Asymptotics of Cholesky GARCH Models and Time-Varying Conditional Betas

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Motivation

Problem: Given some information set $\mathcal{F}_{t-1}$, it is often of interest in financial applications to regress y_t (*asset returns*) on the components of a vector $\boldsymbol{x}_t$ (*factors*)

Solution: $y_t - E(y_t \mid \mathcal{F}_{t-1}) = \boldsymbol{\beta}'_{y\boldsymbol{x},t}\{\boldsymbol{x}_t - E(\boldsymbol{x}_t \mid \mathcal{F}_{t-1})\} + \eta_t,$

with the dynamic conditional beta (DCB) is given by:

$$\boldsymbol{\beta}_{y\boldsymbol{x},t} = \boldsymbol{\Sigma}_{\boldsymbol{xx},t}^{-1}\boldsymbol{\Sigma}_{\boldsymbol{x}y,t}$$

Practical implementations:

1. an ARCH-type model for the conditional variance:

$$\begin{pmatrix} \boldsymbol{\Sigma}_{\boldsymbol{x}\boldsymbol{x},t} & \boldsymbol{\Sigma}_{\boldsymbol{x}y,t} \\ \boldsymbol{\Sigma}_{y\boldsymbol{x},t} & \boldsymbol{\Sigma}_{yy,t} \end{pmatrix}$$

of

$$\epsilon_t = \begin{pmatrix} \boldsymbol{x}_t - E\left(\boldsymbol{x}_t \mid \mathcal{F}_{t-1}\right) \\ y_t - E\left(y_t \mid \mathcal{F}_{t-1}\right) \end{pmatrix}$$

2. or a direct specification of $\beta_{y\boldsymbol{x},t}$?

Notations

Let $\boldsymbol{\epsilon}_t = (\epsilon_{1t}, \ldots, \epsilon_{mt})'$ be a vector of $\boldsymbol{m} \geq 2$ series satisfying a general volatility model of the form

$$\boldsymbol{\epsilon}_t = \boldsymbol{\Sigma}_t^{1/2}(\boldsymbol{\vartheta}_0)\boldsymbol{\eta}_t$$

where $(\boldsymbol{\eta}_t)$ is $i.i.d.(0, I_m)$ and

$$\boldsymbol{\Sigma}_t = \boldsymbol{\Sigma}_t(\boldsymbol{\vartheta}_0) = \boldsymbol{\Sigma}(\boldsymbol{\epsilon}_{t-1}, \boldsymbol{\epsilon}_{t-2}, \ldots; \boldsymbol{\vartheta}_0) > 0$$

where ϑ_0 is a $d \times 1$ vector

Engle (2012) DCC

$$\boldsymbol{\Sigma}_t = \boldsymbol{D}_t \boldsymbol{R}_t \boldsymbol{D}_t = \left(\rho_{ijt}\sqrt{\sigma_{iit}\sigma_{jjt}}\right)$$

where $\boldsymbol{D}_t = diag(\sigma_{11t}^{1/2}, \ldots, \sigma_{mmt}^{1/2})$ contains the volatilities of the individual returns and $\boldsymbol{R}_t = (\rho_{ijt})$ the conditional correlations

The time series model needs to incorporate the complicated constraints of a correlation matrix. One often takes:

$\boldsymbol{R}_t = (diag\ \boldsymbol{Q}_t)^{-1/2}\boldsymbol{Q}_t(diag\ \boldsymbol{Q}_t)^{-1/2}$ where

$$\boldsymbol{Q}_t = (1 - \theta_1 - \theta_2)\boldsymbol{S} + \theta_1 \boldsymbol{u}_{t-1}\boldsymbol{u}'_{t-1} + \theta_2 \boldsymbol{Q}_{t-1}$$

with $\boldsymbol{u}_t = (u_{1t} \ldots u_{mt})'$, $u_{it} = \epsilon_{it}/\sqrt{\sigma_{iit}}$, $\theta_1 + \theta_2 < 1$

Engle (2016) DCB

Assuming

$$\begin{pmatrix} \boldsymbol{x}_t \\ y_t \end{pmatrix} |\mathcal{F}_{t-1} \sim N\left(\begin{pmatrix} \mu_{x,t} \\ \mu_{y,t} \end{pmatrix}, \begin{pmatrix} \Sigma_{xx,t} & \Sigma_{xy,t} \\ \Sigma_{xy,t} & \Sigma_{yy,t} \end{pmatrix}\right)$$

we have

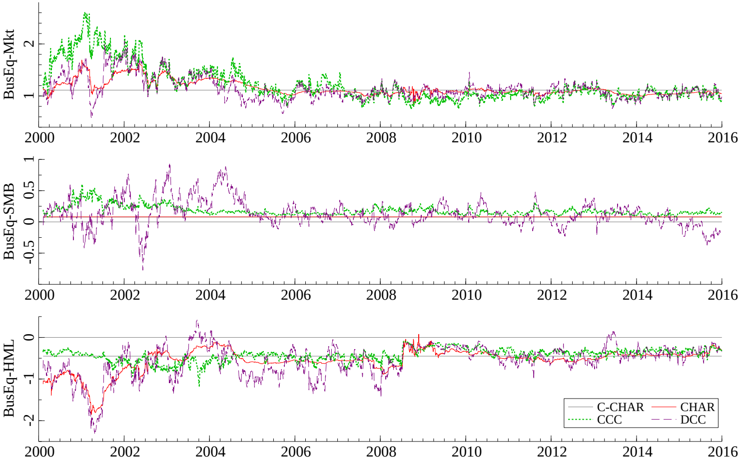
$$y_t|x_t \sim N\left(\mu_{y,t} + \boldsymbol{\Sigma}_{yx,t}\boldsymbol{\Sigma}_{xx,t}^{-1}(x_t - \mu_{x,t}), \Sigma_{yy,t} - \Sigma_{yx,t}\Sigma_{xx,t}^{-1}\Sigma_{xy,t}\right)$$

$\Rightarrow \beta_t = \boldsymbol{\Sigma}_{xx,t}^{-1}\boldsymbol{\Sigma}_{xy,t}$ is obtained in tow steps - by first estimating a DCC GARCH model on (y_t, x_t) - **NOT** a natural way to specify the parameter dynamics !

Drawbacks of DCC-based DCB - 1/2

1) The stationarity and ergodicity conditions of the DCC are not well known.
2) The correlation constraints are complicated.
3) The asymptotic properties of the QMLE are unknown.
4) The effects of the DCC parameters on β_t are hardly interpretable.

Drawbacks of DCC-based DCBI - 2/2



Cholesky Decomposition

We now introduce a class of Cholesky GARCH (CHAR) models that avoids all these drawbacks.

We consider the Cholesky decomposition of Pourahmadi (1999)

$$\boldsymbol{\Sigma}_t = \boldsymbol{L}_t \boldsymbol{G}_t \boldsymbol{L}_t'$$

where $\boldsymbol{G}_t = \text{diag}(g_{1t}, \ldots, g_{mt})$ and $\boldsymbol{L}_t$ is a lower unitriangular matrix (*i.e.* triangular with 1 on the diagonal) with element $\ell_{ij,t}$ at the row i and column j for $i > j$

Example (static case): $\boldsymbol{\Sigma} = \boldsymbol{LGL}'$, $m = 3$

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \qquad \boldsymbol{G} = \begin{bmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} g_{11} & l_{21} g_{11} & l_{31} g_{11} \\ l_{21} g_{11} & l_{21}^2 g_{11} + g_{22} & l_{21} l_{31} g_{11} + l_{32} g_{22} \\ l_{31} g_{11} & l_{21} h_{31} g_{11} + l_{32} g_{22} & l_{31}^2 g_{11} + l_{32}^2 g_{22} + g_{33} \end{bmatrix}$$

Structural Interpretation (static case)

Let us introduce the orthogonal basis assets $\boldsymbol{v} = (v_1, \ldots, v_m)$

with variance $\boldsymbol{G}$ and such that

$$\epsilon = \boldsymbol{\Sigma}^{1/2}\eta_t = \boldsymbol{L}\boldsymbol{G}^{1/2}\eta = \boldsymbol{L}\boldsymbol{v} \qquad (E)$$

Line 1 - $V(\epsilon_1) = \sigma_1^2 = g_{11}$

Line 2 - $Cov(\epsilon_1, \epsilon_2) = \sigma_{21} = \ell_{21} g_{11} \Leftrightarrow \ell_{21} = \frac{\sigma_{21}}{\sigma_1^2} = \beta_2$

We can start to invert (E) : $\epsilon_2 = \beta_{21}\epsilon_1 + v_2 \Leftrightarrow v_2 = \epsilon_2 - \beta_{21}\epsilon_1$

and define:

$$\boldsymbol{B} = \boldsymbol{L}^{-1} = \begin{bmatrix} 1 & 0 & 0 & .... \\ -\beta_{21} & 1 & 0 & ... \\ ... & ... & ... & ... \end{bmatrix}$$

Structural Interpretation (static case)

More generally, we have for *Line i*

$$\begin{aligned} \epsilon_i &= \sum_{j=1}^{i-1} \ell_{ij} v_j + v_i \\ &= \sum_{j=1}^{i-1} \beta_{ij}\epsilon_j + v_i \Leftrightarrow v_i = \epsilon_i - \sum_{j=1}^{i-1} \beta_{ij}\epsilon_j \end{aligned}$$

where v_i is uncorrelated to $v_1, \ldots, v_{i-1}$, and thus uncorrelated to $\epsilon_1, \ldots, \epsilon_{i-1}$

Structural Interpretation (static case)

In matrix form,

$$\epsilon = \boldsymbol{L}\boldsymbol{v} \qquad \text{and} \qquad \boldsymbol{v} = \boldsymbol{B}\epsilon,$$

where $\boldsymbol{L}$ and $\boldsymbol{B} = \boldsymbol{L}^{-1}$ are lower unitriangular and $\boldsymbol{G} := \text{var}(\boldsymbol{v})$ is diagonal

We obtain the (static) Cholesky decomposition $\boldsymbol{\Sigma} = \boldsymbol{L}\boldsymbol{G}\boldsymbol{L}'$ (see Pourahmadi, 1999)

Structural Interpretation (dynamic setting)

Conditioning on $\mathcal{F}_{t-1}$, we get the (dynamic) Choleski decomposition $\boldsymbol{\Sigma}_t = \boldsymbol{L}_t \boldsymbol{G}_t \boldsymbol{L}_t'$ (and $\boldsymbol{\Sigma}_t^{-1} = \boldsymbol{B}_t' \boldsymbol{G}_t^{-1} \boldsymbol{B}_t$)

We thus need ...

- a diagonal ARCH-type model for the factors vector $\boldsymbol{v}_t$

- a time series model for $\boldsymbol{L}_t$ (or $\boldsymbol{B}_t$), without constraint

... Instead of (with the DCC model)

- a diagonal ARCH-type model for the ϵ_t

- a time series model for $\boldsymbol{R}_t$, with constraints

A model for $\boldsymbol{L}_t$ or $\boldsymbol{B}_t$? How ordering the series ?

1. $\boldsymbol{B}_t$ has a direct interpretation

$$\boldsymbol{B}_t = \boldsymbol{L}_t^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & .... \\ -\beta_{21,t} & 1 & 0 & 0 & ... \\ -\beta_{31,t} & -\beta_{32,t} & 1 & 0 & ... \\ -\beta_{41,t} & -\beta_{42,t} & -\beta_{42,t} & 1 & ... \\ ... & ... & ... & ... & ... \end{bmatrix}$$

2. No natural order .... must be driven by the applications

A general model for the factors

Assume

$$\boldsymbol{v}_t = \boldsymbol{G}_t^{1/2}\eta_t, \qquad (\eta_t) \text{ iid } (0, I_n),$$

where $\boldsymbol{G}_t = \mathrm{diag}(\boldsymbol{g}_t)$ follows a GJR-like equation

$$\boldsymbol{g}_t = \omega_0 + \sum_{i=1}^{q} \left\{ \boldsymbol{A}_{0i,+}\boldsymbol{v}_{t-i}^{2+} + \boldsymbol{A}_{0i,-}\boldsymbol{v}_{t-i}^{2-} \right\} + \sum_{j=1}^{p} \boldsymbol{B}_{0j}\boldsymbol{g}_{t-j},$$

with positive coefficients and

$$\boldsymbol{v}_t^{2+} = \left(\left\{ v_{1t}^{+} \right\}^2, \cdots, \left\{ v_{mt}^{+} \right\}^2 \right)', \quad \boldsymbol{v}_t^{2-} = \left(\left\{ v_{1t}^{-} \right\}^2, \cdots, \left\{ v_{mt}^{-} \right\}^2 \right)'.$$

Markovian representation of the factors

Letting $\boldsymbol{z}_t = \left(\boldsymbol{v}^{2+'}_{t:(t-q+1)}, \boldsymbol{v}^{2-'}_{t:(t-q+1)}, \boldsymbol{g}'_{t:(t-p+1)}\right)'$,
$\boldsymbol{h}_t = \left(\omega_0'\boldsymbol{\Upsilon}_t^{+'}, 0'_{m(q-1)}, \omega_0'\boldsymbol{\Upsilon}_t^{-'}, 0'_{(q-1)m}, \omega_0', 0'_{(p-1)m}\right)'$, with
$\boldsymbol{\Upsilon}_t^{+} = \text{diag}\left(\eta_t^{2+}\right)$ $\boldsymbol{\Upsilon}_t^{-} = \text{diag}\left(\eta_t^{2-}\right)$ and obvious notations, we rewrite the model as

$$\boldsymbol{z}_t = \boldsymbol{h}_t + \boldsymbol{H}_t \boldsymbol{z}_{t-1},$$

where, in the case $p = q = 1$,

$$\boldsymbol{H}_t = \begin{pmatrix} \boldsymbol{\Upsilon}_t^{+}\boldsymbol{A}_{01,+} & \boldsymbol{\Upsilon}_t^{+}\boldsymbol{A}_{01,-} & \boldsymbol{\Upsilon}_t^{+}\boldsymbol{B}_{01} \\ \boldsymbol{\Upsilon}_t^{-}\boldsymbol{A}_{01,+} & \boldsymbol{\Upsilon}_t^{-}\boldsymbol{A}_{01,-} & \boldsymbol{\Upsilon}_t^{-}\boldsymbol{B}_{01} \\ \boldsymbol{A}_{01,+} & \boldsymbol{A}_{01,-} & \boldsymbol{B}_{01} \end{pmatrix}.$$

Stationarity of the factors

In view of

$$\boldsymbol{z}_t = \boldsymbol{h}_t + \boldsymbol{H}_t \boldsymbol{z}_{t-1},$$

there exists a stationary and ergodic sequence $(\boldsymbol{v}_t)$ satisfying $\boldsymbol{v}_t = \boldsymbol{G}_t^{1/2} \boldsymbol{\eta}_t$ if and only if

$$\gamma_0 = \inf_{t \geq 1} \frac{1}{t} E(\log \|\boldsymbol{H}_t \boldsymbol{H}_{t-1} \ldots \boldsymbol{H}_1\|) < 0.$$

Stationarity of $\beta_t := -\mathrm{vech}^0 \boldsymbol{B}_t$

If $(\boldsymbol{v}_t)$ is stationary and ergodic ($\gamma_0 < 0$), and

$$\det\left\{\boldsymbol{I}_{m_0} - \sum_{i=1}^{s} \boldsymbol{C}_{0i} z^i\right\} \neq 0 \text{ for all } |z| \leq 1,$$

then

$$\beta_t = \boldsymbol{c}_0\left(\boldsymbol{v}_{t-1}, \ldots, \boldsymbol{v}_{t-r}, \boldsymbol{g}_{t-1}^{1/2}, \ldots, \boldsymbol{g}_{t-r}^{1/2}\right) + \sum_{j=1}^{s} \boldsymbol{C}_{0j} \beta_{t-j}.$$

defines a stationary and ergodic sequence (and thus the existence of a stationary CHAR model)

Existence of moments

If in addition

$$E\|\boldsymbol{\eta}_1\|^{2k_1} < \infty \quad \text{and} \quad \varrho(E\boldsymbol{H}_1^{\otimes k_1}) < 1,$$

for some integer $k_1 > 0$, and

$$\|\boldsymbol{c}_0(\boldsymbol{x}) - \boldsymbol{c}_0(\boldsymbol{y})\| \leq K\,\|\boldsymbol{x} - \boldsymbol{y}\|^a$$

for some constants $K > 0$ and $a \in (0, 1]$, then the CHAR model satisfies $E\,\|\epsilon_1\|^{2k_1} < \infty$.

A simpler triangular parameterization

A tractable submodel is

$$g_{it} = \omega_{0i} + \gamma_{0i+} \left(\epsilon^{+}_{1,t-1}\right)^2 + \gamma_{0i-} \left(\epsilon^{-}_{1,t-1}\right)^2 + \sum_{k=2}^{i} \alpha_{0i}^{(k)} v^2_{k,t-1} + b_{0i} g_{i,t-1}$$

with positivity coefficients, and

$$\beta_{ij,t} = \varpi_{0ij} + \varsigma_{0ij+} \epsilon^{+}_{1,t-1} + \varsigma_{0ij-} \epsilon^{-}_{1,t-1} + \sum_{k=2}^{i} \tau_{0ij}^{(k)} v_{k,t-1} + c_{0ij} \beta_{ij,t-1}$$

without positivity constraints. Notice the triangular structure and note that the asymmetry is introduced via the first (observed) factor only

Stationarity for the previous specification

There exists a strictly stationary and ergodic solution to the CHAR model when

1) $E \log \left\{ \omega_{01} + \gamma_{01+} \left(\eta^{+}_{1,t-1} \right)^2 + \gamma_{01-} \left(\eta^{-}_{1,t-1} \right)^2 + b_{01} \right\} < 0,$
2) $E \log \left\{ \alpha^{(i)}_{0i} \eta^2_{it} + b_{0i} \right\} < 0$ for $i = 2, \ldots, m,$
3) $|c_{0ij}| < 1$ for all (i, j).

Moreover, the stationary solution satisfies $E\|\epsilon_1\|^{2s_0} < \infty$, $E\|\boldsymbol{g}_1\|^{s_0} < \infty$, $E\|\boldsymbol{v}_1\|^{s_0} < \infty$, $E\|\boldsymbol{\beta}_1\|^{s_0} < \infty$ and $E\|\boldsymbol{\Sigma}_1\|^{s_0} < \infty$ for some $s_0 > 0$

Full QMLE of the general CHAR

A QMLE of the CHAR parameter ϑ_0 is

$$\widehat{\vartheta}_n = \arg\min_{\vartheta \in \Theta} \widetilde{O}_n(\vartheta), \qquad \widetilde{O}_n(\vartheta) = n^{-1} \sum_{t=1}^{n} \widetilde{q}_t(\vartheta),$$

where $\widetilde{\boldsymbol{\Sigma}}_t(\vartheta) = \boldsymbol{\Sigma}(\epsilon_{t-1}, \dots, \epsilon_1, \widetilde{\epsilon}_0, \widetilde{\epsilon}_{-1}, \dots; \vartheta)$ and

$$\widetilde{q}_t(\vartheta) \quad = \quad \epsilon_t' \widetilde{\boldsymbol{B}}_t'(\vartheta) \widetilde{\boldsymbol{G}}_t^{-1}(\vartheta) \widetilde{\boldsymbol{B}}_t(\vartheta) \epsilon_t + \sum_{i=1}^{m} \log \widetilde{g}_{it}(\vartheta).$$

- Does not require matrix inversion
- CAN under general regularity conditions

Equation-by-Equation (EbE) estimator

Consider the triangular model. In a first step, the parameter $\boldsymbol{\vartheta}_0^{(1)} = (\omega_{01}, \gamma_{01+}, \gamma_{01-}, b_{01})$ is estimated by

$$\widehat{\boldsymbol{\vartheta}}_n^{(1)} = \underset{\boldsymbol{\vartheta}^{(1)} \in \Theta^{(1)}}{\arg\min} \sum_{t=1}^{n} \widetilde{q}_{1t}(\boldsymbol{\vartheta}^{(1)}),$$

where

$$\widetilde{q}_{1t}(\boldsymbol{\vartheta}^{(1)}) \;\; = \;\; \frac{\epsilon_{1t}^2}{\widetilde{g}_{1t}(\boldsymbol{\vartheta}^{(1)})} + \log \widetilde{g}_{1t}(\boldsymbol{\vartheta}^{(1)}),$$

and $\widetilde{g}_{1t}(\boldsymbol{\vartheta}^{(1)}) = \omega_1 + \gamma_{1+} \left(\epsilon_{1t}^{+}\right)^2 + \gamma_{1-} \left(\epsilon_{1t}^{-}\right)^2 + b_1 \widetilde{g}_{1,t-1}(\boldsymbol{\vartheta}^{(1)})$.

EbE second step

Let $\boldsymbol{\vartheta}_0^{(2)} = (\boldsymbol{\varphi}_0^{(2)}, \boldsymbol{\theta}_0^{(2)})$, where $\widetilde{\beta}_{21,t} = \widetilde{\beta}_{21,t}(\boldsymbol{\varphi}_0^{(2)})$ and $\widetilde{g}_{2t} = \widetilde{g}_{2t}(\boldsymbol{\theta}_0^{(2)})$.

Independently or in parallel to $\boldsymbol{\vartheta}_0^{(1)}$, one can estimate $\boldsymbol{\vartheta}_0^{(2)}$ by

$$\widehat{\boldsymbol{\vartheta}}_n^{(2)} = \arg\min_{\boldsymbol{\vartheta}^{(2)} \in \Theta^{(2)}} \sum_{t=1}^{n} \widetilde{q}_{2t}(\boldsymbol{\vartheta}^{(2)}),$$

where, for $t = 1, \dots, n$,

$$\begin{aligned}
\widetilde{q}_{2t}(\boldsymbol{\vartheta}^{(2)}) &= \frac{\widetilde{v}_{2t}^2(\boldsymbol{\varphi}^{(2)})}{\widetilde{g}_{2t}(\boldsymbol{\vartheta}^{(2)})} + \log \widetilde{g}_{2t}(\boldsymbol{\vartheta}^{(2)}),\\
\widetilde{g}_{2t}(\boldsymbol{\vartheta}^{(2)}) &= \omega_{2,t-1} + \alpha_2^{(2)} \widetilde{v}_{2,t-1}^2(\boldsymbol{\varphi}^{(2)}) + b_2 \widetilde{g}_{2,t-1}(\boldsymbol{\varphi}^{(2)}),\\
\widetilde{v}_{2t}(\boldsymbol{\varphi}^{(2)}) &= \epsilon_{2t} - \widetilde{\beta}_{21,t}(\boldsymbol{\varphi}^{(2)})\epsilon_{1t},\\
\widetilde{\beta}_{21,t}(\boldsymbol{\varphi}^{(2)}) &= \omega_{21,t-1} + \tau_{21}^{(2)} \widetilde{v}_{2,t-1}(\boldsymbol{\varphi}^{(2)}) + c_{21} \widetilde{\beta}_{21,t-1}(\boldsymbol{\varphi}^{(2)}).
\end{aligned}$$

EbE remaining steps

For $i \geq 3$, $\widetilde{\beta}_{ij,t}$ depends on $\varphi_0^{(+i)} = \left(\varphi_0^{(i)}, \varphi_0^{(-i)}\right)$, where $\varphi_0^{(-i)}$ has been estimated in the previous steps. The volatility $\widetilde{g}_{it}$ depends on $\vartheta_0^{(+i)} = (\theta_0^{(i)}, \varphi_0^{(+i)})$, and $\vartheta_0^{(i)} = (\theta_0^{(i)}, \varphi_0^{(i)})$ can be estimated by

$$\widehat{\vartheta}_n^{(i)} = \arg\min_{\vartheta^{(i)} \in \Theta^{(i)}} \sum_{t=1}^{n} \widetilde{q}_{it}(\vartheta^{(i)}, \widehat{\varphi}_n^{(-i)}), \quad \widetilde{q}_{it}(\vartheta^{(+i)}) = \frac{\widetilde{v}_{it}^2(\varphi^{(+i)})}{\widetilde{g}_{it}(\vartheta^{(+i)})} + \log \widetilde{g}_{it}(\vartheta^{(+i)}),$$

$$\widetilde{g}_{it}(\vartheta^{(+i)}) = \omega_{i,t-1} + \sum_{k=2}^{i} \alpha_i^{(k)} \widetilde{v}_{k,t-1}^2(\varphi^{(+k)}) + b_i \widetilde{g}_{i,t-1}(\vartheta^{(+i)}),$$

$$\widetilde{v}_{kt}(\varphi^{(+k)}) = \epsilon_{kt} - \sum_{j=1}^{k-1} \widetilde{\beta}_{kj,t}(\varphi^{(+k)}) \epsilon_{jt},$$

$$\widetilde{\beta}_{ij,t}(\varphi^{(+i)}) = \omega_{ij,t-1} + \sum_{k=2}^{i} \tau_{ij}^{(k)} \widetilde{v}_{k,t-1}(\varphi^{(+k)}) + c_{ij} \widetilde{\beta}_{ij,t-1}(\varphi^{(+i)}),$$

QML vs. EbE

1) If $m = 2$, the one-step full QMLE and the two-step EbEE are exactly the same
2) For $m \geq 3$, the two estimators are generally different
3) The QML and EbE estimators are CAN under similar assumptions
4) The EbEE is simpler, but is not always less efficient than the full QMLE

Applications - Asset Pricing for Industry Portfolios

We consider the 12 industry portfolios returns $r_{j,t}$ used by Engle (2016) examined in the context of the Fama French 3 factor model

$$r_{j,t} = \alpha_j + \beta_{j,m,t} r_{m,t} + \beta_{j,hml,t} r_{hml,t} + \beta_{j,smb,t} r_{smb,t} + u_{j,t}$$

where $r_{smb,t}$ (Small Minus Big) is the average return on small portfolios minus the average return on big portfolios

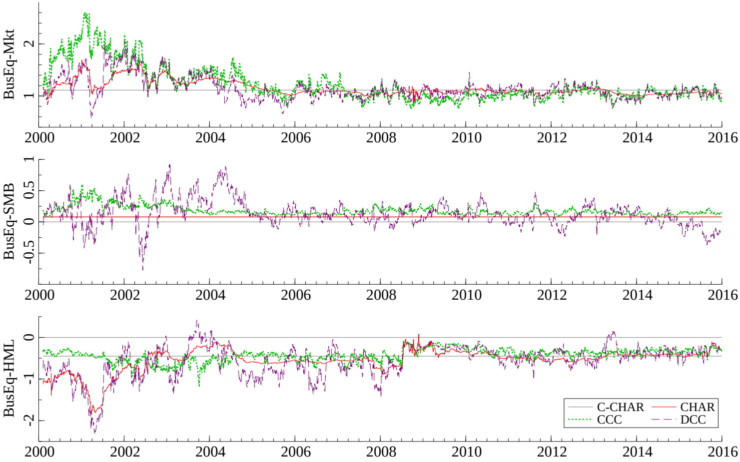
$r_{hml,t}$ (High Minus Low) is the average return on value portfolios minus the average return on growth portfolios

Competing models

Let $\epsilon_t = (\mathbf{x}_t, y_t)'$ with $\mathbf{x}_t = (r_{m,t}, r_{hml,t}, r_{smb,t})'$ and $y_t = r_{j,t}$

- CCC-GARCH(1,1) and DCC-GARCH(1,1). In-sample and out-of-sample one-step-ahead forecasts of conditional betas estimates of the DCB models obtained using $\mathbf{\Sigma}_{yx,t}\mathbf{\Sigma}_{xx,t}^{-1}$ and $\mathbf{\Sigma}_{yx,t+1|t}\mathbf{\Sigma}_{xx,t+1|t}^{-1}$.
- CHAR with constant betas $\beta_{ij,t} = \beta_{ij} \forall t$ and time varying betas $\beta_{ij,t} = \varpi_{ij} + \tau_{ij} v_{i,t-1} v_{j,t-1} + c_{ij}\beta_{ij,t-1}$

Buseq: Business Equipment



Buseq: One-step ahead forecasts

$$Z_{k,t+1|t} = \beta_{k,MKT,t+1|t} MKT_{t+1} + \beta_{k,SMB,t+1|t} SMB_{t+1} + \beta_{k,HML,t+1|t} HML_{t+1},$$

$$TE_{k,t+1} = r_{k,t+1} - Z_{k,t+1|t}$$

Transaction costs : $\frac{\Delta\beta_{CHAR}}{\Delta\beta_{DCC-DCB}}$

	MKT	SMB	HML
BusEq	0.356	0.380	0.341
Chems	0.310	0.263	0.376
Durbl	0.419	0.464	0.693
Enrgy	0.373	0.337	0.456
Hlth	0.461	0.667	0.397
Manuf	0.442	0.402	0.430
Money	0.390	0.397	0.366
NoDur	0.414	0.383	0.296
Other	0.273	0.343	0.335
Shops	0.344	0.297	0.395
Telcm	0.334	0.414	0.640
Utils	0.465	0.408	0.431

$\Delta_{\beta_{k,j}} = \sum_{t=2}^{1,678} |\beta_{k,j,t+1|t} - \beta_{k,j,t|t-1}|$. For each column, the figures correspond to the ratio between the value of $\Delta_{\beta_{k,j}}$ obtained for the CHAR and the DCC-DCB models.

Conclusion

Compare to other multivariate GARCH, the Cholesky-GARCH models introduced here have several advantages.

1) Precise stationarity and moment conditions exist
2) The parameters are directly interpretable in terms of DCB
3) There is no complicated correlation constraint
4) The estimation can be done without matrix invertion
5) The asymptotic theory of the QMLE is available
6) EbE estimation is possible for triangular models
7) The model works nicely in practice, in particuilar for beta hedging