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# Dissipation of Stop-and-Go Waves in Traffic Flows using Controlled Vehicles: a Macroscopic Approach

Paola Goatin

**Abstract**—We study the boundary stabilization of Generic Second Order Macroscopic traffic models in Lagrangian coordinates. These consist in  $2 \times 2$  nonlinear hyperbolic systems of balance equations with a relaxation-type source term. We provide the existence of weak solutions of the Initial Boundary Value problem for generic relaxation terms. In particular, we do not require the *sub-characteristic* stability condition to hold, so that equilibria are unstable and perturbations may lead to the formation of large oscillations, modeling the appearance and persistence of stop-and-go waves. Moreover, since the largest eigenvalue of the system is null, the boundaries are characteristic, and the available results on boundary controllability do not apply. Therefore, we perform a detailed analysis of the Wave Front Tracking approximate solutions to show that weak solutions can be steered to the corresponding equilibrium state by prescribing the equilibrium speed at the right boundary. This corresponds to controlling the speed of one vehicle to stabilize the upstream traffic flow. The result is illustrated through a numerical example.

**Index Terms**—Boundary control, hyperbolic systems with relaxation, macroscopic traffic modeling.

## I. INTRODUCTION

THE foreseen introduction of connected and automated vehicles on public roads offers new perspectives in traffic flow management, traditionally relying on approaches such as ramp metering, traffic light control and variable speed limits, which refer to an Eulerian description of the traffic dynamics. In fact, recent studies have demonstrated the possibility of controlling and stabilizing traffic flows by using a small number of controlled vehicles acting as actuators, both in model based (e.g. [1]–[3]) or machine learning (e.g. [4]) theoretical settings and by real world experiments (e.g. [5], [6]). In particular, several works such as [7]–[9], have focused on the dissipation of stop-and-go waves, which can spontaneously arise due to traffic instabilities [10]. In this framework, model based analytical studies have been restricted mostly to the microscopic setting [9], [11]. Indeed, classical first order macroscopic models [12], [13], cannot capture the formation and persistence of oscillations, due to the intrinsic stabilizing effect of the non-increasing total variation of their solutions. It is therefore mandatory to consider second order models [14]–[16], consisting in  $2 \times 2$  hyperbolic systems of

balance equations with relaxation source terms. Yet, to allow the solution total variation to increase, thus modeling stop-and-go waves, one has to drop the dissipative condition usually assumed to ensure well-posedness and convergence results as the relaxation parameter tends to zero [17]–[19].

Aiming to assess the possibility of dampening stop-and-go waves by controlled vehicles in the role of endogenous actuators, in this work we therefore consider a class of second order models with unstable relaxation terms. We rewrite the system in Lagrangian coordinates, so that the control action of a single vehicle translates into suitable boundary conditions. Note that, due to the change of coordinates, domain boundaries become characteristic, since second family waves have zero speed, and boundary conditions can act only on one system component at the downstream (right) boundary. A similar approach is exploited in [20] to include trajectory data information in a scalar model with time delay accounting for traffic string instability.

Most of the literature about controllability of hyperbolic systems deals with classical solutions of quasilinear systems [21], [22]. Moreover, boundary stabilization is usually derived under the assumption of non-characteristic boundaries (i.e. wave speeds bounded away from zero) [23], see e.g. [24], [25] for traffic flow stabilization results. Nevertheless, solutions to nonlinear hyperbolic equations generally develop jump discontinuities. Therefore, in particular regarding traffic flow applications, it is natural to consider weak solutions, possibly satisfying additional admissibility conditions to single out the physically relevant ones. In this context, control theory is less developed. Lyapunov boundary stabilization is demonstrated in [26] for scalar conservation laws with strictly convex flux functions and in [27]–[29], for systems whose eigenvalues are bounded away from zero. Since available results do not apply to our problem, we are led to perform a detailed analysis of the approximate solutions constructed via the Wave Front Tracking (WFT) algorithm [30], which allows us to prove the asymptotic stabilization of the solutions of the Initial Boundary Value Problem (IBVP) for any initial data of bounded variation to the corresponding equilibrium state.

The remaining of the paper is organized as follows. In Section II we introduce the model under study. In Section III we provide the well-posedness of the corresponding IBVP. The main result on asymptotic boundary stabilization is proved in Section IV and a numerical experiment is provided in Section V. Some supplementary material is collected in the

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Appendix.

## II. GENERIC SECOND ORDER MACROSCOPIC TRAFFIC MODELS

We refer to the Generic Second Order Macroscopic model (GSOM in short) [16] with relaxation. In Eulerian coordinates, it reads

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho w) + \partial_x(\rho w v) = \rho \frac{V(\rho) - v}{\tau}, \end{cases} \quad x \in \mathbb{R}, \quad t > 0. \quad (1)$$

In (1), the average speed of vehicles  $v$  is a function of the density  $\rho = \rho(t, x)$  and a Lagrangian vehicle property  $w = w(t, x)$ , namely  $v = \mathcal{V}(\rho, w)$  for some speed function  $\mathcal{V} : \Omega \rightarrow \mathbb{R}_{\geq 0}$  satisfying [31]:

$$\mathcal{V}(\rho, w) \geq 0, \quad \mathcal{V}(0, w) = w, \quad (2a)$$

$$2\mathcal{V}_\rho(\rho, w) + \rho\mathcal{V}_{\rho\rho}(\rho, w) < 0 \text{ for } w > 0, \quad (2b)$$

$$\mathcal{V}_w(\rho, w) > 0, \quad (2c)$$

$$\forall w > 0 \quad \exists R(w) > 0 : \quad \mathcal{V}(R(w), w) = 0. \quad (2d)$$

Moreover, we assume the equilibrium speed  $V : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is a non-increasing function ( $V'(\rho) \leq 0$ ) such that  $V(\rho) = 0$  for  $\rho \geq \rho_{max} > 0$  and  $\tau > 0$  is the relaxation parameter taking into account the reaction time of drivers. As in [31], we observe that (2b) implies that  $Q(\rho, w) := \rho\mathcal{V}(\rho, w)$  is strictly concave and  $\mathcal{V}_\rho(\rho, w) < 0$  for  $w > 0$ , if  $\mathcal{V}$  is a  $\mathbf{C}^2$  function in  $\rho$ . We also remark that in (2d) we can have  $R(w) = \bar{R}$  for all  $w > 0$ .

Notice that, setting  $\mathcal{V}(\rho, w) = w - p(\rho)$  for a suitable ‘‘pressure’’ function  $p$ , system (1) corresponds to the Aw-Rascle-Zhang (ARZ) model [14], [15]. Also, taking  $w = \bar{w}$  constant, we recover the classical Lighthill-Whitham-Richards (LWR) model [12], [13].

Away from vacuum, system (1) can be rewritten into Lagrangian coordinates  $(t, n)$  ( $n = n(t, x)$  s.t.  $dn = \rho dx - \rho v dt$  [32]) as follows. Denoting by  $s$  the spacing, i.e.  $s = \frac{1}{\rho}$ , and using the notation

$$\tilde{\mathcal{V}}(s, w) = \mathcal{V}\left(\frac{1}{s}, w\right), \quad \tilde{V}(s) = V\left(\frac{1}{s}\right),$$

we obtain the following equivalent hyperbolic system

$$\begin{cases} \partial_t s - \partial_n \tilde{\mathcal{V}} = 0, \\ \partial_t w = \frac{\tilde{V}(s) - \tilde{\mathcal{V}}}{\tau}, \end{cases} \quad (3)$$

whose eigenvalues are

$$\tilde{\lambda}_1(s, w) = -\tilde{\mathcal{V}}_s(s, w) = \mathcal{V}_\rho(1/s, w)/s^2 < 0, \quad (4a)$$

$$\tilde{\lambda}_2(s, w) = 0. \quad (4b)$$

It is known [19] that, under the *sub-characteristic* condition

$$\tilde{\mathcal{V}}_s(s, w) \geq \tilde{V}'(s) \geq 0 \quad \text{for } w \text{ s.t. } \tilde{\mathcal{V}}(s, w) = \tilde{V}(s), \quad (5)$$

(entropy weak) solutions to (3) converge to the (entropy weak) solution of the corresponding LWR equation

$$\partial_t s - \partial_n \tilde{V}(s) = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad (6)$$

as  $\tau \rightarrow 0+$  or  $t \rightarrow +\infty$ , preventing the formation and persistence of large oscillations around equilibria. Aiming at capturing the formation of stop-and-go waves and their damping by boundary controls, in this work we drop condition (5) and we study model (3) for a fixed  $\tau > 0$ . The corresponding Riemann problem and invariant domains are described in the Appendix.

## III. THE INITIAL BOUNDARY VALUE PROBLEM

In this work, we focus on the initial-boundary value problem (IBVP) for (3) on the bounded interval  $I = ]0, N[$  with initial data

$$(s, w)(0, n) = (s_0, w_0)(n), \quad n \in I, \quad (7)$$

and right boundary condition

$$\tilde{\mathcal{V}}(s, w)(t, N) = \tilde{v}_c(t), \quad t > 0. \quad (8)$$

Notice that, as usual with hyperbolic equations, boundary conditions are intended in weak sense. Due to (4), only the speed boundary condition can be imposed at the right boundary  $n = N$ . The IBVP solution cannot depend on conditions on  $w$  and at the left boundary  $n = 0$  [33]. Note that, in Eulerian coordinates, (8) becomes a moving internal condition  $\mathcal{V}(\rho, w)(t, x(t, N)-) = \tilde{v}_c(t)$ , where  $x(t, N)$  is the position at time  $t$  of the vehicle that was initially at  $x_N = x(0, N)$ , i.e.  $N = n(0, x_N)$ . Clearly, this is much harder to treat both analytically and numerically.

*Definition 1:* Given an invariant domain  $\mathbf{E}$  as in (20), let  $U_0 = (s_0, w_0) \in \mathbf{L}^1(I; \mathbf{E})$ ,  $U_c = (s_c, w_c) : [0, T] \rightarrow \mathbf{E}$  and  $T > 0$  be given. Then  $U = (s, w) \in \mathbf{C}^0([0, T], \mathbf{L}^1(I; \mathbf{E}))$  is a weak solution of IBVP (3), (7), (8) if

(i) for all  $\varphi \in \mathbf{C}_c^1(]-\infty, T[ \times I; \mathbb{R})$ , it holds

$$0 = \int_0^T \int_0^N [U \partial_t \varphi + F(U) \partial_n \varphi](t, n) \, dn dt + \int_0^T \int_0^N G(U) \varphi(t, n) \, dn dt + \int_0^L U_0(n) \varphi(0, n) \, dn,$$

where

$$F(U) = \begin{bmatrix} -\tilde{\mathcal{V}}(s, w) \\ 0 \end{bmatrix}, \quad G(U) = \frac{1}{\tau} \begin{bmatrix} 0 \\ \tilde{V}(s) - \tilde{\mathcal{V}}(s, w) \end{bmatrix};$$

(ii) the speed function admits a weak\* trace  $\Psi : [0, T] \rightarrow \mathbb{R}$  at the boundary  $n = N$  (i.e.  $\tilde{\mathcal{V}}(s, w)(\cdot, n) \rightharpoonup^* \Psi$  in  $\mathbf{L}^\infty([0, T])$  as  $n \rightarrow N-$ ) and  $\Psi(t) = \tilde{\mathcal{V}}(s_c, w_c)(t)$  for a.e.  $t \in [0, T]$ . (See [34, Definition 2.2].)

We construct wave-front tracking (WFT) approximate solutions of (3), (7), (8) following a two step process, which iteratively solves approximately the homogenous system (17) and the ODEs (18), as in [19], [35]. Let us consider initial data  $U_0 = (s_0, w_0) : I \rightarrow \mathbf{E}$  such that  $\text{TV}(U_0) := \text{TV}(w_0) + \text{TV}(\tilde{v}_0) < +\infty$  ( $\tilde{v}_0 := \tilde{\mathcal{V}}(s_0, w_0)$ ) and boundary data  $U_c = (s_c, w_c) : [0, T] \rightarrow \mathbf{E}$  such that  $\text{TV}(\tilde{v}_c) < +\infty$  ( $\tilde{v}_c := \tilde{\mathcal{V}}(s_c, w_c)$ ). We consider a sequence of time-steps  $\Delta t^\nu > 0$ ,  $\nu \in \mathbb{N}$ , such that  $\Delta t^\nu \rightarrow 0$  as  $\nu \rightarrow +\infty$  and we partition the interval  $[0, T[$  in intervals of the form  $\Delta T^k := [k\Delta t^\nu, (k+1)\Delta t^\nu[$ ,  $k \in \mathbb{N}$ . We denote with  $U^\nu = (s^\nu, w^\nu) : [0, T] \times I \rightarrow \mathbf{E}$ , the sequence of WFT approximate solutions of (3), (7), (8) constructed as below:

- 1) Define a sequence of piece-wise constant functions  $U_0^\nu = (s_0^\nu, w_0^\nu) : I \rightarrow \mathbf{E}$ ,  $\tilde{v}_0^\nu = \tilde{V}(s_0^\nu, w_0^\nu)$ , and  $U_c^\nu = (s_c^\nu, w_c^\nu) : [0, T] \rightarrow \mathbf{E}$ ,  $\tilde{v}_c^\nu = \tilde{V}(s_c^\nu, w_c^\nu)$ , satisfying

$$\begin{aligned} \text{TV}(\tilde{v}_i^\nu) &\leq \text{TV}(\tilde{v}_i), & \text{TV}(w_0^\nu) &\leq \text{TV}(w_0), \\ \|\tilde{v}_i^\nu - \tilde{v}_i\|_{\mathbf{L}^\infty} &\leq \nu^{-1}, & \|w_0^\nu - w_0\|_{\mathbf{L}^\infty} &\leq \nu^{-1}, \\ \|s_0^\nu - s_0\|_{\mathbf{L}^1} &\leq \nu^{-1}, & \|w_0^\nu - w_0\|_{\mathbf{L}^1} &\leq \nu^{-1}, \end{aligned}$$

for  $i = 0, c$ , and, for each  $\nu \in \mathbb{N}$ , the piece-wise constant function  $U_0^\nu$  has a finite number of discontinuities and  $\tilde{v}_c^\nu$  is constant on  $\Delta T^k$ ,  $k = 0, \dots, \lfloor T/\Delta t^\nu \rfloor$ .

- 2) Solve the homogeneous system (17) corresponding to the Riemann problems arising at each discontinuity and at the right boundary  $n = N$  for  $t \in [0, \Delta t^\nu]$  using the classical WFT method [36] and name  $U^\nu(t, \cdot)$  the corresponding piece-wise constant function.

- 3) At  $t = \Delta t^\nu$ , we define

$$\begin{aligned} s^\nu(\Delta t^\nu, \cdot) &= s^\nu(\Delta t^\nu -, \cdot), \\ w^\nu(\Delta t^\nu, \cdot) &= w^\nu(\Delta t^\nu -, \cdot) \\ &+ \frac{\Delta t^\nu}{\tau} \left[ \tilde{V}(s^\nu(\Delta t^\nu, \cdot)) - \tilde{V}(U^\nu(\Delta t^\nu -, \cdot)) \right]. \end{aligned} \quad (9)$$

Note that  $s$  is conserved during this second step, while  $w$  (and  $\tilde{v} = \tilde{V}(s, w)$ ) is updated according to (18).

- 4) Treat  $U^\nu(\Delta t^\nu, \cdot)$  as a new piece-wise constant initial condition and repeat the previous steps 2–3 to define the solution  $U^\nu(t, \cdot)$  for each  $t \in [0, T]$ .

Following [19], [35], [37], one can prove uniform  $\mathbf{L}^\infty$  and BV estimates that allow to apply Helly's Theorem and infer the convergence (up to a subsequence) of  $U^\nu$  to a weak solution of IBVP.

*Theorem 1:* Given any  $T > 0$ , let  $U_0 : I \rightarrow \mathbf{E}$  with  $\text{TV}(U_0) < +\infty$ ,  $U_c : [0, T] \rightarrow \mathbf{E}$  with  $\text{TV}(\tilde{v}_c) < +\infty$  and let  $(\tilde{v}, w)$  be the limit function of the sequence  $\{\tilde{v}^\nu, w^\nu\}_\nu$  of WFT approximate solutions as  $\nu \rightarrow \infty$ . Then  $U = (s, w)$ , with  $s$  implicitly defined by  $\tilde{V}(s, w) = \tilde{v}$ , is a weak solution of (3), (7), (8) in the sense of Definition 1. In particular,  $\text{TV}(\tilde{v}(t, \cdot)) < +\infty$  for a.e.  $t \in [0, T]$  and  $\tilde{v}$  admits a strong trace at  $n = N$ .

#### IV. ASYMPTOTIC BOUNDARY STABILIZATION

Given any initial state  $U_0 = (s_0, w_0) \in \text{BV}(I; \mathbf{E})$ , the corresponding equilibrium  $U^* = (s^*, w^*)$  of (3) is implicitly given by

$$s^* := \frac{1}{N} \int_0^N s_0(n) dn, \quad \tilde{V}(s^*, w^*) = \tilde{V}(s^*) =: \tilde{v}^*. \quad (10)$$

We seek for a boundary condition (8) that steers  $U_0$  to  $U^*$  under the general assumption that (5) does not hold at  $U^*$ . Therefore, oscillations are not naturally damped by the source term and can persist in time, see e.g. [35], [38] for an analysis of the corresponding travelling waves.

*Theorem 2:* Assume

$$\partial_s[\tilde{V}_w(\tilde{V} - \tilde{v}^*)](U) > 0 \text{ for all } U \in \mathbf{E}. \quad (11)$$

Then, for any  $U_0 : I \rightarrow \mathbf{E}$  with  $\text{TV}(U_0) < +\infty$  and  $\varepsilon > 0$ , there exists  $T > 0$  such that, setting  $\tilde{v}_c(t) = \tilde{v}^*$  for all  $t \in [0, T]$  in (8), the weak solution  $U$  of (3), (7), (8) obtained as limit of WFT approximations satisfies

$$\|U(T, \cdot) - U^*\|_{\mathbf{L}^\infty} \leq \varepsilon. \quad (12)$$

*Proof:* At time  $t = 0$ , let us consider the rightmost Riemann problem corresponding to  $U_L = U_0^\nu(N-)$  and  $U_R = U_c$ , for some  $U_c = (s_c, w_c)$  such that  $\tilde{V}(s_c, w_c) = \tilde{v}^*$ , whose intermediate state is given by  $(s_m, w_L)$  such that  $\tilde{V}(s_m, w_L) = \tilde{v}^*$ . Iterating the argument at possible later wave interactions, we observe that  $\tilde{V}(U^\nu(\Delta t^\nu -, n)) = \tilde{v}^*$  for  $n > N - v_{\min} \Delta t^\nu$ , with  $v_{\min} := \min_{U \in \mathbf{E}} \tilde{V}_s(U) > 0$ . Applying the relaxation step (9), we get

$$\begin{aligned} \tilde{V}(U^\nu(\Delta t^\nu, n)) &= \tilde{v}^* \left( 1 - \frac{\Delta t^\nu}{\tau} \tilde{V}_w \right) \\ &+ \frac{\Delta t^\nu}{\tau} \tilde{V}_w \tilde{V}(s^\nu(\Delta t^\nu, n)) + o(\Delta t^\nu), \end{aligned}$$

with  $\tilde{V}_w = \tilde{V}_w(U^\nu(\Delta t^\nu -, n))$ . Iterating the above process at each time step  $t^k = k \Delta t^\nu$ , we get the (approximate) sequence  $U_k^\nu := U^\nu(t^k, n) = (s_k, w_k)$  implicitly defined by

$$\begin{aligned} \tilde{V}(s_k, w_{k+1}) &= \tilde{v}^* + \frac{\Delta t^\nu}{\tau} \tilde{V}_w(U_k^\nu) \left( \tilde{V}(s_k) - \tilde{v}^* \right), \\ \tilde{V}(s_{k+1}, w_{k+1}) &= \tilde{v}^*. \end{aligned}$$

By the implicit function theorem, the above identities define a function  $\omega$  such that  $w_{k+1} = \omega(s_k)$  and

$$\omega'(s_k) = \frac{1}{\tilde{V}_w} \left( \frac{\Delta t^\nu}{\tau} \left( \tilde{V}_w \tilde{V}' + \tilde{V}_{w,s}(\tilde{V} - \tilde{v}^*) \right) - \tilde{V}_s \right) (U_k^\nu),$$

and a function  $\varphi$  such that  $s_{k+1} = \varphi(s_k)$  satisfying  $\tilde{V}(\varphi(s_k), \omega(s_k)) = \tilde{v}^*$  and

$$\varphi'(s_k) = -\frac{\tilde{V}_w \omega'(s_k)}{\tilde{V}_s} \sim 1 - \frac{\Delta t^\nu}{\tau} \frac{\partial_s[\tilde{V}_w(\tilde{V} - \tilde{v}^*)]}{\tilde{V}_s} (U_k^\nu).$$

Thanks to (11), there exists  $K > 0$  such that  $0 < \partial_s[\tilde{V}_w(\tilde{V} - \tilde{v}^*)]/\tilde{V}_s(U) \leq K$  for all  $U \in \mathbf{E}$ . Thus, the recursive sequence converges to the fixed point  $s^*$  for  $\Delta t^\nu < 2\tau/K$  and

$$|s^\nu(t^k, n) - s^*| \leq (\|\varphi'\|_\infty)^k |s_0(n) - s^*| \leq \varepsilon/2$$

for  $k \geq \ln(\varepsilon/\|s_0 - s^*\|_\infty)/\ln(\|\varphi'\|_\infty)$  and  $n > N - v_{\min} \Delta t^\nu$ . Also,  $\lim_k w_k = \lim_k \omega(s_k) = \omega(s^*) = w^*$ , therefore  $|w_k - w^*| \leq \varepsilon/2$  for  $k$  large enough.

To extend the argument in the whole space domain  $I$ , let us proceed by induction. Assume to have a sequence  $(s_{j,k}, w_{j,k})$  at some point  $n = n_j \in I$  converging to  $(s^*, w^*)$ , so that also  $\tilde{v}_{j,k} := \tilde{V}(s_{j,k}, w_{j,k})$  converges to  $\tilde{v}^*$ . Then, for  $n \in ]n_j - v_{\min} \Delta t^\nu, n_j[$ , the WFT approximate solutions  $U_{j+1,k}^\nu$  satisfy the recursive relation

$$\begin{aligned} \tilde{V}(s_{j+1,k}, w_{j+1,k+1}) &= \tilde{v}_{j,k} \\ &+ \frac{\Delta t^\nu}{\tau} \tilde{V}_w(U_{j+1,k}^\nu) \left( \tilde{V}(s_{j+1,k}) - \tilde{v}_{j,k} \right), \\ \tilde{V}(s_{j+1,k+1}, w_{j+1,k+1}) &= \tilde{v}_{j,k+1}. \end{aligned}$$

Reasoning as above, these relations define a sequence  $s_{j+1,k} = \varphi_{j+1}(s_{j+1,k})$  such that

$$\varphi'_{j+1}(s_{j+1,k}) \sim 1 - \frac{\Delta t^\nu}{\tau} \frac{\partial_s[\tilde{V}_w(\tilde{V} - \tilde{v}_{j,k})]}{\tilde{V}_s} (U_{j+1,k}^\nu),$$

whose limit must satisfy

$$\lim_k \tilde{V}(s_{j+1,k}, w_{j+1,k}) = \lim_k \tilde{v}_{j,k} = \tilde{v}^* = \lim_k \tilde{V}(s_{j+1,k}).$$



Due to the strictly negative propagation speed of first family waves, repeating the argument at most  $\lceil N/(v_{\min}\Delta t^\nu) \rceil$  times allows to cover the whole interval  $I$ , showing that there exists  $\bar{k}$  sufficiently large such that, setting  $T = \bar{k}\Delta t^\nu$ , for any  $n \in I$  it holds  $|U^\nu(T, n) - U^*| \leq \varepsilon$ . ■

*Remark 1:* Assumption (11) is equivalent to

$$[\tilde{\mathcal{V}}_{w,s}(\tilde{V} - \tilde{v}^*) + \tilde{\mathcal{V}}_w \tilde{V}'](U) \geq 0, \quad (13)$$

which is satisfied for  $U$  such that  $\tilde{V}(s) \geq \tilde{v}^*$  (i.e.  $s \geq s^*$ ) if  $\tilde{\mathcal{V}}_{w,s}(U) \geq 0$ . For  $s < s^*$ , (13) is equivalent to

$$[\tilde{\mathcal{V}}_w \tilde{V}'](U) \geq [\tilde{\mathcal{V}}_{w,s}(\tilde{v}^* - \tilde{V})](U) \sim \tilde{\mathcal{V}}_{w,s}(U) \tilde{V}'(s)(s^* - s),$$

i.e.  $2\tilde{\mathcal{V}}_w(s, w) \geq \tilde{\mathcal{V}}_w(s^*, w)$ , showing that assumption (11) is not too restrictive.

## V. NUMERICAL EXPERIMENT

We consider the IBVP (3), (7), (8) with

$$\tilde{v}_c(t) = \begin{cases} \tilde{\mathcal{V}}(s, w)(t, 0+) & \text{for } 0 \leq t < \bar{t}, \\ \tilde{v}^* & \text{for } \bar{t} \leq t \leq T, \end{cases} \quad (14)$$

modeling a ring road where  $\tilde{\mathcal{V}}(s, w)(t, N) = \tilde{\mathcal{V}}(s, w)(t, 0)$  for  $t \in [0, \bar{t}]$  and the control is activated at time  $t = \bar{t}$ . We fix the space and time steps, respectively  $\Delta n = N/J$  for some  $J \in \mathbb{N}$  and  $\Delta t$  satisfying the stability condition [39]

$$\Delta t = 0.9 \min \left\{ \Delta n / \left\| \tilde{\mathcal{V}}_s \right\|_\infty, 2\tau / \left\| \tilde{\mathcal{V}}_w \right\|_\infty \right\},$$

and we set  $n_{j-1/2} = j\Delta n$ ,  $n_j = (j + 1/2)\Delta n$  for  $j = 0, \dots, J$ , and  $t^k = k\Delta t$  for  $k \in \mathbb{N}$ .

We approximate the initial data (7) with piece-wise constant functions  $\{(s_j^0, w_j^0)\}_{j=1}^J$  and we discretize (3) by a finite volume upwind scheme with time-splitting: for  $j = 1, \dots, J$  and  $k \in \mathbb{N}$ , we set  $\tilde{v}_j^k = \tilde{\mathcal{V}}(s_j^k, w_j^k)$  and

$$\begin{aligned} s_j^{k+1} &= s_j^k + \frac{\Delta t}{\Delta n} (\tilde{v}_{j+1}^k - \tilde{v}_j^k), \\ \tilde{v}_j^{k+1/2} &= \tilde{\mathcal{V}}(s_j^{k+1}, w_j^k), \\ w_j^{k+1} &= w_j^k + \frac{\Delta t}{\tau} (\tilde{\mathcal{V}}(s_j^{k+1}) - \tilde{v}_j^{k+1/2}), \\ \tilde{v}_j^{k+1} &= \tilde{\mathcal{V}}(s_j^{k+1}, w_j^{k+1}), \end{aligned}$$

with

$$\tilde{v}_{j+1}^k = \tilde{v}_c(t^k) = \begin{cases} \tilde{v}_1^k & \text{for } 0 \leq k\Delta t < \bar{t}, \\ \tilde{v}^* & \text{for } k\Delta t \geq \bar{t}. \end{cases}$$

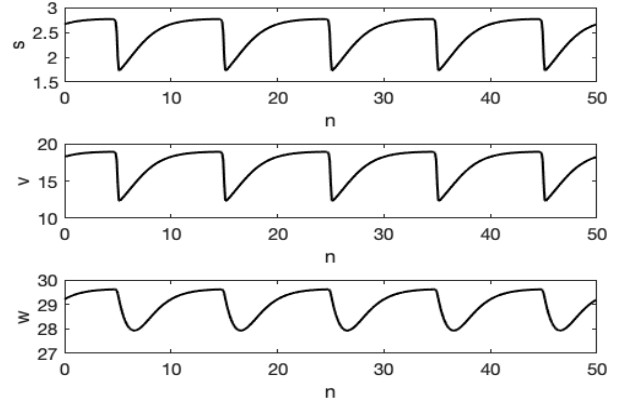
We consider a ring road of length  $L = 125[\text{m}]$  containing  $N = 50$  vehicles of length  $\ell = 1[\text{m}]$  (corresponding to a maximal density  $\bar{R} = \rho_{max} = 1[\text{veh/m}]$ ) with initial spacing and Lagrangian attribute

$$s_0(n) = 2.5[\text{m}], \quad w_0(n) = 29 + 0.1 \sin(10\pi n/N)[\text{m/s}], \quad (15)$$

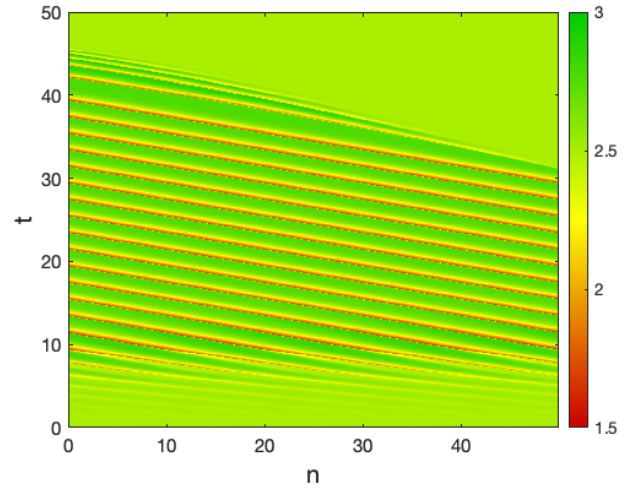
corresponding, under the coordinate transformation  $x(n) = \int_0^n s_0(\nu) d\nu$ , i.e.  $x = 2.5n$ , to the Eulerian quantities

$$\rho_0(x) = 0.4[\text{veh/m}], \quad w_0(x) = 29 + 0.1 \sin(10\pi x/L)[\text{m/s}].$$

Initial conditions (15) represent equidistributed vehicles with small oscillations in the drivers' preferred speed, which, due to the instability of the system, generate persistent stop-and-go waves [35], [38], see Figures 1 and 2.



**Fig. 1:** Spacing  $s(t, \cdot)$  (top), speed  $\tilde{v}(t, \cdot)$  (middle) and Lagrangian marker  $w(t, \cdot)$  (bottom) profiles of the solution to (3), (14), (15) at  $t = 20 < \bar{t}$ .



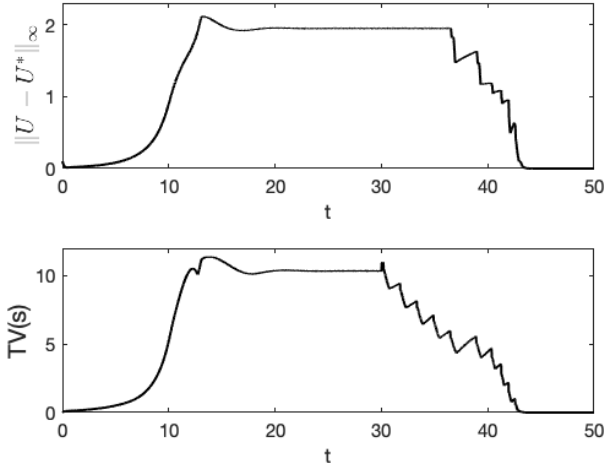
**Fig. 2:** Evolution (heat map) of the spacing component  $s = s(t, n)$  of the solution to (3), (14), (15). Stop-and-go waves moving backward with respect to vehicles are clearly visible in the time interval  $[10\text{s}, 40\text{s}]$ .

The average and equilibrium speed functions are chosen respectively as

$$\begin{aligned} \tilde{\mathcal{V}}(s, w) &= w \left( 1 - \frac{1}{s} \right), \\ \tilde{V}(s) &= V_{\max} (1 - \exp(\alpha(1 - s))), \end{aligned}$$

with  $V_{\max} = 25[\text{m/s}]$  the maximal speed and  $\alpha = 0.8$ . With these choices, we have  $s^* = 2.5[\text{m}]$ ,  $w^* = 29[\text{m/s}]$  and  $\tilde{v}^* = 17.47[\text{m/s}]$ . We also set  $\tau = 0.1[\text{s}]$ .

The space mesh size is set to  $\Delta n = 0.1$  and the simulation time horizon at  $T = 50[\text{s}]$ , with control activation time at  $\bar{t} = 30[\text{s}]$ . In Fig. 2 and 3 (bottom), we observe that the small oscillations in the  $w$  component of the initial datum (15) induce the formation of large stop-and-go waves, which stabilize after  $t = 20[\text{s}]$ . Once the control is activated ( $t = 30[\text{s}]$ ), the spacing total variation  $\text{TV}(s(t, \cdot))$  starts immediately to



**Fig. 3:** Time evolution of the  $L^\infty$  distance to equilibrium  $\|U(t, \cdot) - U^*\|_{L^\infty}$  (top) and spacing total variation  $TV(s(t, \cdot))$  (bottom). The control action is activated at  $t = 30$ s.

decrease, to reach 0 in less than 15[s]. The  $L^\infty$  distance to equilibrium  $\|U(t, \cdot) - U^*\|_{L^\infty}$  starts decreasing later, since the information about the control acting at the right boundary  $n = N$  needs to travel upstream the whole interval  $]0, N[$  to affect the  $L^\infty$  norm, see Fig. 2.

## VI. CONCLUSIONS

We have proved the asymptotic stabilization of the GSOM traffic model by an open loop speed control acting at the downstream boundary, for any number of vehicles and initial conditions with bounded total variation. The result is in line with the conclusions of the microscopic analysis provided in [11]. The asymptotic decay of the total variation will be investigated analytically in future studies. Also, control actions ensuring the respect of the physical constraints, such as  $\int_0^N s(t, n) dn \leq \int_0^N s(0, n) dn$  for all  $t > 0$  in the case of a ring road, should be designed.

We point out that the linearized version of system (3) does not satisfy the conditions for Lyapunov stability [40] when the sub-characteristic condition (5) is violated. This advocates even further for the nonlinear setting adopted in this work.

## APPENDIX: THE RIEMANN PROBLEM

Let us consider (3) with initial data of the form

$$(s, w)(0, n) = \begin{cases} U_L = (s_L, w_L) & \text{if } n < \bar{n}, \\ U_R = (s_R, w_R) & \text{if } n \geq \bar{n}, \end{cases} \quad (16)$$

and set  $\tilde{v}_L = \tilde{V}(s_L, w_L)$ ,  $\tilde{v}_R = \tilde{V}(s_R, w_R)$ . To construct the (approximate) solution, we iteratively solve the homogenous system

$$\begin{cases} \partial_t s - \partial_n \tilde{V} = 0, \\ \partial_t w = 0, \end{cases} \quad (17)$$

which belongs to the Temple class [41] and whose Riemann invariants are  $z_1(s, w) = \tilde{V}(s, w)$  and  $z_2(s, w) = w$ , and the ODE system

$$\begin{cases} \partial_t s = 0, \\ \partial_t w = \frac{\tilde{V}(s) - \tilde{V}}{\tau}. \end{cases} \quad (18)$$

Solutions of (17), (16) consist in general of a first family wave (rarefaction or shock) connecting  $U_L$  to  $U_M = (s_M, w_L)$  travelling with strictly negative speed (with  $s_M$  implicitly defined by  $\tilde{V}(s_M, w_L) = \tilde{v}_R$ ), and a contact discontinuity with zero speed connecting  $U_M$  to  $U_R$ . Notice that, if  $w_L > \tilde{v}_R$ , an intermediate vacuum state appears, where  $s = +\infty$ .

The solution to (18), (16) is instead given by the constant in time component  $s(t, \cdot) = s(0, \cdot)$  and

$$w(t, \cdot) = w(0, \cdot) + \frac{\tilde{V}(s(0, \cdot))}{\tau} t - \int_0^t \frac{\tilde{V}(s(0, \cdot), w(\xi, \cdot))}{\tau} d\xi.$$

Let us now observe that, setting  $s = \mathcal{S}(\tilde{v}, w)$  and differentiating the identity  $\tilde{V}(\mathcal{S}(\tilde{v}, w), w) = \tilde{v}$ , we get the identities

$$\mathcal{S}_{\tilde{v}} = 1/\tilde{V}_s, \quad \mathcal{S}_w = -\tilde{V}_w/\tilde{V}_s. \quad (19)$$

Thus, the equilibrium curve, which is implicitly defined by  $\tilde{V}(s) = \tilde{V}(\mathcal{S}(\tilde{v}, \phi(\tilde{v}))) = \tilde{v}$ , satisfies

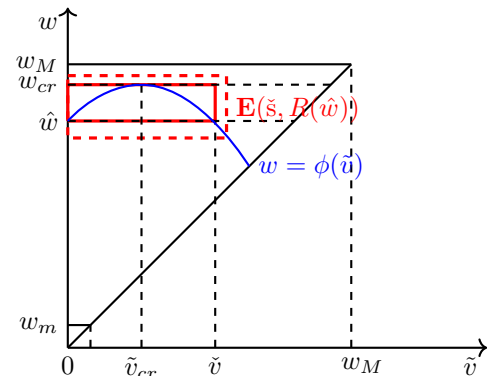
$$\phi'(\tilde{v}) = \frac{1}{\mathcal{S}_w} \left( \frac{1}{\tilde{V}'(s)} - \mathcal{S}_{\tilde{v}} \right) = \frac{1}{\tilde{V}_w} \left( 1 - \frac{\tilde{V}_s}{\tilde{V}'(s)} \right).$$

Since  $\tilde{V}_w > 0$  by (2c), we get  $\phi'(\tilde{v}) \leq 0$  if and only if  $\tilde{V}_s \geq \tilde{V}'(s)$ , i.e. the sub-characteristic condition (5) holds. Let us assume there exists  $\tilde{v}_{cr}$ , resp.  $w_{cr} = \phi(\tilde{v}_{cr})$  and  $s_{cr} = \mathcal{S}(\tilde{v}_{cr}, w_{cr})$ , such that  $\phi'(\tilde{v}_{cr}) = 0$  (i.e.  $\tilde{V}(s_{cr}, w_{cr}) = \tilde{V}'(s_{cr})$ ) and  $\phi'(\tilde{v}) > 0$  if  $\tilde{v} < \tilde{v}_{cr}$ ,  $\phi'(\tilde{v}) < 0$  if  $\tilde{v} > \tilde{v}_{cr}$ . In this case, the equilibrium curve intersects the congestion line  $\tilde{v} = 0$  at  $\hat{w}$  implicitly defined by  $\tilde{V}(1/R(\hat{w}), \hat{w}) = 0$ . We also define  $\check{s}, \check{v}$  such that  $\check{v} = \tilde{V}(\check{s}, \hat{w}) = \tilde{V}(\check{s})$ . See Fig. 4.

By the above analysis, we consider invariant domains for (3) of the form

$$\mathbf{E} = \mathbf{E}[\bar{s}, R(\hat{w})] = \{U = (s, w) : s \in [1/R(w_M), \bar{s}], \tilde{V}(s, w) \in [0, \bar{v}], w \in [\bar{w}, w_M]\}, \quad (20)$$

with  $\bar{s} \geq \check{s}$ ,  $w_M \geq w_{cr}$  and  $\bar{v} = \tilde{V}(\bar{s}, \bar{w}) = \tilde{V}(\bar{s})$ , which do not include vacuum states. We refer to [35] for more details on the structure of Riemann solutions and invariant domains.



**Fig. 4:** Example of minimal invariant domain  $\mathbf{E}$  (continuous red line) and general invariant domain (dashed red line) considered in this paper.

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