aligned} \varphi_1 : \mathcal{R}_1 &\rightarrow \mathbb{F}_q^2 \\ \varphi_1(ae_1 + be_2) &= (a, b) \end{aligned}$$

It can be extended to the length n by:

$$\begin{aligned} \varphi_1 : \mathcal{R}_1^n &\rightarrow \mathbb{F}_q^{2n} \\ \varphi_1((a_1, \dots, a_n)e_1 + (b_1, \dots, b_n)e_2) &= (a_1, \dots, a_n, b_1, \dots, b_n) \end{aligned}$$

Note that it is a linear map. We define the Gray weight of a codeword in \mathcal{R}_1 as:

$$wt_G(ae_1 + be_2) = wt_H(a, b)$$

where wt_H denotes the Hamming weight. If x, y lie in \mathcal{R}_1^n , then their mutual distance is given by:

$$d_G(x, y) = \sum_{i=1}^n wt_G(x_i - y_i) = \sum_{i=1}^{2n} wt_H(\varphi_1(x) - \varphi_1(y)) = d_H(\varphi_1(x), \varphi_1(y)).$$

Hence, φ_1 is a weight preserving map. A non-empty subset \mathcal{C} of \mathcal{R}_i^n is said to be a linear code of length n if \mathcal{C} is \mathcal{R}_i -submodule of \mathcal{R}_i^n .

For $i \in \{1, 2\}$, $A_i \subseteq \mathcal{R}_1$:

$$A_1 \oplus A_2 = \{a_1 + a_2 \mid a_i \in A_i\} \text{ and } A_1 \otimes A_2 = \{(a_1, a_2) \mid a_i \in A_i\}.$$

Let \mathcal{C}_e be a linear code of length n over \mathcal{R}_1 . Then, we define:

$$\mathcal{C}_{e_1} = \{y_1 \in \mathbb{F}_q^n \mid e_1 y_1 + e_2 y_2 \in \mathcal{C}_e, \text{ for some } y_2 \in \mathbb{F}_q^n\}$$

$$\mathcal{C}_{e_2} = \{y_2 \in \mathbb{F}_q^n \mid e_1 y_1 + e_2 y_2 \in \mathcal{C}_e, \text{ for some } y_1 \in \mathbb{F}_q^n\}$$

Therefore, any linear code \mathcal{C}_e over \mathcal{R}_1 can be represented as $\mathcal{C}_e = e_1 \mathcal{C}_{e_1} \oplus e_2 \mathcal{C}_{e_2}$ and $\varphi_1(\mathcal{C}_e) = \mathcal{C}_{e_1} \otimes \mathcal{C}_{e_2}$. Hence, \mathcal{C}_{e_1} and \mathcal{C}_{e_2} are \mathbb{F}_q -linear codes. Also note that $\varphi_1(\mathcal{C}_e^\perp) = \varphi_1(\mathcal{C}_e)^\perp$.

Recall that $\mathcal{R}_2 = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q$, with $u^2 = u, v^2 = v, uv = vu = 0$. Let $o_1 = (1 - u - v), o_2 = u, o_3 = v$ be idempotent orthogonal elements in \mathcal{R}_2 , then:

$$\mathcal{R}_2 = o_1 \mathcal{R}_2 \oplus o_2 \mathcal{R}_2 \oplus o_3 \mathcal{R}_2 \cong o_1 \mathbb{F}_q \oplus o_2 \mathbb{F}_q \oplus o_3 \mathbb{F}_q,$$

where $o_i o_j = 0$ ($i \neq j$), $o_i^2 = o_i$, $o_1 + o_2 + o_3 = 1$. Hence, any element in \mathcal{R}_2 can be written as $ao_1 + bo_2 + co_3$. We now define a weight preserving linear Gray map φ_2 :

$$\begin{aligned} \varphi_2 : \mathcal{R}_2 &\rightarrow \mathbb{F}_q^3 \\ \varphi_2(ao_1 + bo_2 + co_3) &= (a, b, c) \end{aligned}$$

It can be extended to length n by the formula:

$$\varphi_2((a_1, \dots, a_n)o_1 + (b_1, \dots, b_n)o_2 + (c_1, \dots, c_n)o_3) = (a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_n).$$

We define the Gray weight of a codeword in \mathcal{R}_2 as:

$$wt_G(ao_1 + bo_2 + co_3) = wt_H(a, b, c)$$

where wt_H denotes the Hamming weight. If x, y are in \mathcal{R}_2^n , then their Gray distance is given by:

$$d_G(x, y) = \sum_{i=1}^n wt_G(x_i - y_i) = \sum_{i=1}^{3n} wt_H(\varphi_2(x) - \varphi_2(y)) = d_H(\varphi_2(x), \varphi_2(y)).$$

For $i \in \{1, 2, 3\}$, $A_i \subseteq \mathcal{R}_2$:

$$A_1 \oplus A_2 \oplus A_3 = \{a_1 + a_2 + a_3 \mid a_i \in A_i\} \text{ and } A_1 \otimes A_2 \otimes A_3 = \{(a_1, a_2, a_3) \mid a_i \in A_i\}.$$

Let \mathcal{C}_o be a linear code of length n over \mathcal{R}_2 . We define the three codes:

$$\begin{aligned} \mathcal{C}_{o_1} &= \{z_1 \in \mathbb{F}_q^n \mid o_1 z_1 + o_2 z_2 + o_3 z_3 \in \mathcal{C}_o, \text{ for some } z_2, z_3 \in \mathbb{F}_q^n\}, \\ \mathcal{C}_{o_2} &= \{z_2 \in \mathbb{F}_q^n \mid o_1 z_1 + o_2 z_2 + o_3 z_3 \in \mathcal{C}_o, \text{ for some } z_1, z_3 \in \mathbb{F}_q^n\}, \\ \mathcal{C}_{o_3} &= \{z_3 \in \mathbb{F}_q^n \mid o_1 z_1 + o_2 z_2 + o_3 z_3 \in \mathcal{C}_o, \text{ for some } z_1, z_2 \in \mathbb{F}_q^n\}. \end{aligned}$$

Then, any linear code \mathcal{C}_o over \mathcal{R}_2 can be represented as $\mathcal{C}_o = o_1 \mathcal{C}_{o_1} \oplus o_2 \mathcal{C}_{o_2} \oplus o_3 \mathcal{C}_{o_3}$ and $\varphi_2(\mathcal{C}_o) = \mathcal{C}_{o_1} \otimes \mathcal{C}_{o_2} \otimes \mathcal{C}_{o_3}$, where $\mathcal{C}_{o_1}, \mathcal{C}_{o_2}$, and \mathcal{C}_{o_3} are \mathbb{F}_q -linear codes. Also note that $\varphi_2(\mathcal{C}_o^\perp) = \varphi_2(\mathcal{C}_o)^\perp$.

Henceforth, we define the Gray map φ on \mathcal{R} using the maps defined previously:

$$\begin{aligned} \varphi : \mathcal{R} &\rightarrow \mathbb{F}_q^6 \\ \varphi(x, y, z) &= (x, \varphi_1(y), \varphi_2(z)) \end{aligned}$$

now we can extend this map to \mathcal{R}_γ :

$$\varphi(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2}, z_1, \dots, z_{n_3}) = (x_1, \dots, x_{n_1}, \varphi_1(y_1), \dots, \varphi_1(y_{n_2}), \varphi_2(z_1), \dots, \varphi_2(z_{n_3}))$$

then, the Gray weight of an element in \mathcal{R}_γ can be denoted by $wt_G(\alpha) = wt_H(\varphi(\alpha))$. Any linear code \mathcal{C} of \mathcal{R}_γ can be represented by $\mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_e \otimes \mathcal{C}_o$, where $\mathcal{C}_1, \mathcal{C}_e$, and \mathcal{C}_o are linear code over $\mathbb{F}_q, \mathcal{R}_1$, and \mathcal{R}_2 . Let $G_{\mathbb{F}_q}$ be the generator matrix for linear code over \mathbb{F}_q . The generator matrix $G_{\mathcal{R}_1}$ for a linear code over \mathcal{R}_1 is denoted by:

$$G_{\mathcal{R}_1} = \begin{bmatrix} e_1 G_{e_1} \\ e_2 G_{e_2} \end{bmatrix}$$

where G_{e_i} is the generator matrix for the linear code \mathcal{C}_{e_i} , for $i = \{1, 2\}$. The generator matrix $G_{\mathcal{R}_2}$ for the linear code over \mathcal{R}_2 is:

$$G_{\mathcal{R}_2} = \begin{bmatrix} o_1 G_{o_1} \\ o_2 G_{o_2} \\ o_3 G_{o_3} \end{bmatrix}$$

where G_{o_i} is the generator matrix for the linear code \mathcal{C}_{o_i} , for $i = \{1, 2, 3\}$. Using the generator matrices above, we can say that the generator matrix G for the linear code over \mathcal{R} is:

$$G = \begin{bmatrix} G_{\mathbb{F}_q} & 0 & 0 \\ 0 & G_{\mathcal{R}_1} & 0 \\ 0 & 0 & G_{\mathcal{R}_2} \end{bmatrix}.$$

Note that the minimum distance of \mathcal{C} is $\min\{d_H(\mathcal{C}_1), d_H(\varphi_1(\mathcal{C}_e)), d_H(\varphi_2(\mathcal{C}_o))\}$. The following theorem provides the weight preserving nature of the Gray map.

Theorem 1. *The Gray map φ defined above is linear and weight preserving.*

Proof. Let $x = (x_1, x_2, x_3), x' = (x'_1, x'_2, x'_3)$ be in \mathcal{R}_γ where $x_1, x'_1 \in \mathbb{F}_q^{n_1}, x_2, x'_2 \in \mathcal{R}_1^{n_2}, x_3, x'_3 \in \mathcal{R}_2^{n_3}$. We have:

$$\begin{aligned} \varphi(x + x') &= \varphi(x_1 + x'_1, x_2 + x'_2, x_3 + x'_3) \\ &= (x_1 + x'_1, \varphi_1(x_2 + x'_2), \varphi_2(x_3 + x'_3)) \\ &= (x_1, \varphi_1(x_2), \varphi_2(x_3)) + (x'_1, \varphi_1(x'_2), \varphi_2(x'_3)) (\because \varphi_1 \text{ and } \varphi_2 \text{ are linear}) \\ &= \varphi(x) + \varphi(x') \end{aligned}$$

Using the linear map φ ,

$$d_G(x, x') = wt_G(x - x') = wt_H(\varphi(x) - \varphi(x')) = d_H(\varphi(x), \varphi(x'))$$

Hence, φ is a weight preserving linear map. \square

The following theorem gives the parameters of the Gray image of a linear code.

Theorem 2. *If $\mathcal{C} \subseteq \mathcal{R}_\gamma$ is an $(n_1 + n_2 + n_3, d_G)$ linear code then $\varphi(\mathcal{C})$ is an $(n_1 + 2(n_2) + 3(n_3), d_H)$ linear code over \mathbb{F}_q , where $d_G = d_H$.*

Proof. The proof can extended from the proof of Theorem 1. \square

The following Theorem characterizes $\varphi(\mathcal{C})$:

Theorem 3. *If $\mathcal{C} \subseteq \mathcal{R}_\gamma$ is linear, then $\varphi(\mathcal{C}) = \mathcal{C}_1 \otimes_{i=1}^{i=2} \mathcal{C}_{e_i} \otimes_{j=1}^{j=3} \mathcal{C}_{o_j} \mid \mathcal{C} \mid = \mid \mathcal{C}_1 \mid \mid \prod_{i=1}^{i=2} \mid \mathcal{C}_{e_i} \mid \mid \prod_{j=1}^{j=3} \mid \mathcal{C}_{o_j} \mid$.*

Proof. Let:

$$\varphi(x, y, z) = (x, \varphi_1(y), \varphi_2(z)) = (a_1, \dots, a_6) \in \varphi(\mathcal{C}) \subseteq \mathbb{F}_q^6.$$

Note that φ is bijective and $\mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_e \otimes \mathcal{C}_o$ is linear. Thus, $a_1 = x \in \mathcal{C}_1$. Also note that $\varphi_1(y) = (a_2, a_3)$, $\varphi_2(z) = (a_4, a_5, a_6)$. Since φ_i 's are bijective, $a_2e_1 + a_3e_2 \in e_1\mathcal{C}_{e_1} \oplus e_2\mathcal{C}_{e_2} = \mathcal{C}_e$. Hence, $(a_2, a_3) \in \mathcal{C}_{e_1} \otimes \mathcal{C}_{e_2}$ and similarly $(a_4, a_5, a_6) \in \mathcal{C}_{o_1} \otimes \mathcal{C}_{o_2} \otimes \mathcal{C}_{o_3}$. The converse holds in a similar way. The second part of the statement follows from the fact that φ is bijective. \square

The following Theorem furnishes the decomposition of the dual of the linear code \mathcal{C} .

Theorem 4. *If $\mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_e \otimes \mathcal{C}_o$ is a linear code over \mathcal{R} then $\mathcal{C}^\perp = \mathcal{C}_1^\perp \otimes \mathcal{C}_e^\perp \otimes \mathcal{C}_o^\perp$, where \mathcal{C}_1^\perp , \mathcal{C}_e^\perp and \mathcal{C}_o^\perp are duals for the respective linear codes.*

Proof. Let $\mathcal{C}^\perp = \{c' \in \mathcal{R}_\gamma | \langle c, c' \rangle = 0 \text{ for all } c \in \mathcal{C}\} = \{(x', y', z') \in \mathcal{R}_\gamma | x' \in \mathbb{F}_q^{n_1}, y' \in \mathbb{F}_q^{n_2}, z' \in \mathbb{F}_q^{n_3}\}$. Let $c = (x, y, z) \in \mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_e \otimes \mathcal{C}_o$. Then:

$$\langle c, c' \rangle = xx' + yy' + zz' = 0.$$

Thus, $x' \in \mathcal{C}_1^\perp, y' \in \mathcal{C}_e^\perp, z' \in \mathcal{C}_o^\perp$ and so $\mathcal{C}^\perp \subseteq \mathcal{C}_1^\perp \otimes \mathcal{C}_e^\perp \otimes \mathcal{C}_o^\perp$. Since $|\mathcal{C}^\perp| = |\mathcal{C}_1^\perp| |\mathcal{C}_e^\perp| |\mathcal{C}_o^\perp|$, the statement holds. \square

The next result shows that the Gray maps is compatible with duality.

Theorem 5. *If $\mathcal{C} \subseteq \mathcal{R}_\gamma$ is linear, then $\varphi(\mathcal{C}^\perp) = \varphi(\mathcal{C})^\perp$.*

Proof. Let $(x, y, z) \in \mathcal{C}$ and $(x', y', z') \in \mathcal{C}^\perp$, where $x \in \mathcal{C}_1, y \in \mathcal{C}_e, z \in \mathcal{C}_o$ and $x' \in \mathcal{C}_1^\perp, y' \in \mathcal{C}_e^\perp, z' \in \mathcal{C}_o^\perp$, then $\langle (x|y|z, x'|y'|z') \rangle = 0$. Using Theorem 4, $\mathcal{C}_1^\perp, \mathcal{C}_e^\perp$ and \mathcal{C}_o^\perp are duals for $\mathcal{C}_1, \mathcal{C}_e$, and \mathcal{C}_o . Now, we have $\varphi(x, y, z) = (x, \varphi_1(y), \varphi_2(z))$, $\varphi(x', y', z') = (x', \varphi_1(y'), \varphi_2(z'))$, then the inner product is given by:

$$\begin{aligned} \langle \varphi(x, y, z), \varphi(x', y', z') \rangle &= \langle (x, \varphi_1(y), \varphi_2(z)), (x', \varphi_1(y'), \varphi_2(z')) \rangle \\ &= \langle (x, 0, 0), (x', 0, 0) \rangle + \langle (0, \varphi_1(y), 0), (0, \varphi_1(y'), 0) \rangle \\ &\quad + \langle (0, 0, \varphi_2(z)), (0, 0, \varphi_2(z')) \rangle \\ &= 0 \quad (\because \varphi_1(\mathcal{C}^\perp) = \varphi_1(\mathcal{C})^\perp, \varphi_2(\mathcal{C}^\perp) = \varphi_2(\mathcal{C})^\perp) \end{aligned}$$

Thus, $\varphi(\mathcal{C}^\perp) \subseteq \varphi(\mathcal{C})^\perp$. Since the cardinality is the same on both sides, the statement holds. \square

The following result provides the self duality nature of the linear code and its Gray image.

Corollary 1. *If \mathcal{C} is a linear \mathcal{R}_γ -code, then \mathcal{C} is self-dual iff $\varphi(\mathcal{C})$ is self-dual. Moreover, $\varphi(\mathcal{C})$ is a self-orthogonal code over \mathbb{F}_q iff \mathcal{C} is self-orthogonal.*

Proof. Let \mathcal{C} be a self-dual linear code of length n over \mathcal{R} . Thus, $\mathcal{C} = \mathcal{C}^\perp$. Then, $\varphi(\mathcal{C}) = \varphi(\mathcal{C}^\perp)$, and hence, by Theorem 5, we have $\varphi(\mathcal{C}) = (\varphi(\mathcal{C}))^\perp$. Thus, $\varphi(\mathcal{C})$ is a self-dual linear code of length $n_1 + 2n_2 + 3n_3$ over \mathbb{F}_q . Conversely, let $\varphi(\mathcal{C})$ be a self-dual linear code of length $n_1 + 2n_2 + 3n_3$ over \mathbb{F}_q . Then, $\varphi(\mathcal{C}) = (\varphi(\mathcal{C}))^\perp$, and hence, by Theorem 5, we have $\varphi(\mathcal{C}) = \varphi(\mathcal{C}^\perp)$. Since φ is bijection, $\mathcal{C} = \mathcal{C}^\perp$. Therefore, \mathcal{C} is a self-dual linear code over \mathcal{R}_γ . Similarly, the self orthogonal case holds. \square

4. Skew Cyclic \mathcal{R} -Codes

Let θ_t be a non-trivial Frobenius automorphism defined by:

$$\theta_t : \mathbb{F}_q \rightarrow \mathbb{F}_q, \theta_t(a) = a^{p^t},$$

where t divides m . It can be extended to \mathcal{R}_1 and \mathcal{R}_2 by:

$$\theta_t(a + ub) = \theta_t(a) + u\theta_t(b), \theta_t(a + ub + vc) = \theta_t(a) + u\theta_t(b) + v\theta_t(c).$$

Since $t|m$, the order of automorphism θ_t is $\frac{m}{t}$. We define a polynomial ring $\mathcal{R}_i[x, \theta_t]$ ($1 \leq i \leq 2$) as follows:

$$\mathcal{R}_i[x, \theta_t] = \{a_1 + \dots + a_n x^n | a_j \in \mathcal{R}_i, 1 \leq j \leq n\}$$

Clearly, $\mathcal{R}_i[x, \theta_t]$ is a ring with respect to usual addition and the multiplication defined by:

$$ax^m bx^n = a\theta_t^m(b)x^{m+n}$$

Note that it is a non-commutative ring unless θ_t is an identity map. A non-empty set \mathcal{C} is said to be a linear code of length n_i over \mathcal{R}_i if it is a \mathcal{R}_i submodule of $\mathcal{R}_i^{n_i}$. Using the above polynomial rings above, we extend the polynomial ring to \mathcal{R} by:

$$\mathcal{R}[x, \theta_t] = \{(a(x), b(x), c(x)) : a(x) \in \mathbb{F}_q[x], b(x) \in \mathcal{R}_1[x], c(x) \in \mathcal{R}_2[x]\}.$$

It can be seen that $\mathcal{R}[x, \theta_t]$ is a $\mathcal{R}_2[x, \theta_t]$ submodule with respect to usual addition and multiplication defined by:

$$\begin{aligned} \star : \mathcal{R}_2[x] \times \mathcal{R}[x, \theta_t] &\rightarrow \mathcal{R}[x, \theta_t] \\ (ax^s) \star (b_1 x^i, b_2 x^j, b_3 x^k) &= (\eta(a)x^s b_1 x^i, \delta(a)x^s b_2 x^j, ax^s b_3 x^k) \\ &= (\eta(a)\theta_t^s(b_1)x^{s+i}, \delta(a)\theta_t^s(b_2)x^{s+j}, a\theta_t^s(b_3)x^{s+k}) \end{aligned}$$

However, under associative and distributive laws, the multiplication can be extended to $\mathcal{R}_\gamma[x, \theta_t] = \frac{\mathcal{F}_q[x, \theta_t]}{\langle x^{n_1} - 1 \rangle} \times \frac{\mathcal{R}_1[x, \theta_t]}{\langle x^{n_2} - 1 \rangle} \times \frac{\mathcal{R}_2[x, \theta_t]}{\langle x^{n_3} - 1 \rangle}$ as follows:

$$\star : \mathcal{R}_2[x, \theta_t] \times \mathcal{R}_\gamma[x, \theta_t] \rightarrow \mathcal{R}_\gamma[x, \theta_t]$$

$$\begin{aligned} r(x) \star (f_1(x) + \langle x^{n_1} - 1 \rangle, f_2(x) + \langle x^{n_2} - 1 \rangle, f_3(x) + \langle x^{n_3} - 1 \rangle) &= (\eta(r(x))f_1(x) + \langle x^{n_1} - 1 \rangle, \delta(r(x))f_2(x) + \langle x^{n_2} - 1 \rangle, \\ &r(x)f_3(x) + \langle x^{n_3} - 1 \rangle). \end{aligned}$$

Definition 1 ([2]). We say that an \mathcal{R} -submodule \mathcal{C} of \mathcal{R}^n is a θ_t -cyclic code if for any $c = (c_0, c_1, \dots, c_{n-1}) \in \mathcal{C}$, $\sigma_1(c) = (\theta_t(c_{n-1}), \theta_t(c_0), \dots, \theta_t(c_{n-2})) \in \mathcal{C}$. The operator σ_1 is then said to be a θ_t -cyclic shift operator on \mathcal{R}^n .

Definition 2. A non-trivial \mathcal{R}_2 -submodule \mathcal{C} of \mathcal{R}_γ is called a θ_t -cyclic code if for any $c = (c_{0,1}, c_{1,1}, \dots, c_{n_1-1,1}, c_{0,e}, c_{1,e}, \dots, c_{n_2-1,e}, c_{0,o}, c_{1,o}, \dots, c_{n_3-1,o}) \in \mathcal{C}$, $\sigma(c) = (\theta(c_{n_1-1,1}), \theta(c_{0,1}), \dots, \theta(c_{n_1-2,1}), \theta(c_{n_2-1,e}), \theta(c_{0,e}), \dots, \theta(c_{n_2-2,e}), \theta(c_{n_3-1,o}), \theta(c_{0,o}), \dots, \theta(c_{n_3-2,o})) \in \mathcal{C}$. The operator σ is called a θ_t -cyclic shift operator on \mathcal{R}^n .

The following result yields the relationship between the θ_t -cyclic codes over \mathcal{R} and \mathbb{F}_q .

Theorem 6. Let $\mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_e \otimes \mathcal{C}_o \subseteq \mathcal{R}_\gamma$ be linear. Then \mathcal{C} is a θ_t -cyclic code if and only if $\mathcal{C}_1, \mathcal{C}_e$ and \mathcal{C}_o are θ_t -cyclic codes of length n_1, n_2 and n_3 over $\mathbb{F}_q, \mathcal{R}_1$ and \mathcal{R}_2 respectively.

Proof. Let $\mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_e \otimes \mathcal{C}_o$ be a θ_t -cyclic code over \mathcal{R} . Let $z = (z_1, z_e, z_o) \in \mathcal{C}$, that is:

$$z = (z_{0,1}, z_{1,1}, \dots, z_{n_1-1,1}, z_{0,e}, z_{1,e}, \dots, z_{n_2-1,e}, z_{0,o}, z_{1,o}, \dots, z_{n_3-1,o}) \in \mathcal{C}.$$

Then, $\sigma(z) = (\theta(z_{n_1-1,1}), \theta(z_{0,1}), \dots, \theta(z_{n_1-2,1}), \theta(z_{n_2-1,e}), \theta(z_{0,e}), \dots, \theta(z_{n_2-2,e}), \theta(z_{n_3-1,o}), \theta(z_{0,o}), \dots, \theta(z_{n_3-2,o})) = (\sigma(z_1), \sigma(z_e), \sigma(z_o)) \in \mathcal{C}$. From this, we can conclude that:

$$\sigma(z_1) \in \mathcal{C}_1, \sigma(z_e) \in \mathcal{C}_e \text{ and } \sigma(z_o) \in \mathcal{C}_o$$

Hence, $\mathcal{C}_1, \mathcal{C}_e$, and \mathcal{C}_o are θ_t -cyclic code of length n_i . The converse holds in a similar way. \square

We recall the following Theorem from [9].

Theorem 7 ([9]). Let $\mathcal{C}_o = o_1\mathcal{C}_{o_1} \oplus o_2\mathcal{C}_{o_2} \oplus o_3\mathcal{C}_{o_3}$ be a linear code over \mathcal{R}_2 of length n_3 , then \mathcal{C}_o is θ_t -cyclic code iff $\mathcal{C}_{o_i} (1 \leq i \leq 3)$ is a θ_t -cyclic code of length n_3 over \mathbb{F}_q .

The analogue of this result in our setting is as follows.

Theorem 8 ([8]). Let $\mathcal{C}_e = e_1\mathcal{C}_{e_1} \oplus e_2\mathcal{C}_{e_2}$ be a linear code over \mathcal{R}_1 of length n_2 then \mathcal{C}_e is θ_t -cyclic code iff $\mathcal{C}_{e_i} (1 \leq i \leq 2)$ is a θ_t -cyclic code of length n_2 over \mathbb{F}_q .

Theorem 9. If $\mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_e \otimes \mathcal{C}_o$ is a linear code of length $\gamma = n_1 + n_2 + n_3$, then \mathcal{C} is θ_t -cyclic iff $\mathcal{C}_1, \mathcal{C}_{e_i}, \mathcal{C}_{o_j} (1 \leq i \leq 2, 1 \leq j \leq 3)$ are θ_t -cyclic code of length n_1, n_2, n_3 over \mathbb{F}_q respectively.

Proof. We obtain the proof on combining proofs of Theorems 6–8. \square

These notions are well-behaved with respect to duality as the next result shows.

Theorem 10. If \mathcal{C} is a θ_t -cyclic code of length n , then its dual \mathcal{C}^\perp is also a θ_t -cyclic code.

Proof. From Theorem 9, $\mathcal{C}_1, \mathcal{C}_{e_i}, \mathcal{C}_{o_j} (1 \leq i \leq 2, 1 \leq j \leq 3)$ are θ_t -cyclic codes over \mathbb{F}_q . Then, $\mathcal{C}_1^\perp, \mathcal{C}_{e_i}^\perp, \mathcal{C}_{o_j}^\perp (1 \leq i \leq 2, 1 \leq j \leq 3)$ are θ_t -cyclic codes over \mathbb{F}_q from [29] and once again by using Theorem 9, \mathcal{C}^\perp becomes a θ_t -cyclic code. \square

Recall the following result from [4].

Lemma 1 ([4]). Let \mathcal{C} be a θ_t -cyclic code of length n over \mathbb{F}_q . Then, there exists a polynomial $f(x) \in \mathbb{F}_q[x; \theta_t]$ such that $\mathcal{C} = \langle f(x) \rangle$ and $x^n - 1 = g(x)f(x)$ in $\mathbb{F}_q[x; \theta_t]$.

By assuming $o(\theta_t) | n$, the counterpart follows.

Theorem 11. Let $\mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_e \otimes \mathcal{C}_o$ be a θ_t -cyclic code of length n over \mathcal{R} and assume that the order of θ_t divides n . Then, $\mathcal{C} = \langle B_1, B_e, B_o \rangle$, where $B_1 = \langle (f_1(x), 0, 0) \rangle$, $B_e = \langle (0, f_e(x), 0) \rangle$, and $B_o = \langle (b_1(x), b_e(x), f_o(x)) \rangle$, such that $\mathcal{C}_1 = \langle f_1(x) \rangle$, $\mathcal{C}_e = \langle f_e(x) \rangle$, $\mathcal{C}_o = \langle f_o(x) \rangle$, $b_1(x) \in \mathcal{C}_1$ and $b_2(x) \in \mathcal{C}_e$.

Proof. Let $\mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_e \otimes \mathcal{C}_o$ be a θ_t -cyclic code of length $\gamma = n_1 + n_2 + n_3$ over \mathcal{R} . Then, by Theorem 6, $\mathcal{C}_1, \mathcal{C}_e, \mathcal{C}_o$ are θ_t -cyclic codes of length n_i over $\mathbb{F}_q, \mathcal{R}_1$ and \mathcal{R}_2 . Define a homomorphism from \mathcal{C} to \mathcal{R} as follows:

$$\begin{aligned} \psi : \mathcal{C} &\mapsto \mathcal{R} \\ \psi(c_1(x), c_e(x), c_o(x)) &= (0, 0, c_o(x)) \end{aligned}$$

Define:

$$\ker(\psi) = \{(c_1(x), c_e(x), 0) : c_1(x) \in \mathcal{C}_1, c_e(x) \in \mathcal{C}_e\}$$

$$\mathcal{I} = \{(c_1(x), c_e(x)) \in \mathbb{F}_q[x; \theta_t] \times \mathcal{R}_1[x; \theta_t] : (c_1(x), c_e(x), 0) \in \ker(\psi)\}.$$

Clearly, $\mathcal{I} = \mathcal{I}_1 \times \mathcal{I}_e$ forms a submodule of $\mathbb{F}_q[x; \theta_t] \times \mathcal{R}_1[x; \theta_t]$. Therefore, there exist a polynomial $f_1(x)$ and $f_e(x)$ in $\mathbb{F}_q[x; \theta_t]$ and $\mathcal{R}_1[x; \theta_t]$, respectively, generating \mathcal{I}_1 and \mathcal{I}_e with $f_1(x)|x^{n_1} - 1$ and $f_e(x)|x^{n_e} - 1$. Thus, $\mathcal{I} = \langle (f_1(x), 0), (0, f_e(x)) \rangle$, then for any $(c_1(x), c_e(x), 0) \in \ker(\psi)$, $(c_1(x), c_e(x)) = v(x) \star (f_1(x), 0), (0, f_e(x))$ for some $v(x) \in$

$\mathcal{R}_1[x; \theta_t]$. Finally, it leads to $\ker(\psi) = \langle (f_1(x), 0, 0), (0, f_e(x), 0) \rangle$. The fact that \mathcal{C} is a submodule implies that $\psi(\mathcal{C})$ is a submodule. By using the first isomorphism theorem:

$$\mathcal{C} / \ker(\psi) \cong \psi(\mathcal{C}).$$

Let $(b_1(x), b_e(x), f_o(x)) \in \mathcal{C}$, then $\psi(b_1(x), b_e(x), f_o(x)) = (0, 0, f_o(x))$. From this, any θ_t -cyclic code of length n can be represented by $\mathcal{C} = \langle (f_1(x), 0, 0), (0, f_e(x), 0), (b_1(x), b_e(x), f_o(x)) \rangle$, where $f_1(x)|(x^{n_1} - 1)$, $f_e(x)|(x^{n_2} - 1)$ and $f_o(x)|(x^{n_3} - 1)$. \square

Furthermore, we have \mathcal{C} is θ_t -cyclic, then \mathcal{C}_k , where $k \in \{1, e_1, e_2, o_1, o_2, o_3\}$ is skew θ_t -cyclic code over \mathbb{F}_q with respective lengths. From Theorem 3, $|\mathcal{C}| = |\mathcal{C}_1| \prod_{i=1}^{e_2} |\mathcal{C}_{e_i}| \prod_{j=1}^{o_3} |\mathcal{C}_{o_j}|$, since each \mathcal{C}_k is θ_t -cyclic it is generated by a polynomial $f_k(x)$, and thus, $|\mathcal{C}| = q^{\gamma - \sum_{i=1}^6 \epsilon_k}$, where $\gamma = n_1 + 2(n_2) + 3(n_3)$. The following Theorem provides the generator polynomials for θ_t -cyclic codes over \mathbb{F}_q .

Theorem 12. Let $\mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_e \otimes \mathcal{C}_o$ be a skew cyclic code over \mathcal{R} of length $\gamma = n_1 + n_2 + n_3$. Then, there exists a polynomial:

- (i) $f_1(x) \in \mathbb{F}_q[x; \theta_t]$ such that $\mathcal{C}_1 = \langle f_1(x) \rangle$ and $x^{n_1} - 1 = g_1(x)f_1(x)$.
- (ii) $f_e(x) \in \mathcal{R}_1[x; \theta_t]$ such that $\mathcal{C}_e = \langle f_e(x) \rangle$ and $x^{n_2} - 1 = g_e(x)f_e(x)$ where $f_e(x) = \sum_{i=1}^2 e_i f_{e_i}(x)$.
- (iii) $f_o(x) \in \mathcal{R}_2[x; \theta_t]$ such that $\mathcal{C}_o = \langle f_o(x) \rangle$ and $x^{n_3} - 1 = g_o(x)f_o(x)$ where $f_o(x) = \sum_{i=1}^3 o_i f_{o_i}(x)$.

Proof. Let \mathcal{C} be a θ_t -cyclic code of length $\gamma = n_1 + n_2 + n_3$. From Theorem 6, we have that $\mathcal{C}_1, \mathcal{C}_e$, and \mathcal{C}_o are θ_t -cyclic codes. Using Lemma 1, (i) follows.

Then, the proof of (ii) is as follows. Let $\mathcal{C}_e = e_1 \mathcal{C}_{e_1} \oplus e_2 \mathcal{C}_{e_2}$ be a θ_t -cyclic code of length n_2 over \mathcal{R}_1 . Theorem 7 says that, \mathcal{C}_{e_1} and \mathcal{C}_{e_2} are θ_t -cyclic codes of length n_2 over \mathbb{F}_q . Lemma 1 says that we have $\mathcal{C}_i = \langle f_{e_i}(x) \rangle$ and $x^{n_2} - 1 = g_{e_i}(x)f_{e_i}(x)$ in $\mathbb{F}_q[x; \theta_t]$ for $i \in \{1, 2\}$. Then, $e_i f_{e_i}(x) \in \mathcal{C}$ for $i \in \{1, 2\}$. Also, for any $f_e(x) \in \mathcal{C}$, we have $f_e(x) = \sum_{i=1}^2 e_i h_{e_i}(x) f_{e_i}(x)$, where $h_{e_i}(x) \in \mathbb{F}_q[x; \theta_t]$ for $i \in \{1, 2\}$. Thus, $f_e(x) \in \langle e_1 f_{e_1}(x), e_2 f_{e_2}(x) \rangle$. Therefore, $\mathcal{C} = \langle e_1 f_{e_1}(x), e_2 f_{e_2}(x) \rangle$. As $x^{n_2} - 1 = g_{e_i}(x)f_{e_i}(x)$ in $\mathbb{F}_q[x; \theta_t]$ for $i \in \{1, 2\}$. Let $f_e(x) = e_1 f_{e_1}(x) + e_2 f_{e_2}(x) \in \mathcal{R}_1[x; \theta_t]$. Then, $f_e(x) \in \mathcal{C}$. On the other hand $e_i f_{e_i}(x) = e_i f_e(x) \in \langle f_e(x) \rangle$ for $i = 1, 2$. Consequently, $\mathcal{C} = \langle f_e(x) \rangle$. Furthermore, $[\sum_{i=1}^2 e_i g_{e_i}(x)] f_e(x) = \sum_{i=1}^2 e_i g_{e_i}(x) f_{e_i}(x) = \sum_{i=1}^2 e_i (x^{n_2} - 1) = x^{n_2} - 1$. Then, $x^{n_2} - 1 = g_e(x)f_e(x)$ in $\mathcal{R}_1[x; \theta_t]$, where $g_e(x) = \sum_{i=1}^2 e_i g_{e_i}(x)$. Thus, (ii) follows. (iii) is similar to the proof of (ii). \square

5. Skew Constacyclic Code over \mathcal{R}

In this section, we study skew θ_t -constacyclic codes over \mathcal{R} . We choose a unit element $\alpha \in \mathcal{R}_2^*$ such that α satisfies the condition $\alpha^2 = 1$, ($\alpha = 1, -1, \dots$).

Definition 3. Let $\alpha_i \in \mathbb{F}_{p^t} \setminus \{0\}$. A linear code $\mathcal{C} \subseteq \mathcal{R}_\gamma[x, \theta]$ is called skew $\alpha = \alpha_1 + u\alpha_2 + v\alpha_3$ -constacyclic code if it is invariant under the cyclic shift operator λ_α , which is whenever:

$$\begin{aligned} c &= (x_0, x_1, \dots, x_{n_1-1}, y_0, y_1, \dots, y_{n_2-1}, z_0, z_1, \dots, z_{n_3-1}) \in \mathcal{C} \\ \lambda_\alpha(c) &= (\alpha_1 \theta_t(x_{n_1-1}), \theta_t(x_0), \dots, \theta_t(x_{n_1-1}), (\alpha_1 + u\alpha_2) \theta_t(y_{n_1-1}), \theta_t(y_1), \dots, \theta_t(y_{n_2-2}), \\ &\quad (\alpha_1 + u\alpha_2 + v\alpha_3) \theta_t(z_{n_3-1}), \theta_t(z_0), \dots, \theta_t(z_{n_3-2})) \in \mathcal{C} \end{aligned}$$

The following two results translate symmetry conditions into algebraic constraints. We give the first result without proof.

Theorem 13. Let $R_{n,\lambda} = R[x; \theta_t] / \langle x^n - \lambda \rangle$. A linear code \mathcal{C} of length n over R is (θ_t, λ) -cyclic code if and only if \mathcal{C} is a left $R[x; \theta_t]$ -submodule of $R_{n,\lambda}$.

The second result is less immediate.

Theorem 14. A code \mathcal{C} is skew α -cyclic code over $\mathcal{R}_\gamma = \frac{\mathbb{F}_q[x, \theta_1]}{x^{n_1} - \alpha} \times \frac{\mathcal{R}_1[x, \theta_1]}{x^{n_2} - \alpha} \times \frac{\mathcal{R}_2[x, \theta_1]}{x^{n_3} - \alpha}$ iff \mathcal{C} is a left $\mathcal{R}_2[x, \theta_1]$ module over \mathcal{R}_γ .

Proof. Let \mathcal{C} be a skew α -cyclic code. Then, by definition $x \star (f(x)|g(x)|h(x)) \in \mathcal{C}$:

$$x \star (f(x)|g(x)|h(x)) = (\theta_t(f_0)x + \theta_t(f_1)x^2 + \cdots + \alpha_1\theta_t(f_{n_1-1}), \theta_t(g_0)x + \theta_t(g_1)x^2 + \cdots + (\alpha_1 + u\alpha_2)\theta_t(g_{n_2-1}), \\ \theta_t(h_0)x + \theta_t(h_1)x^2 + \cdots + (\alpha_1 + u\alpha_2 + v\alpha_3)\theta_t(h_{n_3-1})) \in \mathcal{C}$$

Moreover, by using linearity of \mathcal{C} :

$$r(x) \star (g_1(x)|g_2(x)|g_3(x)) \in \mathcal{C}$$

for some $r(x) \in \mathcal{R}_2[x, \theta_1]$. Hence, \mathcal{C} is an left $\mathcal{R}_2[x, \theta_1]$ submodule over \mathcal{R}_γ . Conversely, assume that \mathcal{C} is an left $\mathcal{R}_2[x, \theta_1]$ submodule over \mathcal{R}_γ , then we have $x \star (f(x)|g(x)|h(x)) \in \mathcal{C}$ implies \mathcal{C} is skew α -cyclic code. \square

Theorem 15. The code $\mathcal{C}_o \subseteq \mathcal{R}_2^n$ is skew $\alpha = \alpha_1 + u\alpha_2 + v\alpha_3$ -cyclic of length n iff $\mathcal{C}_{o_1}, \mathcal{C}_{o_2}$, and \mathcal{C}_{o_3} are skew $\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3$ -cyclic codes over \mathbb{F}_q of length n .

Proof. Let \mathcal{C}_o be a skew α -cyclic code. Let $a = x_0 + y_0 + z_0 \in \mathcal{C}_o$, where $x = (x_0, x_1, \dots, x_{n-1}) \in \mathcal{C}_{o_1}, y = (y_0, y_1, \dots, y_{n-1}) \in \mathcal{C}_{o_2}$ and $z = (z_0, z_1, \dots, z_{n-1}) \in \mathcal{C}_{o_3}$. Then, we have by definition, $\lambda_\alpha(x) \in \mathcal{C}_{o_1}$:

$$\begin{aligned} \lambda_\alpha(o_1(x_0, x_1, \dots, x_{n-1}) + o_2(y_0, y_1, \dots, y_{n-1}) \\ + o_3(z_0, z_1, \dots, z_{n-1})) &= ((\alpha_1 + u\alpha_2 + v\alpha_3) \star o_1(\theta_t(x_{n-1}), \theta_t(x_0), \dots, \theta_t(x_{n-2})) + \\ &\quad (\alpha_1 + u\alpha_2 + v\alpha_3) \star o_2(\theta_t(y_{n-1}), \theta_t(y_0), \dots, \theta_t(y_{n-2})) + \\ &\quad (\alpha_1 + u\alpha_2 + v\alpha_3) \star o_3(\theta_t(z_{n-1}), \theta_t(z_0), \dots, \theta_t(z_{n-1}))) \\ \implies &\lambda_{\alpha_1}(x) + \lambda_{\alpha_1 + \alpha_2}(y) + \lambda_{\alpha_1 + \alpha_3}(z) \in \mathcal{C}_o \\ \implies &\lambda_{\alpha_1}(x) \in \mathcal{C}_{o_1}, \lambda_{\alpha_1 + \alpha_2}(y) \in \mathcal{C}_{o_2}, \lambda_{\alpha_1 + \alpha_3}(z) \in \mathcal{C}_{o_3} \end{aligned}$$

Hence, $\mathcal{C}_{o_1}, \mathcal{C}_{o_2}$, and \mathcal{C}_{o_3} are skew $\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3$ -cyclic codes over \mathbb{F}_q of length n .

Conversely, assume that $\mathcal{C}_{o_1}, \mathcal{C}_{o_2}$ and \mathcal{C}_{o_3} are skew $\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3$ -cyclic codes over \mathbb{F}_q of length n . Let m_0, m_1, \dots, m_{n-1} be an element in \mathcal{C}_o , where $m_i = o_1x_i + o_2y_i + o_3z_i$ such that $x = (x_0, x_1, \dots, x_{n-1}) \in \mathcal{C}_{o_1}, y = (y_0, y_1, \dots, y_{n-1}) \in \mathcal{C}_{o_2}$ and $z = (z_0, z_1, \dots, z_{n-1}) \in \mathcal{C}_{o_3}$. Then we have $\lambda_{\alpha_1}(x) \in \mathcal{C}_{o_1}, \lambda_{\alpha_1 + \alpha_2}(y) \in \mathcal{C}_{o_2}$ and $\lambda_{\alpha_1 + \alpha_3}(z) \in \mathcal{C}_{o_3}$. So we get,

$$\begin{aligned} o_1\lambda_{\alpha_1}(x) + o_2\lambda_{\alpha_1 + \alpha_2}(y) + o_3\lambda_{\alpha_1 + \alpha_3}(z) &= o_1\lambda_{\alpha_1}(x_0, x_1, \dots, x_{n-1}) + o_2\lambda_{\alpha_1 + \alpha_2}(y_0, y_1, \dots, y_{n-1}) \\ &\quad + o_3\lambda_{\alpha_1 + \alpha_3}(z_0, z_1, \dots, z_{n-1}) \in \mathcal{C} \\ &= \lambda_\alpha(m_0, m_1, \dots, m_{n-1}) \in \mathcal{C} \end{aligned}$$

Hence, \mathcal{C} is skew α -cyclic code over \mathcal{R}_2^n . \square

Theorem 16. \mathcal{C}_e be a skew $\alpha = \alpha_1 + u\alpha_2$ -cyclic code over \mathcal{R}_1 iff \mathcal{C}_{e_1} and \mathcal{C}_{e_2} are skew $\alpha_1 + \alpha_2$ and α_1 -cyclic codes over \mathbb{F}_q .

Proof. The proof is similar to Theorem 15 taking mod v to the above condition. \square

Theorem 17. \mathcal{C} be a skew $\alpha = \alpha_1 + u\alpha_2 + v\alpha_3$ -cyclic code over \mathcal{R} of length $\gamma = n_1 + n_2 + n_3$ iff $\mathcal{C}_1, \mathcal{C}_e$ and \mathcal{C}_o are $\alpha_1, \alpha_1 + u\alpha_2$, and $\alpha_1 + u\alpha_2 + v\alpha_3$ -cyclic codes over $\mathbb{F}_q, \mathcal{R}_1$ and \mathcal{R}_2 , respectively.

Proof. C_1, C_e and C_o be $\alpha_1, \alpha_1 + u\alpha_2$, and $\alpha_1 + u\alpha_2 + v\alpha_3$ -cyclic. Consider $x = (x_0, x_1, \dots, x_{n_1-1})$, $y = (y_0, y_1, \dots, y_{n_2-1})$ and $z = (z_0, z_1, \dots, z_{n_3-1})$. Consider $\alpha_1 + u\alpha_2 = \beta$. Then, we have:

$$(x, y, z) \in \mathcal{C} \implies (\lambda_{\alpha_1}(x), \lambda_{\beta}(y), \lambda_{\alpha}(z)) \in \mathcal{C}$$

Hence, \mathcal{C} is skew α -cyclic. The converse part holds similarly. \square

Theorem 18. \mathcal{C} be a skew α -cyclic code of length $\gamma = n_1 + n_2 + n_3$ iff C_1 is skew α_1 -cyclic code of length n_1 , C_{e_1} , and C_{e_2} are $\alpha_1 + \alpha_2, \alpha_1$ -cyclic codes of length n_2 and C_{o_1}, C_{o_2} , and C_{o_3} are skew $\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3$ -cyclic codes over \mathbb{F}_q of length n_3 .

Proof. Using Theorems 15–17 the result follows. \square

Theorem 19. \mathcal{C} be a skew $\alpha = \alpha_1 + u\alpha_2 + v\alpha_3$ -cyclic code over \mathcal{R} of length $\gamma = n_1 + n_2 + n_3$ iff C_1^\perp, C_e^\perp , and C_o^\perp are $(\alpha_1)^{-1}, (\alpha_1 + u\alpha_2)^{-1}$, and $(\alpha_1 + u\alpha_2 + v\alpha_3)^{-1}$ -cyclic.

Proof. Let \mathcal{C} be a skew α -cyclic code, Lemma 3.1 [5] says that \mathcal{C}^\perp is skew $(\alpha_1 + u\alpha_2 + v\alpha_3)^{-1}$ -cyclic code. From Theorem 17, we have C_1^\perp, C_e^\perp and C_o^\perp are skew $(\alpha_1)^{-1}, (\alpha_1 + u\alpha_2)^{-1}$ and $(\alpha_1 + u\alpha_2 + v\alpha_3)^{-1}$ -cyclic. \square

Corollary 2. Let $\mathcal{C} = C_1 \otimes C_e \otimes C_o$. be a skew $\alpha = \alpha_1 + u\alpha_2 + v\alpha_3$ -cyclic code over \mathcal{R} of length $\gamma = n_1 + n_2 + n_3$. Then, there exist polynomials:

- (i) $f_1(x) \in \mathbb{F}_q[x; \theta_1]$ such that $C_1 = \langle f_1(x) \rangle$ and $x^{n_1} - \alpha_1 = g_1(x)f_1(x)$.
- (ii) $f_e(x) \in \mathcal{R}_1[x; \theta_e]$ such that $C_e = \langle f_e(x) \rangle$ and $x^{n_2} - (\alpha_1 + u\alpha_2) = g_e(x)f_e(x)$.
- (iii) $f_o(x) \in \mathcal{R}_2[x; \theta_o]$ such that $C_o = \langle f_o(x) \rangle$ and $x^{n_3} - (\alpha_1 + u\alpha_2 + v\alpha_3) = g_o(x)f_o(x)$.

Proof. The proof is similar to the proof of Theorem 12. \square

Theorem 20. Let $\mathcal{C} = C_1 \otimes C_e \otimes C_o$ be a θ_i -constacyclic code of length γ over \mathcal{R} . Then $\mathcal{C} = \langle \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_o \rangle$, where $\mathcal{B}_1 = \langle (f_1(x), 0, 0) \rangle$, $\mathcal{B}_2 = \langle (0, f_e(x), 0) \rangle$, and $\mathcal{B}_o = \langle (b_1(x), b_e(x), f_o(x)) \rangle$.

Proof. The proof is similar to the proof of Theorem 11. \square

Example 1. Let $q = 9$ and $\mathbb{F}_9 = \mathbb{F}_3[z]$ with $z^2 + 1 = 0$. Consider the ring $R_\gamma = \frac{\mathbb{F}_9[x, \theta_3]}{x^4-1} \times \frac{\mathcal{R}_1[x, \theta_3]}{x^5-1} \times \frac{\mathcal{R}_2[x, \theta_3]}{x^5-1}$, where θ_3 is the Frobenius automorphism defined by $\theta_3(a) = a^3$ for any $a \in \mathbb{F}_9$. Write:

$$\begin{aligned} x^4 - 1 &= (x+1)(x+2)(x+z)(x+2z) \in \mathbb{F}_9[x, \theta_3] \\ x^5 - 1 &= (x+2)(x^4 + x^3 + x^2 + x + 1) \in \mathbb{F}_9[x, \theta_3] \\ f_1(x) &= \langle (x+1) \rangle, f_e(x) = \langle e_1(x+2) + e_2(x+2) \rangle, f_o(x) = \langle o_1(x+2) + o_2(x+2) + o_3(x+2) \rangle \end{aligned}$$

By Theorem 12, we have that f_i divides $x^{n_i} - 1$ for $(i = 1, e, o)$, yielding a code with parameter $[29, 18, 2]$ over \mathbb{F}_9 .

Example 2. Let $q = 25$ and $\mathbb{F}_{25} = \mathbb{F}_5[z]$ with $z^2 + z + 1 = 0$. Consider the ring $R_\gamma = \frac{\mathbb{F}_{25}[x, \theta_5]}{x^4-1} \times \frac{\mathcal{R}_1[x, \theta_5]}{x^6-1} \times \frac{\mathcal{R}_2[x, \theta_5]}{x^4-1}$, where θ_5 is the Frobenius automorphism defined by $\theta_5(a) = a^5$ for any $a \in \mathbb{F}_{25}$. Write

$$\begin{aligned} x^4 - 1 &= (x+2)(x+3)(x+z)(x+z+1) \in \mathbb{F}_{25}[x, \theta_5] \\ x^6 - 1 &= (x^2-1)(x^2+x+1)(x^2-x+1) \in \mathbb{F}_{25}[x, \theta_5] \\ f_1(x) &= \langle (x+2) \rangle, f_e(x) = \langle e_1(x^2-1) + e_2(x^2-1) \rangle, f_o(x) = \langle o_1(x+z+1) + o_2(x+z+1) + o_3(x+z+1) \rangle \end{aligned}$$

By Theorem 12, we have that f_i divides $x^{n_i} - 1$ for $(i = 1, e, o)$ yielding a code with parameter $[28, 20, 2]$ over \mathbb{F}_{25} .

6. Conclusions and Open Problems

In this note, we have studied the algebraic and metric structure of skew cyclic and skew constacyclic codes over a special mixed alphabet. Thus, our codes have a structure of

module over the largest of the three alphabets \mathcal{R}_2 . Codes over the product ring $\mathbb{F}_q \times \mathcal{R}_1 \times \mathcal{R}_2$ would be modules over that larger ring. The two algebraic structures are different and should not be confused.

The present work leads itself to two paths of generalization: consider different mixed alphabets or replace the concepts of cyclicity by that of quasi-cyclicity. The former path seems easier than the latter, in view of the many examples of rings that have been used as alphabets of cyclic codes in recent years. On the other hand, the structure of quasi-cyclic codes is always more subtle than that of cyclic codes.

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