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Abstract

Let $m \geq 1$ be a rational integer. We give an explicit formula for the mean value

$$\frac{2}{\phi(f)} \sum_{\chi(-1)=(-1)^m} |L(m,\chi)|^2,$$

where χ ranges over the $\phi(f)/2$ Dirichlet characters modulo $f > 2$ with the same parity as m. We then adapt our proof to obtain explicit means values for products of the form $L(m_1,\chi_1)\cdots L(m_{n-1},\chi_{n-1})\overline{L(m_n,\chi_1\cdots\chi_{n-1})}$.

1 Introduction

For $m \geq 1$ and $f > 2$, let

$$M(m,f) = \frac{2}{\phi(f)} \sum_{\chi(-1)=(-1)^m} |L(m,\chi)|^2$$

denote the mean value of $|L(m,\chi)|^2$, where χ ranges over the $\phi(f)/2$ Dirichlet characters modulo f such that $\chi(-1) = (-1)^m$, i.e. over the Dirichlet characters modulo f of the same parity as m.

The first step towards the study of these $M(m,f)$'s was made in [Wal], where the formula for $M(1,f)$ was obtained, but only for prime moduli f. Secondly, in [QiMG] and [Lou93] the formula for $M(1,f)$ for non necessarily prime moduli f was obtained. Thirdly, in [Lou01] a method was developed for obtaining recursively a formula for $M(m,f)$ for a given m. This method was applied to obtain the formulae below for $M(m,f)$ for $m = 1$, 2, 3 and 4. Fourthly, in [Lin] the author developed another way for obtaining recursively a formula for $M(m,f)$ for a given m. S. Louboutin and X. Lin's methods are easy to use with any software for mathematical computation for obtaining a formula for $M(m,f)$ for a given m. But they did not give an explicit formula for these $M(m,f)$'s. Fifthly, in [LZ] the authors gave a rather complicated but explicit formula for $M(m,f)$ and the mean value of the product of the values of two Dirichlet L-functions at positive integers of the same parity.

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Here, inspired by [BY, Proof of Theorem 2.1], we obtain in Theorem 1 an explicit formula for these $M(m,f)$'s. Then, we extend our method developed to obtain Theorem 1 to get in Theorem 7 an explicit formula for the mean value of more general products of the form $L(m_1,\chi_1)\cdots L(m_{n-1},\chi_{n-1})\overline{L(m_n,\chi_1\cdots\chi_{n-1})}$. In particular, for $n=2$ we obtain in Theorem 9 a much simpler formula than the one given in [LZ, Theorem 1.1] for the mean value of the product of the values of two Dirichlet L-functions at positive integers of the same parity.

Theorem 1. *For $m=1$ we have*

$$M(1,f)=\frac{\pi^2}{6}\left(\phi_2(f)-3\frac{\phi_1(f)}{f}\right) \tag{1}$$

and for $m\geq 2$ we have

$$M(m,f)=\zeta(2m)\times\left(\phi_{2m}(f)+(-1)^{m-1}\binom{2m}{m}\sum_{k=1}^{\lfloor m/2\rfloor}\frac{m}{m-k}\binom{m}{2k}\frac{B_{2k}B_{2(m-k)}}{B_{2m}}\frac{\phi_{2k}(f)}{f^{2m-2k}}\right),$$

where $\phi_l(f)=\prod_{p|f}(1-1/p^l)$ and the Bernoulli rational numbers B_k are given by

$$\frac{t}{e^t-1}=\sum_{k\geq 0}\frac{B_k}{k!}t^k.$$

In particular, $M(m,f)$ is asymptotic to $\zeta(2m)\phi_{2m}(f)$ as f goes to infinity.

For example, in accordance with [Lou93, Theorem 2] and [Lin, Page 371] we have:

$$\begin{aligned}
M(2,f) &= \frac{\pi^4}{90}\left(\phi_4(f)+10\frac{\phi_2(f)}{f^2}\right),\\
M(3,f) &= \frac{\pi^6}{945}\left(\phi_6(f)-21\frac{\phi_2(f)}{f^4}\right),\\
M(4,f) &= \frac{\pi^8}{9450}\left(\phi_8(f)+\frac{14}{3}\frac{\phi_4(f)}{f^4}+\frac{200}{3}\frac{\phi_2(f)}{f^6}\right),\\
M(5,f) &= \frac{\pi^{10}}{93555}\left(\phi_{10}(f)-22\frac{\phi_4(f)}{f^6}-231\frac{\phi_2(f)}{f^8}\right).
\end{aligned}$$

2 Proof of Theorem 1

Theorem 1 readily follows from the well known formula

$$\zeta(2m)=(-1)^{m-1}\frac{(2\pi)^{2m}B_{2m}}{2\cdot(2m)!}. \tag{2}$$

and from following Lemmas 2, 3 and 4.

Lemma 2. *(See [Lou01, Proposition 3 and Proof of Proposition 5]). Let $m\geq 1$ and $f>2$ be rational integers. Let $\cot^{(k)}$ denote the kth derivative of $\cot:=\cos/\sin$. We have*

$$M(m,f)=\frac{\pi^{2m}}{2\cdot((m-1)!)^2f^{2m}}\sum_{\substack{d|f\\ d>1}}\mu(f/d)S_m(d),$$

where

$$S_m(d):=\sum_{k=1}^{d-1}\left(\cot^{(m-1)}(\pi k/d)\right)^2\qquad(d>1).$$

Lemma 3. *If $R(X) = \sum_{l=0}^{k} r_{k,l} X^l \in \mathbb{Q}[X]$ is such that $R(1) = 0$, then for $f > 2$ we have*

$$\sum_{\substack{d|f \\ d>1}} \mu(f/d)R(d) = \sum_{d|f} \mu(f/d)R(d) = \sum_{l=1}^{k} r_{k,l}\phi_l(f)f^l.$$

Proof. Notice that $\sum_{d|f} \mu(f/d)d^l = \sum_{d|f} \mu(d)(f/d)^l = f^l\phi_l(f)$. □

Lemma 4. *Set $R_1(X) = (X^2 - 3X + 2)/3$ and for $m \geq 2$ set*

$$R_m(X) = 2^{2m}\cdot\left((-1)^{m-1}\frac{B_{2m}}{m^2\binom{2m}{m}}X^{2m} + \sum_{k=0}^{\lfloor m/2\rfloor}\frac{B_{2k}B_{2(m-k)}}{m(m-k)}\binom{m}{2k}X^{2k} - \frac{1+(-1)^m}{2}\frac{B_m^2}{m^2}\right).$$

Then $R_m(1) = 0$ for $m \geq 1$ and $S_m(d) = R_m(d)$ for $m \geq 1$ and $d \geq 2$.

Proof. For $d \geq 1$, set

$$g_m(d,z) = d\cot(dz)\left(\cot^{(m-1)}(z)\right)^2 = \frac{1}{z^{2m+1}}(dz\cot(dz))(z^m\cot^{(m-1)}(z))^2,$$

a π-periodic meromorphic function of the complex variable z, with a simple pole at each $k\pi/d$, $k \in \mathbb{Z} \setminus d\mathbb{Z}$, of residue

$$\mathrm{Res}_{z=k\pi/d}(g_m(d,z) = \left(\cot^{(m-1)}(k\pi/d)\right)^2 \qquad (k \in \mathbb{Z} \setminus d\mathbb{Z}) \tag{3}$$

and poles of order $2m+1$ at each $k\pi$, $k \in \mathbb{Z}$. For $R > 0$, let C_R be the positively oriented rectangle of width π, height $2R$ and center of gravity at $\pi/2 - \pi/(2d)$.
(i). We claim that

$$\lim_{R\to+\infty}\frac{1}{2\pi i}\int_{C_R} g_m(d,z)\mathrm{d}z = \begin{cases} -d & \text{if } m = 1 \text{ and } d \geq 1, \\ 0 & \text{if } m \geq 2 \text{ and } d \geq 1. \end{cases} \tag{4}$$

Indeed, since g is π-periodic the integrals over the vertical sides of C_R cancel. Moreover, since $|\cot(t+ib) + i\mathrm{sign}(b)| \leq 2/(e^{|b|}-1)$ we have $\lim_{b\to+\infty}\cot(a+ib) = -i$ and $\lim_{R\to+\infty}\cot(a-ib) = i$ uniformly in $a \in I$, where I is a given closed interval. Hence for $m = 1$ we have $\lim_{R\to+\infty} g_1(d, t+iR) = di$ and $\lim_{R\to+\infty} g_1(d,t-iR) = -di$, uniformly in $t \in [-\pi/(2d), \pi - \pi/(2d)]$. The first case of claim (4) follows. Now assume that $m \geq 2$. Then $\cot^{(m-1)} = Q_m(\cot)$ where the polynomials $Q_k(X)$'s are defined inductively by $Q_1(X) = X$ and $Q_{k+1}(X) = -(X^2+1)Q_k'(X)$. Consequently, we have $\lim_{R\to+\infty} g_m(d,t+iR) = Q_m(-i) = 0$ and $\lim_{R\to+\infty} g_m(d,t-iR) = Q_m(i) = 0$, uniformly in $t \in [-\pi/(2d), \pi - \pi/(2d)]$. Hence the integrals over the horizontal sides tend to 0 as R goes to infinity for $m \geq 2$ and the second case of claim (4) follows.
(ii). Now, we claim that

$$R_m(d) = \lim_{R\to+\infty}\frac{1}{2\pi i}\int_{C_R} g_m(d,z)\mathrm{d}z - \mathrm{Res}_{z=0}(g_m(d,z)) \qquad (m \geq 1 \text{ and } d \geq 1). \tag{5}$$

Indeed, if $m = 1$ then $g_1(d,z) = d\cot(dz)\cot^2 z$,

$$\mathrm{Res}_{z=0}(g_1(d,z) = -(d^2+2)/3$$

and the claims follow by (4) and the definition of $R_1(X)$. Now assume that $m \geq 2$. Using the power series expansion

$$z\cot(z) = \sum_{k\geq 0}(-1)^k\frac{2^{2k}B_{2k}}{(2k)!}z^{2k} \qquad (|z| < \pi), \tag{6}$$

we have

$$z^m \cot^{(m-1)}(z) = (-1)^{m-1}(m-1)! + \sum_{k \geq m/2} (-1)^k \frac{2^{2k} B_{2k}}{2k \cdot (2k-m)!} z^{2k} \qquad (|z| < \pi). \quad (7)$$

Using (6) and (7) to compute the coefficient of the term z^{2m} in the power series expansion of the function $(dz \cot(dz))(z^m \cot^{(m-1)}(z))^2$, we readily obtain that

$$\mathrm{Res}_{z=0}(g_m(d,z)) = -R_m(d) \text{ for } m \geq 2 \text{ and } d \geq 1.$$

Hence claim (5) follows from (4).
(iii). Finally, we use the residue theorem. For $d = 1$, noticing that $z = 0$ is the only pole of $g_m(1,z)$ inside C_R and using (5) we obtain $R_m(1) = 0$ for $m \geq 1$. For $d \geq 2$, noticing that $z = 0$ and $z = k\pi/d$, $1 \leq k \leq d-1$, are the only poles of $g_m(d,z)$ inside C_R and using (5) and (3) we obtain $R_m(d) = S_m(d)$ for $d \geq 2$ and $m \geq 1$, as desired. □

In [EL, Corollary 6.8] the authors gave a reformulation of [BY, Corolloray 2.2]. It would be interesting to see if their method can also be applied to a reformulation of our Lemma 4.

3 A general explicit formula

For $f > 2$, $n \geq 2$ and $m_1, \cdots, m_n \geq 1$ such that $m_1 + \cdots + m_n = 2s$ is even, we set $\overrightarrow{m} = (m_1, \cdots, m_n)$ and

$$M(\overrightarrow{m}, f) = \left(\frac{2}{\phi(f)}\right)^{n-1}$$
$$\times \sum_{\chi_1(-1)=(-1)^{m_1}} \cdots \sum_{\chi_{n-1}(-1)=(-1)^{m_{n-1}}} L(m_1,\chi_1)\cdots L(m_{n-1},\chi_{n-1})\overline{L(m_n,\chi_1\cdots\chi_{n-1})}.$$

Hence, $(\chi_1 \cdots \chi_{n-1})(-1) = (-1)^{m_n}$. Using the formula

$$L(m,\chi) = \frac{(-1)^{m-1}\pi^m}{2 \cdot f^m \cdot (m-1)!} \sum_{k=1}^{f-1} \chi(k) \cot^{(m-1)}(\pi k/f) \qquad \text{(whenever } \chi(-1) = (-1)^m\text{)}.$$

proved in [Lou01, (1)], we have

$$M(\overrightarrow{m}, f) = \frac{(-1)^n \pi^{2s}}{2^n f^{2s} \prod_{l=1}^n (m_l - 1)!}$$
$$\times \sum_{k_n=1}^{f-1} \sum_{k_1=1}^{f-1} \cdots \sum_{k_{n-1}=1}^{f-1} \left(\prod_{l=1}^n \cot^{(m_l-1)}(\pi k_l/f)\right) \left(\prod_{l=1}^{n-1} \frac{2}{\phi(f)} \sum_{\chi_l(-1)=(-1)^{m_l}} \chi_l(k_l)\overline{\chi_l(k_n)}\right).$$

We have the following orthogonality relations for the $\phi(f)/2$ characters χ_l's modulo $f > 2$ for which $\chi_l(-1) = (-1)^{m_l}$:

$$\frac{2}{\phi(f)} \sum_{\chi_l(-1)=(-1)^{m_l}} \chi_l(k_l)\overline{\chi_l(k_n)} = \begin{cases} 1 & \text{if } k_l \equiv k_n \pmod f \text{ and } \gcd(k_n, f) = 1, \\ (-1)^{m_l} & \text{if } k_l \equiv -k_n \pmod f \text{ and } \gcd(k_n, f) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Noticing that $\chi_l(-k_n)\cot^{(m_l-1)}(-\pi k_n/f) = \chi_l(k_n)\cot^{(m_l-1)}(\pi k_n/f)$ whenever $\chi_l(-1) = (-1)^{m_l}$, we obtain

$$M(\overrightarrow{m}, f) = \frac{(-1)^n \pi^{2s}}{2f^{2s} \prod_{l=1}^n (m_l-1)!} \sum_{\substack{k_n=1 \\ \gcd(k_n,f)=1}}^{f-1} \prod_{l=1}^n \cot^{(m_l-1)}(\pi k_n/f).$$

The following generalization of Lemma 2 follows:

Lemma 5. *For $f > 2$, $n \geq 2$ and $m_1, \cdots, m_n \geq 1$ such that $m_1 + \cdots + m_n = 2s$ is even we have*

$$M(\overrightarrow{m}, f) = \frac{(-1)^n \pi^{2s}}{2 \cdot (m_1 - 1)! \cdots (m_n - 1)! \cdot f^{2s}} \sum_{\substack{d|f \\ d>1}} \mu(f/d) S_{\overrightarrow{m}}(d),$$

where

$$S_{\overrightarrow{m}}(d) = \sum_{k=1}^{d-1} \prod_{l=1}^{n} \cot^{(m_l-1)}\left(\frac{k\pi}{d}\right) \qquad (d > 1).$$

Now, for $d \geq 1$ a rational integer, we set

$$g_{\overrightarrow{m}}(d, z) = d \cot(dz) \prod_{l=1}^{n} \cot^{(m_l-1)}(z) = \frac{(dz \cot(dz)) \prod_{l=1}^{n} z^{m_l} \cot^{(m_l-1)}(z)}{z^{2s+1}},$$

a π-periodic meromorphic function of the complex variable z, with a simple pole at each $k\pi/d$, $k \in \mathbb{Z} \setminus d\mathbb{Z}$, of residue

$$\mathrm{Res}_{z=k\pi/d}(g_{\overrightarrow{m}}(d, z) = \prod_{l=1}^{n} \cot^{(m_l-1)}(k\pi/d) \qquad (k \in \mathbb{Z} \setminus d\mathbb{Z}) \tag{8}$$

and poles of order $2s + 1$ at each $k\pi$, $k \in \mathbb{Z}$. Let $T_{\overrightarrow{m}}(X)$ denote the coefficient of $z^{m_1+\cdots+m_n}$ in the formal power series expansion

$$\left(\sum_{k\geq 0} (-1)^k \frac{2^{2k} B_{2k}}{(2k)!} X^{2k} z^{2k}\right) \prod_{l=1}^{n} \left((-1)^{m_l-1}(m_l - 1)! + \sum_{k \geq m_l/2} (-1)^k \frac{2^{2k} B_{2k}}{2k \cdot (2k - m_l)!} z^{2k}\right).$$

Hence, $T_{\overrightarrow{m}}(X)$ is a polynomial of degree $2s = m_1 + \cdots + m_n$, with rational coefficients and leading coefficient

$$(-1)^{s-n} \frac{2^{2s} B_{2s}}{(2s)!} \prod_{l=1}^{n} (m_l - 1)!$$

and

$$\mathrm{Res}_{z=0}(g_{\overrightarrow{m}}(d, z)) = T_{\overrightarrow{m}}(d) \text{ for } d \geq 1. \tag{9}$$

With the same proof, we have the following generalization of (4):

$$\lim_{R\to+\infty} \frac{1}{2\pi i} \int_{C_R} g_{\overrightarrow{m}}(d, z) \mathrm{d}z = \begin{cases} (-1)^{n/2} d & (\text{if } m_1 = \cdots = m_n = 1), \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

(n must be even if $m_1 = \cdots = m_n = 1$).
We have the following generalization of Lemma 4:

Lemma 6. *Set*

$$R_{\overrightarrow{m}}(X) = -T_{\overrightarrow{m}}(X) + \begin{cases} (-1)^{n/2} X & \textit{(if } m_1 = \cdots = m_n = 1\textit{)}, \\ 0 & \textit{otherwise.} \end{cases}$$

Then $R_{\overrightarrow{m}}(1) = 0$ and $S_{\overrightarrow{m}}(d) = R_{\overrightarrow{m}}(d)$ for $d \geq 2$.

Proof. By (9) and (10), for $d \geq 1$ we have

$$R_{\overrightarrow{m}}(d) := \frac{1}{2\pi i} \int_{C_R} g_{\overrightarrow{m}}(d, z) \mathrm{d}z - \mathrm{Res}_{z=0}(g_{\overrightarrow{m}}(d, z)).$$

We use the residue theorem. For $d = 1$, noticing that $z = 0$ is the only pole of $g_{\overrightarrow{m}}(1, z)$ inside C_R, here again we get

$$R_{\overrightarrow{m}}(1) = 0. \tag{11}$$

For $d \geq 2$, noticing that $z = 0$ and $z = k\pi/d$, $1 \leq k \leq d - 1$, are the only poles of $g_{\overrightarrow{m}}(d, z)$ inside C_R and using (8), we obtain $R_{\overrightarrow{m}}d) = S_{\overrightarrow{m}}d)$. □

By Lemma 6, Lemma 3 and (2), and noticing that $T_{\overrightarrow{m}}(X)$ is even we end up with the following generalization of Theorem 1:

Theorem 7. *Write*

$$T_{\overrightarrow{m}}(X) = (-1)^{s-n}\frac{2^{2s}B_{2s}}{(2s)!}\left(\prod_{l=1}^{n}(m_l-1)!\right)\left(X^{2s}+\sum_{k=0}^{s-1}r_{\overrightarrow{m},2k}X^{2k}\right),$$

Then, for $f>2$ we have

$$M(\overrightarrow{m},f) = \zeta(2s)\times\left(\phi_{2s}(f)+\sum_{k=1}^{s-1}r_{\overrightarrow{m},2k}\frac{\phi_{2k}(f)}{f^{2s-2k}}\right) + \begin{cases}(-1)^{n/2}\frac{\pi^n\phi_1(f)}{2f^{n-1}} & \text{(if } m_1=\cdots=m_n=1\text{)},\\ 0 & \text{otherwise.}\end{cases}$$

In particular, as f goes to infinity $M(\overrightarrow{m},f)$ is asymptotic to

$$\zeta(2s)\phi_{2s}(f),\ \text{where } 2s=m_1+\cdots+m_n.$$

4 Several explicit examples

4.1 First example.

By Lemma 6 we have $S_{\overrightarrow{(1,1,2)}}(d) = R_{\overrightarrow{(1,1,2)}}(d) = -T_{\overrightarrow{(1,1,2)}}(d)$. By (9), we obtain

$$S_{\overrightarrow{(1,1,2)}}(d) = -T_{\overrightarrow{(1,1,2)}}(d) = -\mathrm{Res}_{z=0}(d\cot(dz)\cot^2(z)\cot'(z)) = -\frac{1}{45}d^4+\frac{1}{9}d^2-\frac{4}{45}.$$

Therefore

$$M(\overrightarrow{(1,1,2)},f) = -\frac{\pi^4}{2f^4}\sum_{d|f}\mu(f/d)S_{\overrightarrow{(1,1,2)}}(d) = \frac{\pi^4}{90}\left(\phi_4(f)-5\frac{\phi_2(f)}{f^2}\right),$$

by Lemma 5 and Lemma 3. We have recovered [Alk, Theorem 1].

4.2 Second example.

Theorem 8. *Let $n\geq 2$ be even. For $f>2$ set*

$$M_n(f) = \left(\frac{2}{\phi(f)}\right)^{n-1}\sum_{\chi_1(-1)=-1}\cdots\sum_{\chi_{n-1}(-1)=-1}L(1,\chi_1)\cdots L(1,\chi_{n-1})\overline{L(1,\chi_1\cdots\chi_{n-1})}.$$

Then

$$M_n(f) = \zeta(n)\times\left(\phi_n(f)+\frac{n!}{B_n}\sum_{k_0=1}^{n/2-1}C_{n,n/2-k_0}\frac{B_{2k_0}}{(2k_0)!}\frac{\phi_{2k_0}(f)}{f^{n-2k_0}}\right)+(-1)^{n/2}\frac{\pi^n\phi_1(f)}{2f^{n-1}},$$

where

$$C_{n,N} = \sum_{\substack{e_1,\cdots,e_N\geq 0\\ e_1+2e_2\cdots+Ne_N=N}}\frac{n!}{(n-(e_1+\cdots+e_N))!}\prod_{l=1}^{N}\frac{1}{e_l!}\left(\frac{B_{2l}}{(2l)!}\right)^{e_l}\qquad (n\geq N\geq 1).$$

Proof. Here

$$T_{\overrightarrow{m}}(d)=(-1)^{n/2}\frac{2^nB_n}{n!}\sum_{\substack{k_0,k_1,\cdots,k_n\geq 0\\ k_0+k_1+\cdots+k_n=n/2}}\frac{n!}{B_n}\left(\prod_{l=1}^{n}\frac{B_{2k_l}}{(2k_l)!}\right)\frac{B_{2k_0}}{(2k_0)!}d^{2k_0}.$$

Now, for $n\geq N\geq 1$ and $F:\mathbb{Z}_{\geq 0}\to\mathbb{C}^*$ such that $F(0)=1$, e.g. for $F(k)=B_{2k}/(2k)!$, we have

$$\sum_{\substack{k_1,\cdots,k_n\geq 0\\ k_1+\cdots+k_n=N}}\prod_{l=1}^{n}F(k_l)=\sum_{\substack{e_1,\cdots,e_N\geq 0\\ e_1+2e_2\cdots+Ne_N=N}}\frac{n!}{(n-(e_1+\cdots+e_N))!}\prod_{l=1}^{N}\frac{F(l)^{e_l}}{e_l!},$$

by letting e_l denote the number of k_i's which are equal to l and by noticing that we have $\binom{n}{e_0}$ possible choices of the k_i's which are equal to 0, then $\binom{n-e_0}{e_1}$ possible choices of the k_i's which are equal to 1,..., and finally by using $e_0+e_1+\cdots e_N=n$ and

$$\binom{n}{e_0}\binom{n-e_0}{e_1}\cdots\binom{n-(e_0+e_1+\cdots+e_{N-1})}{e_N}=\frac{n!}{e_0!\times e_1!\times\cdots\times e_N!}.$$

□

For example,

$$M_2(f)=\frac{\pi^2}{6}\left(\phi_2(f)-3\frac{\phi_1(f)}{f}\right),$$

in accordance with (1),

$$M_4(f)=\frac{\pi^4}{90}\left(\phi_4(f)-20\frac{\phi_2(f)}{f^2}+45\frac{\phi_1(f)}{f^3}\right)$$

and

$$M_6(f)=\frac{\pi^4}{945}\left(\phi_6(f)-21\frac{\phi_4(f)}{f^2}+\frac{483}{2}\frac{\phi_2(f)}{f^4}-\frac{945}{2}\frac{\phi_1(f)}{f^5}\right)$$

For $f=3$, there is only one odd character χ modulo 3, and $L(1,\chi)=\frac{\pi}{3\sqrt{3}}$. Hence, $M_n(3)=\frac{\pi^n}{3^{3n/2}}$. Our formulae do give $M_2(3)=\frac{\pi^2}{3^3}$, $M_4(3)=\frac{\pi^4}{3^6}$ and $M_6(3)=\frac{\pi^6}{3^9}$.

4.3 Third example.

We have the following generalization of Theorem 1:

Theorem 9. *For $m\geq 1$, $n\geq 1$ of the same parity and $(m,n)\neq(1,1)$, we have*

$$\begin{aligned}M(\overrightarrow{(m,n)},f)=&\frac{2}{\phi(f)}\sum_{\chi(-1)=(-1)^m=(-1)^n}L(m,\chi)\overline{L(n,\chi)}\\ =&\zeta(m+n)\times\Bigg(\phi_{m+n}(f)\\ &+(-1)^{m-1}\binom{m+n}{m}\sum_{k=1}^{\lfloor m/2\rfloor}\frac{n}{m+n-2k}\binom{m}{2k}\frac{B_{2k}B_{m+n-2k}}{B_{m+n}}\frac{\phi_{2k}(f)}{f^{m+n-2k}}\\ &+(-1)^{n-1}\binom{m+n}{n}\sum_{k=1}^{\lfloor n/2\rfloor}\frac{m}{m+n-2k}\binom{n}{2k}\frac{B_{2k}B_{m+n-2k}}{B_{m+n}}\frac{\phi_{2k}(f)}{f^{m+n-2k}}\Bigg).\end{aligned}$$

In particular, $M(\overrightarrow{(m,n)},f)$ is asymptotic to $\zeta(m+n)\phi_{m+n}(f)$ as f goes to infinity.

Proof. Use Lemma 6, notice that

$$T_{\overrightarrow{(m,n)}}(X) = 2^{m+n}\times\Bigg((-1)^{\frac{m+n}{2}}\frac{B_{m+n}}{mn\binom{m+n}{n}}X^{m+n} - (-1)^{\frac{m-n}{2}}\sum_{k=1}^{\lfloor m/2\rfloor}\frac{B_{2k}B_{m+n-2k}}{m(m+n-2k)}\binom{m}{2k}X^{2k} - (-1)^{\frac{n-m}{2}}\sum_{k=1}^{\lfloor n/2\rfloor}\frac{B_{2k}B_{m+n-2k}}{n(m+n-2k)}\binom{n}{2k}X^{2k} + (-1)^{\frac{m+n}{2}}\frac{1+(-1)^m}{2}\frac{1+(-1)^m}{2}\frac{B_mB_n}{mn}\Bigg).$$

and finally use Lemma 5. □

Our explicit formula for $M(\overrightarrow{(m,n)},f)$ is much simpler than the ones given in [LZ, Theorem 1.1] and [OO, Theorem 1.2]. For example, the first term in the formula given [LZ, Theorem 1.1] is

$$(-1)^{(m-n)/2}\frac{(2\pi)^{m+n}}{2\cdot m!\cdot n!}\phi_{m+n}(f)\sum_{a=0}^{m}\sum_{b=0}^{n}B_{m-a}B_{n-b}\frac{\binom{m}{a}\binom{n}{b}}{a+b+1}.$$

By (2), this first term being equal to the one $\zeta(m+n)\phi_{m+n}(f)$ of Theorem 9 is tantamount to saying that $B_{m+n}=(-1)^n\binom{m+n}{n}\sum_{a=0}^{m}\sum_{b=0}^{n}B_{m-a}B_{n-b}$. This is probably correct and at least we checked it for several values of m and n.

Notice that for $m\geq 2$ we have $M(\overrightarrow{(m,m)},f)=M(m,f)$ and we recover the formula given in Theorem 1.

4.4 Fourth example.

For $m,n\geq 1$, Theorem 7 gives a formula for

$$M(\overrightarrow{(m,n,m+n)},f)=\frac{4}{\phi(f)^2}\sum_{\chi_1(-1)=(-1)^m}\sum_{\chi_2(-1)=(-1)^n}L(m,\chi_1)L(n,\chi_2)\overline{L(m+n,\chi_1\chi_2)}$$

much simpler than the one given in [OO, Theorem 1.1].

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