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Nonlinear Dynamics in Mechanics: State of the Art and Expected Future Developments

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Starting from a recent classification of the development stages of nonlinear dynamics in mechanics, this review builds on the idea that the level of scientific maturity of the area is now such as to involve a gradual shift of its core interests from the inherent theoretical and practical findings to the application benefits that they can bring to solving dynamic problems in a variety of technological environments. First, an overview of the current state of knowledge and the achievements of the community of relevant scholars in about the last ten years is presented, distinguishing between traditional and emerging themes fully inherent to mechanics, and more hybridized scientific contexts. Then, a “vision” of expected future developments is attempted, by organizing the presentation along some main lines. (i) Identifying modeling, methodological, and computational advancements needed to address challenging, new or updated, research issues, with a view to deepening and further expanding the ranges of theoretical development and practical interest of nonlinear dynamics. (ii) Overlooking directions toward which promoting full exploitation of intrinsic or intentionally added nonlinearities, to the aim of improving and possibly optimizing specific behaviors and general operating conditions of actual systems/structures in a variety of dynamic environments, by also referring to the uncertainty quantification issue. (iii) Pursuing “novel” lines of developments of nonlinear dynamics in a fully hybridized and cross-disciplinary framework, with also possible expectation of new related phenomenologies.

Keywords: nonlinear dynamics, mechanical systems, structures, current and expected lines of development

1 Background and Motivation

Besides causing dramatic losses and meaningfully affecting our general life style, in the last 2 years, the Covid-19 pandemic has also nontrivially re-oriented the research objectives in a number of scientific fields, which certainly include *nonlinear dynamics*. In addition to single papers dealing with the matter, a substantial number of dedicated Special Issues (SIs) have appeared in scientific journals within the relevant environments, as well as in more general journals in the wide area of mechanics. This trend will certainly continue because the conceptual and operational tools of nonlinear dynamics represent powerful means for possibly catching in a reliable way the intricate evolutionary aspects of such a nonlinear event of worldwide importance.

However, moving from about the time (mid-1970s) when the theoretical accomplishments of the dynamical systems community started to be transferred to the engineering realm, the area of mechanics of solids, machines, and structures has also registered a strong increase of interest toward nonlinear dynamic phenomena, in connection with the widespread awareness of their importance in a variety of engineering applications. In a recent review paper [1], four main stages of development of nonlinear dynamics in mechanics have been identified, along with the overall modeling theme of *dimension reduction* that influences and encompasses all of them. They are sketched in the chronological chart of Fig. 1 in which, for each stage, the time window over which those specific scientific developments were so intense to provide nonlinear dynamics with its peculiar and prevailing character is also indicated, schematically.

The first three fundamental stages (*nonlinear oscillations, bifurcations and complex dynamics, experimental nonlinear dynamics*)

reflect the role played in the overall historical development by the three approaches (analytical, geometrical/computational, and experimental) which currently characterize a possibly comprehensive treatment of the nonlinear dynamics of a system or structure. The titles refer, on one hand, to the set of topics that have aroused the greatest interest of scholars over a certain time period, and on the other hand, to the general kind of solution techniques whose development stage actually enabled researchers to successfully afford those topics. Of course, the development of the first three stages has been going on (vanishing colors in Fig. 1) well beyond their nominal periods of major relevance, with novel or revisited advancements, both theoretical and practical, which further enhanced the knowledge in the area.

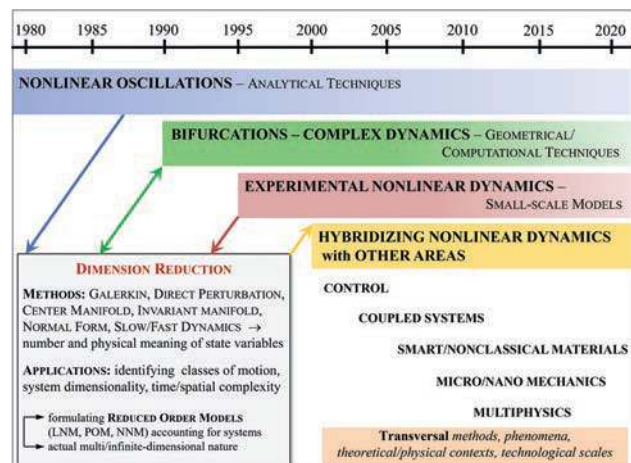


Fig. 1 Main stages of development of nonlinear dynamics in mechanics

The identified fourth stage of development (*hybridizing nonlinear dynamics with other areas*) is the one more properly characterizing the present state of the art. In particular, spreading the by now “traditional” themes of nonlinear dynamics in mechanics toward companion or novel areas of research and technological development, it paves the way to a renewed—and indeed hybridized—character of the whole area. Topics addressed and techniques used in this fourth stage tend often to be cross-disciplinary and cannot be easily categorized in nonlinear dynamics terms, just due to the variety of hybridizing relationships between nonlinear dynamics and other theoretical and application environments. Thus, in Ref. [1], only an incomplete set of existing, and further developing, connections with companion areas was listed, as also reported below for the sake of convenience.

- *Control* Vibration control/suppression. Control of bifurcations: open-loop and feedback. Control of chaos: Ott-Grebogi-Yorke and delayed feedback methods. Targeted energy transfer and nonlinear energy sinks.
- *Macro- and micro/nano-systems* Nonlinear dynamics of cables, beams, plates, shells, composites, rotating systems, shape-memory materials, functionally graded materials, with applications in structural mechanics, machinery, vehicles, system identification, structural health monitoring, atomic force microscopy, micro/nano-electromechanical systems, carbon nanotube.
- *Coupled systems* Chains of nonlinear oscillators, periodic systems. Wave propagation, localization. Structural instabilities. Metamaterials, granular media.
- *Multiphysics* Thermoelasticity; fluid–structure interaction; piezoelectricity; magnetoelasticity; biomechanics. Intelligent systems.

This may be considered a schematic picture of the current state of research topics of interest for the community of scholars of nonlinear dynamics in mechanics, as well as a starting point for making a step forward and dwelling on the predictable lines of future development in this scientific and technological area.

In a more general perspective, the impetuous development of nonlinear dynamics in mechanics and engineering that occurred just before and around the turn of the millennium has definitely brought valuable and distributed knowledge on a rich variety of methodological and phenomenological aspects. The former are concerned (i) with the asymptotic techniques suitable to deal with weakly nonlinear systems, (ii) with the geometric and computational approaches also allowing us to address strongly nonlinear systems, and (iii) with techniques and tools for reconstructing dynamical properties of small scale physical systems from experimental measurements in the nonlinear regime. The latter refers to the rich cornucopia of resonance, interaction, bifurcation, and response features exhibited by mechanical systems and structures in both the realm of classical nonlinear oscillations and the context of bifurcation and complex phenomena to be addressed in local and global dynamics terms. All of this is progressively spreading over a large variety of technological areas across different space and time scales, where nonlinear dynamics has started to be fruitfully exploited in combination with paradigms (both classical and novel) from applied mathematics, control, mechanics, and physics of materials, and general multiphysics frameworks. Yet with a potential that is still dramatically unexplored, also because of being nontrivially limited by the circumstance of considering until now almost solely one-dimensional or fairly low-dimensional models of the physical systems in the background, which have however allowed in-depth investigations and ensuing understanding of some fundamental aspects of the nonlinear behavior.

In this respect, it could even be argued that, after an intense and lasting growth, the trend of research interests and outcomes of nonlinear dynamics in mechanics and engineering is now attaining a sort of plateau corresponding to a stage of *intrinsic full maturity* of the area. Two main items are thus becoming apparent to the community of scholars.

- (1) The transversal nature of dynamical systems theory not only as a proper conceptual framework for dealing with nonlinearity and complexity in different scientific fields, which is a feature already grasped in earlier times of development, but also in terms of associated operational techniques and ensuing phenomenological aspects with respect to a variety of physical contexts and technological scales of application.
- (2) The importance of exploiting nonlinear dynamics for the analysis, control, and design of newly conceived systems both in traditional areas of engineering and at the borders between them or with other scientific and technological areas, this giving rise to mixing of concepts, perspectives, and operational tools which may allow us to attain specific goals in real cross-disciplinary contexts.

In slightly different words it could be said that, after a long time period in which nonlinear dynamics has been a scientific area of main interest in itself, with regard to both the methodological tools used and the phenomena highlighted, the new stage of development is mostly perceived as one in which nonlinear dynamics can play a formidable role in achieving more advanced goals in purposeful technological contexts, even with general societal advancements. This marks a gradual shift from the centrality of conceptual and practical findings inherent in the *scientific* area, to the importance of the *application-oriented* benefits that such findings can bring to solving problems in a variety of technological environments.

Fundamental prerequisites for such a smooth transition consist of complementing the recourse to the low-dimensional models used until now to analyze nonlinear phenomena with the formulation of higher-dimensional models more reliably describing the operating conditions of real systems and structures and, even more, with their effective implementation in specific nonlinear dynamics contexts. This is a definitely challenging task.

Possible lines of development of nonlinear dynamics in mechanics and engineering have to be discussed within the outlined general scenario. To this end, the main subfields of interest—and related specific topics—can be classified according to a variety of possible criteria, also taking into account the strong and ever-growing connections between subfields or topics that do not allow the identification of clearly distinct items in an intrinsically coupled context. In the sequel, the presentation of development lines is organized around some main thematic fields/topics, identified independently of their application area of major interest, thus accepting the risk of repeating characterizing items to be observed from different perspectives. This implicitly entails privileging once more the adoption of a “scientific” and methodologically/phenomenologically oriented perspective versus the “technological” one previously deemed to meaningfully affect future developments of nonlinear dynamics. A position that, in addition to being in the writer’s conceptual vision, is deemed to be anyway the most correct and appropriate one for a possibly comprehensive treatment of an actually cross-disciplinary field.

This paper is not intended to be a detailed review of the current state of research in the area of nonlinear dynamics, whose whole body of scientific topics is indeed enormous and such to make an adequate and comprehensive survey nearly unaffordable. Rather, it aims at generally identifying subfields of major interest within the current scenario, along with some main lines of possible future development, based on a tentative view of what could, and probably should continue to make nonlinear dynamics a central research area, not only in theoretical terms but also with a view to possible practical advantages. Accordingly, the focus of the presentation will be, on the one hand, on some incremental steps to be pursued with respect to the current state of knowledge, and on the other hand, on some possibly novel aspects in terms of systems considered, methods employed, and phenomena highlighted. Moreover, a vision will be adopted in some way preferentially oriented to the last two elements (methods and phenomena), in the

perspective of the mentioned transversality of nonlinear dynamics to different applications and technological environments. Of course, the selection made reflects the writer's interests and expertise and does not imply any greater "objective" importance of the addressed subfields/topics with respect to those dealt with to a lesser extent.

2 The Current Scenario

In about the last twenty years, several reviews have appeared in the literature on different topics of nonlinear dynamics in mechanics, along with a number of general books. Some of them will be cited later when specific topics will be dealt with. To the best of the writer's knowledge, there has been only one general review paper [2] presenting a comprehensive state of the art of nonlinear dynamics in mechanics as of the year 2008, with a rather detailed account of a large number of systems and related nonlinear phenomena. The authors distinguish between traditional and emerging areas, the former involving structural systems, rotating systems and machines, vehicle dynamics, machine and manufacturing systems, and vibration control; the latter including micro- and nano-electromechanical systems, piecewise smooth systems, system identification, and structural health monitoring. Among recently published monographs one can mention, e.g., those by Lacarbonara [3], generally dealing with nonlinear structural mechanics, Amabili [4], focused on the experimental and computational nonlinear dynamics of plates and shells, Younis [5], specifically concerned with micro-electromechanical-systems, Wagg and Neild [6], dealing with nonlinear vibration and control of flexible and adaptive structures. In turn, Awrejcewicz and coauthors published several monographs on a variety of nonlinear dynamics topics, ranging from mathematical modeling, asymptotic techniques, and computational approaches, up to nonlinear and chaotic phenomena [7], mostly referring to elastic structural systems, also in thermal environments [8].

Assuming the review by Shaw and Balachandran [2] as a satisfactory starting point from which moving on, the amount of topics addressed in the literature in the elapsed time is enormous, with variable and generally meaningful achievements obtained as regards modeling, mathematical solution, physical experimentation, description and understanding of nonlinear phenomena, their control, and also engineering design. Thus, trying to come up with a possible representative list according to some criterion appears unfeasible, and only incremental research steps can be sketched, along with some novel aspects in terms of systems considered, methods employed, and phenomena highlighted. Moreover, since about the beginning of the new millennium, the effects of globalization have become enormous in whatever scientific realm, with an overwhelming exchange of information, international collaborations, multiplication of journals, and scientific meetings covering different areas in a wider or stricter sense. At the same time, the opportunity has emerged for a more evident and explicit specification of subfields of interest, aimed at identifying and better grouping topics and involved scientists.

In this situation, instead of committing to present a certainly lacking account of advancements made on the various topics based on a subjective analysis of individual research works, it is worth attempting to overview the most active, and also fashionable, research themes by looking at the lists of some most important related events. These include (i) symposia organized within large international conferences or as single specialized events, (ii) special issues that appeared in archival international journals, and (iii) advanced Ph.D./postdoc schools, along with other similar qualitative (and, indeed, also quantitative) information. In the specific area of interest, one can refer to minisymposia organized within large EUROMECH-ENOC and ASME-IDETC events, to specific IUTAM symposia, to SIs of recognized journals either purposely covering the whole area of nonlinear dynamics or having a more general scope in mechanics and/or dynamics, as well as to CISM advanced schools. In the following, two exemplary

updates of the state of research are presented, according to this criterion.

Table 1 lists the minisymposia held in the last decade within the European nonlinear dynamics conferences (ENOCs brand name) of EUROMECH,¹ which are the largest nonlinear dynamics events held every three years in the field of mechanics. Their original (since 1993) European character of "East meeting West" has later transformed into a conference with the participation of scientists from all over the world, although the record is now being shared with the new (since 2019) NODYCON events (see Table 2), which resume the earlier conferences organized by Ali H. Nayfeh at Virginia Tech for about twenty years. Similar lists of ASME-IDETC minisymposia could be considered, as well. However, the ENOC minisymposia have the nontrivial feature of being preselected by a dedicated committee with the specific aim of identifying the most robust themes, both traditional and novel, in the area (along with a number of recognized scholars to handle them), according to criteria not too much fashion- or academic group-driven.

Browsing through the list, classical themes of well-established importance and spread interest (first and second groups in Table 1) can be distinguished from relatively novel (third group) and even more "emerging" (fourth group) themes. Topics in the first two groups are well-identified in conceptual terms and well-organized in operational terms, according to a vision shared by the community of scholars of nonlinear dynamics in mechanics, albeit to a variable extent as regards both their tradition and the interest toward specific systems and/or aspects (from first to second group). Indeed, they include topics that are continuously revisited, updated, and/or expanded, based on new theoretical achievements on modeling and methodology, further developments of solution techniques, newly observed phenomena, and increased technological interest and usability. Topics in the third group, although also present in earlier times, can be deemed to have undergone a variable explosion in about the last decade as regards an "autonomous" characterization in scientific terms and associated practical interest. Relevant novel symposia have thus become a steady and robust part of the organization and exchange of knowledge over the considered decade. The fourth group of themes is somehow more mixed and "unsteady" because of including both topics of still unequivocally recognized and spread interest for the community of mechanics dynamicists, and more "recent" ones which may be expected to swiftly become of core interest in the years to come. In this respect, it is worth noting that the list of more recently emerging themes is likely to no more reflect the rapidly evolving scenario of research interests in the area, for having been preselected years ago for the 10th ENOC event, later postponed to 2022 due to the Covid-19 pandemic. Overall, themes in the second and third groups can be deemed to possibly require more precise definitions and overall classifications. This is concerned with both the characteristics inherent in the area of nonlinear dynamics and their being sometimes at the crossroad between different scientific and technological areas, which makes a clear assessment of the roles played by the knowledge from contributing disciplinary fields more difficult.

The above scheme identifies the main themes of nonlinear dynamics in mechanics mostly from an evolutionary perspective. A more detailed and current picture is provided, e.g., by the organization of thematic sessions of the online NODYCON 2021 event,² which are grouped into four different classes (concepts/methods, systems/phenomena, dynamics and control, and new trends), with slight differences and specifications. Some thematic sessions more explicitly refer to topics (transient dynamics, aerospace structures, multistable oscillators, constitutive and phenomenological models, multifunctional structures, sensors and actuators, metamaterials, biocapsule robots, stochasticity and noise, ecosystems dynamics) addressed in a relatively small

¹<https://euromech.org/>

²<https://nodycon.org/2021/>

Table 1 ENOC (European Nonlinear Oscillations Conference) Minisymposia in about the last decade

MS title	Seventh ENOC Rome 2011	Eighth ENOC Wien 2014	Ninth ENOC Budapest 2017	10th ENOC Lyon 2020 + 2
Reduced-order modeling and system identification	×	×	×	×
Asymptotic methods	×	×	×	×
Computational methods	×	×	×	×
Dynamics and optimization of multibody systems	×	×	×	×
Nonlinear phenomena in mechanical and structural systems	×	×	×	×
Nonlinear dynamics in engineering systems	×	×	×	×
<hr/>				
Nonsmooth dynamics	×	×	×	×
Systems with time delay	×	×	×	×
Slow-fast systems and phenomena	×	×	×	×
Control and synchronization of chaos and complex dynamics	×	×	×	×
Fluid–structure interaction	×	×	×	×
<hr/>				
Fractional derivatives	×	×	×	×
Experiments in nonlinear dynamics and control	×	×	×	×
Micro- and nano-electromechanical systems	×	×	×	×
Nonlinear dynamics of biological systems	×	×	×	×
Nonlinear dynamics for engineering design	×	×	×	×
Energy transfer and harvesting in nonlinear systems	×	×	×	×
Random dynamical systems: recent advances and new directions	×	×	×	×
<hr/>				
Hybrid systems	×			
Time-periodic systems		×	×	×
Material instabilities	×			
Wave propagation in mechanical systems and nonlinear metamaterials		×	×	×
Traffic and vehicle dynamics			×	
Nonlinear dynamics in acoustics				×

Table 2 NODYCON (Nonlinear Dynamics Conference) 2021 Thematic Sessions

(a) Concepts and methods in nonlinear dynamics		(b) Nonlinear dynamics of mechanical and structural systems	
FSI	Fluid–structure interaction I–II–III–IV	MSS	Mechanical systems and structures I–II–III–IV
NSS	Nonsmooth systems I–II–III–IV	ED	Experimental dynamics I–II
CSU	Chaotic systems and uncertainty I–II–III	SIS	System identification and structural health monitoring I–II–III–IV
BDI	Bifurcation and dynamic instability I–II–III	AS	Aerospace structures
CND	Computational nonlinear dynamics I–II–III–IV–V	MSO	Multistable oscillators
NW	Nonlinear waves I–II–III	CPM	Constitutive and phenomenological models
MBS	Multibody systems	MFS	Multifunctional structures I–II
ROM	Reduced-order models I–II	MIET	Modal interactions and energy transfer I–II
TD	Transient dynamics	RS	Rotating systems I–II
AT	Analytical techniques I–II–III–IV–V	PED	Passive energy damping
(c) Nonlinear dynamics and control		(d) Recent trends in nonlinear dynamics	
NVC	Nonlinear vibration control I–II–III–IV	EH	Energy harvesting I–II–III–IV
SA	Sensors and actuators I–II	MNS	MEMS/NEMS I–II–III–IV
CNS	Control of nonlinear systems I–II–III	MM	Metamaterials I–II
Special sessions		BCR	Biocapsule robots I–II
SS-C19	Complex dynamics of COVID-19 I–II–III	BSD	Biological systems dynamics I–II
SS-LIM	In memory of Prof. L. I. Manevitch	SN	Stochasticity and noise I–II
		NPBED	Nonlinear phenomena in bio- and ecosystems dynamics

number of presentation, which is sometimes a mark of actually emerging character within the community of mechanics dynamicists.

Further meaningful information on the research state of the art can be obtained by looking at the evolutionary trends of single events, like symposia or journal SIs. They highlight both the

persistence of traditional, yet updated, themes, and the emergence of novel ones that institutional/professional bodies of mechanics of worldwide relevance or renowned international journals in dynamics and mechanics deemed to require dedicated exchanges of ideas among expert scientists and related research efforts.

2.1 Traditional and Emerging Themes Inherent to Mechanics. Many journal special issues associated with anniversary and memorial circumstances or selectively ensuing from symposia/conference papers deal comprehensively, though variably, with traditional and emerging topics in nonlinear dynamics and vibrations of mechanical systems/structures in terms of modeling, techniques, analysis, stability, bifurcation, behavior, control, and applications. A relevant, though certainly not exhaustive, list is included in the bibliography, with a summary of addressed issues given below by grouping them into some main items. Of course, topics discussed and references cited are far from adequately representing the huge amount of studies also spread through regular journal issues.

- Systems and structures with *geometric or constitutive nonlinearities*, or deriving from nonsmooth interactions, and possibly amplified by structural flexibility or triggered by damage, are considered. Buckling, dynamic instabilities, and nonlinear vibrations of structures are paid attention to not only in themselves but also in view of assessing their load carrying capacity. This is indeed a primary and challenging objective in modern structural engineering, owed to the enhanced use of innovative materials, increasingly demanding structures, and new control technologies. In the last ten or more years, interest toward nonlinear phenomena in macrosystems/structures for applications in classical engineering fields (mechanical, civil, aeronautical, offshore) has been meaningfully complemented with a focus on micro- and nano-systems/structures, whose specific and diversified new sources of nonlinearities often govern the dynamic behavior [9–13].
- *Mathematical models* suitable to describe physical behaviors under combined actions of different nature, nonstandard materials, materials with memory, and nonlinear damping, are developed. Increasing attention is paid to a variety of multiphysics contexts in which the mechanical model is coupled to, e.g., thermal loads, electrodynamic shaker, fluid domain, fringing electrostatic fields, and/or magnetic forces [9,10,14].
- Reference *approximate analyses* furnishing comprehensive descriptions of the nonlinear response are always widely accomplished through classical asymptotic techniques, in particular the method of multiple time scales (see, e.g., Ref. [15]), and semi-analytical procedures. However, numerical simulations and a variety of computational approaches are used extensively to analyze not only bifurcations of the solutions but also general phenomena of global dynamics whose adequate description is necessary for an overall understanding of a system response.
- General loss of *stability and bifurcations* of equilibrium and dynamic solutions due to the occurrence of nonlinear resonances is addressed in local and global terms, with the ensuing phenomena of dynamic buckling, possible malfunctioning, unwanted or wanted dynamic snap-through, fatigue, and collapse. Special attention is paid to instability and bifurcations arising in fluid-structure problems in view of various practical applications, with emphasis on aeroelastic systems, flutter analysis in classical or supersonic conditions, vortex-induced vibrations, galloping instabilities, and more theoretical issues [16]. Specific phenomena such as synchronization in the dynamics of coupled oscillators, e.g., pendula systems [17,18], and their applications in engineering are investigated, too.
- *Nonlinear normal modes* (NNMs) deserve specific consideration. Upon being originally formulated in the 1960s [19], the NNM concept was meaningfully redefined in the 1990s in terms of invariant manifold theory [20], and was given further important theoretical and practical contributions [21]. Theoretical developments [22] and applications [23] have been reviewed, dwelling on how NNMs represent a useful

framework for the analysis of nonlinear vibrating structures [24], also in terms of their practical computation [25,26]. In the reduced order modeling perspective [27], to be dealt with more extensively in Sec. 3.1 forward, NNMs also play a fundamental role [28]. Recently, a unified mathematical approach to NNMs in dissipative systems has been proposed in the framework of the invariant manifold theory, with their slow spectral submanifolds (SSMs), which are the smoothest nonlinear continuations of spectral subspaces of the linearized dynamics, being also relevant for model reduction [29]. In the new millennium, nonlinear modal analysis has been revisited for *essentially nonlinear systems*, based on the introduction of the complexification-averaging method [30], which is instrumental in the exploration of targeted energy transfer (TET), and of the notion of limiting phase trajectory, which can be used for the description of nonstationary processes with strong energy exchange [31]. All of the above has led to establishing a new series of so-called NNM conferences, which have swiftly incorporated, by virtue of a number of subtle connections, the emerging issue of *wave propagation* in discrete or continuous nonlinear systems (e.g., see the following link cited in footnote³). Related features of localization, solitons' occurrence, pass/stop bands characterization, and their exploitation for vibration control purposes have become of major interest.

- In their 2008 review paper, Shaw and Balachandran considered the nonlinear dynamics-based *control of vibrations* “yet to be fully explored,” and predicted the area “to be an active and fascinating one of continued interest” [2]. This has indeed been the case. Active and passive control based on a variety of techniques and mechanisms, and different added devices, are largely investigated [10,15,16]. However, in the field of passive control, the major interest is paid to the use of nonlinear coupling for realizing an irreversible TET (also known as energy pumping) from the main system to a nonlinear energy sink (NES), i.e., to a subsystem with purely nonlinear stiffness acting as a broadband vibration absorber [32,33]. Over the last decade, performances of TET-NES control in terms of energy dissipation across a possible wide range of frequencies have been investigated in a variety of practical applications, as also overviewed in Ref. [34]. General conceptual and operational issues of nonlinear energy transfer in dynamical and acoustical systems are summarized in Ref. [35].
- *Other systems and topics* of interest in theoretical terms or in view of specific applications are the subjects of focused editorial initiatives, surveys, or scientific events. They include damage and cracking problems [10]; vibro-impact and friction problems modeled by discontinuous differential equations [36–38]; general nonsmooth systems with an emphasis on bifurcation theory [39]; nonsmooth modal analysis in discrete and continuous systems [40]; nonlinear system identification in structural dynamics [41]; drill-string dynamics [42]; time-periodic systems [43]; nonlinear rotordynamics [44]; delayed dynamics [45], with also a focus on mechatronic systems [46]; aeroacoustics and nonlinear vibrations in turbomachinery [47]. A list that, to the best of the writer's knowledge, certainly overlooks many other important topics that do not seem to have been the subject of widespread dedicated initiatives.

As already mentioned, starting from studies carried out in the first decade of the 2000s, and summarized in earlier special issues [48] and reviews [49], nonlinear dynamics of *micro/nano-electromechanical systems* (M/NEMS) has swiftly become a field of main interest. Compared to macromechanical systems, M/NEMS exhibit important peculiarities in terms of extremely small scales, types of nonlinearity also associated with electric actuation,

³<https://nnm2021.ch>

multiphysics environments, and the role played by noise and uncertainty. The main addressed topics are the following. (i) Modeling, which ranges from nonclassical continuum to atomistic-based theories, with also the inclusion of couple stress theory, nonlocal formulations, and molecular dynamics. (ii) Analysis of novel materials, like carbon nanotubes and graphene sheets. (iii) Multifield frameworks including piezoelectric, electrostatic, and electromagnetic sensing/actuation, as well as fluid–structure interaction. (iv) Analysis of instabilities and nonlinear dynamics under parametric excitation and external/internal resonances, with non-smooth and chaos phenomena, too [50–53].

In about the last ten years or more, the importance of promoting the great *potential of nonlinear dynamics* for better describing/understanding the systems' behavior in a variety of engineering realms, as well as improving and possibly optimizing their operational conditions, has originated scientific initiatives in many directions. Earlier [54] and recent [55] or planned⁴ IUTAM Symposia in the area of dynamics generally deal with the exploitation of nonlinear dynamics for developing advanced technologies and reformulating criteria of engineering design of systems and structures across different length and time scales. Advanced schools at the International Centre for Mechanical Sciences (CISM) have been devoted to a variety of related general prospects, which include nonlinear strategies for vibration mitigation and system identification [56], use of nonlinear behavior in structural dynamics [57], modal analysis of nonlinear systems [58], exploitation of global dynamics for engineering design [59]. Journal SIs aimed at pointing out how to exploit nonlinearity for practical gain in M/NEMS are planned, too.

A specific theme of major importance in terms of dynamics and control is the *smart design* of mechanical systems/devices and, more generally, of *composite* and *adaptive* structures, in view of the variety of possible applications due to an advantageous balance between weight, stiffness, and strength. Their nonlinear operating regime, also due to multiphysics environmental conditions, is often pursued on purpose to obtain intelligent and effective behaviors. Typical examples are shape memory alloys, magnetostrictive, thermoelectric, and piezoelectric devices used in smart sensors and actuators, morphing structures made from composite and/or functionally graded materials, for applications in MEMS, medicine, industry, system control, energy harvesting, health monitoring, and aerospace [60–64]. Improving energy harvesting by exploiting system nonlinearities and the ensuing dynamic behavior to possibly optimize the conversion of ambient mechanical vibrations into electrical energy is a major issue in macro- and micro-engineering, pursued via different electromechanical mechanisms and using active materials, too [61,65]. Indeed, nonlinear effects of smart and composite materials and structures may ensure a broadband frequency, which is important in the area of energy conversion to be realized via electromagnetic and piezoelectric transducers [66]. The general role and the potential benefits/drawbacks of nonlinearities in energy harvesting are reviewed and discussed [67], by considering monostable and bistable devices with continuous nonlinear restoring forces as well as harvesters with piecewise-linear restoring force. Energy harvesters can be used also for sensing purposes, in connection with the general issue of structural health monitoring (SHM) [68].

2.2 More Hybridized Contexts. Taking up the general idea of a marked hybridization of nonlinear dynamics with different scientific areas in the contemporary scenario, *fractional dynamical systems* can be deemed a wide (mainly theoretical) research field recently ensuing from the encounter between the mathematics of fractional calculus and the nonlinear dynamics of systems in a variety of application environments. Indeed, after being investigated for a long time mostly from a mathematical point of view,

in the last few decades, fractional calculus dealing with integrals and derivatives of arbitrary (i.e., noninteger) order has emerged as a useful tool to address seemingly anomalous behaviors. Its applications cover many areas of science and engineering, including physics, chemistry, viscoelasticity, electronics, electromagnetism, robotics, mechanics, telecommunications, rheology, control theory, biology, signal processing, and finance. Realizing that the dynamic behavior of many complex systems can be properly described through fractional-order models, research on fractional dynamical systems has exploded in about the last decade, as also witnessed by the constant inclusion of a related minisymposium in all ENOC events (Table 1). Besides being paid increasing interest in regular issues of archival journals in dynamics and vibrations, this “new” field has become the subject of a huge number of dedicated SIs [69–76] or parts of them [77], with articles variably spanning through mathematical theory, modeling, implementation, and applications. Topics are grouped according to different viewpoints, roughly distinguished between theoretical developments, fractional derivatives in time or space, fractional-order modeling and identification, dynamical analysis of fractional differential equations and systems, fractional-order control theory, numerical computation, and applications. Focusing on aspects more clearly related to mechanics, one can mention, among others, fractional modeling for viscoelastic behavior, for broadband energy harvesting, for contact, impact, and impulse problems, for human motion tracking, for propagation and damping of vibrations in complex and heterogeneous materials, along with fractional wave equation, discrete dynamic behaviors, fractional chaotic systems, and their control and synchronization.

Referring to the encounter of mechanics with other disciplinary contexts, nonlinear dynamics of *biological systems* is another hybridized area under strong development, which is constantly included in the ENOC programs since 2011 with a dedicated minisymposium (Table 1). An introductory review of the most typical phenomena associated with nonlinearity, dissipation, and openness in biological systems is presented in Ref. [78], with illustrative examples of damped oscillations, self-oscillations, synchronization, chaotic and noise-induced oscillations. Several SIs appeared in archival journals, focusing on nonlinear modeling and analysis of biological systems and processes, including the human brain, heart, tumor growth, neuronal, and cellular systems, human locomotion (to name a few), with the related dynamic phenomena and their control. Scientific contributions are at the crossroad between life science, mathematics of dynamical systems, and mechanics, with a view to numerical simulation, physical experimentation, information techniques, and engineering practices [79–82]. The need for more innovative and integrated approaches to biomedical research from a dynamical systems point of view is also highlighted [83], with computational modeling, dynamic analysis, nonlinear phenomena, and control clearly aimed at clinical diagnosis, patient health monitoring, drug administration, and biosignal-assisted rehabilitation. Objectives likely achievable only through broader cross-disciplinary collaborations and deeper exchanges of knowledge between complementary areas.

A few points about *chaos in mechanical and structural systems* and, more generally, on the role of the related research within the broad phenomenology of nonlinear dynamic behaviors, are also in order. After the earlier time of dominant studies on classical nonlinear oscillations, with phenomena such as nonlinear resonances and ensuing modal interactions, chaos has been an important research focus in nonlinear dynamics in mechanics, if not even its actual core, since about the end of the 1980s until the whole first decade of the new millennium. This occurred in connection with the great interest in local and global bifurcations in theoretical and topological terms, and the tremendous development of geometrical techniques and, mostly, computational tools. A huge amount of studies on minimal (single-degree-of-freedom (dof)/mode) models of forced nonlinear systems and structures, or anyway low-dimensional models (often with no more than the two-dof/modes involved in some internal resonance), shed light on a

⁴<https://iutam.org/event/iutam-symposium-on-nonlinear-dynamics-for-design-of-mechanical-systems-across-different-length-time-scales/>

variety of issues. These include qualitative and quantitative characterization of complexity, bifurcation mechanisms and routes to chaos, construction of basins of attraction, and the analysis of their evolution with varying control parameters. Meaningful regions in the system and/or excitation parameter spaces have been detected where complex dynamics (e.g., in-well or cross-well chaos, and escape phenomena, in potential systems) occur. Recent surveys on chaos in one-dimensional mechanics can be found in a monograph [7] and a review paper [84]. It could be said that, if suitably modifying a system or excitation parameter, chaotic dynamics achieved through different transition mechanisms may be virtually found in nearly whatever forced reduced order model (ROM). Yet, the parameter values considered in the analysis do not always correspond to realistic ranges for actual mechanical systems and structures, so that the detected chaos turns out to be often an essentially “numerical” one. It is thus fundamental to complement the analytical and, mostly, computational detection of chaos with refined experimental investigations of small-scale physical models of systems and structures aimed at detecting “real” chaotic responses. These require reliable, and indeed sophisticated, techniques of reconstruction of the dynamical features and possibly underlying chaotic attractors, such as delay embedding, and spatial coherence analysis for identification of the number and shape of configuration variables mostly contributing to nonregular responses [85,86]. But experiments in nonlinear dynamics and control [87] are quite challenging and do require more systematic efforts than the scattered, though sometimes meaningful, ones accomplished until now (e.g., Ref. [88]), mostly as regards detecting and characterizing chaos and the underlying bifurcation mechanisms (e.g., Refs. [89] and [90]). Overall, the whole issue of chaotic dynamics seems to have reached a sort of plateau—perhaps only temporary—as regards being the main research goal of nonlinear dynamics in mechanics. This is also evidenced by the quite low number of dedicated scientific initiatives [91–93] undertaken in about the last decade within the mechanics environment and by the attention paid to the interdisciplinary aspects of chaos and its applications in other sectors (mathematics, physics, economics, biology) rather than to those in engineering. Nevertheless, a renewed interest in chaos in mechanical systems and structures, fostered by novel perspectives opened by its relation and/or confrontation with concepts of wave turbulence [94,95], stochastically perturbed regular orbits [96], and transient dynamics [97], is worth to be mentioned, too.

Besides specific interest to the dynamics of the novel (e.g., phononic [98,99]) materials and structures, a series of entirely new research fields in which *nonlinear dynamics* is significantly *integrated with disciplinary knowledge* from other areas are now emerging, with the aim of realizing a more comprehensive and effective framework to cooperatively face challenging issues. These include evolving complex networks [100], modeling and prediction of climate changes or epidemiological events (also in connection with the recent explosion of Covid-related research), sensitivity analysis and uncertainty quantification [101], potentially fruitful relationships between machine learning and dynamical systems [102]. This is also causing a nontrivial modification of the core topics of interest in archival journals of nonlinear dynamics traditionally paying major attention to mechanics, as it can be easily verified by looking at the high percentage of papers not strictly referable to the latter published in some of them (e.g., *Nonlinear Dynamics*) in the last few years. Some of the mentioned hybridized contexts will be resumed in Sec. 3.

3 Expected Future Developments

Most of the thematic sectors discussed in Sec. 2, if not even all of them, are expected to remain of interest to the community of nonlinear dynamicists in mechanics in the incoming future, although possibly to a variable extent based on the perception of the specific needs for a deeper understanding and the existing margins for novel acquisitions. The related prospects could thus

be discussed for each sector separately. However, independent of the writer’s obvious lack of the needed trans-disciplinary expertise, it seems more useful to tentatively identify a few general lines of possible development of the whole area of nonlinear dynamics, into which also embedding new achievements predictable, or anyway to be pursued, for specific sectors. Actually, clearly distinguishing and categorizing topics that are meaningfully interconnected with each other is a quite difficult and challenging task. Nonetheless, in the sequel, the discussion on development prospects is organized along the following main lines. (i) Identifying *modeling, methodological, and computational advancements* needed to address challenges, novel, or updated, research issues, with a view to deepening and further expanding the ranges of theoretical development and practical interest of nonlinear dynamics (Secs. 3.1 and 3.2). (ii) Overviewing directions toward which promoting a *full exploitation of systems’ inherent or intentionally added nonlinearities*, to the aim of improving and possibly optimizing their specific behaviors or general operating conditions in different dynamic environments (Sec. 3.3). (iii) Pursuing novel lines of developments of nonlinear dynamics in a *fully hybridized and cross-disciplinary framework*, with also possible expectation of new related phenomenologies (Secs. 3.4 and 3.5).

3.1 Some General Challenging Issues of Modeling, Analysis, Control, and Design. To foster further theoretical acquisitions of nonlinear dynamics and complement them with real-world applications, major challenging fields of interest in terms of *modeling, analysis, control, and design* are likely concerned with (i) model order reduction, (ii) multistability, (iii) global dynamics, and (iv) nonlinear systems in a stochastic environment. Some main relevant research issues are addressed in the sequel, as per the current scenario of knowledge.

Model order reduction aims at reducing the computational efforts needed to deal with high-dimensional dynamical systems, while also getting otherwise unobtainable, and thus unexploitable, conceptual information on some main features of a structure’s local and global dynamics, also possibly in a multiphysics context. From the practical viewpoint, low-order models of systems and structures—possibly also extracted/identified from nonlinear experimental data—may be used as complementary tools to richer computational models in the analysis and design stage. The considerable amount of related research pursued in about the last two decades has led to the use of various projection-based model reduction nonlinear techniques for geometrically nonlinear structures, see the comprehensive review [103]. Moreover, due to the widespread use of finite element procedures in engineering, reduction techniques applicable to the ensuing structural models are becoming of major importance, as well, see the earlier review [104] and a recent specific comparison [105]. Yet, the most used techniques in nonlinear dynamics still rely on ad hoc selections of solely active modes, that may produce a priori unknown errors. The problem is overcome in various possible ways, e.g., by enslaving seemingly nonessential modes to the active/dominant ones, as in the invariant manifolds/NNMs method [20,21,58], or by applying nonlinear near-identity coordinate transformations (normal form method [106–109]) to the full/many-mode discretized model, while retaining only essential normal coordinates. The use of a perturbation-driven low-order elimination technique [110] in which passive patterns are taken into account via the elimination of corresponding low-order sources allows us to further elucidate the theoretical correspondence/equivalence between the mentioned two reduction paradigms [111], already dealt with in Ref. [103] within the mathematical framework of different methods of parameterization of invariant manifolds [112]. Within a unified mathematical approach to NNMs in dissipative systems, a refined method is provided by the local theory of spectral submanifolds [29], possibly used sequentially to a slow-fast decomposition that results in a global ROM containing only the slow

degrees-of-freedom of the full dynamical system [113]. Spectral submanifolds enable a mathematically exact reduction of high-dimensional nonlinear systems to two-dimensional invariant manifolds of any vibration mode of interest in a mechanical problem [114], that allows us to assess the influence of nonlinearities in forced response curves, backbone curves, and detached resonance curves (isolas) with remarkable computational speed.

Multistability, i.e., the coexistence of different attractors, is a common feature in nonlinear systems, whose final states strongly depend on initial conditions. Its analysis is of great interest, in relation to the possibility of switching between coexisting states representing different dynamic behaviors of the system, which also include chaos. From the practical viewpoint, multistability either needs to be avoided or can be exploited for specific application purposes. In the former case, the desired state has to be stabilized against a generally noisy environment, in the latter case the coexistence of different stable states offers great flexibility in terms of system performance without major parameter changes. In both situations, challenging and suitable strategies of control of multistability have to be applied in order to avoid unwanted regimes or to switch between coexisting ones. Meaningful relevant research has been accomplished in applied mathematics and physics, with several control strategies (feedback, nonfeedback, stochastic) being developed and used in a huge variety of application contexts. Mechanical systems and structures were not yet meaningfully present in earlier corresponding lists (e.g., Ref. [115]), although bistability, and general multistability are encountered in a wide number of applications from macro- to micromechanics, with important prospects to avoid or exploit it. The increasing interest toward morphing structures capable to adapt their geometrical configuration to specific application purposes, with different driving methods used to trigger the snap from one equilibrium configuration to the other, is just one major example in this direction [116,117], with nontrivial steps forward yet to be made in dynamic, and mostly nonlinear, environments. The field is thus expected to become of major relevance in the mechanics area, too, also in connection with the increased attention paid to the analysis of the influence of imperfections, uncertainties, and noise on the nonlinear response, and with the development of higher-dimensional models exhibiting multiple attractors.

Global dynamics of systems and structures is another challenging field, along with its control and exploitation for design. Global analysis enhances the understanding of system dynamics not only in incremental terms, as highlighted in many circumstances, but also allows us to detect specifically important phenomena. This is certainly the case of multiphysics systems, where the meaningful steady effects produced, e.g., in a model of two-field coupled structure by the slow transient dynamics of one field (i.e., thermal) on the much faster dynamics of another field (i.e., mechanical) is only unveiled through refined analyses of basins of attraction in multidimensional phase space [118]. Global dynamics also play a fundamental role as regards evaluation and control of response robustness of multistable systems [119]. Indeed, the sudden erosion of a safe basin of technical interest, with final ending up of the system to a different bounded dynamics or unbounded escape as a control parameter changes, is triggered by some global bifurcation event, whose possibly optimal control allows us to enhance the system safety in operating conditions. Reliable safety evaluations in uncontrolled or controlled situations are obtained by using dynamical integrity (DI) concepts and tools, as highlighted for a large variety of single-mode models of systems and structures in different application contexts [59]. The DI approach also opens challenging perspectives as regards revisiting the safety paradigm currently used in the design of engineering systems, via a proper account of the expected disturbances and uncertainties, and conscious exploitation of the system's actual load carrying capacity [120]. Of course, DI evaluations become much more challenging for multidimensional models.

Uncertainties of systems and excitations play a fundamental role in the analysis, control, and design of multistable systems.

Indeed, the inherent variability of initial conditions and model parameters, along with imprecise measurements, may have a meaningful impact on the dynamics, blurring the solution features with respect to those of deterministic systems. This is of great importance to design, which must be robust with respect to imperfections and uncertainties, and maintain or even optimize system performances and operational safety. Influence of initial conditions' or parameters' randomness, as well as excitation noise entailing time-varying randomness, are to be addressed with reference to local and global dynamics, as earlier accomplished, e.g., via stochastic Galerkin methods with generalized polynomial chaos expansion for ROMs of structures liable to unstable buckling [121,122]. Yet, more theoretically based stochastic methods are likely to be employed, especially in the case of noise. Steps ahead to deal with also global dynamics in a fully probabilistic perspective should be made based on transfer operators approaches (e.g., Refs. [123] and [124]), with the likely need to also revisit DI tools due to the topological characteristics of diffuse attractors and blurred basin boundaries produced by the noisy and/or parameter uncertainty context [125,126]. Overall, this requires transferring and implementing in mechanics a whole basket of concepts and acquisitions from functional analysis, probability theory, and statistical physics. Overall, uncertainty quantification tools for assessing the influence of parameter variations and uncertain operating conditions, but also of errors stemming from mathematical idealizations and model approximations, play increasing roles in solving engineering problems [101].

3.2 Needed Computational Advancements. Suitably facing the mentioned issues requires *computational* advancements as regards both reduced modeling and high-dimensional investigations. Indeed, on the one hand, universal nonlinear phenomena of dynamical systems may be investigated fruitfully mostly in low-dimensional models generally requiring relatively low computational resources. On the other hand, the computational power needed for deeper analyses of nonlinear phenomena in also low-dimensional multistable systems, and most of global features and effects of uncertainties in higher-dimensional systems, can increase by orders of magnitude, with major problems of efficiency. This occurs in a variety of situations, which include the following. (i) When exploring the bifurcation structure in parameter space with high resolution, to account for the influence of "secondary" control parameters. (ii) When considering many initial conditions in phase space, as mandatory for investigating multistability and localizing coexisting attractors (including rare/hidden ones [127]), also by possibly exploiting the recently developed concept of perpetual points [128]. (iii) When constructing multidimensional basins of attraction and possibly controlling global events which govern their robustness. In the sequel, a few computational items related to model order reduction, multistability, and global dynamics in a deterministic or stochastic framework are dealt with.

Local and global computation of invariant manifolds. Invariant manifolds in the neighborhood of invariant sets such as fixed points, periodic orbits, or invariant tori are computed through local approximations. Based on a reformulation of the parameterization method for local approximations that facilitates a rigorous nonlinear model reduction, advancements in the computation of invariant manifolds and their reduced dynamics are being obtained for very high-dimensional nonlinear systems arising from spatial (i.e., finite-element) discretization of the governing partial differential equations [129]. The methodology has been implemented in a numerical package [130] that enables the computation of invariant manifolds in finite element-based discretized problems via an integrated finite element solver and bifurcation analysis of the reduced dynamics on the invariant manifolds via its Coco [131] integration. Applications of interest in dissipative mechanical systems include the computation of two-dimensional spectral submanifolds and their associated forced response curves

[129]. By virtue of computational benefits ensuing from parallelization, the procedure is expected to work also for higher-dimensional spectral submanifolds, whose reduced dynamics can be used for obtaining force response curves in internally resonant mechanical systems. In contrast, advancements with global methods, which seek invariant manifolds globally in the phase space by employing numerical continuation for growing them from local approximations, are much more challenging for high-dimensional systems, since numerical continuation of multidimensional invariant manifolds is nearly computationally intractable. This currently prevents from dealing with global dynamical structures (e.g., homoclinic and heteroclinic connections) and associated phenomena of high-dimensional systems in a dimension reduction perspective.

Detailing bifurcation scenarios and controlling multistability. To obtain valuable and detailed information on bifurcation structures, boundary value problem solvers requiring an already found orbit to initiate the computation must be combined with initial value problem simulations. High-resolution multidimensional parameter scans via initial value problem solvers based on the high power of graphic processing units [132] may allow discovering new features of dynamical systems that are hard to find by other means, e.g., the direct selection of a desired attractor in the parameter space of a multistability control problem, or the detection of new focal points of grazing lines in the initial conditions space of a nonsmooth problem [133]. However, in order to fully exploit the advantages of such “brute force” computational techniques, a massively parallel hardware architecture has to be used, with nontrivial difficulties in distributing the workload evenly to tens of thousands of parallel threads.

Constructing multidimensional basins of attraction. Parallelized implementations of the cell mapping method are also needed to construct multidimensional basins of attraction with a sufficient resolution, steps forward being made in the last few years to analyze a large set of initial conditions in a reasonable computational time [134,135]. Cell mapping methods are applied to global analysis and control design of stochastic and fuzzy dynamical systems, too [136]. Efficient implementations on massive parallel architectures require a proper balance of the algorithmic processes between memory operations and numerical integrations [137]. An important issue is concerned with efficient analyses of how basins of attraction change with a varying control parameter, parallel computation with reasonable CPU time allowing to consider its nearly continuous variation and to highlight fine aspects of basins’ topology changes. Geometrical aspects of representation/visualization of global dynamics outcomes in multidimensional phase space are also quite complicated. The whole matter is of considerable importance in the perspective of a reliable evaluation of systems’ dynamical integrity [138], also in view of possibly implementing algorithms that, while purposely giving up to study whole basins of attraction, may represent viable alternatives to more complex and computationally demanding techniques, thus being more appealing for design purposes and also industrial applications [139].

Basin stability in a deterministic framework. Detecting all possible solutions and basins of attraction of multistable systems is quite complicated already in single-dof systems. For multi-DOF ones, it becomes nearly practically impossible, mostly if the initial conditions cannot be precisely predicted and a wide range of system parameters has to be analyzed. Broadly speaking, it can be said that a transition from the brute force computational approach aimed at constructing the whole basins of attraction to a “sample-based” approach solely aiming to obtain some general measures of relevant stability is in order. In this perspective, the basin stability method [140] uses Bernoulli trials with randomly chosen initial conditions to estimate the volume of a basin of attraction, which gives information on the attractor occurrence probability. Its extended version [141] randomly chooses selected parameters, too, in order to estimate the influence of their uncertainties on the probability to reach a given attractor of interest. The method is applied successfully to multistable mechanical systems [142].

Based on parallelization strategies of efficient Monte Carlo simulations, an open-source toolbox for computing basin stability of multistable systems with minimal efforts and in a highly automated manner is now available, too [143]. Overall, sample-based methods of dynamic analysis may allow (i) to detect all possible solutions with basin stability greater than some threshold; (ii) to investigate ranges of stability in multiple parameters space and their susceptibility to perturbations. Of course, giving up a detailed identification, description, and visualization of bifurcational and topological aspects behind instability phenomena, sample-based approaches may lack information about the compactness of basins, which strongly affects their robustness.

Basin stability in a stochastic framework. If the system is stochastic due to a random parameter or a stochastic excitation, the topological notions of attractor and basin of attraction are no longer meaningful. Stochastic basins of attraction for deterministic systems with noise are defined based on the notion of escape probability from the basin of attraction of the underlying deterministic system, which quantifies the stability of a metastable state by specifying what and how other states in the state space are dynamically associated with the metastable one under stochastic dynamical evolution [144]. With no need to preliminarily know the basins’ structure, other time-scale dependent notions of stochastic basins are based on expected mean sojourn times and absorption probabilities, with the transfer operator approach providing a conceptual framework for stochastic basin stability, and sampling procedures allowing its evaluation in generalized terms [145]. For low-dimensional systems, the influence of random noise or uncertain parameters in global dynamics terms is analyzed through algorithms approximating transfer operators as a probability transition matrix with the Ulam (or generalized cell-mapping) method via Monte Carlo simulations [146], or set-oriented numerical methods accounting for long-term uncertainty propagation [147]. However, for systems with long transient states, trajectories may be very expensive to compute. For high-dimensional systems, the transfer operator framework is expected to facilitate the development of efficient algorithms for estimating basin stability and related measures.

3.3 Full Technical Exploitation of Nonlinear Features and Phenomenologies. Nonlinear phenomena may adversely affect the performance of mechanical systems and structures at whatever size-scale. Thus, in most applications, it is accepted that these can safely operate only in linear regimes. Yet, it has become progressively clear that linear design severely limits the usability and effectiveness of all engineering systems whose intrinsic nonlinear phenomena are of great relevance, as it happens for flexible structures in macromechanics and micro/nanodevices generally operating in a marked multiphysics context. However, based on accurate modeling, analysis, and description, nonlinear phenomena may also be harnessed to design structures and devices with new principles of operation and enhanced performances in a variety of application fields across different length and time scales. The *exploitation of nonlinearities* intrinsic to systems/structures, or intentionally added to them, can allow improvements in performance and more general design efficiency. In this respect, reference can be made to the involved nonlinear features or dynamic phenomena, as well as to practical goals to be achieved within different technological fields.

Both regular and complex *features/phenomena* of nonlinear dynamics can be suitably exploited, as per the following, albeit certainly incomplete, list. Instability regions of parametric resonance and ensuing parametric amplification; self-excitations effects; the influence of even nonlinearities; competing nonlinear behaviors in given dynamic environments (e.g., mechanical hardening and electrostatic softening in MEMS); inherent or induced internal resonances and their specific effects (e.g., saturation of a directly excited mode and increased response of an indirectly excited one, in 2:1 case); resonance capture entailing irreversible

energy transfer; multimode interactions of different nature; multi-stability; bifurcation-based amplification; the role of high-order nonlinearities; impact-induced energy dissipation; nonlinear mode localization; dynamic stabilization under high-frequency excitation; topological features of chaotic attractors; use of chaotic signals/excitations for parametric identification (e.g., probing a nonlinear system) and SHM.

Generally speaking, the important role that *strong nonlinearities* (of geometrical, kinematical, material nature, or due to clearances, impacts, friction) can play in the improvement of system response, is still to be adequately explored. As a matter of fact, intentional use of strong nonlinearity for controlling energy transfers can lead to enhanced performance of dynamical and acoustical systems [35]. This has been already highlighted in vibration and shock isolation, vibroimpact, and self-adaptive absorbers, energy harvesting, passive energy redirection and localization, wave propagation, modulation and arrest, blast and seismic mitigation, acoustic systems capable of extreme energy transfers across scales, micro-electronics, nanotechnology, phononic crystals and nanoresonators, granular media, and biomimetic systems. In many cases, with also fascinating aspects having no analogs in linear or weakly nonlinear dynamics [148].

Among the main *practical goals* toward which nonlinearities can be exploited through different mechanisms for improved/optimized behavior, the following can be mentioned.

- *Vibration suppression* makes use of nonlinear response through passive or active approaches. This includes: auto-parametric vibration absorbers, with the nonlinearities of a secondary system (tuned absorber), added to the primary one used to maximize the frequency bandwidth for vibration suppression; inducing nonlinear effects in the structure to be controlled through active (e.g., piezoelectric) materials; adding essentially nonlinear systems (NES) to effectively dissipate energy across a wide range of frequencies; mitigating vibratory amplitudes under multiple resonance conditions and high forcing amplitudes; dissipating energy of a primary system through impacts with an absorber or particle impact dampers. An updated review of mechanical inerters as passive control devices can be found in Ref. [149], with also a focus on nonlinear applications. Of course, when coupling a secondary element to a primary one and exploiting some induced nonlinear phenomenon, disturbance of parameter values can entail unwanted nonlinear effects, and a control aimed at shifting bifurcation points out of the parameter range of interest has to be applied. From the practical viewpoint, vibration absorption objectives depend on the specific operational and technological context.
- *Vibration energy harvesting*, with the conversion of ambient mechanical vibrations to electrical energy by exploiting, e.g., oscillatory or rotary motions of a floating pendulum system parametrically excited by sea wave excitation [150–152], or, more generally, by using electromagnetic, piezoelectric, and electrostatic methods, as in a huge variety of applications. Piezoelectric transducers, in particular, can be easily integrated into, e.g., elastic multilayer microbeams, with the main advantage of furnishing high power output for a given device volume and excitation, thus being efficient for applications to small electronic devices and sensors. Again, nonlinearities may allow us to overcome the narrow band limitation associated with resonant tuning and provide a suitable broadband frequency effect, through multimodal systems also possibly operating under multidirectional excitation sources [153]. In general terms, the advantages that nonlinearities could bring in terms of developing innovative technologies and designs in the rapidly expanding area of *renewable energy* and *sustainable* systems/structures in a dynamic environment are still to be explored.
- *Duality* or possible *combination* of vibration control and energy harvesting. The former as observed, e.g., in a rod

assemblage subjected to parametric excitation [154]. The latter as realized, e.g., in a rotational NES harnessing a significant amount of energy from the motion of a cylinder in a submarine current, while also suppressing its vortex-induced vibrations [155], in a simply supported beam with a tunable bistable magnetopiezoelectric absorber [156], or in an internally resonant tuned mass damper-cable system even by merely exploiting nonlinear coupling in a passive control framework [157].

- *Dynamic morphing* of light structures for the enhancement of structural performance through active, shape-changing, surfaces, as achievable in instability regions of parametric resonance. This is of main interest not only in aeronautical applications for optimum flight performance under different flight conditions [158,159], but also in smart truss structures, deployable mechanisms, tensegrity systems, and origami structures (e.g., Ref. [160]).
- *Controlling wave propagation* in a great variety of systems of coupled oscillators, like nonlinear chains and lattices, with a view to manipulating and guiding different features and mechanisms. For example, to modify pass/stop-bands, to affect the occurrence of traveling breathers, to realize localized responses of technical interest (even possibly in bio-inspired hierarchically organized oscillatory models [161]), all of this depending on the characteristics of some main system parameters.
- Using concepts and tools of *perpetual* points and manifolds [162] to investigate the nonlinear dynamics of discrete systems in a possibly great variety of mechanical applications targeted at rigid body motions.
- Fully exploiting *nonlinear* and *global* [119,163] *dynamics*, as well as *chaos/complexity*, for different purposes. Generally speaking, using it to pursue a *novel* and/or *tailored design* of safe, yet lighter, faster, smaller, more robust, and high-performing engineering systems for applications in a variety of technological contexts (from turbomachinery and automotive or aerospace structures, up to multiphysical/multiscale environments), also capable to operate in ever harsher, uncertain, and even poorly predictable conditions.

All of this requires a paradigm shift from the current view of avoiding strongly nonlinear effects and the ensuing cross-scale energy transfers to a novel approach also involving, and even intentionally implementing, *strong nonlinearity for design*, in order to achieve unprecedented objectives in an effective and robust way. In this respect, despite the occurrence of major methodological challenges in the analysis of strongly nonlinear systems—with regard, e.g., to modeling and prediction of nonstationary dynamical and acoustical processes—, research based on a concept of “mechanical turbulence” [148] can be potentially transformative for a wide class of application fields, where energy transfer is critical and broader impacts are far-reaching. Some main related topics will be the subject of a CISM Advanced School⁵ to be held in the summer of 2022, after two consecutive postponements due to the Covid pandemic.

3.4 Mechanical Metamaterials and Architected Meta-structures. Novel materials and structures have been the subject of intense research over the last few decades, and deserve specific considerations in the nonlinear dynamics perspective. Broadly speaking, *mechanical metamaterials* are artificial materials at macro- or micro/nanoscale with mechanical properties defined by their structure rather than by their composition, and designed to have values that cannot be found in nature. They are a meaningful part of the current extraordinary era of research on new materials. Based on different pursued perspectives, which encompass theoretical conception, mathematical modeling, analysis, characterization, functional purposes, design, optimization, and fabrication,

⁵<https://www.cism.it/en/activities/courses/C2110/>

mechanical metamaterials include a great variety of specific materials with superior and/or unusual properties. Acoustic/phononic metamaterials, auxetics (materials with negative Poisson's ratio), materials with negative longitudinal and volumetric compressibility transitions, origami and kirigami-inspired metamaterials, chiral metamaterials, micropolar metamaterials, are among the most important examples. A number of relevant reviews, highlighting relationships among materials, structures, properties, and applications, have appeared in the literature (e.g., Refs. [164] and [165]). In more general terms, *architected materials* and *metastructures* are to be mentioned. They consist of/exhibit multiphase and/or cellular materials/organizations in which the topological distribution of the phases and/or the cells is carefully controlled and optimized, possibly also based on hierarchical assumptions, to the aim of achieving specific mechanical properties or (multi-)functional characteristics of interest in different physical or industrial applications [166]. Mechanical metamaterials are being considered also in terms of dynamic performances, with regard to, e.g., the interaction between matter distribution and energy flow propagation [167].

However, extended research on the dynamic behavior of mechanical metamaterials and architected metastructures, which is a field at the crossroads of vibration and acoustics engineering and condensed matter physics, is only relatively recent, with an overall acquired knowledge that is still far away from being fully satisfactory and/or comprehensive, mostly as regards the nonlinear regime. Interest in the propagation of elastic waves through periodic mechanical systems, earlier studied in a multiplicity of different scientific branches, has indeed been renewed by the extraordinary developments that occurred in the micro/nano-engineering of architected materials and acoustic metamaterials. Within the more mechanics-oriented realm of dynamics, the role played by techniques developed for the treatment of damping and nonlinearities in structural dynamics for addressing wave propagation problems in periodic media was overviewed earlier [99]. A more recent review paper on tunable and active phononic crystals and metamaterials deal with bandgap engineering, anomalous behaviors of wave propagation, and tunable manipulation of waves based on different regulation mechanisms [168].

Effects of nonlinearities on periodic and localized solutions have been investigated in lattice models and acoustic metamaterials (e.g., Refs. [169–171]). Recently, earlier predictions of amplitude-dependent dispersion relationships obtained for architectures with weakly nonlinear stiffness via perturbation or homotopy [172] approaches have been complemented with asymptotic and numerical results on nonlinear wavefrequencies and also nonlinear waveforms of mechanical metamaterials with specific internal arrangements (e.g., Ref. [173]). A general framework has also been proposed to analytically and numerically obtain unified pictures of the nonlinear dynamics of flexible mechanical metamaterials, and to systematically investigate propagation and stability of phonons, solitons, amplitude–frequency gaps, and large amplitude oscillatory traveling waves [174].

Yet, extended analyses of architected *metamaterials* in full dynamic environments are still lacking, and the strong relationships occurring between their mechanical properties and inherent organization, on the one hand, and the ensuing *nonlinear dynamics*, on the other hand, are topics that definitely require intense investigations and are expected to be major research subjects in the near future.

3.5 Data-Driven Approaches. Since Poincaré and Lyapunov, qualitative behavior of dynamical systems is addressed via mathematical models consisting of a system of ordinary/partial, deterministic/stochastic, differential/difference equations, as well as control equations, which approximate the observed reality. Models are generally quite reliable to describe many dynamical processes, such as those occurring in mechanical systems and structures. However, for some of the most challenging applications, which include climate dynamics and extreme events, brain dynamics, pattern formation in biological systems, financial

markets, the formulation of reliable models is considerably difficult. Thus, in these fields, the ever-increasing amount of physical observations and data collection that occurred in the last few years has stimulated advanced research as to the role they can play in developing novel approaches to describe dynamical processes of the real world. Indeed, the availability of larger and more complex data sets not only enables improving a system's understanding but also originates research challenges and a wide basket of new possibilities. Data-driven approaches are currently revolutionizing the modeling, prediction, and control of complex systems.

This strongly evolving area combines complex systems science, machine learning, engineering mathematics, and mathematical physics to integrate modeling and control of dynamical systems with modern methods in data science, which can be applied to a diverse range of complex and stochastic systems. *Machine learning* methods, in particular, are concerned with algorithms designed to accomplish a certain task, whose performance improves with the input of more data. By way of example, a machine learning approach blending classical tools from numerical analysis (multistep time-stepping schemes) with powerful nonlinear function approximators (deep neural networks) may allow extracting nonlinear dynamical systems from time-series data [175]. The machine learning approach is invaluable in settings (computer vision, stock market analysis, speech recognition, recommender systems, sentiment analysis in social media) where no explicit model is formulated but measurement data is available. Yet, the intersection of dynamical systems and machine learning is still largely unexplored, as recently highlighted in a special issue [102] dwelling on a variety of topics including reservoir computing, learning theory, Gaussian processes, data-driven modeling, geometric and topological data analysis.

Leaving aside data-assisted contributions originated by the Covid pandemic in about the last couple of years, and focusing on mechanics where data-driven approaches have been used earlier mostly for fluids, research and initiatives are increasingly being undertaken with regard to structures and mechanical systems, too. They are mostly concerned with the identification and inverse problems and include topics like nonparametric data analysis toward model structure selection; input design and machine learning mappings in nonlinear data-driven modeling; analysis, reduction, and interpretation of nonlinear data-driven models; nonlinear model-based control and design [176]. Data-assisted architectures are anyway proposed in various environments and to different aims. For reduced-order aeroelastic modeling of morphing wings suitable for control and valid over a wide range of operating conditions [177]; for prediction of extreme transient events in complex systems with high-dimensional attractors, to complement imperfect ROMs with data-streams integrated through a recurrent neural network architecture [178]; for origami systems in a noisy vibration environment and with no mathematical model in the background, to characterize and predict chaotic dynamics [179]; for nonlinear modal identification (NNMs with invariant manifolds, energy-dependent nonlinear modal spectrum, future-state prediction) of unknown nonlinear systems from response data only [180].

Quite systematic work is being accomplished as to the meaningful contribution that the recent theory of SSMs can provide to formulate an effective data-driven model reduction methodology for essentially nonlinear (or nonlinearizable) systems. Indeed, SSMs theory offers a systematic basis for model reduction and allows a simplification of the reduced dynamics via normal forms, thus enabling to handle multimodal interactions, to identify amplitude-dependent damping and frequency, and to predict the forced structural response of mechanical systems [181,182]. Relevant results are implemented also in an open-source package [130] that highlights the power of data-driven SSM-reduction on high-dimensional numerically generated data sets and experimental data.

Data-driven methods are being proposed also for automatically reducing the system dimension of nonlinear randomly vibrating

systems [183] and, generally, for engineering systems with stochastic properties under stochastic forcing. Simultaneous real-time prediction and uncertainty quantification of fatigue failure is a critical topic for complementing the prediction of time to failure with fast and operational decision-making. A hybrid architecture of artificial neural networks and Gaussian process regression may allow extending the forecasting capabilities of existing time-domain and machine learning-based methods, paving the way toward the development of predictive systems providing real-time safety and operational instructions, as well as insights for SHM purposes [184]. Another topic is the data-driven optimization of sensor placement for enhanced accuracy of response prediction at a selected set of locations of interest, and for early prediction capabilities in complex dynamical systems with hundreds and millions of dofs, which cannot be described by reduced order modeling [185].

Data-driven approaches can also contribute to global analysis, which becomes nearly impractical when the underlying system model is too complex to analyze or even unavailable. A data-driven generalized cell mapping has been proposed to investigate the global properties of nonlinear systems from a sequence of measurement data, without prior knowledge of the underlying system. The method includes the estimation of the system state dimension and the time-step for creating a mapping from the data. This allows computing a one-step transition probability matrix via a statistical approach and uncovering the global properties of the underlying system [186].

In general terms, the *great potential* of data-driven approaches for reduced-order modeling, identification of nonlinear and stochastic systems, understanding, prediction, and control of the dynamical response, uncertainty quantification, and complexity evaluation, is far away from being adequately explored. Indeed, it is expected to be a major research item of nonlinear dynamics in mechanics and high-tech engineering over the next few years, with the aim of describing real-life applications. Its importance is becoming fully apparent also within top-level international associations involved with mechanics. The first IUTAM Symposium on data-driven nonlinear and stochastic dynamics with control⁶ will be held in China in 2022, with the synergic participation of specialists from different fields, including mechanics, mechanical engineering, physics, biology, mathematics, and statistics. Thus making apparent once more how a meaningful part of the advanced research on nonlinear dynamics in mechanics in the years to come will be conducted at actually cooperative and widely *cross-disciplinary* levels.

4 Conclusions

This perspective overview on nonlinear dynamics in mechanics has been constructed along two separate but necessarily interconnected lines. First, a summary of the themes of interest to the community of related scholars has been presented, based on the current state of knowledge and achievements, distinguishing between traditional and emerging themes inherent to mechanics, on one side, and more hybridized contexts, on the other side. Then, a “vision” of expected future developments has been attempted, along the following lines. (i) Presenting some general modeling, analysis, control, and design problems, and the computational advances needed to tackle them. (ii) Emphasizing the existing tension toward full technical exploitation of the great potential of nonlinear characteristics and phenomenologies. (iii) Focusing on specific issues of foreseeable development, such as the nonlinear dynamics of mechanical metamaterials and architected metastructures, and the approach to dynamical systems in mechanics based on the analysis and use of large amounts of data.

Irrespective of the rich variety of specific systems and/or topics to be possibly considered, the challenging new general frontiers of

nonlinear and complex dynamics in mechanics, and beyond, can be traced and summarized as follows.

- (1) As regards *modeling* and ensuing *methods of analysis*, an appropriately balanced perspective must be adopted between two apparently opposite scenarios. Indeed, on one hand, multi-degree-of-freedom models more reliable than the single/few-dof ones typically used for obtaining solutions (usually approximate) to the nonlinear dynamics of systems/structures/processes in mechanics at whatever scale have to be formulated. On the other hand, the advantages of a thorough understanding and a systematic description of the main nonlinear effects, made possible by the conscious use of reliable reduced order models, cannot be lost, especially in the multiphysics and uncertain environments in which real systems operate. This reaffirms the importance of an effective, reliable and controllable dimension reduction, obtained through an intelligent combination of novel theoretical achievements, data-assisted computational techniques, and even experimental acquisitions, to be pursued in order to frame, and appropriately integrate, the enormous mass of results provided by numerical analyses of companion multidimensional models.
- (2) As regards *response phenomenology* and the consequent *control* and *design* problems, the main expectations concern full exploitation of nonlinear and complex dynamics aimed at significantly improving the operating conditions of mechanical systems and structures in a variety of technological environments. To this end, two main aspects deserve specific attention. (i) Thoroughly exploring the enormous potential of inherent or intentionally added nonlinearities for the purpose of advanced and innovative modeling, analysis, and control of systems in a huge variety of traditional and novel applications, to be assessed by also referring to the uncertainty quantification issue. (ii) Overcoming the consolidated, but too conservative, design criteria based on linear or weakly nonlinear behavior of systems and structures, still variably adopted in practically all fields of engineering. This requires the development of novel conceptual and operational paradigms that skillfully exploit also strong nonlinearities—and the related phenomenologies—to achieve new results in terms of local and, mostly, global dynamics, by taking into account also real-world imperfections.

All of the above to be cooperatively carried out in the increasingly hybridized and cross-disciplinary contexts in which most phenomena of nonlinear dynamics in mechanics are now embedded, in a “natural” or “intentional” way.

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⁶<https://iutam.org/event/iutam-symposium-on-data-driven-nonlinear-and-stochastic-dynamics-with-the-control/>

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