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# A priori and a posteriori analysis of the discontinuous Galerkin approximation of the time-harmonic Maxwell's equations under minimal regularity assumptions

T. Chaumont-Frelet\*, A. Ern†

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## Abstract

We derive a priori and a posteriori error estimates for the discontinuous Galerkin (dG) approximation of the time-harmonic Maxwell's equations. Specifically, we consider an interior penalty dG method, and establish error estimates that are valid under minimal regularity assumptions and involving constants that do not depend on the frequency for sufficiently fine meshes. The key result of our a priori error analysis is that the dG solution is asymptotically optimal in an augmented energy norm that contains the dG stabilization. Specifically, up to a constant that tends to one as the mesh is refined, the dG solution is as accurate as the best-approximation in the same norm. We also show that for fine meshes, the inf-sup stability constant is as good as the continuous one up to a factor two. Concerning the a posteriori analysis, we consider a residual-based error estimator under the assumption of piecewise constant material properties. We derive a global upper bound and local lower bounds on the error with constants that (i) only depend on the shape-regularity of the mesh if it is sufficiently refined and (ii) are independent of the stabilization bilinear form.

**Keywords.** Time-harmonic Maxwell's equations, discontinuous Galerkin, Interior penalty, Duality argument, Asymptotic optimality, A posteriori error analysis

**MSC.** 65N30, 78M10, 65N15

## 1 Introduction

Let  $D \subset \mathbb{R}^d$ ,  $d = 3$ , be an open, bounded, Lipschitz polyhedron with boundary  $\partial D$  and outward unit normal  $\mathbf{n}_D$ . We do not make any simplifying assumption on the topology of  $D$ . We use boldface fonts for vectors, vector fields, and functional spaces composed of such fields. More details on the notation are given in Section 2.

Given a positive real number  $\omega > 0$  representing a frequency and a source term  $\mathbf{J} : D \rightarrow \mathbb{R}^3$ , and focusing for simplicity on homogeneous Dirichlet boundary conditions (a.k.a. perfect electric conductor boundary conditions), the model problem consists in finding  $\mathbf{E} : D \rightarrow \mathbb{R}^3$  such that

$$-\omega^2 \epsilon \mathbf{E} + \nabla \times (\boldsymbol{\mu}^{-1} \nabla \times \mathbf{E}) = \mathbf{J} \quad \text{in } D, \quad (1.1a)$$

$$\mathbf{E} \times \mathbf{n}_D = \mathbf{0} \quad \text{on } \partial D, \quad (1.1b)$$

where  $\epsilon$  represents the electric permittivity of the materials contained in  $D$  and  $\boldsymbol{\mu}$  their magnetic permeability. Both material properties can vary in  $D$  and take symmetric positive-definite values with eigenvalues uniformly bounded from above and from below away from zero. We

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assume that  $\omega$  is not a resonant frequency, so that (1.1) is uniquely solvable in  $\mathbf{H}_0(\mathbf{curl}; D)$  for every  $\mathbf{J}$  in the topological dual space  $\mathbf{H}_0(\mathbf{curl}; D)'$ . The time-harmonic Maxwell's equations (1.1) are one of the central models of electrodynamics. Therefore, efficient discretizations are a cornerstone for the computational modelling of electromagnetic wave propagation [26, 36]. In this work, we focus on the discontinuous Galerkin (dG) method.

The dG method employs approximation spaces composed of nonconforming (discontinuous, broken) polynomials on the mesh. DG schemes are attractive since they easily allow for more flexibility in the mesh and for local variations of the polynomial degree. Besides, dG schemes are of special interest in time-dependent wave propagation problems as they lead to block-diagonal mass matrices [25]. Notice that time-dependent wave propagation problems are also relevant here since (1.1) can arise for instance if a Laplace transform is (abstractly) employed to analyze a numerical scheme for the time-dependent Maxwell's equations as in [10, 17], or if a controllability method is used to solve (1.1) with a time-dependent discretization scheme as in [13, 37].

DG methods exist in many flavors. One popular approach for the Poisson model problem is the interior penalty dG method, which hinges on a consistency term involving the mean-value of the normal flux at the mesh faces, possibly a symmetry term, and a stabilization term penalizing the jumps across the mesh interfaces and the values at the mesh boundary faces (see, e.g., [2, 20] and the references therein). The expression of the consistency term involving the mean-value of the normal flux is convenient for efficient implementation, but for the analysis, it is useful to consider an (equivalent) reformulation involving jump liftings. The approach was first considered in [4] and analyzed in [6]. One important outcome is the notion of discrete gradient obtained by adding the jump liftings to the broken (piecewise) gradient. Indeed, the discrete gradient enjoys a compactness property that plays a central role in various nonlinear problems [9, 7, 19]. Another attractive feature is that the discrete gradient admits a bounded extension to  $H^1$ , whereas the standard consistency term can only be extended to  $H^{1+s}$  with  $s > \frac{1}{2}$ . It is also possible to define bounded extensions of the consistency term to  $H^{1+s}$  for  $s > 0$  arbitrarily small by proceeding as in [23].

In the context of the time-harmonic Maxwell's equations, interior penalty dG methods were devised and analyzed in [39, 28], and the notion of discrete curl, obtained by adding the liftings of the tangential jumps to the broken curl, has been considered in the method formulation and analysis. However, the combined use of discrete curls (allowing for minimal regularity requirements) with Schatz's duality argument seems to be lacking in the literature, contrary to the case of the Helmholtz equation where such a result has been recently achieved in [11]. Our main contribution to interior penalty dG methods is to show that an *asymptotically optimal* error estimate holds true with an augmented energy norm including a nonconformity measure. Asymptotic optimality means that the ratio between the approximation error and the best-approximation error tends to one as the mesh size  $h$  is sent to zero. Quite importantly, we establish such asymptotically optimal result under minimal regularity assumptions on the exact solution, thereby allowing for a general setting regarding the domain and the material coefficients. Concerning the source term, we assume that it sits in  $\mathbf{L}^2(D)$  (rather than in the dual space  $\mathbf{H}_0(\mathbf{curl}; D)'$ ) with no further assumption on  $\nabla \cdot \mathbf{J}$ . The second salient contribution of the paper is an a posteriori error analysis leading to global upper bounds and local lower bounds on the error, where the constants are independent of the frequency, again in the limit as the mesh is refined. The a posteriori error analysis is of residual-type and requires to tighten slightly the assumption on the source term so that  $\nabla \cdot \mathbf{J} \in L^2(D)$ . Moreover, to avoid technicalities, we assume that the material properties are piecewise constant.

Let us put our results in perspective with the literature. Concerning the a priori error analysis, a quasi-optimal (but not asymptotically optimal) error estimate under minimal regularity is derived in [32], but for a different interior penalty dG method, where a Lagrange multiplier related to the divergence constraint is introduced together with the corresponding stabilization term. Moreover, asymptotically optimal error estimates for the time-harmonic Maxwell's equations approximated using conforming edge elements have been derived quite

recently in [34] and in [12]. The analysis in [34] considers impedance boundary conditions, is explicit in the frequency, but requires the domain boundary to be smooth and connected. Instead, the analysis in [12] considers Dirichlet boundary conditions and allows for general domains, material coefficients, and right-hand side, but, for brevity, is not made explicit in the frequency. The present analysis leverages the ideas developed in [12], but needs to address two additional, nontrivial difficulties: (i) the lack of strong consistency under minimal regularity, leading to the appearance of new terms in the analysis related to the consistency defect; (ii) the nonconforming nature of the approximation, which calls for a careful handling of the stabilization. In particular, we notice that we allow for a rather general stabilization bilinear form and provide explicit design assumptions for the analysis to hold true.

Concerning the a posteriori error analysis, we leverage the ideas proposed in [14] for the conforming edge finite element approximation of the time-harmonic Maxwell's equations. Here, the novelty is twofold. First, we additionally deal with the consistency defect and the presence of stabilization in the discontinuous Galerkin setting, by extending ideas introduced in [11] for the Helmholtz equation. Moreover, we tighten some arguments from [14] in the proof of the error upper bound so that the involved constants only depend on the shape-regularity parameter of the mesh, whereas in [14] some constants are frequency-dependent in the low-frequency regime. Specifically, instead of invoking the regular decomposition results from [27, Theorem 2.1] as in [14], we make use of Galerkin orthogonality on conforming test functions to invoke the regular decomposition results from [41, Theorem 1].

The paper is organized as follows. In Section 2, we briefly present the continuous setting, and in Section 3, we do the same for the discrete setting. In Section 4, we establish a key estimate on the weak consistency of the dG approximation and also introduce various (nondimensional) approximation and divergence-conformity factors to be used in the analysis. In Section 5, we deal with the a priori error analysis and inf-sup stability. The main results in this section are Theorem 5.5 and Theorem 5.8. In Section 6, we perform the a posteriori residual-based error analysis. The main results in this section are Theorem 6.4 and Theorem 6.5. Finally, in Section 7, we establish bounds on the approximation and divergence conformity factors introduced in Section 4. These bounds prove that these factors tend to zero with the mesh size.

## 2 Continuous setting

In this section, we briefly recall the functional setting for the time-harmonic Maxwell's equations and formulate the model problem.

### 2.1 Functional spaces

We use standard notation for Lebesgue and Sobolev spaces. To alleviate the notation, the inner product and associated norm in the spaces  $L^2(D)$  and  $\mathbf{L}^2(D)$  are denoted by  $(\cdot, \cdot)$  and  $\|\cdot\|$ , respectively. The material properties  $\epsilon$  and  $\nu := \mu^{-1}$  are measurable functions that take symmetric positive-definite values in  $D$  with eigenvalues uniformly bounded from above and from below away from zero. It is convenient to introduce the inner product and associated norm weighted by either  $\epsilon$  or  $\nu$ , leading to the notation  $(\cdot, \cdot)_\epsilon$ ,  $\|\cdot\|_\epsilon$ ,  $(\cdot, \cdot)_\nu$  and  $\|\cdot\|_\nu$ . Whenever no confusion can arise, we use the symbol  $^\perp$  to denote orthogonality with respect to the inner product  $(\cdot, \cdot)_\epsilon$ . Moreover, all the projection operators denoted using the symbol  $\Pi$  are meant to be orthogonal with respect to this inner product; we say that the projections are  $\mathbf{L}_\epsilon^2$ -orthogonal.

We consider the Hilbert Sobolev spaces

$$\mathbf{H}(\mathbf{curl}; D) := \{\mathbf{v} \in \mathbf{L}^2(D) \mid \nabla \times \mathbf{v} \in \mathbf{L}^2(D)\}, \quad (2.1a)$$

$$\mathbf{H}(\mathbf{curl} = \mathbf{0}; D) := \{\mathbf{v} \in \mathbf{H}(\mathbf{curl}; D) \mid \nabla \times \mathbf{v} = \mathbf{0}\}, \quad (2.1b)$$

$$\mathbf{H}_0(\mathbf{curl}; D) := \{\mathbf{v} \in \mathbf{H}(\mathbf{curl}; D) \mid \gamma_{\partial D}^{\mathbf{c}}(\mathbf{v}) = \mathbf{0}\}, \quad (2.1c)$$

$$\mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D) := \{\mathbf{v} \in \mathbf{H}_0(\mathbf{curl}; D) \mid \nabla_0 \times \mathbf{v} = \mathbf{0}\}, \quad (2.1d)$$

where the tangential trace operator  $\gamma_{\partial D}^{\mathbf{c}} : \mathbf{H}(\mathbf{curl}; D) \rightarrow \mathbf{H}^{-\frac{1}{2}}(\partial D)$  is the extension by density of the tangent trace operator such that  $\gamma_{\partial D}^{\mathbf{c}}(\mathbf{v}) = \mathbf{v}|_{\partial D} \times \mathbf{n}_D$  for smooth fields. The subscript  $0$  indicates the curl operator acting on fields respecting homogeneous Dirichlet conditions. Notice that  $\nabla \times$  and  $\nabla_0 \times$  are adjoint to each other, i.e.,  $(\nabla_0 \times \mathbf{v}, \mathbf{w}) = (\mathbf{v}, \nabla \times \mathbf{w})$  for all  $(\mathbf{v}, \mathbf{w}) \in \mathbf{H}_0(\mathbf{curl}; D) \times \mathbf{H}(\mathbf{curl}; D)$ . We equip the space  $\mathbf{H}(\mathbf{curl}; D)$  and its subspaces defined in (2.1) with the following (dimensionally consistent) energy norm:

$$\|\mathbf{v}\|^2 := \omega^2 \|\mathbf{v}\|_{\epsilon}^2 + \|\nabla \times \mathbf{v}\|_{\nu}^2. \quad (2.2)$$

We consider the subspace

$$\mathbf{X}_0^{\mathbf{c}} := \mathbf{H}_0(\mathbf{curl}; D) \cap \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^{\perp}, \quad (2.3)$$

and we introduce the  $\mathbf{L}^2_{\epsilon}$ -orthogonal projection

$$\mathbf{\Pi}_0^{\mathbf{c}} : \mathbf{L}^2(D) \rightarrow \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D). \quad (2.4)$$

Since  $\nabla H_0^1(D) \subset \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)$ , any field  $\boldsymbol{\xi} \in \mathbf{X}_0^{\mathbf{c}}$  is such that  $\nabla \cdot (\epsilon \boldsymbol{\xi}) = 0$  in  $D$ . Hence,  $\mathbf{X}_0^{\mathbf{c}}$  compactly embeds into  $\mathbf{L}^2(D)$  [42].

**Remark 2.1** (Topology of  $D$ ). *We have  $\mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^{\perp} \subset \{\mathbf{v} \in \mathbf{L}^2(D), \nabla \cdot (\epsilon \mathbf{v}) = 0\}$  with equality if only if  $\partial D$  is connected (see, e.g., [1]).*

## 2.2 Model problem

Given a positive real number  $\omega > 0$  and a source term  $\mathbf{J} \in (\mathbf{H}_0(\mathbf{curl}; D))'$  (the topological dual space of  $\mathbf{H}_0(\mathbf{curl}; D)$ ), the model problem amounts to finding  $\mathbf{E} \in \mathbf{H}_0(\mathbf{curl}; D)$  such that

$$b(\mathbf{E}, \mathbf{w}) = \langle \mathbf{J}, \mathbf{w} \rangle \quad \forall \mathbf{w} \in \mathbf{H}_0(\mathbf{curl}; D), \quad (2.5)$$

with the bilinear form defined on  $\mathbf{H}_0(\mathbf{curl}; D) \times \mathbf{H}_0(\mathbf{curl}; D)$  such that

$$b(\mathbf{v}, \mathbf{w}) := -\omega^2 (\mathbf{v}, \mathbf{w})_{\epsilon} + (\nabla_0 \times \mathbf{v}, \nabla_0 \times \mathbf{w})_{\nu}, \quad (2.6)$$

and where the brackets on the right-hand side of (2.5) denote the duality product between  $(\mathbf{H}_0(\mathbf{curl}; D))'$  and  $\mathbf{H}_0(\mathbf{curl}; D)$ . In what follows, we assume that  $\omega^2$  is not an eigenvalue of the  $\epsilon^{-1} \nabla \times (\nu \nabla_0 \times \cdot)$  operator in  $D$ . As a result, the model problem (2.5) is well-posed. We observe that the bilinear form  $b$  satisfies  $|b(\mathbf{v}, \mathbf{w})| \leq \|\mathbf{v}\| \|\mathbf{w}\|$ . The following inf-sup stability result is established in [12, Lemma 2].

**Lemma 2.2** (Inf-sup stability). *The following holds:*

$$\frac{1}{1 + 2\beta_{\text{st}}} \leq \inf_{\substack{\mathbf{v} \in \mathbf{H}_0(\mathbf{curl}; D) \\ \|\mathbf{v}\|=1}} \sup_{\substack{\mathbf{w} \in \mathbf{H}_0(\mathbf{curl}; D) \\ \|\mathbf{w}\|=1}} |b(\mathbf{v}, \mathbf{w})| \leq \frac{1}{\beta_{\text{st}}}, \quad (2.7)$$

with the (nondimensional) stability constant

$$\beta_{\text{st}} := \sup_{\substack{\mathbf{g} \in \mathbf{H}_0(\mathbf{curl}=\mathbf{0}; D)^{\perp} \\ \|\mathbf{g}\|_{\epsilon}=1}} \omega \|\mathbf{v}_{\mathbf{g}}\|. \quad (2.8)$$

Here, for all  $\mathbf{g} \in \mathbf{L}^2(D)$ ,  $\mathbf{v}_{\mathbf{g}} \in \mathbf{H}_0(\mathbf{curl}; D)$  denotes the unique solution to (2.5) with right-hand side  $(\mathbf{g}, \mathbf{w})_{\epsilon}$ , i.e.,  $b(\mathbf{v}_{\mathbf{g}}, \mathbf{w}) = (\mathbf{g}, \mathbf{w})_{\epsilon}$  for all  $\mathbf{w} \in \mathbf{H}_0(\mathbf{curl}; D)$ .

### 3 Discontinuous Galerkin approximation

In this section, we introduce the discrete setting and formulate the interior penalty dG approximation of the model problem (2.5). To avoid technicalities, we assume from now on that  $\mathbf{J} \in \mathbf{L}^2(D)$ ; notice though that we do not make any further assumption on  $\nabla \cdot \mathbf{J}$  for the a priori error analysis and that the sole assumption on the material properties is uniform boundedness from above and from below away from zero.

#### 3.1 Discrete setting

Let  $\mathcal{T}_h$  be an affine simplicial mesh covering  $D$  exactly. A generic mesh cell is denoted  $K$ , its diameter  $h_K$  and its outward unit normal  $\mathbf{n}_K$ . We define the piecewise constant functions  $\tilde{h}$  and  $\tilde{\nu}$  such that

$$\tilde{h}|_K := h_K, \quad \tilde{\nu}|_K := \min_{\substack{\mathbf{u} \in \mathbb{R}^d \\ |\mathbf{u}|=1}} \boldsymbol{\nu}|_K \mathbf{u} \cdot \mathbf{u}, \quad \forall K \in \mathcal{T}_h. \quad (3.1)$$

We write  $\mathcal{F}_h$  for the set of mesh faces,  $\mathcal{F}_h^\circ$  for the subset of mesh interfaces (shared by two distinct mesh cells,  $K_l, K_r$ ), and  $\mathcal{F}_h^\partial$  for the subset of mesh boundary faces (shared by one mesh cell,  $K_l$ , and the boundary,  $\partial D$ ). Every mesh interface  $F \in \mathcal{F}_h^\circ$  is oriented by the unit normal,  $\mathbf{n}_F$ , pointing from  $K_l$  to  $K_r$  (the orientation is arbitrary, but fixed). Every boundary face  $F \in \mathcal{F}_h^\partial$  is oriented by the unit normal  $\mathbf{n}_F := \mathbf{n}_D|_F$ . For all  $K \in \mathcal{T}_h$ ,  $\mathcal{F}_K$  is the collection of the mesh faces composing  $\partial K$ .

Let  $k \geq 1$  be the polynomial degree. Let  $\mathbb{P}_{k,d}$  be the space composed of  $d$ -variate polynomials of total degree at most  $k$  and set  $\mathbf{P}_{k,d} := [\mathbb{P}_{k,d}]^d$ . The dG approximation hinges on the following broken polynomial space:

$$\mathbf{P}_k^b(\mathcal{T}_h) := \{\mathbf{v}_h \in \mathbf{L}^2(D) \mid \mathbf{v}_h|_K \in \mathbf{P}_{k,d}, \forall K \in \mathcal{T}_h\}. \quad (3.2)$$

Moreover, the error analysis makes use of the following subspaces:

$$\mathbf{P}_k^c(\mathcal{T}_h) := \mathbf{P}_k^b(\mathcal{T}_h) \cap \mathbf{H}(\mathbf{curl}; D), \quad (3.3a)$$

$$\mathbf{P}_{k,0}^c(\mathcal{T}_h) := \mathbf{P}_k^b(\mathcal{T}_h) \cap \mathbf{H}_0(\mathbf{curl}; D), \quad (3.3b)$$

$$\mathbf{P}_{k,0}^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h) := \mathbf{P}_k^b(\mathcal{T}_h) \cap \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D). \quad (3.3c)$$

The superscript  $^c$  in the above subspaces is meant to remind us that all these subspaces are  $\mathbf{H}(\mathbf{curl}; D)$ -conforming. The  $\mathbf{L}_\epsilon^2$ -orthogonal projection

$$\mathbf{\Pi}_{h0}^c : \mathbf{L}^2(D) \rightarrow \mathbf{P}_{k,0}^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h), \quad (3.4)$$

plays a key role in what follows. In particular, we introduce the subspace

$$\mathbf{X}_h^b := \mathbf{P}_k^b(\mathcal{T}_h) \cap \mathbf{P}_{k,0}^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)^\perp, \quad (3.5)$$

which is composed of fields  $\mathbf{v}_h \in \mathbf{P}_k^b(\mathcal{T}_h)$  such that  $\mathbf{\Pi}_{h0}^c(\mathbf{v}_h) = \mathbf{0}$ .

#### 3.2 Jumps and discrete curl operator

For all  $K \in \mathcal{T}_h$ , all  $F \in \mathcal{F}_K$ , and all  $\mathbf{v}_h \in \mathbf{P}_k^b(\mathcal{T}_h)$ , we define the local trace operators such that  $\gamma_{K,F}^g(\mathbf{v}_h)(\mathbf{x}) := \mathbf{v}_h|_K(\mathbf{x})$ ,  $\gamma_{K,F}^c(\mathbf{v}_h)(\mathbf{x}) := \mathbf{v}_h|_K(\mathbf{x}) \times \mathbf{n}_F$ , for a.e.  $\mathbf{x} \in F$ . Then, for all  $F \in \mathcal{F}_h^\circ$  and  $\mathbf{x} \in \{g, c\}$ , we define the jump and average operators such that

$$\llbracket \mathbf{v}_h \rrbracket_F^x := \gamma_{K_l, F}^x(\mathbf{v}_h) - \gamma_{K_r, F}^x(\mathbf{v}_h), \quad \{\!\!\{ \mathbf{v}_h \}\!\!\}_F^x := \frac{1}{2} (\gamma_{K_l, F}^x(\mathbf{v}_h) + \gamma_{K_r, F}^x(\mathbf{v}_h)). \quad (3.6)$$

We also set  $\llbracket \mathbf{v}_h \rrbracket_F^x := \{\!\!\{ \mathbf{v}_h \}\!\!\}_F^x := \gamma_{K_l, F}^x(\mathbf{v}_h)$  for all  $F \in \mathcal{F}_h^\partial$ .

For every field  $\mathbf{v}_h \in \mathbf{P}_k^b(\mathcal{T}_h)$ ,  $\nabla_h \times \mathbf{v}_h$  denotes the broken curl of  $\mathbf{v}_h$  (evaluated cellwise). Let  $\ell \geq k - 1 \geq 0$ . We define the discrete curl operator  $\mathbf{C}_{h,0}^{k,\ell} : \mathbf{P}_k^b(\mathcal{T}_h) \rightarrow \mathbf{P}_\ell^b(\mathcal{T}_h)$  such that, for all  $\mathbf{v}_h \in \mathbf{P}_k^b(\mathcal{T}_h)$ ,

$$\mathbf{C}_{h,0}^{k,\ell}(\mathbf{v}_h) := \nabla_h \times \mathbf{v}_h + \mathbf{L}_{h,0}^\ell(\mathbf{v}_h), \quad (3.7)$$

where the jump lifting operator  $\mathbf{L}_{h,0}^\ell(\mathbf{v}_h) \in \mathbf{P}_\ell^b(\mathcal{T}_h)$  is defined by requiring that

$$(\mathbf{L}_{h,0}^\ell(\mathbf{v}_h), \phi_h) := \sum_{F \in \mathcal{F}_h} (\llbracket \mathbf{v}_h \rrbracket_F^c, \{\!\{ \phi_h \}\!\}_F^s)_{\mathbf{L}^2(F)} \quad (3.8)$$

for all  $\phi_h \in \mathbf{P}_\ell^b(\mathcal{T}_h)$ . Taking the polynomial degree  $\ell$  larger than  $k - 1$  is useful to improve the consistency property of the discrete curl operator; see Lemma 4.2 below.

It is convenient to introduce the infinite-dimensional space

$$\mathbf{V}_\sharp := \mathbf{H}_0(\mathbf{curl}; D) + \mathbf{P}_k^b(\mathcal{T}_h), \quad (3.9)$$

where the error  $(\mathbf{E} - \mathbf{E}_h)$  lives. Although the sum in (3.9) is not direct, any field  $\mathbf{v}_h \in \mathbf{H}_0(\mathbf{curl}; D) \cap \mathbf{P}_k^b(\mathcal{T}_h)$  satisfies  $\llbracket \mathbf{v}_h \rrbracket_F^c = \mathbf{0}$  for all  $F \in \mathcal{F}_h$ , as well as  $\mathbf{C}_{h,0}^{k,\ell}(\mathbf{v}_h) = \nabla_0 \times \mathbf{v}_h$ . It is therefore legitimate to extend the curl and jump operators to  $\mathbf{V}_\sharp$  by setting, for all  $\mathbf{v} = \tilde{\mathbf{v}} + \mathbf{v}_h \in \mathbf{V}_\sharp$  with  $\tilde{\mathbf{v}} \in \mathbf{H}_0(\mathbf{curl}; D)$  and  $\mathbf{v}_h \in \mathbf{P}_k^b(\mathcal{T}_h)$ ,

$$\mathbf{C}_{h,0}^{k,\ell}(\mathbf{v}) := \nabla_0 \times \tilde{\mathbf{v}} + \nabla_h \times \mathbf{v}_h + \mathbf{L}_{h,0}^\ell(\mathbf{v}_h), \quad \llbracket \mathbf{v} \rrbracket_F^c := \llbracket \mathbf{v}_h \rrbracket_F^c. \quad (3.10)$$

### 3.3 Discrete bilinear forms and discrete problem

We consider a stabilization bilinear form  $s_\sharp$  defined on  $\mathbf{V}_\sharp \times \mathbf{V}_\sharp$  and which satisfies the following assumptions:

$$(i) \quad s_\sharp \text{ is symmetric positive semidefinite,} \quad (3.11a)$$

$$(ii) \quad s_\sharp(\mathbf{v}, \cdot) = s_\sharp(\cdot, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}; D). \quad (3.11b)$$

We define the bilinear forms  $b_\sharp$  and  $b_{\sharp s}$  on  $\mathbf{V}_\sharp \times \mathbf{V}_\sharp$  such that

$$b_\sharp(\mathbf{v}, \mathbf{w}) := -\omega^2(\mathbf{v}, \mathbf{w})_\epsilon + (\mathbf{C}_{h,0}^{k,\ell}(\mathbf{v}), \mathbf{C}_{h,0}^{k,\ell}(\mathbf{w}))_\nu, \quad (3.12a)$$

$$b_{\sharp s}(\mathbf{v}, \mathbf{w}) := b_\sharp(\mathbf{v}, \mathbf{w}) + s_\sharp(\mathbf{v}, \mathbf{w}). \quad (3.12b)$$

The bilinear form  $b_{\sharp s}$  is used to define the discrete problem and perform the a priori error analysis; the bilinear form  $b_\sharp$  is useful in the a posteriori error analysis. Owing to (3.10) and (3.11b), we have the following (minimal) consistency property:

$$b_{\sharp s}(\mathbf{v}, \mathbf{w}) = b_\sharp(\mathbf{v}, \mathbf{w}) = b(\mathbf{v}, \mathbf{w}) \quad \forall \mathbf{v}, \mathbf{w} \in \mathbf{H}_0(\mathbf{curl}; D). \quad (3.13)$$

We extend the  $\|\cdot\|$ -norm defined in (2.2) to  $\mathbf{V}_\sharp$  by setting, for all  $\mathbf{v} \in \mathbf{V}_\sharp$ ,

$$\|\mathbf{v}\|_\sharp^2 := \omega^2 \|\mathbf{v}\|_\epsilon^2 + \|\mathbf{C}_{h,0}^{k,\ell}(\mathbf{v})\|_\nu^2, \quad (3.14a)$$

$$\|\mathbf{v}\|_{\sharp s}^2 := \|\mathbf{v}\|_\sharp^2 + |\mathbf{v}|_s^2, \quad |\mathbf{v}|_s^2 := s_\sharp(\mathbf{v}, \mathbf{v}). \quad (3.14b)$$

(The definition of  $|\mathbf{v}|_s$  is legitimate owing to (3.11a).) This leads to the following boundedness properties on the bilinear form  $b_{\sharp s}$ :

$$|b_{\sharp s}(\mathbf{v}, \mathbf{w})| \leq \|\mathbf{v}\|_{\sharp s} \|\mathbf{w}\|_{\sharp s} \quad \forall (\mathbf{v}, \mathbf{w}) \in \mathbf{V}_\sharp \times \mathbf{V}_\sharp, \quad (3.15a)$$

$$|b_{\sharp s}(\mathbf{v}, \mathbf{w})| \leq \|\mathbf{v}\|_\sharp \|\mathbf{w}\| \quad \forall (\mathbf{v}, \mathbf{w}) \in \mathbf{V}_\sharp \times \mathbf{H}_0(\mathbf{curl}; D). \quad (3.15b)$$

The discrete problem reads as follows: Find  $\mathbf{E}_h \in \mathbf{P}_k^b(\mathcal{T}_h)$  such that

$$b_{\sharp s}(\mathbf{E}_h, \mathbf{w}_h) = (\mathbf{J}, \mathbf{w}_h)_{\mathbf{L}^2(D)} \quad \forall \mathbf{w}_h \in \mathbf{P}_k^b(\mathcal{T}_h). \quad (3.16)$$

Notice that we use here the assumption that  $\mathbf{J} \in \mathbf{L}^2(D)$ .

### 3.4 Example: interior penalty discontinuous Galerkin method

The classical interior penalty dG formulation for the model problem (2.5) is based upon the following discrete bilinear form [39]: For all  $\mathbf{v}_h, \mathbf{w}_h \in \mathbf{P}_k^b(\mathcal{T}_h)$ ,

$$b_h(\mathbf{v}_h, \mathbf{w}_h) := -\omega^2(\mathbf{v}_h, \mathbf{w}_h)_\epsilon + (\nabla_h \times \mathbf{v}_h, \nabla_h \times \mathbf{w}_h)_\nu + \eta_* s_h(\mathbf{v}_h, \mathbf{w}_h) + \sum_{F \in \mathcal{F}_h} \left\{ (\llbracket \nu \nabla_h \times \mathbf{v}_h \rrbracket_F^g, \llbracket \mathbf{w}_h \rrbracket_F^c)_{\mathbf{L}^2(F)} + (\llbracket \mathbf{v}_h \rrbracket_F^c, \llbracket \nu \nabla_h \times \mathbf{w}_h \rrbracket_F^g)_{\mathbf{L}^2(F)} \right\}, \quad (3.17)$$

with the stabilization bilinear form

$$s_h(\mathbf{v}_h, \mathbf{w}_h) := \sum_{F \in \mathcal{F}_h} \frac{\tilde{\nu}_F}{h_F} (\llbracket \mathbf{v}_h \rrbracket_F^c, \llbracket \mathbf{w}_h \rrbracket_F^c)_{\mathbf{L}^2(F)}, \quad (3.18)$$

and the user-dependent parameter  $\eta_* > 0$  is to be taken large enough. In (3.18),  $h_F$  denotes the diameter of  $F \in \mathcal{F}_h$  and  $\tilde{\nu}_F := \max_{K \in \mathcal{T}_F} \tilde{\nu}_K$  with  $\mathcal{T}_F := \{K \in \mathcal{T}_h \mid F \in \mathcal{F}_K\}$ .

The discrete bilinear form  $b_h$  defined in (3.17) can be extended to  $\mathbf{V}_\# \times \mathbf{V}_\#$  using the bilinear form  $b_{\#s}$  defined in (3.12b) provided the polynomial degree for the jump lifting satisfies  $\ell \geq k - 1 \geq 0$  and provided the material property  $\nu$  is piecewise constant on the mesh. In this situation,  $b_{\#s}$  is indeed an extension of  $b_h$ , i.e.,  $b_{\#s}|_{\mathbf{P}_k^b(\mathcal{T}_h) \times \mathbf{P}_k^b(\mathcal{T}_h)} = b_h$ , provided the stabilization bilinear form  $s_\#$  is defined as follows:

$$s_\#(\mathbf{v}, \mathbf{w}) := \eta_* s_h(\mathbf{v}, \mathbf{w}) - (\mathbf{L}_{h,0}^\ell(\mathbf{v}), \mathbf{L}_{h,0}^\ell(\mathbf{w}))_\nu \quad \forall (\mathbf{v}, \mathbf{w}) \in \mathbf{V}_\# \times \mathbf{V}_\#. \quad (3.19)$$

(Notice that  $s_\#$  is not an extension of  $s_h$ .) The bilinear form  $s_\#$  defined in (3.19) trivially satisfies (3.11b) and is symmetric. It is positive semidefinite, i.e., (3.11a) also holds true, if the factor  $\eta_*$  is chosen large enough [38]. The minimal threshold classically depends on the mesh shape-regularity and the polynomial degree  $\ell$ .

**Remark 3.1** (Weighted averages). *Whenever the jumps of (the eigenvalues of)  $\nu$  are large across the mesh interfaces, it can be useful to consider weighted  $\nu$ -dependent averages to evaluate the last two terms on the right-hand side of (3.17). We refer the reader, e.g., to [24] for an example in the context of scalar diffusion problems. Such weighted averages can be handled in our framework by modifying the definition of the lifting operator accordingly.*

## 4 Weak consistency and approximation factors

In this section, we introduce two important notions for the analysis of the dG approximation: a suitable consistency measure and various approximation factors.

### 4.1 Galerkin orthogonality on conforming test functions

Our first observation is that Galerkin orthogonality holds true whenever the discrete test functions are required to be  $\mathbf{H}_0(\mathbf{curl}; D)$ -conforming.

**Lemma 4.1** (Galerkin orthogonality). *If  $\mathbf{E}_h$  solves (3.16), the following holds true:*

$$b_{\#s}(\mathbf{E} - \mathbf{E}_h, \mathbf{v}_h^c) = 0 \quad \forall \mathbf{v}_h^c \in \mathbf{P}_{k,0}^c(\mathcal{T}_h). \quad (4.1)$$

*In particular, we have*

$$\mathbf{\Pi}_{h,0}^c(\mathbf{E} - \mathbf{E}_h) = \mathbf{0}. \quad (4.2)$$

*Proof.* The property (4.1) follows from the definition of  $b_{\#s}$  which implies that  $b_{\#s}(\mathbf{E}, \mathbf{v}_h^c) = b(\mathbf{E}, \mathbf{v}_h^c) = (\mathbf{J}, \mathbf{v}_h^c)$  for all  $\mathbf{v}_h^c \in \mathbf{P}_{k,0}^c(\mathcal{T}_h)$ . Moreover, since  $\mathbf{P}_{k,0}^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h) \subset \mathbf{P}_{k,0}^c(\mathcal{T}_h)$ , (4.1) implies that  $(\mathbf{E} - \mathbf{E}_h, \mathbf{w}_h)_\epsilon = 0$  for all  $\mathbf{w}_h \in \mathbf{P}_{k,0}^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)$ , which proves (4.2).  $\square$



## 4.2 Weak consistency

We now consider the consistency error produced by the discrete curl operator when tested against smooth fields. For all  $\Psi \in \mathbf{H}_0(\mathbf{curl}; D)$  such that  $\nu \nabla_0 \times \Psi \in \mathbf{H}(\mathbf{curl}; D)$  and for all  $\mathbf{v}_h \in \mathbf{P}_k^b(\mathcal{T}_h)$ , we define the weak consistency error as

$$\delta_{\text{wkc}}(\mathbf{v}_h, \Psi) := (\mathbf{v}_h, \nabla \times (\nu \nabla_0 \times \Psi)) - (\mathbf{C}_{h,0}^{k,\ell}(\mathbf{v}_h), \nabla_0 \times \Psi)_\nu. \quad (4.3)$$

We consider the following two norms:

$$\|\mathbf{v}_h\|_{\text{ap}}^2 := \|\tilde{\nu}^{\frac{1}{2}} \tilde{h}^{-1} \mathbf{v}_h\|^2 + \|\mathbf{C}_{h,0}^{k,\ell}(\mathbf{v}_h)\|_\nu^2 \quad \forall \mathbf{v}_h \in \mathbf{P}_k^b(\mathcal{T}_h), \quad (4.4a)$$

$$\|\mathbf{v}\|_{\text{ap}^*}^2 := \|\mathbf{v}\|_{\nu^{-1}}^2 + \|\tilde{\nu}^{-\frac{1}{2}} \tilde{h} \nabla \times \mathbf{v}\|^2 \quad \forall \mathbf{v} \in \mathbf{H}(\mathbf{curl}; D). \quad (4.4b)$$

(Recall that  $\tilde{h}$  and  $\tilde{\nu}$  are defined in (3.1).) We introduce the following nonconformity measure:

$$|\mathbf{v}|_{\text{nc}} := \min_{\mathbf{v}_h^c \in \mathbf{P}_{k,0}^c(\mathcal{T}_h)} \|\mathbf{v}_h - \mathbf{v}_h^c\|_{\text{ap}} \quad \forall \mathbf{v} := \tilde{\mathbf{v}} + \mathbf{v}_h \in \mathbf{V}_\sharp. \quad (4.5)$$

Notice that the definition (4.5) is independent of the decomposition  $\mathbf{v} := \tilde{\mathbf{v}} + \mathbf{v}_h$ .

**Lemma 4.2** (Weak consistency). *For all  $\Psi \in \mathbf{H}_0(\mathbf{curl}; D)$  such that  $\nu \nabla_0 \times \Psi \in \mathbf{H}(\mathbf{curl}; D)$  and for all  $\mathbf{v}_h \in \mathbf{P}_k^b(\mathcal{T}_h)$ , the following holds true:*

$$|\delta_{\text{wkc}}(\mathbf{v}_h, \Psi)| \leq |\mathbf{v}_h|_{\text{nc}} \min_{\Phi_h^c \in \mathbf{P}_\ell^c(\mathcal{T}_h)} \|\nu \nabla_0 \times \Psi - \Phi_h^c\|_{\text{ap}^*}, \quad (4.6)$$

with  $\mathbf{P}_\ell^c(\mathcal{T}_h) := \mathbf{P}_\ell^b(\mathcal{T}_h) \cap \mathbf{H}(\mathbf{curl}; D)$ .

*Proof.* We follow the idea of [11] for the Helmholtz problem. For any field  $\Phi_h^c \in \mathbf{P}_\ell^c(\mathcal{T}_h)$ , integration by parts gives

$$(\mathbf{v}_h, \nabla \times \Phi_h^c) = (\mathbf{C}_{h,0}^{k,\ell}(\mathbf{v}_h), \Phi_h^c).$$

We infer that

$$\delta_{\text{wkc}}(\mathbf{v}_h, \Psi) = (\mathbf{v}_h, \nabla \times (\nu \nabla_0 \times \Psi - \Phi_h^c)) - (\mathbf{C}_{h,0}^{k,\ell}(\mathbf{v}_h), \nu \nabla_0 \times \Psi - \Phi_h^c).$$

Let  $\mathbf{v}_h^c \in \mathbf{P}_{k,0}^c(\mathcal{T}_h)$  and observe that  $\mathbf{C}_{h,0}^{k,\ell}(\mathbf{v}_h^c) = \nabla_0 \times \mathbf{v}_h^c$ . Integration by parts using  $\zeta := \nu \nabla_0 \times \Psi - \Phi_h^c \in \mathbf{H}(\mathbf{curl}; D)$  gives

$$(\mathbf{v}_h^c, \nabla \times \zeta) = (\nabla_0 \times \mathbf{v}_h^c, \zeta) = (\mathbf{C}_{h,0}^{k,\ell}(\mathbf{v}_h^c), \zeta).$$

Putting everything together yields

$$\begin{aligned} \delta_{\text{wkc}}(\mathbf{v}_h, \Psi) &= (\mathbf{v}_h - \mathbf{v}_h^c, \nabla \times (\nu \nabla_0 \times \Psi - \Phi_h^c)) - (\mathbf{C}_{h,0}^{k,\ell}(\mathbf{v}_h - \mathbf{v}_h^c), \nu \nabla_0 \times \Psi - \Phi_h^c) \\ &\leq \|\mathbf{v}_h - \mathbf{v}_h^c\|_{\text{ap}} \|\nu \nabla_0 \times \Psi - \Phi_h^c\|_{\text{ap}^*}, \end{aligned}$$

where we used the Cauchy–Schwarz inequality and the definitions (4.4a) and (4.4b) of the norms  $\|\cdot\|_{\text{ap}}$  and  $\|\cdot\|_{\text{ap}^*}$ , respectively. Taking the infimum over  $\mathbf{v}_h^c \in \mathbf{P}_{k,0}^c(\mathcal{T}_h)$  and over  $\Phi_h^c \in \mathbf{P}_\ell^c(\mathcal{T}_h)$  completes the proof.  $\square$

## 4.3 Approximation and divergence conformity factors

Here, we introduce three factors to be used in the error analysis. We prove in Section 7 that these factors tend to zero (with a certain rate) as the mesh size tends to zero.

For all  $\theta \in \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp$ , we consider the adjoint problem consisting of finding  $\zeta_\theta \in \mathbf{H}_0(\mathbf{curl}; D)$  such that

$$b(\mathbf{w}, \zeta_\theta) = (\mathbf{w}, \theta)_\epsilon \quad \forall \mathbf{w} \in \mathbf{H}_0(\mathbf{curl}; D). \quad (4.7)$$

Taking any test function  $\mathbf{w} \in \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D) \subset \mathbf{H}_0(\mathbf{curl}; D)$  shows that

$$\omega^2(\mathbf{w}, \boldsymbol{\zeta}_\theta)_\epsilon = b(\mathbf{w}, \boldsymbol{\zeta}_\theta) = (\mathbf{w}, \boldsymbol{\theta})_\epsilon = 0, \quad (4.8)$$

where the first equality follows from  $\nabla_0 \times \mathbf{w} = \mathbf{0}$ , the second from the definition of the adjoint solution, and the third from the assumption  $\boldsymbol{\theta} \in \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp$ . Since  $\mathbf{w}$  is arbitrary in  $\mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)$ , this proves that  $\boldsymbol{\zeta}_\theta \in \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp$ . Thus,  $\boldsymbol{\zeta}_\theta \in \mathbf{X}_0^c$ . We introduce the (nondimensional) approximation factors

$$\gamma_{\text{ap}} := \sup_{\substack{\boldsymbol{\theta} \in \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp \\ \|\boldsymbol{\theta}\|_\epsilon = 1}} \min_{\substack{\mathbf{v}_h^c \in \mathbf{P}_{k,0}^c(\mathcal{T}_h) \\ \|\mathbf{v}_h^c\|_\epsilon = 1}} \omega \|\boldsymbol{\zeta}_\theta - \mathbf{v}_h^c\|, \quad (4.9a)$$

$$\gamma_{\text{ap}^*} := \sup_{\substack{\boldsymbol{\theta} \in \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp \\ \|\boldsymbol{\theta}\|_\epsilon = 1}} \min_{\mathbf{\Phi}_h^c \in \mathbf{P}_k^c(\mathcal{T}_h)} \omega \|\boldsymbol{\nu} \nabla_0 \times \boldsymbol{\zeta}_\theta - \mathbf{\Phi}_h^c\|_{\text{ap}^*}. \quad (4.9b)$$

Notice that  $\gamma_{\text{ap}}$  uses the triple norm  $\|\cdot\|$  related to the boundedness property (3.15b) on  $b_{\sharp\text{s}}$ . Moreover, we introduce the (nondimensional) divergence conformity factor

$$\gamma_{\text{dv}} := \sup_{\substack{\mathbf{v}_h \in \mathbf{X}_h^b \\ \|\mathbf{C}_{h,0}^{k,\ell}(\mathbf{v}_h)\|_\nu^2 + |\mathbf{v}_h|_{\text{nc}}^2 = 1}} \omega \|\boldsymbol{\Pi}_0^c(\mathbf{v}_h)\|_\epsilon. \quad (4.10)$$

Loosely speaking,  $\gamma_{\text{dv}}$  measures how much discretely divergence-free fields depart from being exactly divergence-free.

## 5 A priori error analysis and inf-sup stability

This section is devoted to the error analysis of the dG approximation. As usual with Schatz-like arguments, we first establish an error estimate by assuming that the discrete solution  $\mathbf{E}_h$  exists and then we prove that the discrete problem (3.16) is indeed well-posed.

### 5.1 Error decomposition and best-approximation

We define the approximation error  $\mathbf{e} := \mathbf{E} - \mathbf{E}_h$  and consider the error decomposition

$$\mathbf{e} = \boldsymbol{\theta}_0 + \boldsymbol{\theta}_\Pi, \quad (5.1)$$

with

$$\boldsymbol{\theta}_0 := (I - \boldsymbol{\Pi}_0^c)(\mathbf{e}) \in \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp, \quad \boldsymbol{\theta}_\Pi := \boldsymbol{\Pi}_0^c(\mathbf{e}) \in \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D). \quad (5.2)$$

Let us define the bilinear forms  $b_\sharp^+$  and  $b_{\sharp\text{s}}^+$  on  $\mathbf{V}_\sharp \times \mathbf{V}_\sharp$  such that

$$b_\sharp^+(\mathbf{v}, \mathbf{w}) := \omega^2(\mathbf{v}, \mathbf{w})_\epsilon + (\mathbf{C}_{h,0}^{k,\ell}(\mathbf{v}), \mathbf{C}_{h,0}^{k,\ell}(\mathbf{w}))_\nu, \quad (5.3a)$$

$$b_{\sharp\text{s}}^+(\mathbf{v}, \mathbf{w}) := b_\sharp^+(\mathbf{v}, \mathbf{w}) + s_\sharp(\mathbf{v}, \mathbf{w}). \quad (5.3b)$$

The difference with  $b_\sharp$  and  $b_{\sharp\text{s}}$  lies in the sign of the zero-order term. We define the best-approximation operator  $\mathcal{B}_h^b : \mathbf{V}_\sharp \rightarrow \mathbf{P}_k^b(\mathcal{T}_h)$  as follows: For all  $\mathbf{v} \in \mathbf{V}_\sharp$ ,  $\mathcal{B}_h^b(\mathbf{v}) \in \mathbf{P}_k^b(\mathcal{T}_h)$  is such that

$$b_{\sharp\text{s}}^+(\mathbf{v} - \mathcal{B}_h^b(\mathbf{v}), \mathbf{w}_h) = 0 \quad \forall \mathbf{w}_h \in \mathbf{P}_k^b(\mathcal{T}_h). \quad (5.4)$$

The best-approximation error is defined to be

$$\boldsymbol{\eta} := \mathbf{E} - \mathcal{B}_h^b(\mathbf{E}). \quad (5.5)$$

**Lemma 5.1** (Properties of  $\mathcal{B}_h^b$ ). *The best-approximation operator  $\mathcal{B}_h^b$  defined in (5.4) enjoys the following two properties:*

$$\|\mathcal{B}_h^b(\mathbf{v})\|_{\sharp s} \leq \|\mathbf{v}\|_{\sharp s}, \quad \forall \mathbf{v} \in \mathbf{V}_{\sharp}, \quad (5.6a)$$

$$\mathcal{B}_h^b(\mathbf{v}) \in \mathbf{X}_h^b, \quad \forall \mathbf{v} \in \mathbf{P}_{k,0}^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)^\perp. \quad (5.6b)$$

In particular, the error  $\mathbf{e} = \mathbf{E} - \mathbf{E}_h$  satisfies

$$\mathcal{B}_h^b(\mathbf{e}) \in \mathbf{X}_h^b. \quad (5.6c)$$

*Proof.* (5.6a) follows from the fact that the bilinear form  $b_{\sharp s}^+$  is the inner product associated with the  $\|\cdot\|_{\sharp s}$ -norm. To prove (5.6b), consider any  $\mathbf{v} \in \mathbf{P}_{k,0}^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)^\perp$ . Take any  $\mathbf{w}_h \in \mathbf{P}_{k,0}^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)$  in (5.4) and observe that  $\mathbf{C}_{h,0}^{k,\ell}(\mathbf{w}_h) = \mathbf{0}$  and  $s_{\sharp}(\cdot, \mathbf{w}_h) = 0$ . Since

$$\omega^2(\mathcal{B}_h^b(\mathbf{v}), \mathbf{w}_h)_\epsilon = \omega^2(\mathcal{B}_h^b(\mathbf{v}) - \mathbf{v}, \mathbf{w}_h)_\epsilon = b_{\sharp s}^+(\mathcal{B}_h^b(\mathbf{v}) - \mathbf{v}, \mathbf{w}_h) = 0,$$

we infer that  $\mathcal{B}_h^b(\mathbf{v}) \in \mathbf{P}_{k,0}^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)^\perp$ . Moreover,  $\mathcal{B}_h^b(\mathbf{v}) \in \mathbf{P}_k^b(\mathcal{T}_h)$  by construction. This proves (5.6b). Finally, (5.6c) follows from (4.2) and (5.6b).  $\square$

## 5.2 Preliminary bounds

**Lemma 5.2** (Bound on  $\boldsymbol{\theta}_0$ ). *We have*

$$\omega \|\boldsymbol{\theta}_0\|_\epsilon \leq \gamma_{\text{ap}} \|\mathbf{e}\|_{\sharp} + \gamma_{\text{ap}^*} |\mathbf{e}|_{\text{nc}}, \quad (5.7)$$

with the approximation factors  $\gamma_{\text{ap}}$  and  $\gamma_{\text{ap}^*}$  defined in (4.9a) and (4.9b), respectively.

*Proof.* Let  $\boldsymbol{\zeta}_\theta \in \mathbf{H}_0(\mathbf{curl}; D)$  solve the adjoint problem (4.7) with data  $\boldsymbol{\theta} := \boldsymbol{\theta}_0$ . Since  $\mathbf{E}, \boldsymbol{\zeta}_\theta \in \mathbf{H}_0(\mathbf{curl}; D)$ , we infer from (3.13) and the definition of the adjoint solution that

$$b_{\sharp s}(\mathbf{E}, \boldsymbol{\zeta}_\theta) = b(\mathbf{E}, \boldsymbol{\zeta}_\theta) = (\mathbf{E}, \boldsymbol{\theta}_0)_\epsilon.$$

Moreover, recalling the definition (3.12b) of  $b_{\sharp s}$ , the assumption (3.11b) on  $s_{\sharp}$ , and the definition (4.3) of the weak consistency error, we infer that

$$\begin{aligned} b_{\sharp s}(\mathbf{E}_h, \boldsymbol{\zeta}_\theta) &= -\omega^2(\mathbf{E}_h, \boldsymbol{\zeta}_\theta)_\epsilon + (\mathbf{C}_{h,0}^{k,\ell}(\mathbf{E}_h), \nabla_0 \times \boldsymbol{\zeta}_\theta)_\nu \\ &= (\mathbf{E}_h, -\omega^2 \boldsymbol{\zeta}_\theta + \nabla \times (\boldsymbol{\nu} \nabla_0 \times \boldsymbol{\zeta}_\theta)) - \delta_{\text{wkc}}(\mathbf{E}_h, \boldsymbol{\zeta}_\theta) \\ &= (\mathbf{E}_h, \boldsymbol{\theta}_0)_\epsilon - \delta_{\text{wkc}}(\mathbf{E}_h, \boldsymbol{\zeta}_\theta). \end{aligned}$$

Combining the above two identities and using  $(\boldsymbol{\theta}_0, \boldsymbol{\theta}_\Pi)_\epsilon = 0$  gives

$$\omega \|\boldsymbol{\theta}_0\|_\epsilon^2 = \omega(\mathbf{e}, \boldsymbol{\theta}_0)_\epsilon = \omega b_{\sharp s}(\mathbf{e}, \boldsymbol{\zeta}_\theta) - \omega \delta_{\text{wkc}}(\mathbf{E}_h, \boldsymbol{\zeta}_\theta).$$

Owing to Galerkin orthogonality (see (4.1)), we infer that, for all  $\mathbf{v}_h^c \in \mathbf{P}_{k,0}^c(\mathcal{T}_h)$ ,

$$\omega \|\boldsymbol{\theta}_0\|_\epsilon^2 = \omega b_{\sharp s}(\mathbf{e}, \boldsymbol{\zeta}_\theta - \mathbf{v}_h^c) - \omega \delta_{\text{wkc}}(\mathbf{E}_h, \boldsymbol{\zeta}_\theta). \quad (5.8)$$

Invoking the boundedness property (3.15b) on  $b_{\sharp s}$ , and using the definition of the approximation factor  $\gamma_{\text{ap}}$  gives

$$\omega |b_{\sharp s}(\mathbf{e}, \boldsymbol{\zeta}_\theta - \mathbf{v}_h^c)| \leq \|\mathbf{e}\|_{\sharp} \omega \|\boldsymbol{\zeta}_\theta - \mathbf{v}_h^c\| \leq \|\mathbf{e}\|_{\sharp} \gamma_{\text{ap}} \|\boldsymbol{\theta}_0\|_\epsilon.$$

Moreover, invoking Lemma 4.2 to bound the weak consistency error and recalling the definition of the approximation factor  $\gamma_{\text{ap}^*}$  gives

$$\omega |\delta_{\text{wkc}}(\mathbf{E}_h, \boldsymbol{\zeta}_\theta)| \leq |\mathbf{E}_h|_{\text{nc}} \gamma_{\text{ap}^*} \|\boldsymbol{\theta}_0\|_\epsilon = |\mathbf{e}|_{\text{nc}} \gamma_{\text{ap}^*} \|\boldsymbol{\theta}_0\|_\epsilon.$$

Putting the above two bounds together proves (5.7).  $\square$

**Lemma 5.3** (Bound on  $\boldsymbol{\theta}_\Pi$ ). *We have*

$$\omega \|\boldsymbol{\theta}_\Pi\|_\epsilon \leq \omega \|\boldsymbol{\Pi}_0^c(\boldsymbol{\eta})\|_\epsilon + \gamma_{\text{dv}} \{ \|\mathbf{C}_{h,0}^{k,\ell}(\mathcal{B}_h^b(\mathbf{e}))\|_\nu^2 + |\mathcal{B}_h^b(\mathbf{e})|_{\text{nc}}^2 \}^{\frac{1}{2}}. \quad (5.9)$$

*Proof.* We observe that

$$\boldsymbol{\theta}_\Pi + \boldsymbol{\theta}_0 = \mathbf{e} = \mathcal{B}_h^b(\mathbf{e}) + (I - \mathcal{B}_h^b)(\mathbf{e}) = \mathcal{B}_h^b(\mathbf{e}) + \boldsymbol{\eta},$$

since  $(I - \mathcal{B}_h^b)(\mathbf{E}_h) = \mathbf{0}$ . This gives  $\boldsymbol{\theta}_\Pi = \mathcal{B}_h^b(\mathbf{e}) + \boldsymbol{\eta} - \boldsymbol{\theta}_0$ . Since  $(\boldsymbol{\theta}_\Pi, \boldsymbol{\theta}_0)_\epsilon = 0$ , we infer that

$$\begin{aligned} \|\boldsymbol{\theta}_\Pi\|_\epsilon^2 &= (\boldsymbol{\theta}_\Pi, \mathcal{B}_h^b(\mathbf{e}))_\epsilon + (\boldsymbol{\theta}_\Pi, \boldsymbol{\eta})_\epsilon \\ &= (\boldsymbol{\theta}_\Pi, \boldsymbol{\Pi}_0^c(\mathcal{B}_h^b(\mathbf{e})))_\epsilon + (\boldsymbol{\theta}_\Pi, \boldsymbol{\Pi}_0^c(\boldsymbol{\eta}))_\epsilon =: \Theta_1 + \Theta_2, \end{aligned}$$

where we used that  $\boldsymbol{\theta}_\Pi = \boldsymbol{\Pi}_0^c(\boldsymbol{\theta}_\Pi)$  and that  $\boldsymbol{\Pi}_0^c$  is self-adjoint for the inner product  $(\cdot, \cdot)_\epsilon$ . We bound  $\Theta_1$  as follows:

$$\begin{aligned} |\Theta_1| &\leq \|\boldsymbol{\theta}_\Pi\|_\epsilon \|\boldsymbol{\Pi}_0^c(\mathcal{B}_h^b(\mathbf{e}))\|_\epsilon \\ &\leq \|\boldsymbol{\theta}_\Pi\|_\epsilon \gamma_{\text{dv}} \omega^{-1} \{ \|\mathbf{C}_{h,0}^{k,\ell}(\mathcal{B}_h^b(\mathbf{e}))\|_\nu^2 + |\mathcal{B}_h^b(\mathbf{e})|_{\text{nc}}^2 \}^{\frac{1}{2}}, \end{aligned}$$

where we used the divergence conformity factor defined in (4.10) (this is legitimate since  $\mathcal{B}_h^b(\mathbf{e}) \in \mathbf{X}_h^b$  owing to (5.6c)). Moreover, the Cauchy–Schwarz inequality gives

$$|\Theta_2| \leq \|\boldsymbol{\theta}_\Pi\|_\epsilon \|\boldsymbol{\Pi}_0^c(\boldsymbol{\eta})\|_\epsilon.$$

Putting the above two bounds together proves the assertion.  $\square$

**Lemma 5.4** (Bound on  $\|\boldsymbol{\theta}_0\|_{\sharp\text{s}}$ ). *We have*

$$\begin{aligned} \|\boldsymbol{\theta}_0\|_{\sharp\text{s}}^2 &\leq \|(I - \boldsymbol{\Pi}_0^c)(\boldsymbol{\eta})\|_{\sharp\text{s}}^2 + 2\omega \|\boldsymbol{\theta}_\Pi\|_\epsilon \gamma_{\text{dv}} \{ \|\mathbf{C}_{h,0}^{k,\ell}(\mathcal{B}_h^b(\mathbf{e}))\|_\nu^2 + |\mathcal{B}_h^b(\mathbf{e})|_{\text{nc}}^2 \}^{\frac{1}{2}} \\ &\quad + 2|\mathcal{B}_h^b(\mathbf{e})|_{\text{nc}} \min_{\boldsymbol{\Phi}_h^c \in \mathbf{P}_\ell^c(\mathcal{T}_h)} \|\boldsymbol{\nu} \nabla_0 \times \mathbf{E} - \boldsymbol{\Phi}_h^c\|_{\text{ap}^*} + 4\omega^2 \|\boldsymbol{\theta}_0\|_\epsilon^2. \end{aligned} \quad (5.10)$$

*Proof.* Since  $\mathbf{e} = \boldsymbol{\eta} + \mathcal{B}_h^b(\mathbf{e})$  as shown in the above proof, we have  $\boldsymbol{\theta}_0 = (I - \boldsymbol{\Pi}_0^c)(\mathbf{e}) = (I - \boldsymbol{\Pi}_0^c)(\boldsymbol{\eta}) + (I - \boldsymbol{\Pi}_0^c)(\mathcal{B}_h^b(\mathbf{e}))$ . This gives

$$\begin{aligned} b_{\sharp\text{s}}(\boldsymbol{\theta}_0, \boldsymbol{\theta}_0) &= b_{\sharp\text{s}}(\boldsymbol{\theta}_0, (I - \boldsymbol{\Pi}_0^c)(\boldsymbol{\eta})) + b_{\sharp\text{s}}(\boldsymbol{\theta}_0, (I - \boldsymbol{\Pi}_0^c)(\mathcal{B}_h^b(\mathbf{e}))) \\ &= b_{\sharp\text{s}}(\boldsymbol{\theta}_0, (I - \boldsymbol{\Pi}_0^c)(\boldsymbol{\eta})) + b_{\sharp\text{s}}(\mathbf{e}, (I - \boldsymbol{\Pi}_0^c)(\mathcal{B}_h^b(\mathbf{e}))), \end{aligned}$$

where the second equality follows from  $b_{\sharp\text{s}}(\boldsymbol{\theta}_\Pi, (I - \boldsymbol{\Pi}_0^c)(\cdot)) = 0$ . The first term on the right-hand side is bounded by invoking the continuity property (3.15a), giving

$$|b_{\sharp\text{s}}(\boldsymbol{\theta}_0, (I - \boldsymbol{\Pi}_0^c)(\boldsymbol{\eta}))| \leq \|\boldsymbol{\theta}_0\|_{\sharp\text{s}} \|(I - \boldsymbol{\Pi}_0^c)(\boldsymbol{\eta})\|_{\sharp\text{s}}.$$

The second term is decomposed as  $b_{\sharp\text{s}}(\mathbf{e}, (I - \boldsymbol{\Pi}_0^c)(\mathcal{B}_h^b(\mathbf{e}))) = \beta_1 + \beta_2$  with

$$\beta_1 := b_{\sharp\text{s}}(\mathbf{e}, \mathcal{B}_h^b(\mathbf{e})), \quad \beta_2 := -b_{\sharp\text{s}}(\mathbf{e}, \boldsymbol{\Pi}_0^c(\mathcal{B}_h^b(\mathbf{e}))) = \omega^2 (\boldsymbol{\theta}_\Pi, \boldsymbol{\Pi}_0^c(\mathcal{B}_h^b(\mathbf{e})))_\epsilon.$$

A straightforward calculation shows that

$$\begin{aligned} \beta_1 &= b_{\sharp\text{s}}(\mathbf{E}, \mathcal{B}_h^b(\mathbf{e})) - b_{\sharp\text{s}}(\mathbf{E}_h, \mathcal{B}_h^b(\mathbf{e})) = b_{\sharp\text{s}}(\mathbf{E}, \mathcal{B}_h^b(\mathbf{e})) - (\mathbf{J}, \mathcal{B}_h^b(\mathbf{e})) \\ &= (\nabla_0 \times \mathbf{E}, \mathbf{C}_{h,0}^{k,\ell}(\mathcal{B}_h^b(\mathbf{e})))_\nu - (\nabla \times (\boldsymbol{\nu} \nabla_0 \times \mathbf{E}), \mathcal{B}_h^b(\mathbf{e})) = -\delta_{\text{wkc}}(\mathcal{B}_h^b(\mathbf{e}), \mathbf{E}). \end{aligned}$$

(Notice that we need to consider here all possible test functions in  $\mathbf{P}_k^b(\mathcal{T}_h)$  in the discrete problem (3.16).) Hence, invoking Lemma 4.2 gives

$$|\beta_1| \leq |\mathcal{B}_h^b(\mathbf{e})|_{\text{nc}} \min_{\boldsymbol{\Phi}_h^c \in \mathbf{P}_\ell^c(\mathcal{T}_h)} \|\boldsymbol{\nu} \nabla_0 \times \mathbf{E} - \boldsymbol{\Phi}_h^c\|_{\text{ap}^*}.$$

Moreover, using the Cauchy–Schwarz inequality and since  $\mathcal{B}_h^b(\mathbf{e}) \in \mathbf{X}_h^b$  (see (5.6c)), we have

$$|\beta_2| \leq \omega \|\boldsymbol{\theta}_\Pi\|_\epsilon \gamma_{\text{dv}} \left\{ \|\mathbf{C}_{h,0}^{k,\ell}(\mathcal{B}_h^b(\mathbf{e}))\|_\nu^2 + |\mathcal{B}_h^b(\mathbf{e})|_{\text{nc}}^2 \right\}^{\frac{1}{2}}.$$

Altogether, this gives

$$\begin{aligned} b_{\sharp\text{s}}(\boldsymbol{\theta}_0, \boldsymbol{\theta}_0) &\leq \|\boldsymbol{\theta}_0\|_{\sharp\text{s}} \|(I - \boldsymbol{\Pi}_0^c)(\boldsymbol{\eta})\|_{\sharp\text{s}} + \omega \|\boldsymbol{\theta}_\Pi\|_\epsilon \gamma_{\text{dv}} \left\{ \|\mathbf{C}_{h,0}^{k,\ell}(\mathcal{B}_h^b(\mathbf{e}))\|_\nu^2 + |\mathcal{B}_h^b(\mathbf{e})|_{\text{nc}}^2 \right\}^{\frac{1}{2}} \\ &\quad + |\mathcal{B}_h^b(\mathbf{e})|_{\text{nc}} \min_{\boldsymbol{\Phi}_h^c \in \mathcal{P}_\ell^c(\mathcal{T}_h)} \|\boldsymbol{\nu} \nabla_0 \times \mathbf{E} - \boldsymbol{\Phi}_h^c\|_{\text{ap}*}. \end{aligned}$$

Since  $\|\boldsymbol{\theta}_0\|_{\sharp\text{s}}^2 = b_{\sharp\text{s}}(\boldsymbol{\theta}_0, \boldsymbol{\theta}_0) + 2\omega^2 \|\boldsymbol{\theta}_0\|_\epsilon^2$ , we infer that

$$\begin{aligned} \|\boldsymbol{\theta}_0\|_{\sharp\text{s}}^2 &\leq \|\boldsymbol{\theta}_0\|_{\sharp\text{s}} \|(I - \boldsymbol{\Pi}_0^c)(\boldsymbol{\eta})\|_{\sharp\text{s}} + \omega \|\boldsymbol{\theta}_\Pi\|_\epsilon \gamma_{\text{dv}} \left\{ \|\mathbf{C}_{h,0}^{k,\ell}(\mathcal{B}_h^b(\mathbf{e}))\|_\nu^2 + |\mathcal{B}_h^b(\mathbf{e})|_{\text{nc}}^2 \right\}^{\frac{1}{2}} \\ &\quad + |\mathcal{B}_h^b(\mathbf{e})|_{\text{nc}} \min_{\boldsymbol{\Phi}_h^c \in \mathcal{P}_\ell^c(\mathcal{T}_h)} \|\boldsymbol{\nu} \nabla_0 \times \mathbf{E} - \boldsymbol{\Phi}_h^c\|_{\text{ap}*} + 2\omega^2 \|\boldsymbol{\theta}_0\|_\epsilon^2. \end{aligned}$$

Dealing with the first term on the right-hand side by Young's inequality gives (5.10).  $\square$

### 5.3 A priori error estimate

To derive our a priori error estimate, we make the following assumption on the stabilization bilinear form:

$$\exists \rho > 0 \quad \text{s.t.} \quad \rho |\mathbf{v}|_{\text{nc}} \leq |\mathbf{v}|_{\text{s}}, \quad \forall \mathbf{v} \in \mathbf{V}_\sharp. \quad (5.11)$$

In (5.11), the positive constant  $\rho$  is independent of the mesh size. The value of  $\rho$  can depend on the mesh shape-regularity and the polynomial degree; see Remark 5.11 below for further insight.

**Theorem 5.5** (A priori error estimate and discrete well-posedness). *Assume (5.11). The following holds:*

$$(1 - c_\gamma) \|\mathbf{e}\|_{\sharp\text{s}}^2 \leq (1 + 4\gamma_{\text{dv}}) \|\boldsymbol{\eta}\|_{\sharp\text{s}}^2 + 2\rho^{-1} \|\mathbf{e}\|_{\sharp\text{s}} \min_{\boldsymbol{\Phi}_h^c \in \mathcal{P}_\ell^c(\mathcal{T}_h)} \|\boldsymbol{\nu} \nabla_0 \times \mathbf{E} - \boldsymbol{\Phi}_h^c\|_{\text{ap}*}. \quad (5.12)$$

with  $c_\gamma := 8 \max(\gamma_{\text{ap}}^2, \rho^{-2} \gamma_{\text{ap}*}^2) + \max(1, \rho^{-2})(\gamma_{\text{dv}} + 3\gamma_{\text{dv}}^2)$ . Consequently, if the mesh size is small enough so that  $c_\gamma < 1$ , the discrete problem (3.16) is well-posed.

*Proof.* We square (5.9) and add the result to (5.10). Since

$$\|\mathbf{e}\|_{\sharp\text{s}}^2 = \|\boldsymbol{\theta}_0\|_{\sharp\text{s}}^2 + \omega^2 \|\boldsymbol{\theta}_\Pi\|_\epsilon^2, \quad \|\boldsymbol{\eta}\|_{\sharp\text{s}}^2 = \|(I - \boldsymbol{\Pi}_0^c)(\boldsymbol{\eta})\|_{\sharp\text{s}}^2 + \omega^2 \|\boldsymbol{\Pi}_0^c(\boldsymbol{\eta})\|_\epsilon^2,$$

we obtain

$$\begin{aligned} \|\mathbf{e}\|_{\sharp\text{s}}^2 &\leq \|\boldsymbol{\eta}\|_{\sharp\text{s}}^2 + 4\omega \|\boldsymbol{\Pi}_0^c(\boldsymbol{\eta})\|_\epsilon \gamma_{\text{dv}} \left\{ \|\mathbf{C}_{h,0}^{k,\ell}(\mathcal{B}_h^b(\mathbf{e}))\|_\nu^2 + |\mathcal{B}_h^b(\mathbf{e})|_{\text{nc}}^2 \right\}^{\frac{1}{2}} \\ &\quad + 3\gamma_{\text{dv}}^2 \left\{ \|\mathbf{C}_{h,0}^{k,\ell}(\mathcal{B}_h^b(\mathbf{e}))\|_\nu^2 + |\mathcal{B}_h^b(\mathbf{e})|_{\text{nc}}^2 \right\} \\ &\quad + 2|\mathcal{B}_h^b(\mathbf{e})|_{\text{nc}} \min_{\boldsymbol{\Phi}_h^c \in \mathcal{P}_\ell^c(\mathcal{T}_h)} \|\boldsymbol{\nu} \nabla_0 \times \mathbf{E} - \boldsymbol{\Phi}_h^c\|_{\text{ap}*} + 4\omega^2 \|\boldsymbol{\theta}_0\|_\epsilon^2. \end{aligned}$$

We deal with the second term on the right-hand side by Young's inequality. Since  $\omega \|\boldsymbol{\Pi}_0^c(\boldsymbol{\eta})\|_\epsilon \leq \omega \|\boldsymbol{\eta}\|_\epsilon \leq \|\boldsymbol{\eta}\|_{\sharp\text{s}}$ , this gives

$$\begin{aligned} \|\mathbf{e}\|_{\sharp\text{s}}^2 &\leq (1 + 4\gamma_{\text{dv}}) \|\boldsymbol{\eta}\|_{\sharp\text{s}}^2 + (\gamma_{\text{dv}} + 3\gamma_{\text{dv}}^2) \left\{ \|\mathbf{C}_{h,0}^{k,\ell}(\mathcal{B}_h^b(\mathbf{e}))\|_\nu^2 + |\mathcal{B}_h^b(\mathbf{e})|_{\text{nc}}^2 \right\} \\ &\quad + 2|\mathcal{B}_h^b(\mathbf{e})|_{\text{nc}} \min_{\boldsymbol{\Phi}_h^c \in \mathcal{P}_\ell^c(\mathcal{T}_h)} \|\boldsymbol{\nu} \nabla_0 \times \mathbf{E} - \boldsymbol{\Phi}_h^c\|_{\text{ap}*} + 4\omega^2 \|\boldsymbol{\theta}_0\|_\epsilon^2. \end{aligned}$$

We invoke (5.7) to bound the last term on the right-hand side. This yields

$$\begin{aligned} \|\mathbf{e}\|_{\sharp s}^2 &\leq (1 + 4\gamma_{\text{dv}})\|\boldsymbol{\eta}\|_{\sharp s}^2 + (\gamma_{\text{dv}} + 3\gamma_{\text{dv}}^2)\{\|\mathbf{C}_{h,0}^{k,\ell}(\mathcal{B}_h^{\text{b}}(\mathbf{e}))\|_{\nu}^2 + |\mathcal{B}_h^{\text{b}}(\mathbf{e})|_{\text{nc}}^2\} \\ &\quad + 2|\mathcal{B}_h^{\text{b}}(\mathbf{e})|_{\text{nc}} \min_{\Phi_h^c \in \mathbf{P}_\ell^c(\mathcal{T}_h)} \|\boldsymbol{\nu} \nabla_0 \times \mathbf{E} - \Phi_h^c\|_{\text{ap}^*} + 8(\gamma_{\text{ap}}^2 \|\mathbf{e}\|_{\sharp}^2 + \gamma_{\text{ap}^*}^2 |\mathbf{e}|_{\text{nc}}^2). \end{aligned}$$

We observe that

$$\begin{aligned} \|\mathbf{C}_{h,0}^{k,\ell}(\mathcal{B}_h^{\text{b}}(\mathbf{e}))\|_{\nu}^2 + |\mathcal{B}_h^{\text{b}}(\mathbf{e})|_{\text{nc}}^2 &\leq \max(1, \rho^{-2})\{\|\mathbf{C}_{h,0}^{k,\ell}(\mathcal{B}_h^{\text{b}}(\mathbf{e}))\|_{\nu}^2 + |\mathcal{B}_h^{\text{b}}(\mathbf{e})|_{\text{s}}^2\} \\ &\leq \max(1, \rho^{-2})\|\mathcal{B}_h^{\text{b}}(\mathbf{e})\|_{\sharp s}^2 \leq \max(1, \rho^{-2})\|\mathbf{e}\|_{\sharp s}^2, \end{aligned}$$

where the last bound follows from (5.6a). Moreover, we have

$$\gamma_{\text{ap}}^2 \|\mathbf{e}\|_{\sharp}^2 + \gamma_{\text{ap}^*}^2 |\mathbf{e}|_{\text{nc}}^2 \leq \max(\gamma_{\text{ap}}^2, \rho^{-2}\gamma_{\text{ap}^*}^2)(\|\mathbf{e}\|_{\sharp}^2 + |\mathbf{e}|_{\text{s}}^2) = \max(\gamma_{\text{ap}}^2, \rho^{-2}\gamma_{\text{ap}^*}^2)\|\mathbf{e}\|_{\sharp s}^2.$$

Combining the above bounds shows that

$$(1 - c_\gamma)\|\mathbf{e}\|_{\sharp s}^2 \leq (1 + 4\gamma_{\text{dv}})\|\boldsymbol{\eta}\|_{\sharp s}^2 + 2|\mathcal{B}_h^{\text{b}}(\mathbf{e})|_{\text{nc}} \min_{\Phi_h^c \in \mathbf{P}_\ell^c(\mathcal{T}_h)} \|\boldsymbol{\nu} \nabla_0 \times \mathbf{E} - \Phi_h^c\|_{\text{ap}^*}.$$

Since  $\rho|\mathcal{B}_h^{\text{b}}(\mathbf{e})|_{\text{nc}} \leq |\mathcal{B}_h^{\text{b}}(\mathbf{e})|_{\text{s}} \leq \|\mathcal{B}_h^{\text{b}}(\mathbf{e})\|_{\sharp s} \leq \|\mathbf{e}\|_{\sharp s}$ , this readily gives (5.12).  $\square$

**Remark 5.6** (Assumption (5.11)). *In the context of the interior penalty dG method, where the stabilization bilinear form  $s_\sharp$  is defined in (3.19), it is possible to choose the parameter  $\eta_* > 0$  large enough so that (5.11) holds true. To this purpose, one can, for instance, bound  $|\mathbf{v}_h|_{\text{nc}}$  by taking  $\mathbf{v}_h^c := \mathcal{I}_{h0}^{c,\text{av}}(\mathbf{v}_h)$  defined using the  $\mathbf{H}_0(\mathbf{curl}; D)$ -conforming averaging operator analyzed in [21]. Furthermore, we notice that the converse bound to (5.11) is not required in the analysis. It is, however, reasonable to assume that the converse bound  $\tilde{\rho}|\mathbf{v}_h|_{\text{s}} \leq |\mathbf{v}_h|_{\text{nc}}$  holds for some  $\tilde{\rho} > 0$ , in order to avoid ill-conditioned linear systems.*

**Remark 5.7** (Error estimate (5.12)). *The last term on the right-hand side of (5.12) stems from the weak consistency of the discrete formulation and somewhat pollutes the asymptotic optimality of the a priori error estimate. We notice that this term can be made superconvergent already with the choice  $\ell = k$  (provided  $\boldsymbol{\nu} \nabla_0 \times \mathbf{E}$  is smooth enough). The (slight) price to pay is to choose the stabilization factor  $\eta_*$  large enough so that  $s_\sharp$  is indeed positive semidefinite for  $\ell = k$  (see (3.19)).*

## 5.4 Inf-sup stability

Here, we establish the discrete inf-sup stability of the bilinear form  $b_{\sharp s}$  on  $\mathbf{P}_k^{\text{b}}(\mathcal{T}_h) \times \mathbf{P}_k^{\text{b}}(\mathcal{T}_h)$ .

**Theorem 5.8** (Inf-sup stability). *Under assumption (5.11), we have*

$$\min_{\substack{\mathbf{v}_h \in \mathbf{P}_k^{\text{b}}(\mathcal{T}_h) \\ \|\mathbf{v}_h\|_{\sharp s} = 1}} \max_{\substack{\mathbf{w}_h \in \mathbf{P}_k^{\text{b}}(\mathcal{T}_h) \\ \|\mathbf{w}_h\|_{\sharp s} = 1}} |b_{\sharp s}(\mathbf{v}_h, \mathbf{w}_h)| \geq \frac{1 - c'_\gamma}{1 + 2\beta_{\text{st}}}, \quad (5.13)$$

with  $c'_\gamma := 2(\gamma_{\text{ap}} + \frac{1}{2}\rho^{-1}\gamma_{\text{ap}^*} + \max(1, \rho^{-2})\gamma_{\text{dv}}^2)$ .

*Proof.* Let  $\mathbf{v}_h \in \mathbf{P}_k^{\text{b}}(\mathcal{T}_h)$ . We build a suitable  $\mathbf{w}_h \in \mathbf{P}_k^{\text{b}}(\mathcal{T}_h)$  so that  $\|\mathbf{w}_h\|_{\sharp s} \leq (1 + 2\beta_{\text{st}})\|\mathbf{v}_h\|_{\sharp s}$  and  $b_{\sharp s}(\mathbf{v}_h, \mathbf{w}_h) \geq (1 - c'_\gamma)\|\mathbf{v}_h\|_{\sharp s}^2$ .

(1) Set  $\mathbf{v}_{h0} := (I - \Pi_{h0}^c)(\mathbf{v}_h) \in \mathbf{X}_h^{\text{b}}$  and  $\mathbf{v}_{h\Pi} := \Pi_{h0}^c(\mathbf{v}_h) \in \mathbf{P}_{k,0}^c(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)$ , so that  $\mathbf{v}_h = \mathbf{v}_{h0} + \mathbf{v}_{h\Pi}$ . We further decompose  $\mathbf{v}_{h0}$  as  $\mathbf{v}_{h0} = \phi_0 + \phi_\Pi$  with  $\phi_0 := (I - \Pi_0^c)(\mathbf{v}_{h0})$  and  $\phi_\Pi := \Pi_0^c(\mathbf{v}_{h0})$ . Let  $\boldsymbol{\xi}_0 \in \mathbf{H}_0(\mathbf{curl}; D)$  be the unique adjoint solution such that  $b(\mathbf{w}, \boldsymbol{\xi}_0) = \omega^2(\mathbf{w}, \phi_0)_\epsilon$  for all  $\mathbf{w} \in \mathbf{H}_0(\mathbf{curl}; D)$ . We set  $\boldsymbol{\xi}_{h0} := \mathcal{B}_{h0}^c(\boldsymbol{\xi}_0)$ , where  $\mathcal{B}_{h0}^c : \mathbf{H}_0(\mathbf{curl}; D) \rightarrow$

$\mathbf{P}_{k,0}^c(\mathcal{T}_h)$  is uniquely defined by requiring that  $b_{\sharp}^+(\mathbf{v} - \mathcal{B}_{h0}^c(\mathbf{v}), \mathbf{w}_h) = 0$ , for all  $\mathbf{v} \in \mathbf{H}_0(\mathbf{curl}; D)$  and all  $\mathbf{w}_h \in \mathbf{P}_{k,0}^c(\mathcal{T}_h)$ . Finally, we set

$$\mathbf{w}_h := \mathbf{v}_{h0} + 2\boldsymbol{\xi}_{h0} - \mathbf{v}_{h\Pi} \in \mathbf{P}_k^b(\mathcal{T}_h).$$

(2) Upper bound on  $\|\mathbf{w}_h\|_{\sharp s}$ . The same argument as in the proof of Lemma 5.1 shows that  $\|\boldsymbol{\xi}_{h0}\| \leq \|\boldsymbol{\xi}_0\|$ . Moreover, since  $\boldsymbol{\phi}_0 \in \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp$ , we have

$$\beta_{\text{st}}^{-1} \|\boldsymbol{\xi}_0\| \leq \omega \|\boldsymbol{\phi}_0\|_\epsilon \leq \omega \|\mathbf{v}_{h0}\|_\epsilon \leq \omega \|\mathbf{v}_h\|_\epsilon \leq \|\mathbf{v}_h\|_{\sharp} \leq \|\mathbf{v}_h\|_{\sharp s}.$$

This gives

$$\begin{aligned} \|\mathbf{w}_h\|_{\sharp s}^2 &= \|\mathbf{v}_{h0} + 2\boldsymbol{\xi}_{h0}\|_{\sharp s}^2 + \omega^2 \|\mathbf{v}_{h\Pi}\|_\epsilon^2 \leq (\|\mathbf{v}_{h0}\|_{\sharp s} + 2\|\boldsymbol{\xi}_{h0}\|)^2 + \omega^2 \|\mathbf{v}_{h\Pi}\|_\epsilon^2 \\ &\leq (1 + 2\beta_{\text{st}})^2 \|\mathbf{v}_{h0}\|_{\sharp s}^2 + \omega^2 \|\mathbf{v}_{h\Pi}\|_\epsilon^2 \leq (1 + 2\beta_{\text{st}})^2 \|\mathbf{v}_h\|_{\sharp s}^2. \end{aligned}$$

(3) Lower bound on  $b_{\sharp s}(\mathbf{v}_h, \mathbf{w}_h)$ . We first observe that

$$b_{\sharp s}(\mathbf{v}_h, \boldsymbol{\xi}_{h0}) = b_{\sharp s}(\mathbf{v}_{h0}, \boldsymbol{\xi}_{h0}) + b_{\sharp s}(\mathbf{v}_{h\Pi}, \boldsymbol{\xi}_{h0}) = b_{\sharp s}(\mathbf{v}_{h0}, \boldsymbol{\xi}_{h0}) = b_{\sharp s}(\mathbf{v}_{h0}, \boldsymbol{\xi}_0) + b_{\sharp s}(\mathbf{v}_{h0}, \boldsymbol{\xi}_{h0} - \boldsymbol{\xi}_0).$$

We have

$$b_{\sharp s}(\mathbf{v}_{h0}, \boldsymbol{\xi}_0) = \omega^2 (\mathbf{v}_{h0}, \boldsymbol{\phi}_0)_\epsilon - \delta_{\text{wkc}}(\mathbf{v}_{h0}, \boldsymbol{\xi}_0) = \omega^2 \|\boldsymbol{\phi}_0\|_\epsilon^2 - \delta_{\text{wkc}}(\mathbf{v}_{h0}, \boldsymbol{\xi}_0).$$

Invoking Lemma 4.2 and the definition (4.9b) of the approximation factor  $\gamma_{\text{ap}^*}$  gives

$$b_{\sharp s}(\mathbf{v}_{h0}, \boldsymbol{\xi}_0) \geq \omega^2 \|\boldsymbol{\phi}_0\|_\epsilon^2 - |\mathbf{v}_{h0}|_{\text{nc}} \gamma_{\text{ap}^*} \omega \|\boldsymbol{\phi}_0\|_\epsilon.$$

Using the above bound on  $\|\boldsymbol{\phi}_0\|_\epsilon$  together with  $|\mathbf{v}_{h0}|_{\text{nc}} = |\mathbf{v}_h|_{\text{nc}}$  and assumption (5.11), we infer that

$$b_{\sharp s}(\mathbf{v}_{h0}, \boldsymbol{\xi}_0) \geq \omega^2 \|\boldsymbol{\phi}_0\|_\epsilon^2 - \rho^{-1} \gamma_{\text{ap}^*} |\mathbf{v}_h|_s \|\mathbf{v}_h\|_{\sharp} \geq \omega^2 \|\boldsymbol{\phi}_0\|_\epsilon^2 - \frac{1}{2} \rho^{-1} \gamma_{\text{ap}^*} \|\mathbf{v}_h\|_{\sharp s}^2, \quad (5.14)$$

where the last bound follows from Young's inequality. Furthermore, using the definition (4.9a) of the approximation factor  $\gamma_{\text{ap}}$  together with the boundedness property (3.15b) gives

$$b_{\sharp s}(\mathbf{v}_{h0}, \boldsymbol{\xi}_{h0} - \boldsymbol{\xi}_0) \geq -\|\mathbf{v}_h\|_{\sharp s} \gamma_{\text{ap}} \omega \|\boldsymbol{\phi}_0\|_\epsilon \geq -\gamma_{\text{ap}} \|\mathbf{v}_h\|_{\sharp s}^2.$$

Combining this lower bound with (5.14), we infer that

$$b_{\sharp s}(\mathbf{v}_{h0}, \boldsymbol{\xi}_0) \geq \omega^2 \|\boldsymbol{\phi}_0\|_\epsilon^2 - (\gamma_{\text{ap}} + \frac{1}{2} \rho^{-1} \gamma_{\text{ap}^*}) \|\mathbf{v}_h\|_{\sharp s}^2. \quad (5.15)$$

Furthermore, using the divergence conformity factor  $\gamma_{\text{dv}}$  yields

$$\begin{aligned} \omega^2 \|\boldsymbol{\phi}_\Pi\|_\epsilon^2 &= \omega^2 \|\boldsymbol{\Pi}_0^c(\mathbf{v}_{h0})\|_\epsilon^2 \leq \gamma_{\text{dv}}^2 \{ \|\mathbf{C}_{h,0}^{k,\ell}(\mathbf{v}_{h0})\|_\nu^2 + |\mathbf{v}_{h0}|_{\text{nc}}^2 \} \\ &\leq \max(1, \rho^{-2}) \gamma_{\text{dv}}^2 \|\mathbf{v}_{h0}\|_{\sharp s}^2 \leq \max(1, \rho^{-2}) \gamma_{\text{dv}}^2 \|\mathbf{v}_h\|_{\sharp s}^2. \end{aligned}$$

Since  $\|\mathbf{v}_{h0}\|_\epsilon^2 = \|\boldsymbol{\phi}_0\|_\epsilon^2 + \|\boldsymbol{\phi}_\Pi\|_\epsilon^2$ , combining this bound with (5.15) gives

$$b_{\sharp s}(\mathbf{v}_{h0}, \boldsymbol{\xi}_{h0}) \geq \omega^2 \|\mathbf{v}_{h0}\|_\epsilon^2 - \frac{1}{2} c'_\gamma \|\mathbf{v}_h\|_{\sharp s}^2.$$

Finally,  $b_{\sharp s}(\mathbf{v}_h, \mathbf{v}_{h0} - \mathbf{v}_{h\Pi}) = \|\mathbf{v}_h\|_{\sharp s}^2 - 2\omega^2 \|\mathbf{v}_{h0}\|_\epsilon^2$ , we infer that

$$b_{\sharp s}(\mathbf{v}_h, \mathbf{w}_h) = b_{\sharp s}(\mathbf{v}_h, \mathbf{v}_{h0} + 2\boldsymbol{\xi}_{h0} - \mathbf{v}_{h\Pi}) \geq (1 - c'_\gamma) \|\mathbf{v}_h\|_{\sharp s}^2.$$

This completes the proof.  $\square$

**Remark 5.9** (Discrete inf-sup constant). *The discrete inf-sup constant appearing on the left-hand side of (5.13) tends to  $(1 + 2\beta_{\text{st}})^{-1}$  as the mesh is refined, thus approaching, by up to a factor of two at most, the inf-sup constant from the continuous setting.*

## 6 A posteriori residual-based error analysis

In this section, we estimate the error  $\mathbf{e} := \mathbf{E} - \mathbf{E}_h$  by means of local residual-based quantities called error indicators. We derive both a global upper error bound (reliability) and local lower error bounds (local efficiency). The only property required for the discrete object  $\mathbf{E}_h$  in the a posteriori error analysis is to satisfy the Galerkin orthogonality (4.1) on conforming test functions, i.e.,  $b_{\sharp}(\mathbf{E} - \mathbf{E}_h, \mathbf{v}_h^c) = b_{\sharp}(\mathbf{E} - \mathbf{E}_h, \mathbf{v}_h^c) = 0$  for all  $\mathbf{v}_h^c \in \mathbf{P}_{k,0}^c(\mathcal{T}_h)$ . Lemma 4.1 shows that the dG solution solving (3.16) satisfies this property. For simplicity, we keep the notation  $\mathbf{E}_h$  in this section. For the a posteriori error analysis, we assume that  $\nabla \cdot \mathbf{J} \in L^2(D)$  and that the material properties are piecewise constant on the mesh.

### 6.1 Notation

For all  $K \in \mathcal{T}_h$ , the element patch  $\tilde{K}$  (resp.,  $\hat{K}$ ) denotes the domain covered by all the cells  $K' \in \mathcal{T}_h$  sharing at least one vertex (resp., one edge) with  $K$ . Similarly, the extended patch  $\tilde{\tilde{K}}$  is the domain covered by all the cells  $K'' \in \mathcal{T}_h$  sharing at least one vertex with a cell  $K' \in \tilde{K}$ . For a face  $F \in \mathcal{F}_h$ ,  $\tilde{F}$  is the domain covered by the one or two cells sharing  $F$ . Whenever no confusion can arise, we also employ the symbols  $\tilde{K}$ ,  $\hat{K}$ ,  $\tilde{\tilde{K}}$  and  $\tilde{F}$  for the set of cells covering the domains. We employ the symbol  $\kappa_{\mathcal{T}_h}$  for the shape-regularity parameter of the mesh  $\mathcal{T}_h$ , and  $C(\kappa_{\mathcal{T}_h})$  denotes any generic constant solely depending on  $\kappa_{\mathcal{T}_h}$  and whose value can change at each occurrence. For all  $\mathcal{T} \subset \mathcal{T}_h$ , we introduce the notation

$$\varepsilon_{\max, \mathcal{T}} := \max_{K \in \mathcal{T}} \max_{\mathbf{x} \in K} \max_{\substack{\mathbf{u} \in \mathbb{R}^d \\ |\mathbf{u}|=1}} \max_{\substack{\mathbf{v} \in \mathbb{R}^d \\ |\mathbf{v}|=1}} \boldsymbol{\epsilon}(\mathbf{x}) \mathbf{u} \cdot \mathbf{v}, \quad \varepsilon_{\min, \mathcal{T}} := \min_{T \in \mathcal{T}} \min_{\mathbf{x} \in K} \min_{\substack{\mathbf{u} \in \mathbb{R}^d \\ |\mathbf{u}|=1}} \boldsymbol{\epsilon}(\mathbf{x}) \mathbf{u} \cdot \mathbf{u}, \quad (6.1)$$

and define  $\nu_{\max, \mathcal{T}}$  and  $\nu_{\min, \mathcal{T}}$  similarly. Then,  $\vartheta_{\mathcal{T}} := (\nu_{\min, \mathcal{T}} / \varepsilon_{\max, \mathcal{T}})^{\frac{1}{2}}$  stands for the minimum velocity in the subdomain covered by the cells in  $\mathcal{T}$ . Finally, we write  $\|v\|_{\mathcal{T}}^2 := \sum_{K \in \mathcal{T}} \|v\|_{L^2(K)}^2$  and employ a similar notation if  $v$  is vector-valued. We also write  $\|v\|_{\mathcal{F}}^2 := \sum_{F \in \mathcal{F}} \|v\|_{L^2(F)}^2$  for every subset  $\mathcal{F} \subset \mathcal{F}_h$ .

### 6.2 Estimator and error measure

The a posteriori error estimator is written as the sum over the mesh cells of local error indicators  $\eta_K$  for all  $K \in \mathcal{T}_h$ . The local error indicator consists of three pieces. The first two respectively measure the residuals of the divergence constraint and of Maxwell's equations:

$$\eta_{K, \text{div}}^2 := \varepsilon_{\min, \tilde{K}}^{-1} \left\{ \frac{h_K^2}{\omega^2 k^2} \|\nabla \cdot (\mathbf{J} + \omega^2 \boldsymbol{\epsilon} \mathbf{E}_h)\|_K^2 + \frac{\omega^2 h_K}{k} \|[\boldsymbol{\epsilon} \mathbf{E}_h]_{\partial K}^d\|_{\partial K \setminus \partial \Omega}^2 \right\}, \quad (6.2a)$$

$$\eta_{K, \text{curl}}^2 := \nu_{\min, \tilde{K}}^{-1} \left\{ \frac{h_K^2}{k^2} \|\mathbf{J} + \omega^2 \boldsymbol{\epsilon} \mathbf{E}_h - \nabla \times (\boldsymbol{\nu} \mathbf{C}_{h,0}^{k,\ell}(\mathbf{E}_h))\|_K^2 + \frac{h_K}{k} \|[\boldsymbol{\nu} \mathbf{C}_{h,0}^{k,\ell}(\mathbf{E}_h)]_{\partial K}^c\|_{\partial K \setminus \partial \Omega}^2 \right\}, \quad (6.2b)$$

where  $[\boldsymbol{\epsilon} \mathbf{E}_h]_{\partial K}^d|_F := [\boldsymbol{\epsilon} \mathbf{E}_h]_F^g \cdot \mathbf{n}_F$  and  $[\boldsymbol{\nu} \nabla \times \mathbf{E}_h]_{\partial K}^c|_F := [\boldsymbol{\nu} \nabla \times \mathbf{E}_h]_F^g \times \mathbf{n}_F$  for all  $F \in \mathcal{F}_K$ . The last part of the estimator controls the nonconformity of the discrete field  $\mathbf{E}_h$  as follows:

$$\eta_{K, \text{nc}}^2 := \frac{\psi_K(\nu, k)}{h_K} \|[\mathbf{E}_h]_{\partial K}^c\|_{\partial K}^2, \quad \psi_K(\nu, k) := \max(\nu_{\max, K} k^2, \nu_{\min, \hat{K}} k^4), \quad (6.2c)$$

where  $[\mathbf{E}_h]_{\partial K}^c|_F := [\mathbf{E}_h]_F^g \times \mathbf{n}_F$  for all  $F \in \mathcal{F}_K$ . For shortness, we also introduce the following notation:

$$\eta_K^2 := \eta_{K, \text{div}}^2 + \eta_{K, \text{curl}}^2 + \eta_{K, \text{nc}}^2, \quad \eta_{\bullet, K}^2 := \sum_{K \in \mathcal{T}_h} \eta_{\bullet, K}^2, \quad \eta^2 := \sum_{K \in \mathcal{T}_h} \eta_K^2, \quad (6.3)$$



with  $\bullet \in \{\text{div}, \text{curl}, \text{nc}\}$ .

For all  $\mathcal{T} \subset \mathcal{T}_h$ , we define the error measure

$$\|\mathbf{e}\|_{\dagger, \mathcal{T}}^2 := \sum_{K \in \mathcal{T}} \left\{ \omega^2 \|\mathbf{e}\|_{\epsilon, K}^2 + \|\mathbf{C}_{h,0}^{k,\ell}(\mathbf{e})\|_{\nu, K}^2 + \frac{\psi_K(\nu, k)}{h_K} \|\llbracket \mathbf{e} \rrbracket_{\partial K}^c\|_{\partial K}^2 \right\}, \quad (6.4)$$

and we omit the subscript  $\mathcal{T}$  whenever  $\mathcal{T} = \mathcal{T}_h$ . A crucial observation is that the last term in the norm measuring the nonconformity can be chosen independently of the stabilization in the dG scheme. In particular, it does not have to be large enough.

### 6.3 Error upper bound (reliability)

We employ the quasi-interpolation operators from [31] (see also [33]). Specifically, there exists an operator  $\mathcal{I}_{h,0}^g : H_0^1(D) \rightarrow \mathcal{P}_{k+1}^b(\mathcal{T}_h) \cap H_0^1(D)$  such that, for all  $q \in H_0^1(D)$  and all  $K \in \mathcal{T}_h$ ,

$$\frac{k^2}{h_K^2} \|q - \mathcal{I}_{h,0}^g(q)\|_K^2 + \frac{k}{h_K} \|q - \mathcal{I}_{h,0}^g(q)\|_{\partial K}^2 \leq C(\kappa_{\mathcal{T}_h}) \ell_D^{-2} \|q\|_{H^1(\tilde{K})}^2, \quad (6.5)$$

where  $\|q\|_{H^1(\tilde{K})}^2 := \sum_{K' \in \tilde{K}} \{\|q\|_{K'}^2 + \ell_D^2 \|\nabla q\|_{K'}^2\}$  and  $\ell_D$  is a global length scale (e.g., the diameter of  $D$ ) which is introduced here for dimensional consistency. Similarly, there exists an operator  $\mathcal{I}_{h,0}^c : \mathbf{H}_0^1(D) \rightarrow \mathbf{P}_{k,0}^c(\mathcal{T}_h)$  such that, for all  $\mathbf{w} \in \mathbf{H}_0^1(D)$  and all  $K \in \mathcal{T}_h$ ,

$$\frac{k^2}{h_K^2} \|\mathbf{w} - \mathcal{I}_{h,0}^c(\mathbf{w})\|_K^2 + \frac{k}{h_K} \|(\mathbf{w} - \mathcal{I}_{h,0}^c(\mathbf{w})) \times \mathbf{n}_K\|_{\partial K}^2 \leq C(\kappa_{\mathcal{T}_h}) \ell_D^{-2} \|\mathbf{w}\|_{H^1(\tilde{K})}^2. \quad (6.6)$$

We will also need the quasi-interpolation averaging operator  $\mathcal{I}_{h,0}^{c,\text{av}} : \mathbf{P}_k^b(\mathcal{T}_h) \rightarrow \mathbf{P}_{k,0}^c(\mathcal{T}_h)$  from [21] which is such that there is  $c_k^{\text{av}} \geq 1$  so that, for all  $\mathbf{v}_h \in \mathbf{P}_k^b(\mathcal{T}_h)$  and all  $K \in \mathcal{T}_h$ ,

$$h_K^{-1} \|\mathbf{v}_h - \mathcal{I}_{h,0}^{c,\text{av}}(\mathbf{v}_h)\|_K + \|\nabla \times (\mathbf{v}_h - \mathcal{I}_{h,0}^{c,\text{av}}(\mathbf{v}_h))\|_K \leq C(\kappa_{\mathcal{T}_h}) c_k^{\text{av}} \sqrt{\frac{k^2}{h_K}} \|\llbracket \mathbf{v}_h \rrbracket_{\partial K}^c\|_{\partial K}. \quad (6.7)$$

Previous work on averaging operators for scalar-valued functions [8, 29] indicates that one could expect, at least for  $d = 2$ , that  $c_k^{\text{av}}$  is bounded uniformly in  $k$ , but we keep here the dependency on  $k$  as the question remains, to our knowledge, open for  $d = 3$ . Invoking a discrete trace inequality (see, e.g., [22, Lem. 12.10]) yields

$$\|\nabla \times \mathbf{v}_h - \mathbf{C}_{h,0}^{k,\ell}(\mathbf{v}_h)\|_K = \|\mathbf{L}_{h,0}^\ell(\mathbf{v}_h)\|_K \leq C(\kappa_{\mathcal{T}_h}) \sqrt{\frac{k^2}{h_K}} \|\llbracket \mathbf{v}_h \rrbracket_{\partial K}^c\|_{\partial K}. \quad (6.8)$$

Combining (6.7) and (6.8) gives

$$h_K^{-1} \|\mathbf{v}_h - \mathcal{I}_{h,0}^{c,\text{av}}(\mathbf{v}_h)\|_K + \|\mathbf{C}_{h,0}^{k,\ell}(\mathbf{v}_h - \mathcal{I}_{h,0}^{c,\text{av}}(\mathbf{v}_h))\|_K \leq C(\kappa_{\mathcal{T}_h}) c_k^{\text{av}} \sqrt{\frac{k^2}{h_K}} \|\llbracket \mathbf{v}_h \rrbracket_{\partial K}^c\|_{\partial K}, \quad (6.9)$$

and

$$\|\mathbf{v}_h - \mathcal{I}_{h,0}^{c,\text{av}}(\mathbf{v}_h)\|_{\#}^2 \leq C(\kappa_{\mathcal{T}_h}) (c_k^{\text{av}})^2 \left( 1 + \max_{K \in \mathcal{T}_h} \frac{\omega h_K}{k \vartheta_{\min, \tilde{K}}} \right)^2 \sum_{K \in \mathcal{T}_h} \nu_{\min, \tilde{K}} \frac{k^4}{h_K^2} \|\llbracket \mathbf{v}_h \rrbracket_{\partial K}^c\|_{\partial K}^2. \quad (6.10)$$

**Remark 6.1** (Broken curl). *In view of (6.8), we can freely replace the discrete curl  $\mathbf{C}_{h,0}^{k,\ell}$  by the broken curl in the definition of the estimator  $\eta$  and the error measure  $\|\cdot\|_{\dagger}$ .*

We start by controlling the PDE residual in Lemma 6.2. Lemma 6.2 is similar to [14, Lemma 3.2], but the result proposed here is sharper. In particular, the constant only depends on the shape-regularity of the mesh. Notice also that we consider here only conforming test functions so that we can work with the bilinear form  $b_{\sharp}$  rather than  $b_{\sharp s}$ .

**Lemma 6.2** (Residual). *For all  $\mathbf{v} \in \mathbf{H}_0(\mathbf{curl}; D)$ , we have*

$$|b_{\sharp}(\mathbf{e}, \mathbf{v})| \leq C(\kappa_{\mathcal{T}_h}) \eta_{\text{dc}} \|\mathbf{v}\|, \quad (6.11)$$

with  $\eta_{\text{dc}}^2 := \eta_{\text{div}}^2 + \eta_{\text{curl}}^2$ .

*Proof.* Here, we invoke [41, Theorem 1], which states that, given any  $\mathbf{w} \in \mathbf{H}_0(\mathbf{curl}; D)$ , there exists  $\mathbf{S}_{h0}^c(\mathbf{w}) \in \mathbf{P}_{k,0}^c(\mathcal{T}_h)$ , such that

$$\mathbf{w} - \mathbf{S}_{h0}^c(\mathbf{w}) = \nabla q + \phi, \quad (6.12)$$

with  $q \in H_0^1(D)$ ,  $\phi \in \mathbf{H}_0^1(D)$  such that, for all  $K \in \mathcal{T}_h$ ,

$$h_K^{-1} \|q\|_K + \|\nabla q\|_K \leq C(\kappa_{\mathcal{T}_h}) \|\mathbf{w}\|_{\tilde{K}}, \quad h_K^{-1} \|\phi\|_K + \|\nabla \phi\|_K \leq C(\kappa_{\mathcal{T}_h}) \|\nabla_0 \times \mathbf{w}\|_{\tilde{K}}. \quad (6.13)$$

We now pick an arbitrary test function  $\mathbf{v} \in \mathbf{H}_0(\mathbf{curl}; D)$ . We have

$$b_{\sharp}(\mathbf{e}, \mathbf{v}) = b_{\sharp}(\mathbf{e}, \mathbf{v} - \mathbf{S}_{h0}^c(\mathbf{v})) = b_{\sharp}(\mathbf{e}, \nabla q + \phi),$$

where  $q$  and  $\phi$  are the components of the decomposition in (6.12). We then estimate the two parts of the residual associated with the decomposition separately.

For the gradient part, we write

$$\begin{aligned} b_{\sharp}(\mathbf{e}, \nabla q) &= b_{\sharp}(\mathbf{e}, \nabla(q - \mathcal{I}_{h0}^g(q))) \\ &= -\omega^2(\boldsymbol{\epsilon}(\mathbf{e}), \nabla(q - \mathcal{I}_{h0}^g(q))) \\ &= \sum_{K \in \mathcal{T}_h} \omega^2(\nabla \cdot (\boldsymbol{\epsilon}(\mathbf{e})), q - \mathcal{I}_{h0}^g(q))_K - \sum_{F \in \mathcal{F}_h^{\text{int}}} \omega^2([\boldsymbol{\epsilon}(\mathbf{e})]_F^{\text{d}}, q - \mathcal{I}_{h0}^g(q))_F \\ &= \sum_{K \in \mathcal{T}_h} -(\nabla \cdot (\mathbf{J} + \omega^2 \boldsymbol{\epsilon} \mathbf{E}_h), q - \mathcal{I}_{h0}^g(q))_K + \sum_{F \in \mathcal{F}_h^{\text{int}}} \omega^2([\boldsymbol{\epsilon} \mathbf{E}_h]_F^{\text{d}}, q - \mathcal{I}_{h0}^g(q))_F \\ &\leq \sum_{K \in \mathcal{T}_h} \{ \|\nabla \cdot (\mathbf{J} + \omega^2 \boldsymbol{\epsilon} \mathbf{E}_h)\|_K \|q - \mathcal{I}_{h0}^g(q)\|_K + \omega^2 \|[\boldsymbol{\epsilon} \mathbf{E}_h]_{\partial K}^{\text{d}}\|_{\partial K \setminus \partial \Omega} \|q - \mathcal{I}_{h0}^g(q)\|_{\partial K} \} \\ &\leq \sum_{K \in \mathcal{T}_h} \eta_{K, \text{div}} \varepsilon_{\min, \tilde{K}}^{\frac{1}{2}} \omega \left\{ \frac{k}{h_K} \|q - \mathcal{I}_{h0}^g(q)\|_K + \sqrt{\frac{k}{h_K}} \|q - \mathcal{I}_{h0}^g(q)\|_{\partial K} \right\}. \end{aligned}$$

For all  $K \in \mathcal{T}_h$ , invoking (6.5) and (6.13) and since  $h_K \leq \ell_D$ , we have

$$\begin{aligned} \frac{k}{h_K} \|q - \mathcal{I}_{h0}^g(q)\|_K + \sqrt{\frac{k}{h_K}} \|q - \mathcal{I}_{h0}^g(q)\|_{\partial K} &\leq C(\kappa_{\mathcal{T}_h}) \ell_D^{-1} \|q\|_{H^1(\tilde{K})} \\ &\leq C(\kappa_{\mathcal{T}_h}) \|\mathbf{v}\|_{\tilde{K}} \leq C(\kappa_{\mathcal{T}_h}) \varepsilon_{\min, \tilde{K}}^{-\frac{1}{2}} \|\mathbf{v}\|_{\boldsymbol{\epsilon}, \tilde{K}}. \end{aligned}$$

Summing over  $K \in \mathcal{T}_h$  and since the number of overlaps is uniformly controlled by  $\kappa_{\mathcal{T}_h}$ , we obtain

$$|b_{\sharp}(\mathbf{e}, \nabla q)| \leq C(\kappa_{\mathcal{T}_h}) \eta_{\text{div}} \omega \|\mathbf{v}\|_{\boldsymbol{\epsilon}}. \quad (6.14)$$

For the  $\mathbf{H}_0^1(D)$ -part, proceeding similarly gives

$$\begin{aligned}
b_{\sharp}(\mathbf{e}, \phi) &= b_{\sharp}(\mathbf{e}, \phi - \mathcal{I}_{h0}^c(\phi)) \\
&= (\mathbf{J} + \omega^2 \boldsymbol{\epsilon} \mathbf{E}_h, \phi - \mathcal{I}_{h0}^c(\phi)) - (\mathbf{C}_{h,0}^{k,\ell}(\mathbf{E}_h), \nabla_0 \times (\phi - \mathcal{I}_{h0}^c(\phi)))_{\nu} \\
&= \sum_{K \in \mathcal{T}_h} (\mathbf{J} + \omega^2 \boldsymbol{\epsilon} \mathbf{E}_h - \nabla \times (\nu \mathbf{C}_{h,0}^{k,\ell}(\mathbf{E}_h)), \phi - \mathcal{I}_{h0}^c(\phi))_K \\
&\quad - \sum_{F \in \mathcal{F}_h} ([\nu \mathbf{C}_{h,0}^{k,\ell}(\mathbf{E}_h)]_F^c, \phi - \mathcal{I}_{h0}^c(\phi))_F \\
&\leq \sum_{K \in \mathcal{T}_h} \eta_{K, \text{curl}} \nu_{\min, \tilde{K}}^{\frac{1}{2}} \left\{ \frac{k}{h_K} \|\phi - \mathcal{I}_{h0}^c(\phi)\|_K + \sqrt{\frac{k}{h_K}} \|(\phi - \mathcal{I}_{h0}^c(\phi)) \times \mathbf{n}\|_{\partial K} \right\} \\
&\leq C(\kappa_{\mathcal{T}_h}) \sum_{K \in \mathcal{T}_h} \eta_{K, \text{curl}} \nu_{\min, \tilde{K}}^{\frac{1}{2}} \ell_D^{-1} \|\phi\|_{\mathbf{H}^1(\tilde{K})} \\
&\leq C(\kappa_{\mathcal{T}_h}) \sum_{K \in \mathcal{T}_h} \eta_{K, \text{curl}} \nu_{\min, \tilde{K}}^{\frac{1}{2}} \|\nabla_0 \times \mathbf{v}\|_{\tilde{K}} \\
&\leq C(\kappa_{\mathcal{T}_h}) \sum_{K \in \mathcal{T}_h} \eta_{K, \text{curl}} \|\nabla_0 \times \mathbf{v}\|_{\nu, \tilde{K}},
\end{aligned}$$

so that

$$|b_{\sharp}(\mathbf{e}, \phi)| \leq C(\kappa_{\mathcal{T}_h}) \eta_{\text{curl}} \|\nabla_0 \times \mathbf{v}\|_{\nu}. \quad (6.15)$$

Combining (6.14) and (6.15) concludes the proof.  $\square$

The next step is an Aubin–Nitsche-type duality argument to estimate the  $\mathbf{L}_{\boldsymbol{\epsilon}}^2$ -norm of the error. Here, the weak consistency estimate from Lemma 4.2 is crucial to treat the nonconformity of the dG solution.

**Lemma 6.3** ( $\mathbf{L}_{\boldsymbol{\epsilon}}^2$ -norm reliability estimate). *We have*

$$\omega \|\mathbf{e}\|_{\boldsymbol{\epsilon}} \leq C(\kappa_{\mathcal{T}_h}) (1 + \gamma_{\text{ap}} + c_k^{\text{av}} \gamma_{\text{ap}^*}) \eta. \quad (6.16)$$

*Proof.* Recall the  $\mathbf{L}_{\boldsymbol{\epsilon}}^2$ -orthogonal decomposition  $\mathbf{e} = \boldsymbol{\theta}_0 + \boldsymbol{\theta}_{\Pi}$  with  $\boldsymbol{\theta}_0 \in \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^{\perp}$  and  $\boldsymbol{\theta}_{\Pi} \in \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)$  (see (5.1)).

For the first component, we have

$$\omega \|\boldsymbol{\theta}_0\|_{\boldsymbol{\epsilon}}^2 = \omega(\mathbf{e}, \boldsymbol{\theta}_0)_{\boldsymbol{\epsilon}} = \omega b_{\sharp}(\mathbf{e}, \boldsymbol{\zeta}_{\boldsymbol{\theta}}) - \omega \delta_{\text{wkc}}(\mathbf{E}_h, \boldsymbol{\zeta}_{\boldsymbol{\theta}}). \quad (6.17)$$

Since  $\mathbf{E}_h$  satisfies the Galerkin orthogonality for conforming test functions and invoking the bound (6.11) established in Lemma 6.2, we have, for all  $\mathbf{v}_h^c \in \mathbf{P}_{k,0}^c(\mathcal{T}_h)$ ,

$$\omega b_{\sharp}(\mathbf{e}, \boldsymbol{\zeta}_{\boldsymbol{\theta}}) = \omega b_{\sharp}(\mathbf{e}, \boldsymbol{\zeta}_{\boldsymbol{\theta}} - \mathbf{v}_h^c) \leq C(\kappa_{\mathcal{T}_h}) \eta \omega \|\boldsymbol{\zeta}_{\boldsymbol{\theta}} - \mathbf{v}_h^c\| \leq C(\kappa_{\mathcal{T}_h}) \gamma_{\text{ap}} \eta_{\text{dc}} \|\boldsymbol{\theta}_0\|_{\boldsymbol{\epsilon}}, \quad (6.18)$$

where we used the definition (4.9a) of  $\gamma_{\text{ap}}$  in the last inequality (recall that  $\mathbf{v}_h^c$  is arbitrary in  $\mathbf{P}_{k,0}^c(\mathcal{T}_h)$ ). Moreover, owing to the estimate (4.6) from Lemma 4.2, we have

$$\omega |\delta_{\text{wkc}}(\mathbf{E}_h, \boldsymbol{\zeta}_{\boldsymbol{\theta}})| \leq \|\mathbf{E}_h\|_{\text{nc}} \gamma_{\text{ap}^*} \|\boldsymbol{\theta}_0\|_{\boldsymbol{\epsilon}}.$$

Recalling the definition (4.5) of the  $|\cdot|_{\text{nc}}$ -seminorm and using (6.9), we infer that

$$\|\mathbf{E}_h\|_{\text{nc}} \leq \|\mathbf{E}_h - \mathcal{I}_{h0}^{\text{c,av}}(\mathbf{E}_h)\|_{\text{ap}} \leq C(\kappa_{\mathcal{T}_h}) c_k^{\text{av}} \eta_{\text{mc}},$$

where we used that  $\nu_{\max, K} k^2 \leq \psi_K(\nu, k)$  (see (6.2c)). This gives

$$\omega |\delta_{\text{wkc}}(\mathbf{E}_h, \boldsymbol{\zeta}_{\boldsymbol{\theta}})| \leq C(\kappa_{\mathcal{T}_h}) c_k^{\text{av}} \eta_{\text{nc}} \gamma_{\text{ap}^*} \|\boldsymbol{\theta}_0\|_{\boldsymbol{\epsilon}}. \quad (6.19)$$

Combining (6.17), (6.18) and (6.19), we arrive at

$$\omega \|\boldsymbol{\theta}_0\|_\epsilon \leq C(\kappa_{\mathcal{T}_h})(\gamma_{\text{ap}} + c_k^{\text{av}} \gamma_{\text{ap}^*})\eta. \quad (6.20)$$

For the other part of the error, since  $\boldsymbol{\theta}_\Pi \in \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)$ , we can use (6.11) to write

$$\omega^2 \|\boldsymbol{\theta}_\Pi\|_\epsilon^2 = -b_\sharp(\mathbf{e}, \boldsymbol{\theta}_\Pi) \leq C(\kappa_{\mathcal{T}_h})\eta_{\text{dc}} \|\boldsymbol{\theta}_\Pi\| = C(\kappa_{\mathcal{T}_h})\eta_{\text{dc}}\omega \|\boldsymbol{\theta}_\Pi\|_\epsilon. \quad (6.21)$$

Combining (6.20) and (6.21) proves (6.16).  $\square$

We are now ready to establish a reliability estimate with an argument similar to the one used in [11] for the scalar Helmholtz problem.

**Theorem 6.4** (Reliability). *We have*

$$\|\mathbf{e}\|_\dagger \leq C(\kappa_{\mathcal{T}_h})c_k^{\text{av}} \left( 1 + \max_{K \in \mathcal{T}_h} \frac{\omega h_K}{k \vartheta_{\min, \widehat{K}}} + \gamma_{\text{ap}} + \gamma_{\text{ap}^*} \right) \eta. \quad (6.22)$$

*Proof.* Since  $\|\mathbf{e}\|_\dagger^2 = \|\mathbf{e}\|_\sharp^2 + \eta_{\text{nc}}^2$ , we only need to estimate  $\|\mathbf{e}\|_\sharp^2$ . To this purpose, recall that the bilinear form  $b_\sharp^+$  defined in (5.3b) is the inner product associated with the  $\|\cdot\|_\sharp$ -norm. We introduce the conforming  $\mathbf{H}_0(\mathbf{curl}; D)$ -conforming projection  $\mathcal{B}_0^c : \mathbf{V}_\sharp \rightarrow \mathbf{H}_0(\mathbf{curl}; D)$  such that  $b_\sharp^+(\mathbf{v} - \mathcal{B}_0^c(\mathbf{v}), \mathbf{w}) = 0$  for all  $\mathbf{v} \in \mathbf{V}_\sharp$  and all  $\mathbf{w} \in \mathbf{H}_0(\mathbf{curl}; D)$ . Reasoning as in the proof of Lemma 5.1 proves the following Pythagorean identity:

$$\|\mathbf{e}\|_\sharp^2 = \|\mathbf{E} - \mathcal{B}_0^c(\mathbf{E}_h)\|_\sharp^2 + \|\mathbf{E}_h - \mathcal{B}_0^c(\mathbf{E}_h)\|_\sharp^2.$$

We estimate separately the two terms on the right-hand side.

For the first term, we observe that  $\mathbf{E} - \mathcal{B}_0^c(\mathbf{E}_h) \in \mathbf{H}_0(\mathbf{curl}; D)$ , and write that

$$\begin{aligned} \|\mathbf{E} - \mathcal{B}_0^c(\mathbf{E}_h)\|_\sharp^2 &= b_\sharp^+(\mathbf{E} - \mathcal{B}_0^c(\mathbf{E}_h), \mathbf{E} - \mathcal{B}_0^c(\mathbf{E}_h)) \\ &= b_\sharp^+(\mathbf{e}, \mathbf{E} - \mathcal{B}_0^c(\mathbf{E}_h)) \\ &= b_\sharp(\mathbf{e}, \mathbf{E} - \mathcal{B}_0^c(\mathbf{E}_h)) + 2\omega^2(\mathbf{e}, \mathbf{E} - \mathcal{B}_0^c(\mathbf{E}_h))_\epsilon. \end{aligned}$$

Since the second argument in the first term on the right-hand side is conforming, this term can be estimated by means of the estimate (6.11) from Lemma 6.2. This gives

$$|b_\sharp(\mathbf{e}, \mathbf{E} - \mathcal{B}_0^c(\mathbf{E}_h))| \leq C(\kappa_{\mathcal{T}_h})\eta_{\text{dc}} \|\mathbf{E} - \mathcal{B}_0^c(\mathbf{E}_h)\|.$$

We apply the Cauchy–Schwarz inequality to bound the second term, leading to

$$\omega^2 |(e, \mathbf{E} - \mathcal{B}_0^c(\mathbf{E}_h))_\epsilon| \leq \omega \|e\|_\epsilon \omega \|\mathbf{E} - \mathcal{B}_0^c(\mathbf{E}_h)\|_\epsilon \leq \omega \|e\|_\epsilon \|\mathbf{E} - \mathcal{B}_0^c(\mathbf{E}_h)\|.$$

This yields the following estimate:

$$\|\mathbf{E} - \mathcal{B}_0^c(\mathbf{E}_h)\|_\sharp \leq C(\kappa_{\mathcal{T}_h})\eta_{\text{dc}} + \omega \|e\|_\epsilon \leq C(\kappa_{\mathcal{T}_h})c_k^{\text{av}}(1 + \gamma_{\text{ap}} + \gamma_{\text{ap}^*})\eta,$$

where we employed the  $\mathbf{L}_\epsilon^2$ -estimate (6.16) from Lemma 6.3.

For the second term, invoking (6.10) gives

$$\|\mathbf{E}_h - \mathcal{B}_0^c(\mathbf{E}_h)\|_\sharp \leq \|\mathbf{E}_h - \mathcal{I}_{h0}^{\text{c,av}}(\mathbf{E}_h)\|_\sharp \leq C(\kappa_{\mathcal{T}_h})c_k^{\text{av}} \left( 1 + \max_{K \in \mathcal{T}_h} \frac{\omega h_K}{k \vartheta_{\min, \widehat{K}}} \right) \eta_{\text{nc}},$$

where we used that  $\nu_{\min, \widehat{K}} k^4 \leq \psi_K(\nu, k)$  (see (6.2c)). Putting everything together yields the assertion.  $\square$

## 6.4 Local error lower bound (local efficiency)

We now derive efficiency estimates. To do so, we will need bubble functions (see [35] and [14, Section 2.7]). Specifically, for all  $K \in \mathcal{T}_h$ , there exists a function  $b_K \in H_0^1(K)$  with  $b_K \leq C(\kappa_{\mathcal{T}_h})$  such that, for all  $\mathbf{v}_K \in \mathbf{P}_{k,d}$ ,

$$\|\mathbf{v}_K\|_K \leq C(\kappa_{\mathcal{T}_h})k \|b_K^{\frac{1}{2}} \mathbf{v}_K\|_K, \quad (6.23)$$

and

$$\|\nabla(b_K \mathbf{v}_K)\|_K \leq C(\kappa_{\mathcal{T}_h}) \frac{k}{h_K} \|b_K^{\frac{1}{2}} \mathbf{v}_K\|_K. \quad (6.24)$$

Similarly, for all  $F \in \mathcal{F}_h$ , there exists a function  $b_F \in H_0^1(F)$  such that, for all  $\mathbf{v}_F \in \mathbf{P}_{k,d-1}$ ,

$$\|\mathbf{v}_F\|_F \leq C(\kappa_{\mathcal{T}_h})k \|b_F^{\frac{1}{2}} \mathbf{v}_F\|_F, \quad (6.25)$$

and an extension operator  $\mathcal{E}_F : \mathbf{P}_{k,d-1} \rightarrow \mathbf{H}_0^1(\tilde{F})$  such that, for all  $\mathbf{v}_F \in \mathbf{P}_{k,d-1}$ ,  $\mathcal{E}_F(\mathbf{v}_F)|_F = b_F \mathbf{v}_F$  and

$$kh_F^{-\frac{1}{2}} \|\mathcal{E}_F(\mathbf{v}_h)\|_{\tilde{F}} + k^{-1} h_F^{\frac{1}{2}} \|\nabla \mathcal{E}_F(\mathbf{v}_h)\|_{\tilde{F}} \leq C(\kappa_{\mathcal{T}_h}) \|b_F^{\frac{1}{2}} \mathbf{v}_h\|_F. \quad (6.26)$$

**Theorem 6.5** (Local efficiency). *For all  $K \in \mathcal{T}_h$ , we have*

$$\eta_K \leq C(\kappa_{\mathcal{T}_h}) \mathcal{K}_K^{\frac{1}{2}} k^{\frac{3}{2}} \left\{ \left( 1 + \frac{\omega h_K}{k \vartheta_{\min, \tilde{K}}} \right) \|\mathbf{E} - \mathbf{E}_h\|_{\dagger, \tilde{K}} + \text{osc}_{\tilde{K}} \right\}, \quad (6.27)$$

with the data oscillation term

$$\text{osc}_{\tilde{K}}^2 := \frac{1}{\omega^2} \varepsilon_{\min, \tilde{K}}^{-1} \sum_{K' \in \tilde{K}} \min_{\mathbf{J}_h \in \mathbf{P}_k^b(\mathcal{T}_h)} \left\{ \frac{\omega^2 h_{K'}^2}{k^2 \vartheta_{\min, \tilde{K}}} \|\mathbf{J} - \mathbf{J}_h\|_{K'}^2 + \frac{h_{K'}^2}{k^2} \|\nabla \cdot (\mathbf{J} - \mathbf{J}_h)\|_{K'}^2 \right\}, \quad (6.28)$$

and the contrast coefficient  $\mathcal{K}_K := \max \left\{ \frac{\varepsilon_{\max, \tilde{K}}}{\varepsilon_{\min, \tilde{K}}}, \frac{\nu_{\max, \tilde{K}}}{\nu_{\min, \tilde{K}}} \right\}$ .

*Proof.* Fix  $K \in \mathcal{T}_h$ . The proof contains three parts, respectively dedicated to providing upper bounds for  $\eta_{K, \text{div}}$ ,  $\eta_{K, \text{curl}}$  and  $\eta_{K, \text{nc}}$ .

(i) The proof that  $\eta_{K, \text{div}} \leq C(\kappa_{\mathcal{T}_h}) k^{\frac{3}{2}} (\omega \|\mathbf{E} - \mathbf{E}_h\|_{\varepsilon, \tilde{K}} + \text{osc}_{\tilde{K}})$  can be found in [14, Lemma 3.5]. This proof is established for conforming edge finite elements, but it holds verbatim in the discontinuous Galerkin setting.

(ii) For  $\eta_{K, \text{curl}}$ , we need to slightly adapt the proof from [14, Lemma 3.6]. The volumic residual and jump term in  $\eta_{K, \text{curl}}$  are estimated separately.

(iii) For the volume term, we introduce  $\mathbf{r}_K := \mathbf{J}_K + \omega^2 \boldsymbol{\varepsilon} \mathbf{E}_h - \nabla \times (\boldsymbol{\nu} \mathbf{C}_{h,0}^{k,\ell}(\mathbf{E}_h))|_K$  and  $\mathbf{v}_K = b_K \mathbf{r}_K$ , with  $\mathbf{J}_K$  arbitrary in  $\mathbf{P}_{k,d}$ . We observe that  $\mathbf{v}_K$  vanishes on  $\partial K$ , so that, letting  $\mathbf{v}_h$  be the zero-extension of  $\mathbf{v}_K$  to  $D$ , we infer that

$$\|b_K^{\frac{1}{2}} \mathbf{r}_K\|_K^2 = (\mathbf{r}_h, \mathbf{v}_K)_K = (\mathbf{J}_K, \mathbf{v}_K)_K - b_{\text{ps}}(\mathbf{E}_h, \mathbf{v}_h) = b_{\text{ps}}(\mathbf{E} - \mathbf{E}_h, \mathbf{v}_h) - (\mathbf{J} - \mathbf{J}_K, \mathbf{v}_K)_K.$$

For the first term, we employ (6.24) to show that

$$\begin{aligned} & |b_{\text{ps}}(\mathbf{E} - \mathbf{E}_h, \mathbf{v}_h)| \\ & \leq \omega \|\mathbf{E} - \mathbf{E}_h\|_{\varepsilon, K} \omega \varepsilon_{\max, K}^{\frac{1}{2}} \|\mathbf{v}_K\|_K + \|\mathbf{C}_{h,0}^{k,\ell}(\mathbf{E} - \mathbf{E}_h)\|_{\nu, K} \nu_{\max, K}^{\frac{1}{2}} \|\nabla \times \mathbf{v}_K\|_K \\ & \leq C(\kappa_{\mathcal{T}_h}) \left( (\omega \varepsilon_{\max, K}^{\frac{1}{2}} \omega \|\mathbf{E} - \mathbf{E}_h\|_{\varepsilon, K} + \nu_{\max, K}^{\frac{1}{2}} \frac{k}{h_K} \|\mathbf{C}_{h,0}^{k,\ell}(\mathbf{E} - \mathbf{E}_h)\|_{\nu, K}) \|b_K^{\frac{1}{2}} \mathbf{r}_K\|_K \right) \\ & \leq C(\kappa_{\mathcal{T}_h}) \nu_{\max, K}^{\frac{1}{2}} \frac{k}{h_K} \left( \frac{\omega h_K}{k \vartheta_K} \omega \|\mathbf{E} - \mathbf{E}_h\|_{\varepsilon, K} + \|\mathbf{C}_{h,0}^{k,\ell}(\mathbf{E} - \mathbf{E}_h)\|_{\nu, K} \right) \|b_K^{\frac{1}{2}} \mathbf{r}_K\|_K. \end{aligned}$$

The second term is simply estimated as follows

$$|(\mathbf{J} - \mathbf{J}_K, \mathbf{v}_K)_K| \leq \|\mathbf{J} - \mathbf{J}_K\|_K \|\mathbf{v}_K\|_K \leq C(\kappa_{\mathcal{T}_h}) \left( \nu_{\max, K}^{\frac{1}{2}} \frac{k}{h_K} \right) \nu_{\max, K}^{-\frac{1}{2}} \frac{h_K}{k} \|\mathbf{J} - \mathbf{J}_K\|_K \|b_{\frac{1}{2}}^{\frac{1}{2}} \mathbf{r}_K\|_K.$$

Combining these two bounds leads to

$$\nu_{\max, K}^{-\frac{1}{2}} \frac{h_K}{k} \|b_{\frac{1}{2}}^{\frac{1}{2}} \mathbf{r}_K\|_K \leq C(\kappa_{\mathcal{T}_h}) \left\{ \left( 1 + \frac{\omega h_K}{k \vartheta_K} \right) \|\mathbf{E} - \mathbf{E}_h\|_{\#, K} + \nu_{\max, K}^{-\frac{1}{2}} \frac{h_K}{k} \|\mathbf{J} - \mathbf{J}_K\|_K \right\},$$

and therefore

$$\begin{aligned} \nu_{\max, K}^{-\frac{1}{2}} \frac{h_K}{k} \|\mathbf{r}_K\|_K &\leq k \nu_{\max, K}^{-\frac{1}{2}} \frac{h_K}{k} \|b_{\frac{1}{2}}^{\frac{1}{2}} \mathbf{r}_K\|_K \\ &\leq C(\kappa_{\mathcal{T}_h}) k \left\{ \left( 1 + \frac{\omega h_K}{k \vartheta_K} \right) \|\mathbf{E} - \mathbf{E}_h\|_{\#, K} + \nu_{\max, K}^{-\frac{1}{2}} \frac{h_K}{k} \|\mathbf{J} - \mathbf{J}_K\|_K \right\}. \end{aligned}$$

Invoking the triangle inequality leads to

$$\begin{aligned} \nu_{\max, K}^{-\frac{1}{2}} \frac{h_K}{k} \|\mathbf{J} + \omega^2 \boldsymbol{\epsilon} \mathbf{E}_h - \nabla \times (\boldsymbol{\nu} \mathbf{C}_{h,0}^{k,\ell}(\mathbf{E}_h))\|_K \\ \leq C(\kappa_{\mathcal{T}_h}) k \left\{ \left( 1 + \frac{\omega h_K}{k \vartheta_K} \right) \|\mathbf{E} - \mathbf{E}_h\|_{\#, K} + \nu_{\max, K}^{-\frac{1}{2}} \frac{h_K}{k} \|\mathbf{J} - \mathbf{J}_K\|_K \right\}. \end{aligned} \quad (6.29)$$

(iib) For the jump term, we introduce  $\mathbf{r}_F := \llbracket \boldsymbol{\nu} \mathbf{C}_{h,0}^{k,\ell}(\mathbf{E}_h) \rrbracket_F^c$  and  $\mathbf{v}_F := \mathcal{E}_F(\mathbf{r}_F) \in \mathbf{H}_0^1(\tilde{F})$ . Since  $\mathbf{v}_F \in \mathbf{H}_0^1(\tilde{F})$ , we have

$$\begin{aligned} \|b_{\tilde{F}}^{\frac{1}{2}} \mathbf{r}_F\|_{\tilde{F}}^2 &= (\mathbf{r}_F, \mathbf{v}_F)_F \\ &= (\nabla \times (\boldsymbol{\nu} \mathbf{C}_{h,0}^{k,\ell}(\mathbf{E}_h)), \mathbf{v}_F)_{\tilde{F}} - (\boldsymbol{\nu} \mathbf{C}_{h,0}^{k,\ell}(\mathbf{E}_h), \nabla \times \mathbf{v}_F)_{\tilde{F}} \\ &= (\boldsymbol{\nu} \mathbf{C}_{h,0}^{k,\ell}(\mathbf{E} - \mathbf{E}_h), \nabla \times \mathbf{v}_F)_{\tilde{F}} - (\nabla \times (\boldsymbol{\nu} \mathbf{C}_{h,0}^{k,\ell}(\mathbf{E} - \mathbf{E}_h)), \mathbf{v}_F)_{\tilde{F}}. \end{aligned} \quad (6.30)$$

For the first term, we can immediately write that

$$\begin{aligned} |(\boldsymbol{\nu} \mathbf{C}_{h,0}^{k,\ell}(\mathbf{E} - \mathbf{E}_h), \nabla \times \mathbf{v}_F)_{\tilde{F}}| &\leq \nu_{\max, \tilde{F}}^{\frac{1}{2}} \|\mathbf{C}_{h,0}^{k,\ell}(\mathbf{E} - \mathbf{E}_h)\|_{\nu, \tilde{F}} \|\nabla \times \mathbf{v}_F\|_{\tilde{F}} \\ &\leq C(\kappa_{\mathcal{T}_h}) k h_F^{-\frac{1}{2}} \nu_{\max, \tilde{F}}^{\frac{1}{2}} \|\mathbf{C}_{h,0}^{k,\ell}(\mathbf{E} - \mathbf{E}_h)\|_{\nu, \tilde{F}} \|b_{\tilde{F}}^{\frac{1}{2}} \mathbf{r}_F\|_{\tilde{F}}, \end{aligned}$$

where we employed (6.26). We infer that

$$\nu_{\max, \tilde{F}}^{-\frac{1}{2}} \sqrt{\frac{h_F}{k}} |(\boldsymbol{\nu} \mathbf{C}_{h,0}^{k,\ell}(\mathbf{E} - \mathbf{E}_h), \nabla \times \mathbf{v}_F)_{\tilde{F}}| \leq C(\kappa_{\mathcal{T}_h}) k^{\frac{1}{2}} \|\mathbf{C}_{h,0}^{k,\ell}(\mathbf{E} - \mathbf{E}_h)\|_{\nu, \tilde{F}} \|b_{\tilde{F}}^{\frac{1}{2}} \mathbf{r}_F\|_{\tilde{F}}. \quad (6.31)$$

For the second term, we first observe that

$$\begin{aligned} \nabla \times (\boldsymbol{\nu} \mathbf{C}_{h,0}^{k,\ell}(\mathbf{E} - \mathbf{E}_h)) &= \mathbf{J} + \omega^2 \boldsymbol{\epsilon} \mathbf{E} - \nabla \times (\boldsymbol{\nu} \mathbf{C}_{h,0}^{k,\ell} \mathbf{E}_h) \\ &= \omega^2 \boldsymbol{\epsilon} (\mathbf{E} - \mathbf{E}_h) + \mathbf{J} + \omega^2 \boldsymbol{\epsilon} \mathbf{E}_h - \nabla \times (\boldsymbol{\nu} \mathbf{C}_{h,0}^{k,\ell} \mathbf{E}_h), \end{aligned}$$

and therefore,

$$\begin{aligned} &\|\nabla \times (\boldsymbol{\nu} \mathbf{C}_{h,0}^{k,\ell}(\mathbf{E} - \mathbf{E}_h))\|_{\tilde{F}} \\ &\leq \omega \epsilon_{\max, \tilde{F}}^{\frac{1}{2}} \|\mathbf{E} - \mathbf{E}_h\|_{\epsilon, \tilde{F}} + \|\mathbf{J} + \omega^2 \boldsymbol{\epsilon} \mathbf{E}_h - \nabla \times (\boldsymbol{\nu} \mathbf{C}_{h,0}^{k,\ell} \mathbf{E}_h)\|_{\tilde{F}} \\ &\leq \nu_{\max, \tilde{F}}^{\frac{1}{2}} \frac{k}{h_K} \left( \frac{\omega h_K}{k \vartheta_{\tilde{F}}} \|\mathbf{E} - \mathbf{E}_h\|_{\epsilon, \tilde{F}} + \nu_{\max, \tilde{F}}^{-\frac{1}{2}} \frac{h_K}{k} \|\mathbf{J} + \omega^2 \boldsymbol{\epsilon} \mathbf{E}_h - \nabla \times (\boldsymbol{\nu} \mathbf{C}_{h,0}^{k,\ell} \mathbf{E}_h)\|_{\tilde{F}} \right). \end{aligned}$$

Combining this last estimate with (6.26) and invoking the Cauchy–Schwarz inequality gives

$$\begin{aligned} & \nu_{\max, \tilde{F}}^{-\frac{1}{2}} \sqrt{\frac{h_K}{k}} |(\nabla \times (\boldsymbol{\nu} \mathbf{C}_{h,0}^{k,\ell}(\mathbf{E} - \mathbf{E}_h)), \mathbf{v}_F)_{\tilde{F}}| \leq C(\kappa_{\mathcal{T}_h}) \times \\ & \left( \frac{\omega h_K}{\vartheta_{\tilde{F}} k} \omega \|\mathbf{E} - \mathbf{E}_h\|_{\epsilon, \tilde{F}} + \nu_{\max, \tilde{F}}^{-\frac{1}{2}} \frac{h_K}{k} \|\mathbf{J} + \omega^2 \boldsymbol{\epsilon} \mathbf{E}_h - \nabla \times (\boldsymbol{\nu}(\mathbf{C}_{h,0}^{k,\ell} \mathbf{E}_h))\|_{\tilde{F}} \right) k^{\frac{1}{2}} \|b_{\tilde{F}}^{\frac{1}{2}} \mathbf{r}_F\|_F. \end{aligned} \quad (6.32)$$

We can now plug (6.31) and (6.32) in (6.30), leading to

$$\begin{aligned} & \nu_{\max, \tilde{F}}^{-\frac{1}{2}} \sqrt{\frac{h_K}{k}} \|b_{\tilde{F}}^{\frac{1}{2}} \mathbf{r}_F\|_F \\ & \leq C(\kappa_{\mathcal{T}_h}) k^{\frac{1}{2}} \left( \frac{\omega h_K}{\vartheta_{\tilde{F}} k} \omega \|\mathbf{E} - \mathbf{E}_h\|_{\epsilon, \tilde{F}} + \nu_{\max, \tilde{F}}^{-\frac{1}{2}} \frac{h_K}{k} \|\mathbf{J} + \omega^2 \boldsymbol{\epsilon} \mathbf{E}_h - \nabla \times (\boldsymbol{\nu}(\mathbf{C}_{h,0}^{k,\ell} \mathbf{E}_h))\|_{\tilde{F}} \right) \\ & \quad + C(\kappa_{\mathcal{T}_h}) k^{\frac{1}{2}} \|\mathbf{C}_{h,0}^{k,\ell}(\mathbf{E} - \mathbf{E}_h)\|_{\boldsymbol{\nu}, \tilde{F}} \\ & \leq C(\kappa_{\mathcal{T}_h}) k^{\frac{1}{2}} \left( 1 + \frac{\omega h_K}{\vartheta_{\tilde{F}} k} \right) \|\mathbf{E} - \mathbf{E}_h\|_{\#, \tilde{F}} \\ & \quad + C(\kappa_{\mathcal{T}_h}) k^{-\frac{1}{2}} \nu_{\max, \tilde{F}}^{-\frac{1}{2}} \frac{h_K}{k} \|\mathbf{J} + \omega^2 \boldsymbol{\epsilon} \mathbf{E}_h - \nabla \times (\boldsymbol{\nu}(\mathbf{C}_{h,0}^{k,\ell} \mathbf{E}_h))\|_{\tilde{F}} \\ & \leq C(\kappa_{\mathcal{T}_h}) k^{\frac{1}{2}} \left( \left( 1 + \frac{\omega h_F}{\vartheta_{\tilde{F}} k} \right) \|\mathbf{E} - \mathbf{E}_h\|_{\#, \tilde{F}} + \nu_{\max, \tilde{F}}^{-\frac{1}{2}} \frac{h_F}{k} \|\mathbf{J} - \mathbf{J}_h\|_{\tilde{F}}, \right) \end{aligned}$$

owing to (6.29). It follows from (6.25) that

$$\eta_{K, \text{curl}} \leq C(\kappa_{\mathcal{T}_h}) k^{\frac{3}{2}} \left( \left( 1 + \frac{\omega h_K}{\vartheta_{\hat{K}} k} \right) \|\mathbf{E} - \mathbf{E}_h\|_{\#, \hat{K}} + \nu_{\max, \hat{K}}^{-\frac{1}{2}} \frac{h_K}{k} \|\mathbf{J} - \mathbf{J}_h\|_{\hat{K}} \right).$$

(iii) For the last part of the estimator, we simply use that  $\eta_{K, \text{nc}}^2 \leq \|\mathbf{E} - \mathbf{E}_h\|_{\#, K}^2 + \eta_{K, \text{nc}}^2 = \|\mathbf{E} - \mathbf{E}_h\|_{\dagger, K}^2$ .  $\square$

## 7 Bound on approximation and divergence conformity factors

In this section, we show that the factors introduced in Section 4.3 tend to zero as the mesh is refined. For positive real numbers  $A$  and  $B$ , we abbreviate as  $A \lesssim B$  the inequality  $A \leq CB$  with a generic (nondimensional) constant  $C$  whose value can change at each occurrence as long as it is independent of the mesh size, the frequency parameter  $\omega$ , and, whenever relevant, any function involved in the bound. The constant  $C$  can depend on the shape-regularity of the mesh, the polynomial degree  $k$ , the (global) contrast in the coefficients (i.e.  $\epsilon_{\max}/\epsilon_{\min}$  and  $\nu_{\max}/\nu_{\min}$ ), and the shape of the domain  $D$  (but not on its size).

For simplicity, we focus on the case where the parameters  $\boldsymbol{\epsilon}$  and  $\boldsymbol{\nu}$  are piecewise constant on a polyhedral partition of  $D$ . Under this assumption (see [18, 30, 5]), there exists  $s \in (0, \frac{1}{2})$  such that, for all  $\mathbf{v} \in \mathbf{H}_0(\mathbf{curl}; D)$  with  $\boldsymbol{\epsilon} \mathbf{v} \in \mathbf{H}(\text{div} = 0; D)$ , and for all  $\mathbf{w} \in \mathbf{H}(\mathbf{curl}; D)$  with  $\boldsymbol{\nu}^{-1} \mathbf{w} \in \mathbf{H}_0(\text{div} = 0; D)$ , we have  $\mathbf{v}, \mathbf{w} \in \mathbf{H}^s(D)$  with

$$|\mathbf{v}|_{\mathbf{H}^s(D)} \lesssim \ell_D^{1-s} \nu_{\min}^{-\frac{1}{2}} \|\nabla \times \mathbf{v}\|_{\boldsymbol{\nu}}, \quad |\mathbf{w}|_{\mathbf{H}^s(D)} \lesssim \ell_D^{1-s} \epsilon_{\max}^{\frac{1}{2}} \|\nabla \times \mathbf{w}\|_{\boldsymbol{\epsilon}^{-1}}. \quad (7.1)$$

Here again, the length scale  $\ell_D$ , e.g., the diameter of  $D$ , is introduced for dimensional consistency. We will also need commuting (quasi-)interpolation operators,  $\mathcal{J}_h^c : \mathbf{H}(\mathbf{curl}; D) \rightarrow \mathbf{P}_k^c(\mathcal{T}_h)$  and  $\mathcal{J}_h^d : \mathbf{H}(\text{div}; D) \rightarrow \mathbf{P}_k^d(\mathcal{T}_h) := \mathbf{P}_k^b(\mathcal{T}_h) \cap \mathbf{H}(\text{div}; D)$  such that  $\nabla \times (\mathcal{J}_h^c(\mathbf{v})) = \mathcal{J}_h^d(\nabla \times \mathbf{v})$  for all  $\mathbf{v} \in \mathbf{H}(\mathbf{curl}; D)$  and

$$\|\mathbf{v} - \mathcal{J}_h^c(\mathbf{v})\| \lesssim h^s |\mathbf{v}|_{\mathbf{H}^s(D)}, \quad \|\mathbf{w} - \mathcal{J}_h^d(\mathbf{w})\| \lesssim \|\mathbf{w}\|, \quad (7.2)$$

for all  $\mathbf{v} \in \mathbf{H}(\mathbf{curl}; D) \cap \mathbf{H}^s(D)$  and all  $\mathbf{w} \in \mathbf{H}(\mathbf{div}; D)$ . Since we are working on simplicial meshes, we can invoke the operators devised in [22, Section 20] (see also [40, 3, 15, 16]) using edge (Nédélec) and Raviart–Thomas finite elements.

**Proposition 7.1** (Primal approximation factor). *Let  $\gamma_{\text{ap}}$  be defined in (4.9a). We have*

$$\gamma_{\text{ap}} \lesssim (1 + \beta_{\text{st}}) \left( \frac{\omega \ell_D}{\vartheta_{\min}} \right)^{1-s} \left( \frac{\omega h}{\vartheta_{\min}} \right)^s,$$

where  $\beta_{\text{st}}$  is the stability constant introduced in (2.8).

*Proof.* See [12, Lemma 5.1]. □

**Lemma 7.2** (Dual approximation factor). *Let  $\gamma_{\text{ap}^*}$  be defined in (4.9b). We have*

$$\gamma_{\text{ap}^*} \lesssim (1 + \beta_{\text{st}}) \left( \frac{\omega \ell_D}{\vartheta_{\min}} \right)^{1-s} \left( \frac{\omega h}{\vartheta_{\min}} \right)^s. \quad (7.3)$$

*Proof.* Consider a right-hand side  $\boldsymbol{\theta} \in \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp$  and the associated adjoint solution  $\boldsymbol{\zeta}_\theta \in \mathbf{H}_0(\mathbf{curl}; D)$  defined in (4.7). Set  $\boldsymbol{\phi}_\theta := \boldsymbol{\nu} \nabla_0 \times \boldsymbol{\zeta}_\theta$ . The strong form of Maxwell’s equations ensures that

$$\nabla \times \boldsymbol{\phi}_\theta = \boldsymbol{\epsilon} \boldsymbol{\theta} + \omega^2 \boldsymbol{\epsilon} \boldsymbol{\zeta}_\theta,$$

so that  $\boldsymbol{\phi}_\theta \in \mathbf{H}(\mathbf{curl}; D)$  with

$$\|\nabla \times \boldsymbol{\phi}_\theta\|_{\epsilon^{-1}} \leq \|\boldsymbol{\theta}\|_\epsilon + \omega^2 \|\boldsymbol{\zeta}_\theta\|_\epsilon \leq (1 + \beta_{\text{st}}) \|\boldsymbol{\theta}\|_\epsilon. \quad (7.4)$$

Besides, since  $\boldsymbol{\nu}^{-1} \boldsymbol{\phi}_\theta \in \mathbf{H}_0(\mathbf{div} = 0; D)$ , we infer that  $\boldsymbol{\phi}_\theta \in \mathbf{H}^s(D)$  with

$$|\boldsymbol{\phi}_\theta|_{\mathbf{H}^s(D)} \lesssim \ell_D^{1-s} \epsilon_{\max}^{\frac{1}{2}} \|\nabla \times \boldsymbol{\phi}_\theta\|_{\epsilon^{-1}}. \quad (7.5)$$

We are now ready to bound  $\gamma_{\text{ap}^*}$ . We notice that

$$\begin{aligned} \gamma_{\text{ap}^*}^2 &\leq \omega^2 \|\boldsymbol{\phi}_\theta - \mathcal{J}_h^c(\boldsymbol{\phi}_\theta)\|_{\text{ap}^*}^2 \\ &= \omega^2 \|\boldsymbol{\phi}_\theta - \mathcal{J}_h^c(\boldsymbol{\phi}_\theta)\|_{\boldsymbol{\nu}^{-1}}^2 + \omega^2 \|\tilde{\nu}^{-\frac{1}{2}} \tilde{h} (\nabla \times \boldsymbol{\phi}_\theta - \mathcal{J}_h^d(\nabla \times \boldsymbol{\phi}_\theta))\|^2, \end{aligned}$$

where we employed the commuting property satisfied by  $\mathcal{J}_h^c$  and  $\mathcal{J}_h^d$ . We bound the two terms on the right-hand side. For the first term, invoking (7.2) and (7.5), we have

$$\begin{aligned} \omega \|\boldsymbol{\phi}_\theta - \mathcal{J}_h^c(\boldsymbol{\phi}_\theta)\|_{\boldsymbol{\nu}^{-1}} &\lesssim \omega \nu_{\min}^{-\frac{1}{2}} h^s |\boldsymbol{\phi}_\theta|_{\mathbf{H}^s(D)} \lesssim \omega \ell_D^{1-s} \frac{h^s}{\vartheta_{\min}} \|\nabla \times \boldsymbol{\phi}_\theta\|_{\epsilon^{-1}} \\ &= \left( \frac{\omega \ell_D}{\vartheta_{\min}} \right)^{1-s} \left( \frac{\omega h}{\vartheta_{\min}} \right)^s \|\nabla \times \boldsymbol{\phi}_\theta\|_{\epsilon^{-1}}. \end{aligned} \quad (7.6)$$

For the second term, using (7.2), we can write

$$\begin{aligned} \omega \|\tilde{\nu}^{-\frac{1}{2}} \tilde{h} (\nabla \times \boldsymbol{\phi}_\theta - \mathcal{J}_h^d(\nabla \times \boldsymbol{\phi}_\theta))\| &\leq \omega h \nu_{\min}^{-\frac{1}{2}} \|\nabla \times \boldsymbol{\phi}_\theta - \mathcal{J}_h^d(\nabla \times \boldsymbol{\phi}_\theta)\| \\ &\lesssim \omega h \nu_{\min}^{-\frac{1}{2}} \|\nabla \times \boldsymbol{\phi}_\theta\| \lesssim \frac{\omega h}{\vartheta_{\min}} \|\nabla \times \boldsymbol{\phi}_\theta\|_{\epsilon^{-1}}. \end{aligned} \quad (7.7)$$

Combining (7.4), (7.6) and (7.7) and observing that  $h \leq \ell_D$  proves the assertion. □

**Lemma 7.3** (Divergence conformity factor). *Let  $\gamma_{\text{dv}}$  be defined in (4.10). We have*

$$\gamma_{\text{dv}} \lesssim \left( \frac{\omega \ell_D}{\vartheta_{\min}} \right)^{1-s} \left( \frac{\omega h}{\vartheta_{\min}} \right)^s. \quad (7.8)$$



*Proof.* Let  $\mathbf{v}_h \in \mathbf{X}_h^b$  and consider an arbitrary  $\mathbf{v}_h^c \in \mathbf{P}_{k,0}^c(\mathcal{T}_h)$ . Since  $\mathbf{\Pi}_{h,0}^c(\mathbf{v}_h) = \mathbf{0}$  by assumption, we have

$$\begin{aligned}\mathbf{\Pi}_0^c(\mathbf{v}_h) &= \mathbf{\Pi}_0^c(\mathbf{v}_h - \mathbf{v}_h^c) + \mathbf{\Pi}_0^c(\mathbf{v}_h^c) \\ &= \mathbf{\Pi}_0^c((I - \mathbf{\Pi}_{h,0}^c)(\mathbf{v}_h - \mathbf{v}_h^c)) + \mathbf{\Pi}_0^c(\mathbf{v}_h^c - \mathbf{\Pi}_{h,0}^c(\mathbf{v}_h^c)).\end{aligned}$$

Multiplying by  $\omega$ , and invoking the triangle inequality and the  $\mathbf{L}_\epsilon^2$ -stability of the projection operators, we infer that

$$\omega \|\mathbf{\Pi}_0^c(\mathbf{v}_h)\|_\epsilon \leq \omega \|\mathbf{v}_h - \mathbf{v}_h^c\|_\epsilon + \omega \|\mathbf{\Pi}_0^c(\mathbf{v}_h^c - \mathbf{\Pi}_{h,0}^c(\mathbf{v}_h^c))\|_\epsilon.$$

For the second term on the right-hand side, we invoke [12, Lemma 5.2] which gives

$$\begin{aligned}\omega \|\mathbf{\Pi}_0^c(\mathbf{v}_h^c - \mathbf{\Pi}_{h,0}^c(\mathbf{v}_h^c))\|_\epsilon &\lesssim \left(\frac{\omega \ell_D}{\vartheta_{\min}}\right)^{1-s} \left(\frac{\omega h}{\vartheta_{\min}}\right)^s \|\nabla_0 \times (\mathbf{v}_h^c - \mathbf{\Pi}_{h,0}^c(\mathbf{v}_h^c))\|_\nu \\ &\lesssim \left(\frac{\omega \ell_D}{\vartheta_{\min}}\right)^{1-s} \left(\frac{\omega h}{\vartheta_{\min}}\right)^s (\|\mathbf{C}_{h,0}^{k,\ell}(\mathbf{v}_h)\|_\nu + \|\mathbf{C}_{h,0}^{k,\ell}(\mathbf{v}_h - \mathbf{v}_h^c)\|_\nu),\end{aligned}$$

where we used that  $\nabla_0 \times \mathbf{\Pi}_{h,0}^c(\mathbf{v}_h^c) = \mathbf{0}$  and the triangle inequality. For the first term, we observe that

$$\omega \|\mathbf{v}_h - \mathbf{v}_h^c\|_\epsilon \leq \frac{\omega h}{\vartheta_{\min}} \|\tilde{\nu}^{\frac{1}{2}} \tilde{h}^{-1}(\mathbf{v}_h - \mathbf{v}_h^c)\| \leq \frac{\omega h}{\vartheta_{\min}} \|\mathbf{v}_h - \mathbf{v}_h^c\|_{\text{ap}}.$$

Combining this bound with the above two bounds and since  $h \leq \ell_D$ , this gives

$$\omega \|\mathbf{\Pi}_0^c(\mathbf{v}_h)\|_\epsilon \lesssim \left(\frac{\omega \ell_D}{\vartheta_{\min}}\right)^{1-s} \left(\frac{\omega h}{\vartheta_{\min}}\right)^s (\|\mathbf{C}_{h,0}^{k,\ell}(\mathbf{v}_h)\|_\nu^2 + \|\mathbf{v}_h - \mathbf{v}_h^c\|_{\text{ap}}^2)^{\frac{1}{2}}.$$

The bound (7.8) follows by taking the minimum over  $\mathbf{v}_h^c \in \mathbf{P}_{k,0}^c(\mathcal{T}_h)$ .  $\square$

**Remark 7.4** (Other settings). *It is possible to show that the above factors tend to zero under less restrictive assumptions on the coefficients. Similarly, one can show improved convergence rates by assuming extra regularity. We refer the reader to the discussions in [12, Section 5] for more details.*

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