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AN ORIGINAL METHOD FOR COMPUTING THE RESPONSE OF A PARAMETRICALLY EXCITED FORCED SYSTEM

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An original method to compute the steady state forced response of linear systems with periodically varying parameters under external excitations is proposed. The procedure is based on a modal approach with developments in the frequency domain. By using an iterative scheme to construct the approximate response, the spectra and the root mean square values of the forced response at every degree of freedom can be directly obtained. The technique has been successfully tested on illustrative examples and compared to the standard Newmark method. Its efficiency is a consequence of the very short computing times needed. A practical application concerning a geared transmission system which is excited by periodic meshing stiffness is presented to demonstrate that this method is very effective in analyzing systems having many degrees of freedom.

1. INTRODUCTION

Because of the periodic variation of some characteristics, such as inertia, stiffness or damping, the dynamic behaviour of various mechanical systems can be modelled by a coupled set of second order linear differential equations with periodic coefficients. Many practical applications of such parametric systems can be found. Some classical examples are structures under vertical ground motion or periodic loads, cracked rotors, mechanisms (such as gears), elastic linkage systems (such as slider–crank mechanisms), liquid sloshing in flexible containers, hydroelastic systems (such as pipes conveying pulsating fluids), aeroelastic systems (such as rotor blade flutter), etc. A great number of current problems may be found in references [1–4] and several books are devoted to parametric phenomena (see, for example, references [5, 6]).

As the literature shows, much has been done to analyze the nature of the free response, which is the solution to homogeneous equations with periodic coefficients associated with the parametric system. Most of these investigations based on Floquet's theory which reveals the unbounded free response, concern stability analysis. Because closed form analytical solutions do not exist, approximate or numerical methods must be employed. Many different methods have been developed, which can be classified into two groups: those determining the boundaries of parametric instability regions, such as Hill's infinite determinant method [6, 7] and Bolotin's method [5, 8]; and those examining the eigenvalues (characteristic multipliers) of the state transition (monodromy) matrix [9, 10, 12]. The stability problem is thus relatively well mastered.

It is well known that if the solution of the homogeneous equations is asymptotically stable then the particular solution which represents the forced response of the periodic

system under external excitations is also stable [6]. As for the homogeneous problem, closed form analytical solutions do not exist, so an important key is to establish a suitable method for this kind of forced problem. In contrast to the extensive efforts which have been devoted to parametric stability, few studies exist in the area of the forced response of parametric systems. In particular few specific computing methods have been developed. The existing methods comprise perturbation methods, numerical time integration schemes, and those based on Fourier series expansion by the Ritz averaging method or Galerkin's procedure.

Classical perturbation methods [5, 6, 13, 14] can be used to solve the forced problem [15]. However, these methods are essentially based on the assumption of small periodic coefficients, but as in many applications this assumption cannot be justified these methods are not always valid.

Commonly used procedures for finding forced responses are classical numerical time integration schemes like central-difference, Runge–Kutta and Newmark's methods. Other numerical schemes have also been developed for treating the parametric system [10, 11, 16, 17]. The principal advantages of these well known methods are their robustness and the ease of their numerical implementation, so they can be used to obtain reference solutions in order to test other methods. Generally, these numerical integration time methods have some disadvantages because of the requirement to eliminate the free response and to choose the integration time step carefully. This is particularly true when the spectral content of the steady state response is broad and marked by both low and high frequencies. For these reasons the computation times can increase drastically, in particular when the number of degrees of freedom of the system is high.

Methods based on Fourier series expansion (requiring the Ritz averaging method, Galerkin's procedure or others) present advantages compared to numerical integration time methods. They do not require the choice of an integration time step, and they allow the error between exact and approximate solutions to be estimated. However, these methods have not been extensively employed [18, 19]. This is certainly due to the fact that these methods are not very convenient to implement. They also require the spectral contents of the solutions to be fixed, which is not always simple. On the other hand, these procedures are not well-suited to the treatment of systems having a large number of degrees of freedom.

In the present paper an original method is introduced for determining the steady state response of a linear mechanical system under both periodic parametric and external excitations. The approach can be seen to be a generalization of modal decomposition on the spectral domain for the treatment of classical constant linear systems. The proposed procedure is based on an iterative scheme developed in the frequency domain, which is why this method is called the Iterative Spectral Method. After the description of the principles of the method, some numerical results concerning single- and two-degree-of-freedom systems are given. In addition, a practical application is considered in order to show the capability of the proposed method for treating large multi-degree-of-freedom systems.

2. THE ITERATIVE SPECTRAL METHOD

2.1. EQUATIONS OF MOTION: A MODAL APPROACH IN THE SPECTRAL DOMAIN

Consider an n -degree-of-freedom linear system governed by n equations of motion, written in matrix form as

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + ([K] + [P(t)])\{x\} = \{F(t)\}, \quad (1)$$

where matrices are denoted by brackets, column matrices by braces, and a superscript dot represents differentiation with respect to the time t .

Here $\{x\}$ is an n -vector representing the generalized co-ordinates used to describe the motion of the system. $[M]$, $[C]$ and $[K]$ are the time-invariant parts of the mass, damping and stiffness matrices of the system, and $\{F(t)\}$ is the external force vector. For the sake of simplicity, the parametric excitation is introduced via the stiffness matrix $[P(t)]$, the elements of which are zero-mean-time-value periodic functions. Without loss of generality, one may assume the parametric excitation to be in the form

$$[P(t)] = g(t)[D], \quad (2)$$

where $[D]$ is a square matrix with constant elements, and $g(t)$ is a zero-mean-time-value periodic function.

Therefore, the matrix equation of motion, describing the dynamic behaviour of the parametric system, can be rewritten as

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} + g(t)[D]\{x\} = \{F(t)\}. \quad (3)$$

It is easy to prove that the parametric excitation does not generally allow the equations of motion to be uncoupled. However, the corresponding conservative time-invariant homogenous equation

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\} \quad (4)$$

can be used in order to diagonalize the time-invariant stiffness and mass matrices. It is clear that the eigenvalues ω_j^2 and the associated eigenvectors $\{\Phi_j\}$ obtained from equation (4) correspond to the natural frequencies and the associated modal base of the system in the absence of parametric excitation.

By using the modal transformation

$$\{x\} = [\Phi]\{q\}, \quad (5)$$

where $\{q\}$ is the modal co-ordinates vector, and $[\Phi]$ is the matrix of the eigenvectors (that is, the modal base), and assuming a damping matrix which satisfies the Rayleigh condition (i.e., the classical hypothesis of proportional damping, $[C] = \alpha[M] + \beta[K]$), the matrix equation (3) is changed into

$$[m]\{\ddot{q}\} + [c]\{\dot{q}\} + [k]\{q\} + g(t)[d]\{q\} = \{s(t)\}, \quad (6)$$

where $[m]$, $[c]$ and $[k]$ are the usual modal mass, damping and stiffness diagonal matrices, and $\{s(t)\}$ is the modal force vector. $[d]$ is a non-diagonal matrix, introduced by the parametric excitation, which couples up the equations of motion. This matrix is given by

$$[d] = [\Phi]^T[D][\Phi], \quad (7)$$

where the superscript T denotes the transpose of a matrix.

By normalizing the eigenvector in such a way that the modal masses have unit values, the equation of motion simplifies to

$$[I]\{\ddot{q}(t)\} + 2[\zeta_j\omega_j]\{\dot{q}(t)\} + [\omega_j^2]\{q(t)\} + g(t)[d]\{q(t)\} = \{s(t)\}, \quad (8)$$

where $[I]$ is the identity matrix, ω_j is the j th natural frequency previously introduced, and ζ_j is an equivalent viscous damping for the j th mode. The matrix equation (8) can be rearranged by transferring the parametric excitation terms to the right side,

$$[I]\{\ddot{q}(t)\} + 2[\zeta_j\omega_j]\{\dot{q}(t)\} + [\omega_j^2]\{q(t)\} = \{s(t)\} - g(t)[d]\{q(t)\}, \quad (9)$$

and then the left side corresponds to the uncoupled part of the equations of motion.

If one wants to find a bounded steady state response of the externally forced system, the homogeneous time-varying counterpart of equation (8) must be asymptotically stable, which will be assumed later. In this case of free response vanishes as time increases. Thus the Fourier transforms of both sides of equation (9) can be obtained by retaining only stationary terms, and yields

$$(-\omega^2[I] + 2i\omega[\zeta_j\omega_j] + [\omega_j^2])\{\tilde{q}(\omega)\} = \{\tilde{s}(\omega)\} - \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega t} \mathbf{g}(t)[d]\{q(t)\} dt, \quad (10)$$

where the tilde denotes the Fourier transforms of time functions.

By using the convolution theorem (i.e., the Fourier transform of the product of two time functions is the convolution of the Fourier transforms of the time functions), the integral which appears in equation (10) can be expressed as

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega t} \mathbf{g}(t)[d]\{q(t)\} dt = (\tilde{\mathbf{g}} \otimes [d]\{\tilde{q}\})(\omega), \quad (11)$$

where \otimes denotes convolution. From substituting expression (11) in equation (10), one obtains

$$(-\omega^2[I] + 2i\omega[\zeta_j\omega_j] + [\omega_j^2])\{\tilde{q}(\omega)\} = \{\tilde{s}(\omega)\} - (\tilde{\mathbf{g}} \otimes [d]\{\tilde{q}\})(\omega). \quad (12)$$

By introducing the diagonal matrix $[H_j(\omega)]$ which represents the modal frequency response functions, classically defined as

$$H_j(\omega) = 1/(\omega_j^2 - \omega^2 + 2i\zeta_j\omega_j\omega), \quad (13)$$

the matrix equation expressed in the frequency domain and which governs the steady state response of the dynamic system under combined parametric and external excitations can be changed to its final form

$$\{\tilde{q}(\omega)\} = [H_j(\omega)]\{\tilde{s}(\omega)\} - [H_j(\omega)](\tilde{\mathbf{g}} \otimes [d]\{\tilde{q}\})(\omega). \quad (14)$$

2.2. SOLUTION PROCEDURE: AN ITERATIVE TECHNIQUE

It is now proposed to solve the matrix equation of motion (14) iteratively by using successive approximations:

$$\{\tilde{q}_{p+1}(\omega)\} = \{\tilde{q}_0(\omega)\} - [H_j(\omega)](\tilde{\mathbf{g}} \otimes [d]\{\tilde{q}_p\})(\omega). \quad (15)$$

For the initial estimate, the approximate solution is obtained by cancelling out the parametric excitation; it then matches that of the corresponding “time-invariant” system:

$$\{\tilde{q}_0(\omega)\} = [H_j(\omega)]\{\tilde{s}(\omega)\}. \quad (16)$$

In practice, equation (15) is iterated until convergence is reached. Some stop criteria is introduced. First, one must impose a maximum number of iterations, beyond which divergence of the solution is assumed. Second, one makes the convergence test on the measurement of the incremental vector. In practice, at the n th step, one calculates the relative error ϵ , defined by

$$\epsilon = |\{\sigma_n\} - \{\sigma_{n-1}\}|/|\{\sigma_n\}|, \quad (17)$$

which is compared to a sufficiently small value previously imposed. In equation (17), σ_n denotes the root mean square value of the co-ordinates q_n , and $|\{u\}|$ denotes a norm of the vector $\{u\}$. Actually, this criterion is not rigorous but it has two advantages, as follows.

First, the root mean square values σ_n can be obtained directly in the frequency domain. Therefore, the efficiency of the method in respect to computing time is not affected by one calculation of the time histories at each iteration.

Second, one compares only two vectors of the scalar co-ordinates which simplifies the programming of the stop criterion.

Finally, it appeared that using this criterion was always sufficient in the numerical simulations. Thus no other convergence tests have been tried.

The iterative spectral method allows the response spectra for each degree of freedom to be computed directly. The first advantage of the proposed method is that the approximate solution can be constructed, as and when iterations progress, without previous knowledge of their spectral contents. This assumes the choice of a programming language which permits dynamic data storage. In addition, by using dynamic data one can consider only non-zero spectral components, and so work over large bands of frequency without loss of efficiency. Finally, if the convolution products is directly obtained on the frequency domain with a reasonable number of iterations, one can hope for a far from negligible gain in computing time compared to those required other methods.

2.3. REMARKS

One should establish conditions for which the iterative scheme converges. This is a difficult problem because these conditions depend on the frequency contents of both external and internal excitations.

Concerning single-degree-of-freedom systems, an additional study allows one make the two following conclusions: in most cases, conditions leading to divergence of the iterative process are close to the instability conditions associated with unbounded free response of the system: depending on the frequency content of the external force, the method can converge even if the system is unstable.

Concerning multi-degree-of-freedom systems, good convergence behaviour of the proposed method has been noted in all applications to date.

According to these remarks, it can be interesting to know in addition the parametric instability regions associated with the unbounded free response. As noted in the introduction, many works have been concerned with stability analysis and numerous methods have been developed. Unfortunately, no method has been used for systems having a very large number of degrees of freedom. More precisely, the most complex systems which have been studied, have never, to the author's knowledge, exceeded sizes larger than about 50 degrees of freedom. Therefore, the determination of instability conditions for systems with a very large number of degrees of freedom is still a problem in the author's opinion.

3. ILLUSTRATIVE EXAMPLES

In order to validate the proposed method, a number of periodic systems were studied, the numerical results being systematically compared with those obtained by using a classical integration time scheme. To choose among these time methods, four numerical schemes have been used in the case of a single-degree-of-freedom system (Newmark-beta, Runge-Kutta, central-difference and Houbolt methods). Finally, a Newmark-beta method was selected, when relatively large integration time steps were found to be sufficient to achieve convergence. In this section, two examples, a single- and a two-degree-of-freedom system, illustrate this comparative study between the proposed method and the Newmark-beta method. Finally, the third example given afterwards provides a test of the

efficiency of the Iterative Spectral Method for treating large multi-degree-of-freedom systems.

3.1. SINGLE-DEGREE-OF-FREEDOM SYSTEM

For the first example, consider the steady state response of a linear single-degree-of-freedom system excited by both periodic parametric and periodic external excitations. The parametric internal excitation is induced by periodically time-varying stiffness. Since it is a linear system, only the harmonic external force is retained. Thus, the differential equation of motion may be written in dimensionless form as

$$\ddot{x} + 2\zeta\dot{x} + [1 + \mu p(\bar{\omega}\tau)]x = \cos(\omega\tau), \quad (18)$$

where $\tau = \Omega t$, $\zeta = c/(2m\Omega)$ and $\dot{x} = dx/d\tau$, with $\Omega^2 = k/m$. Here, m , c and k are, respectively, the mass, damping and average stiffness values. By cancelling out the parametric excitation, Ω is the natural dimensionless frequency of the system. $\bar{\omega}$ and ω are the dimensionless internal and external excitation frequencies. $p(\bar{\omega}\tau)$ is a zero-mean-value periodic function, and the magnitude of this function is controlled by constant μ .

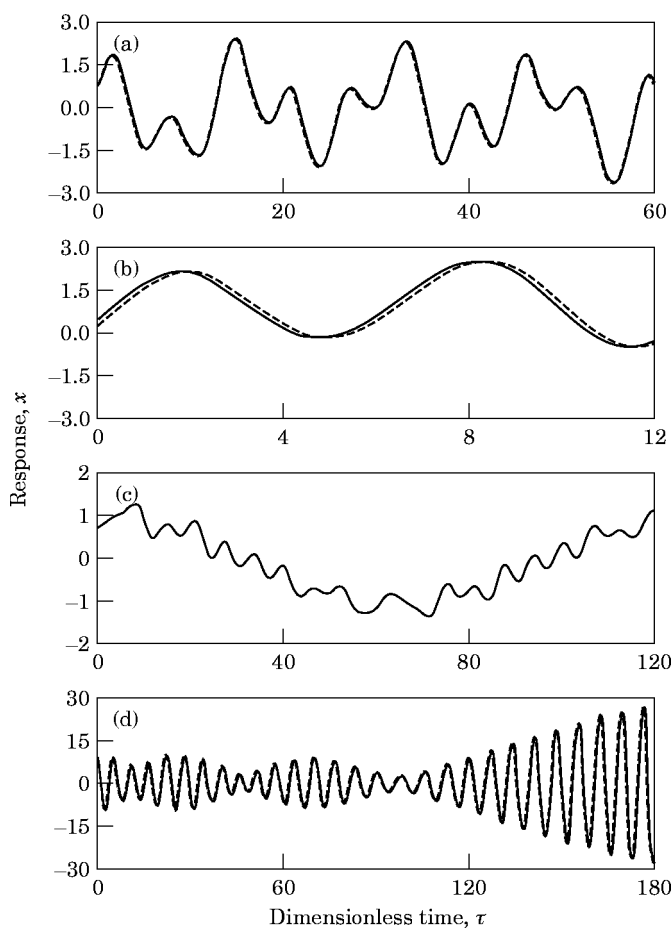


Figure 1. Steady state forced response histories; $\zeta = 0.01$, $\mu = 0.2$; (a) $\bar{\omega}_i = 0.3$, $\bar{\omega}_e = 0.4$; (b) $\bar{\omega}_i = 0.5$, $\bar{\omega}_e = 0$; (c) $\bar{\omega}_i = 0.5$, $\bar{\omega}_e = 0.05$; (d) $\bar{\omega}_i = 0.02$, $\bar{\omega}_e = 0.9$; —, Iterative Spectral Method; ---, Newmark's method.

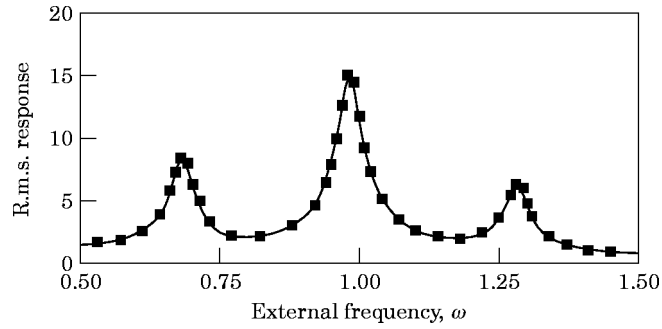


Figure 2. R.m.s. steady state forced response versus external frequency ω ; $\bar{\omega} = 0.3$, $\zeta = 0.02$, $\mu = 0.5$. —, Iterative Spectral Method; ■, Newmark's method.

Typical time histories of the steady stated forced response for different internal and external frequencies $\bar{\omega}$ and ω are shown in Figure 1. One can clearly observe the influence of these frequencies on the nature of the responses. Whatever the kind of response, comparisons of results obtained by the Iterative Spectral Method and Newmark's method illustrate the very good agreement which has always been observed between the two methods. In some cases, one can observe, however, a slight difference in phase between the two sets of results. After some numerical experiments, this difference has been ascribed to Newmark's method. In fact, it is well known that most integration time methods induce errors in the periods of the responses. This problem can easily be controlled for systems with constant coefficients, which is not the case for periodic systems. Moreover, this method was seen to be very dependent on the integration time steps, and was penalized by the need to introduce small steps. Identical problems were also observed with an explicit scheme (central difference method).

In Figure 2 is shown one example of the evolution of the root mean square displacement response versus the external frequency, exhibiting three marked parametric resonances, close to $\omega = 1$, $1 + \bar{\omega}$ and $1 - \bar{\omega}$. These parametric phenomena have been previously explained in reference [22]. Again, there is good agreement between the Iterative Spectral Method and Newmark's method. Moreover, the proposed method behaves particularly well with regard to the parameters which mainly control the vibratory level of the forced responses at the resonances. This result is illustrated in Figure 3.

The approximate steady state response waveforms obtained by the Iterative Spectral Method for different iteration numbers are compared in Figure 4. In this case, it is clear

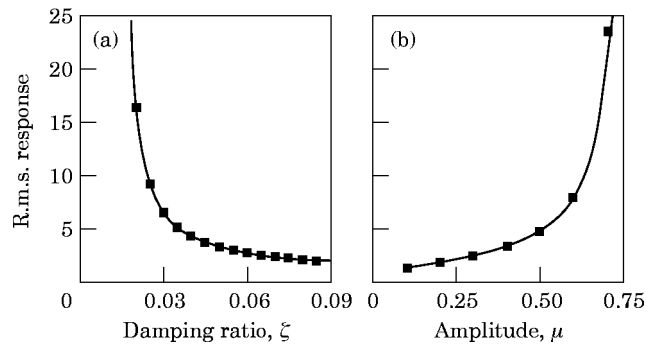


Figure 3. R.m.s. steady state forced response at $\omega = 0$ and $\bar{\omega} = 1$. (a) versus damping ratio ζ with $\mu = 0.4$; (b) versus amplitude μ with $\zeta = 0.05$. —, Iterative Spectral Method; ■, Newmark's method.

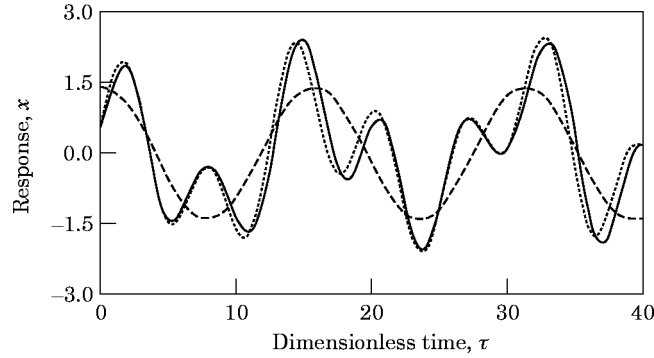


Figure 4. Time-histories of the computed forced response achieved with the Iterative Spectral Method for different numbers N of iterations: ---, $N = 0$ (first estimate); ..., $N = 1$; —, $N = 5$ and 50. $\mu = 0.2$, $\zeta = 0.01$, $\bar{\omega} = 0.3$, $\omega = 0.4$.

that the convergence rate is rapid. To give an idea of this, the numbers of iterations required to obtain a relative difference of less than $1/1000$ between two successive approximate solutions were respectively 4, 7, 4 and 21 for the four cases presented in Figure 1. More generally, and for all the numerous tests run by the author, convergence was always achieved with a small number of iterations, usually around ten. Therefore, the computation time needed for the proposed method was always far shorter than that needed for Newmark's method. With a stop criterion based on a relative error of less than $1/1000$ between two successive approximate solutions, computation times 10–100 times shorter were obtained.

3.2. TWO-DEGREE-OF-FREEDOM SYSTEM

As an example of a multi-degree-of-freedom system, consider the steady state response of the linear two-degree-of-freedom parametric system shown in Figure 5. The parametric internal excitation is introduced via harmonically time-varying stiffnesses as follows (note that the time is shifted by the phase angle φ):

$$\begin{aligned} k_1(t) &= \bar{k}_1(1 + \alpha_1 \cos(\omega t)) = \bar{k}_1(1 + g_1(t)), \\ k_2(t) &= \bar{k}_2(1 + \alpha_2 \cos(\omega t + \varphi)) = \bar{k}_2(1 + g_2(t)). \end{aligned} \quad (19)$$

Here \bar{k}_i denotes the time-mean value of $k_i(t)$. The external force is constant and applied to the second mass m_2 . The matricial equation of motion is then

$$\begin{aligned} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} \bar{k}_1 + \bar{k}_2 & -\bar{k}_2 \\ -\bar{k}_2 & \bar{k}_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \\ + g_1(t) \begin{bmatrix} \bar{k}_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + g_2(t) \begin{bmatrix} \bar{k}_2 & -\bar{k}_2 \\ -\bar{k}_2 & \bar{k}_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F \end{Bmatrix}. \end{aligned} \quad (20)$$

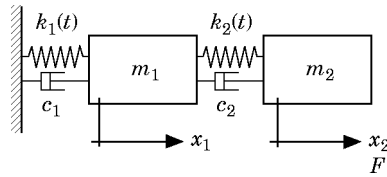


Figure 5. The two-degree-of-freedom parametric system studied.

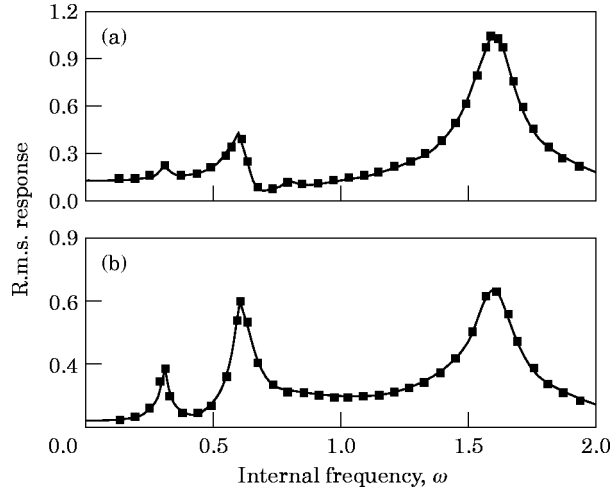


Figure 6. Typical r.m.s. steady state forced responses versus internal frequency ω . (a) x_1 ; (b) x_2 ; $\varphi = 180^\circ$, $\alpha_1 = \alpha_2 = 0.2$, $\zeta = 0.05$; —, Iterative Spectral Method; ■, Newmark's method.

The numerical values of the mass, and mean stiffnesses are chosen as

$$[m] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad [\bar{k}] = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}. \quad (21)$$

From these matrices, the eigenvalues and eigenvectors normalized by the mass matrix are:

$$\omega_1^2 = 0.382, \quad \omega_2^2 = 2.618 \quad \text{and} \quad [\Phi] = [\{\Phi_1\} \{\Phi_2\}] = \begin{bmatrix} 0.526 & -0.851 \\ 0.851 & 0.526 \end{bmatrix}. \quad (22)$$

Therefore, upon introducing viscous modal damping, the equation of motion in the modal base is expressed as

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} 2\zeta_1\omega_1 & 0 \\ 0 & 2\zeta_2\omega_2 \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \\ + \left\{ g_1(t) \begin{bmatrix} 0.277 & -0.448 \\ -0.448 & 0.724 \end{bmatrix} + g_2(t) \begin{bmatrix} 0.106 & 0.448 \\ 0.448 & 1.896 \end{bmatrix} \right\} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}. \quad (23)$$

An approximate solution can therefore be obtained by applying the Iterative Spectral Method. Here, the proposed iterative scheme differs from equation (15) by the fact that the parametric excitation is introduced by a sum of harmonic matrices. Therefore, the successive approximations are obtained as follows:

$$\{\tilde{q}_{p+1}(\omega)\} = \{\tilde{q}_0(\omega)\} - [H_j(\omega)] \sum_{n=1}^{n=2} (\tilde{g}_n \otimes [d_n] \{\tilde{q}_p\})(\omega). \quad (24)$$

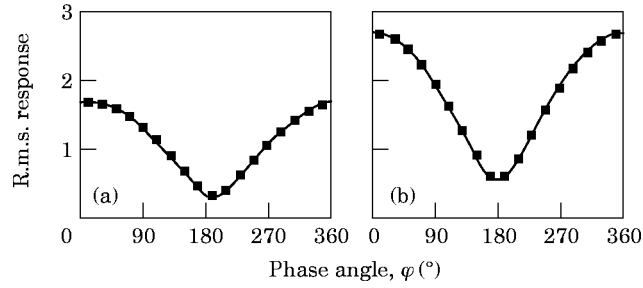


Figure 7. R.m.s. steady state forced responses versus phase angle φ . (a) x_1 ; (b) x_2 ; $\omega = \omega_1$, $\alpha_1 = \alpha_2 = 0.2$, $\zeta = 0.05$; —, Iterative Spectral Method; ■, Newmark's method.

The steady state responses calculated by numerical time integration with those obtained by the Iterative Spectral Method have been compared. These comparisons, performed for numerous cases, have always shown very good agreement between the two methods. Figure 6, in which typical root mean square response curves versus ω are shown illustrates this agreement. Another result is shown in Figure 7, which again demonstrates the accuracy of the Iterative Spectral Method and in particular its ability to account correctly for the phase angle φ .

As mentioned above, it has also been found that few iterations were required. The number of iterations for the two examples presented in Figures 6 and 7 never exceeded 6. This means that using the Iterative Spectral Method leads to an appreciable reduction in computation time compared to Newmark's method. Typically, gains of around 50, and sometimes up to 100, have been obtained.

3.3. DYNAMIC BEHAVIOUR OF A GEARED TRANSMISSION

The ability of the proposed method to describe a large multi-degree-of-freedom system is illustrated with some results regarding the steady state response of a parallel helical gear transmission system. This practical case is concerned with many numerical simulations of the overall dynamic behaviour of a simplified truck gearbox test rig previously developed [20, 21].

The entire transmission (gears, shafts, bearings and housing) was discretized by using the finite element method and ANSYS software. A discrete model with a total number of 1059 degrees of freedom was obtained. This model is schematically described in Figure 8. The numerical results were obtained by retaining the first 30 modes over the frequency range 0–5 kHz.

This system is excited both by the linear time-varying mesh stiffness and by the unloaded static transmission error. The mesh stiffness variation, which is generally assumed to be

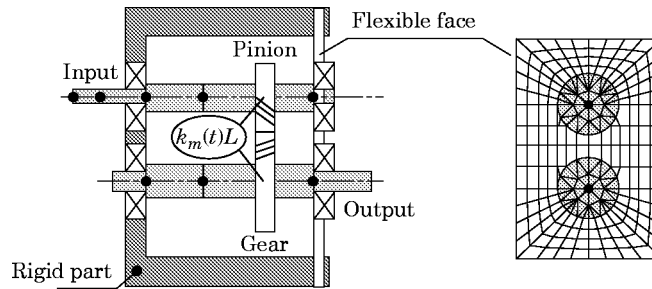


Figure 8. A sketch of the geared transmission model used for the application example.

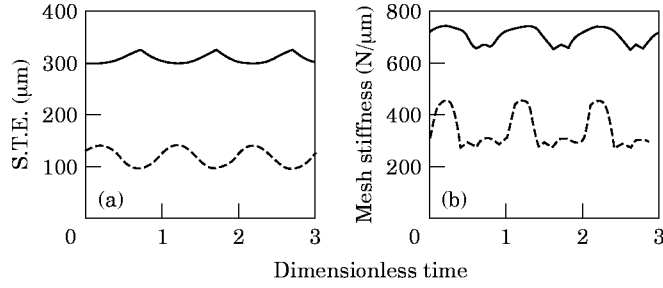


Figure 9. Typical histories of the Static Transmission Error (S.T.E.) under load (a) and meshing stiffness (b). Input torque: —, 600Nm; ---, 100Nm.

periodic for a constant shaft speed, essentially results from the variation in the number of teeth in contact during meshing action. The unloaded static transmission error, which is induced by geometric errors (manufactured and assembly errors), can be modeled by external forces which are also periodic.

Therefore, the governing equation of the steady state motion of the multi-degree-of-freedom geared transmission system is given by the matrix equation

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} + k_m(t)[L]\{x\} = \{F\} + e(t)\{R\}. \quad (25)$$

In this equation, $k_m(t)$ represents the periodic mesh stiffness, which is the parametric excitation, and $e(t)\{R\}$ is an equivalent force vector induced by the unloaded static transmission error. The square matrix $k_m[L]$ describes the elastic coupling occurring between the pinion and the wheel of the gear. Both $[L]$ and $\{R\}$ are defined from the geometrical characteristics of the gear pair. The parametric excitation frequency is the well known meshing frequency, which is equal to the pinion rotational frequency multiplied by the number of its teeth. Depending on the manufactured gear errors selected for the numerical simulations, the external excitation frequency can be equal to or much lower than the meshing frequency. In the present analysis, both $k_m(t)$ and $e(t)$ were predicted by an existing tooth contact model. Their time-variations computed for various input torques are sketched in Figure 9. For computation of purposes, only the first five Fourier components of $k_m(t)$ and $e(t)$ were kept.

Because of the short computation time required by the proposed method, it was possible to make many numerical simulations of the overall dynamic behaviour of this gearbox. These numerical results allow one to reach conclusions about the main characteristics of the transfer between the loaded static transmission error and the vibrational response of the gearbox for many shaft speeds [20].

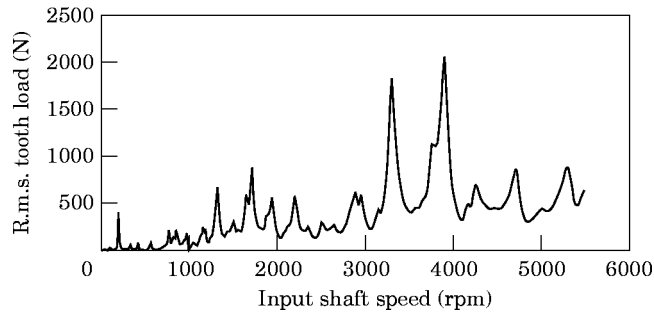


Figure 10. A typical plot of the r.m.s. dynamic tooth load versus input shaft speed.

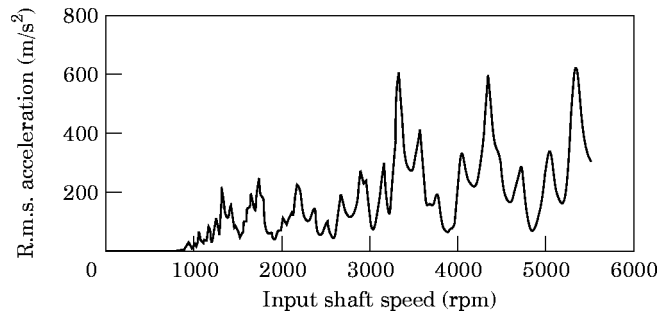


Figure 11. A typical plot of the r.m.s. acceleration response on the housing versus input shaft speed.

As an example, in Figure 10 is shown the evolution of the root mean square value of the dynamic tooth load rotational shaft speed. This result was obtained with a damping ratio at every mode of 1 per cent. Some critical rotational speeds can clearly be seen in this figure. A detailed analysis showed that these were parametric resonances. For example, the strongest critical speeds which appear close to 3300 rpm and 3900 rpm are associated with resonances induced respectively by the first harmonic and by the fundamental components of the static transmission error. One can note that all modes involved in these critical speeds are characterized by large modal displacement through the meshing stiffness. Further parametric resonances can also be observed at every degree of freedom, and in particular for the housing acceleration responses as displayed in Figure 11.

For all the numerous simulations performed by the author, the convergence rate was very rapid: never more than ten iterations with a required tolerance on the relative error between two successive approximate solutions of less than 10^{-4} . The computational time was very short, but depended of course on the hardware and software used. To give an idea, for the numerical simulations presented here, the time needed to evaluate the root mean square values discretized over the speed range with 500 points never exceeded 90 seconds. In this study, the hardware used was a personal computer, with a 486 SX coprocessor rated at 66 MHz under DOS, and a code written in Turbo Pascal.

4. CONCLUSIONS

The Iterative Spectral Method described in this paper allows one to obtain a direct spectral description of the forced response of a linear system with periodic characteristics under external excitations for every degree of freedom. For all the dynamic systems studied and through the numerous numerical simulations undertaken, a very good agreement of the proposed method with Newmark's method has always been obtained. Although the convergence of the iterative scheme was not rigorously proved, all these comparative studies validate the introduced method. Quick convergence was always observed with few iterations. Typically, about ten iterations was sufficient to obtain solutions with the required precision of 1/1000. With account taken of this small number of required iterations, the computation time was significantly less than that of Newmark's method, with gains between 10 and 100. Because of the short computing time associated with the numerical implementation of the method, it has been possible to analyze in detail a practical application concerning a discrete model of geared transmission having many degrees of freedom and for which a large number of modes was retained.

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