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STACKELBERG-NASH EXACT CONTROLLABILITY FOR THE BENNEY-LIN TYPE EQUATION WITH MIXED BOUNDARY CONDITIONS

MANISH KUMAR [†], AND SUBRATA MAJUMDAR [‡]

ABSTRACT. This paper deals with a bi-objective control problem for a *strongly dissipative fifth-order Korteweg-de Vries (KdV) equation* with Dirichlet-periodic mixed boundary conditions by means of distributed controls. More precisely, we address a *Stackelberg-Nash* control problem, where our first priority is to achieve the “controllability” type objective with a secondary objective of “optimal control” type. The control is assumed to be localized in three mutually disjoint subsets of the spatial domain and the control restricted to each region has different objectives to accomplish. One of the localized controls is responsible for local controllability to a free trajectory of the concerned nonlinear model, while the other two controls have to act as a *Nash equilibrium* for a pair of given functionals.

To study such a problem, we give a hierarchy to the localized controls. More precisely, assuming the existence of the localized control (*called leader*) responsible for the “controllability” type objective, we establish the existence of the other two controls (*called followers*), which depend on the choice of leader and whose goal is to keep the state close to two given trajectories on two different regions of the domain. In order to find the leader control, a controllability result is first established for a linearized forward-backward coupled system by deriving a weighted observability estimate for the associated adjoint system. The local inversion mapping theorem is then employed to guarantee the existence of the leader control for the nonlinear problem. To obtain the desired weighted observability estimate, a new Carleman estimate is derived.

1. INTRODUCTION AND MAIN RESULTS

Let $T, L > 0$, and set $Q := (0, T) \times (0, L)$. In this paper, we address a bi-objective control problem for the following *Benney-Lin type equation*

$$\partial_t u + \partial_x^3 u + \mu_0 \partial_x^4 u + \partial_x^5 u + u \partial_x u = 0, \text{ on } Q, \quad (1.1)$$

with $\mu_0 > 0$, by means of localized interior control, where the primary goal is of “controllability” type and the secondary goal of “optimization” type. This equation describes the evolution of one-dimensional small but finite amplitude long waves in various physical systems in fluid dynamics (see [Ben66], [Lin74], [TDK18]), where $u(t, x)$ denotes height of the wave at time instant t and point x . The coefficient $\mu_0 > 0$ adds nonconservative dissipative effects in the *dispersive Kawahara equation* ((1.1) with $\mu_0 = 0$), and so it is sometimes referred as *strongly dissipative Kawahara equation* (see [Zho19]) or *fifth-order Korteweg-de Vries equation* (see [ZZF18]) or *generalized Kawahara equation* (see [CDLW22]). Concerning the well-posedness results for this equation on the real line \mathbb{R} , we cite the works [BL97], [CL08], [ZC09], [CdR22] and for the bounded interval see the works [ZZF18], [ZWB23].

For the controllability perspective on the higher order dispersive equation, let us mention the works [Che19, CFG21], both inspired by the work [CFPR15] for the Korteweg-de Vries equation. More precisely, the authors in [Che19] investigated the local null controllability of the Kawahara equation ((1.1) with $\mu_0 = 0$)

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x^5 u + u \partial_x u = f \chi_\omega, & (t, x) \in Q, \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = \partial_x^2 u(t, 0) = 0, & t \in (0, T), \\ u(0, x) = u_0(x), & x \in (0, L), \end{cases} \quad (1.2)$$

by means of distributed control, while the work [CFG21] deals with the well-posedness and exact controllability of the same equation in some weighted Sobolev spaces. The authors in [CFG21] also introduced the *regional controllability* of the aforementioned equation in the space L^2 . Concerning the boundary

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controllability for this equation, let us first cite the work [GG09], where the authors used two boundary control on the right side of the boundary to establish the local controllability to the trajectories. In the case of the control problem for Kawahara equation with periodic boundary, we refer to the works [ZZ15], [ZZ12], [PV23]. Moreover, in a most recent work [KM24], the authors discussed the existence of an insensitizing control for the Kawahara equation.

It is worth noting that the authors in [CMZ20] have explored the local exact controllability to the trajectories of the *Korteweg-de Vries-Burgers equation* under mixed boundary conditions

$$\begin{cases} \partial_t u + \partial_x^3 u - \mu(t)\partial_x^2 u + u\partial_x u = f\chi_\omega, & (t, x) \in Q, \\ u(t, 0) = u(t, L) = 0, \partial_x u(t, 0) = \partial_x u(t, L), & t \in (0, T), \\ u(0, x) = u_0(x), & x \in (0, L). \end{cases} \quad (1.3)$$

To the best of the authors' knowledge, no prior research has addressed controllability issues for the Benney-Lin type equation (1.1) with the following mixed boundary conditions

$$u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = 0, \partial_x^2 u(t, 0) = \partial_x^2 u(t, L).$$

Our work not just fills this gap concerning the controllability result, but also answers a more general question of Stackelberg-Nash exact controllability.

1.1. Problem formulation and main results. Let \mathcal{O}_0 , \mathcal{O}_1 , and \mathcal{O}_2 be mutually disjoint open subsets of $(0, L)$, and assume them to be sufficiently small. Consider the control system

$$\begin{cases} \partial_t u + \partial_x^3 u + \mu_0 \partial_x^4 u + \partial_x^5 u + u\partial_x u = \chi_{\mathcal{O}_0} f^0 + \chi_{\mathcal{O}_1} f^1 + \chi_{\mathcal{O}_2} f^2, & (t, x) \in Q, \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = 0, \partial_x^2 u(t, L) = \partial_x^2 u(t, 0), & t \in (0, T), \\ u(0, x) = u_0(x), & x \in (0, L), \end{cases} \quad (1.4)$$

where u_0 is the given initial data, and the function $f := \chi_{\mathcal{O}_0} f^0 + \chi_{\mathcal{O}_1} f^1 + \chi_{\mathcal{O}_2} f^2$ is the interior control, localized on the subsets \mathcal{O}_0 , \mathcal{O}_1 , \mathcal{O}_2 .

Let \bar{u} be a free trajectory associated to the system (1.4), i.e., for a given $\bar{u}_0 \in L^2(0, L)$, let \bar{u} solves the system

$$\begin{cases} \partial_t \bar{u} + \partial_x^3 \bar{u} + \mu(t)\partial_x^4 \bar{u} + \partial_x^5 \bar{u} + \bar{u}\partial_x \bar{u} = 0, & (t, x) \in Q, \\ \bar{u}(t, 0) = \bar{u}(t, L) = \partial_x \bar{u}(t, 0) = \partial_x \bar{u}(t, L) = 0, \partial_x^2 \bar{u}(t, L) = \partial_x^2 \bar{u}(t, 0), & t \in (0, T), \\ \bar{u}(0, x) = \bar{u}_0(x), & x \in (0, L). \end{cases} \quad (1.5)$$

Further, for given open subsets $\mathcal{O}_{1,d}$, $\mathcal{O}_{2,d}$ of $(0, L)$, and given functions $u_{1,d}$, $u_{2,d} \in L^2(0, L)$, consider the functional

$$J_i(f; f^1, f^2) := \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d} \times (0, T)} |u - u_{i,d}|^2 + \frac{\mu_i}{2} \iint_{\mathcal{O}_i \times (0, T)} |f^i|^2, \text{ for } i \in \{1, 2\}. \quad (1.6)$$

The main problem addressed in this article is to achieve the following two objectives in the specified order of priority.

1. Find $f^0 \in L^2((0, T) \times \mathcal{O}_0)$ such that

$$u(T, \cdot) = \bar{u}(T, \cdot). \quad (1.7)$$

This is referred to as the problem of exact controllability to the free trajectory, which is the primary objective of this article. The function f^0 is known as primary control, and the region \mathcal{O}_0 as primary control region.

2. The second objective is to find the controls $f^i \in L^2((0, T) \times \mathcal{O}_i)$ s.t.

$$\begin{cases} J_1(f^0; f^1, f^2) = \min_{\hat{f}^1 \in L^2((0, T) \times \mathcal{O}_1)} J_1(f^0; \hat{f}^1, f^2), \\ \text{and } J_2(f^0; f^1, f^2) = \min_{\hat{f}^2 \in L^2((0, T) \times \mathcal{O}_2)} J_2(f^0; f^1, \hat{f}^2). \end{cases} \quad (1.8)$$

The controls f^1 , f^2 are termed as secondary controls and the regions \mathcal{O}_1 , \mathcal{O}_2 as secondary control regions. Such pair of controls (f^1, f^2) satisfying (1.8) is referred as *Nash equilibrium* for the cost functionals J_1 , J_2 .

Such *hierarchical control problem* is well-known in the literature by the name of *Stackelberg-Nash exact controllability problem*. To solve such hierarchical control problem, we first find the controls f^1, f^2 which acts as a Nash equilibrium for the functionals J_1, J_2 , treating the control f^0 as a known function. Next, we establish the existence of the control f^0 , responsible for the controllability goal (1.10). Note that the

way this problem is dealt, the controls f^1, f^2 depends on the choice of control f^0 . This is why the control f^0 is referred as the *leader*, while the controls f^1, f^2 as *followers*.

The first problem concerning controllability to trajectory can be reformulated as a null controllability problem. More precisely, if we introduce the new variables

$$v = u - \bar{u}, \text{ and } v_{i,d} = u_{i,d} - \bar{u}, \text{ for } i \in \{1, 2\},$$

then the controllability problem of finding $f^0 \in L^2((0, T) \times \mathcal{O}_0)$ so that the solution u of (1.4) satisfies (1.7) reduces to find $f^0 \in L^2((0, T) \times \mathcal{O}_0)$ such that the solution v of system

$$\begin{cases} \partial_t v + \partial_x^3 v + \mu_0 \partial_x^4 v + \partial_x^5 v + v \partial_x v + \partial_x(\bar{u}v) = \chi_{\mathcal{O}_0} f + \chi_{\mathcal{O}_1} f^1 + \chi_{\mathcal{O}_2} f^2, & (t, x) \in Q, \\ v(t, 0) = v(t, L) = \partial_x v(t, 0) = \partial_x v(t, L) = 0, \partial_x^2 v(t, L) = \partial_x^2 v(t, 0), & t \in (0, T), \\ v(0, x) = v_0(x) := u_0(x) - \bar{u}_0(x), & x \in (0, L). \end{cases} \quad (1.9)$$

satisfies

$$v(T, \cdot) = 0 \text{ on } (0, L), \quad (1.10)$$

which is indeed a question of null controllability for the system (1.9). Moreover, the functionals J_i can be rewritten in terms of new state variables as

$$J_i(f^0; f^1, f^2) := \frac{\nu_i}{2} \iint_{\mathcal{O}_{i,d} \times (0, T)} |v - v_{i,d}|^2 + \frac{\mu_i}{2} \iint_{\mathcal{O}_i \times (0, T)} |f^i|^2, \text{ for } i \in \{1, 2\}. \quad (1.11)$$

Thus, our aim now reduces to find the controls $f^i \in L^2((0, T) \times \mathcal{O}_i)$ such that

- (i) the solution v of (1.9) satisfies $v(T, \cdot) = 0$, and
- (ii) the control pair (f^1, f^2) satisfies (1.8), where the functionals J_1, J_2 are given by (1.11).

To accomplish the Nash equilibrium objective, we assume that the leader control $f^0 \in L^2((0, T) \times \mathcal{O}_0)$ is known, and characterize the followers f^1, f^2 as mentioned in the theorem below.

Theorem 1.1 (Characterization of Nash equilibrium). *For given $f^0 \in L^2((0, T) \times \mathcal{O}_0)$, the pair (f^1, f^2) is a Nash equilibrium for the functionals J_1, J_2 iff*

$$f^i = -\frac{1}{\mu_i} \varphi^i \chi_{\mathcal{O}_i} \text{ for } i \in \{1, 2\}, \quad (1.12)$$

where $(v, \varphi^1, \varphi^2)$ solves the optimality system

$$\begin{cases} \partial_t v + \partial_x^5 v + \mu_0 \partial_x^4 v + \partial_x^3 v + v \partial_x v + \partial_x(\bar{u}v) = f^0 \chi_{\mathcal{O}_0} - \frac{1}{\mu_1} \varphi^1 \chi_{\mathcal{O}_1} - \frac{1}{\mu_2} \varphi^2 \chi_{\mathcal{O}_2}, & (t, x) \in Q, \\ -\partial_t \varphi^i - \partial_x^5 \varphi^i + \mu_0 \partial_x^4 \varphi^i - \partial_x^3 \varphi^i - (v + \bar{u}) \partial_x \varphi^i = \chi_{\mathcal{O}_{i,d}} \nu_i (v - v_{i,d}), & (t, x) \in Q, \\ v(t, 0) = v(t, L) = \partial_x v(t, 0) = \partial_x v(t, L) = 0, \partial_x^2 v(t, L) = \partial_x^2 v(t, 0), & t \in (0, T), \\ \varphi^i(t, 0) = \varphi^i(t, L) = \partial_x \varphi^i(t, 0) = \partial_x \varphi^i(t, L) = 0, \partial_x^2 \varphi^i(t, L) = \partial_x^2 \varphi^i(t, 0), & t \in (0, T), \\ v(0, x) = v_0(x), \varphi^i(T, x) = 0, & x \in (0, L). \end{cases} \quad (1.13)$$

Remark 1.2. Note that the followers f^1 and f^2 given by (1.12) depend on the choice of f^0 due to the coupling of v in the equation of φ^1 and φ^2 .

The above proposition establishes the existence of the followers f^1 and f^2 , under the assumption that the system (1.13) is well posed, which is indeed true (see Proposition 2.10). Thus, our problem now reduces to a single objective control problem of ‘‘null controllability’’ type. More precisely, the problem boils down to find the leader $f^0 \in L^2((0, T) \times \mathcal{O}_0)$ such that the v component of the solution of the extended system (1.13) satisfies (1.10). Corresponding to this null controllability problem we obtained the following answer, which is the main result of this paper.

Theorem 1.3. *For sufficiently large real number $\hat{\mu} > 0$, assume that $\mu_1, \mu_2 \geq \hat{\mu}$, and suppose*

$$\mathcal{O}_{i,d} \cap \mathcal{O}_0 \neq \emptyset, \text{ for } i \in \{1, 2\}. \quad (1.14)$$

Further, assume that the subsets $\mathcal{O}_{i,d}$ satisfy one of the following conditions:

$$\mathcal{O}_{1,d} = \mathcal{O}_{2,d} \quad (1.15)$$

or

$$\mathcal{O}_{1,d} \cap \mathcal{O}_0 \neq \mathcal{O}_{2,d} \cap \mathcal{O}_0. \quad (1.16)$$

For any given free trajectory $\bar{u} \in X_T^0$ which solves (1.5), there exist a $\delta > 0$ and a positive function $\rho(t)$ blowing up at $t = T$ such that if the functions v_0 and $v_{i,d}$ satisfy the smallness condition

$$\|v_0\|_{L^2(0,L)} + \sum_{i=1}^2 \int_0^T \int_{\mathcal{O}_{i,d}} \rho^2(t) |v_{i,d}|^2 < \delta,$$

then there exist a control $f^0 \in L^2((0,T) \times \mathcal{O}_0)$ such that the v component of the associated solution of the optimality system (1.13) satisfies (1.10).

1.2. Methodology adopted to prove Theorem 1.3. We establish the partial null controllability result for the optimality system (1.13) as stated above, by investigating a similar partial null controllability problem for the following linear system, obtained by linearizing the system (1.13) about $v = 0$:

$$\begin{cases} \partial_t v + \partial_x^5 v + \mu_0 \partial_x^4 v + \partial_x^3 v + \partial_x(\bar{u}v) = f^0 \chi_{\mathcal{O}_0} - \sum_{i=1}^2 \frac{1}{\mu_i} \varphi^i \chi_{\mathcal{O}_i} + g^0, & (t, x) \in Q, \\ -\partial_t \varphi^i - \partial_x^5 \varphi^i + \mu_0 \partial_x^4 \varphi^i - \partial_x^3 \varphi^i - \bar{u} \partial_x \varphi^i = \nu_i v \chi_{\mathcal{O}_{i,d}} + g^i, & (t, x) \in Q, \\ v(t, 0) = v(t, L) = \partial_x v(t, 0) = \partial_x v(t, L) = 0, \partial_x^2 v(t, L) = \partial_x^2 v(t, 0), & t \in (0, T), \\ \varphi^i(t, 0) = \varphi^i(t, L) = \partial_x \varphi^i(t, 0) = \partial_x \varphi^i(t, L) = 0, \partial_x^2 \varphi^i(t, L) = \partial_x^2 \varphi^i(t, 0), & t \in (0, T), \\ v(0, x) = v_0(x), \varphi^i(T, x) = 0, & x \in (0, L), \end{cases} \quad (1.17)$$

and then using the well-known *local inversion mapping theorem*, we conclude the main result stated in [Theorem 1.3](#).

To establish the null controllability of the component v of solution of the linear system (1.17), we follow the duality approach where we prove an observability estimate for the following associated adjoint system:

$$\begin{cases} -\partial_t \eta - \partial_x^5 \eta + \mu_0 \partial_x^4 \eta - \partial_x^3 \eta - \bar{u} \eta_x = \sum_{i=1}^2 \nu_i \psi^i \chi_{\mathcal{O}_{i,d}} + h^0, & (t, x) \in Q, \\ \partial_t \psi^i + \partial_x^5 \psi^i + \mu_0 \partial_x^4 \psi^i + \partial_x^3 \psi^i + \partial_x(\bar{u} \psi^i) = -\frac{1}{\mu_i} \eta \chi_{\mathcal{O}_i} + h^i, & (t, x) \in Q, \\ \eta(t, 0) = \eta(t, L) = \partial_x \eta(t, 0) = \partial_x \eta(t, L) = 0, \partial_x^2 \eta(t, L) = \partial_x^2 \eta(t, 0), & t \in (0, T), \\ \psi^i(t, 0) = \psi^i(t, L) = \partial_x \psi^i(t, 0) = \partial_x \psi^i(t, L) = 0, \partial_x^2 \psi^i(t, L) = \partial_x^2 \psi^i(t, 0), & t \in (0, T), \\ \eta(T, x) = \eta_T(x), \psi^i(0, x) = 0, & x \in (0, L). \end{cases} \quad (1.18)$$

To prove the preferred observability estimate for the system (1.18), we first derive a Carleman estimate for the following adjoint system

$$\begin{cases} -\partial_t w - \partial_x^5 w + \mu_0 \partial_x^4 w - \partial_x^3 w - \bar{u} \partial_x w = g, & (t, x) \in Q, \\ w(t, 0) = w(t, L) = \partial_x w(t, 0) = \partial_x w(t, L) = 0, \partial_x^2 w(t, L) = \partial_x^2 w(t, 0), & t \in (0, T), \\ w(T, x) = w_T, & x \in (0, L), \end{cases} \quad (1.19)$$

and then use it to obtain a suitable Carleman estimate for main adjoint system (1.18). Finally, using the standard technique we deduce the desired observability inequality for (1.18) from this Carleman estimate. Thanks to the observability inequality for the adjoint problem, we prove the desired partial null controllability for the linearized system (1.17) with some suitable weighted source term. Finally, we use the local inversion theorem to conclude the main result.

1.3. Literature survey on the Stackelberg strategy. In practical situations, it is common to encounter such control problems where different objectives have to be achieved simultaneously. Due to its practical significance, such a control problem has been studied extensively in recent times for different dynamical systems modelled by PDEs. To start with, let us mention the works [Lio86] and [Lio94], where the hierarchical control problem for evolutionary PDE was first introduced to deal with multiple objectives. The work [Lio86] studies the Pareto optimal control notion for parabolic and hyperbolic type PDEs, whereas in [Lio94], the authors studied the notion of Stackelberg's optimal control for the parabolic type system.

Concerning the multi-objective control problems based on Stackelberg-Nash notion, let us cite the initial works [D02, DL04] where the primary objective is to find a leader control responsible for approximate controllability of the system, while the secondary one is to find a tuple of N followers which act as a Nash equilibrium for given N functionals. Motivated by these works, several such Stackelberg-Nash control problems for various linear and non-linear PDE systems have been addressed in the literature, for instance, see [RGP02a] for heat equation, [RGP02b] for Burgers equation, [GGMLRM13] for Stokes equation, [AdMRM14] for linearized micropolar fluid equation. Later on, many other Stackelberg-Nash control

problems have been studied with same secondary objective but with primary objective of exact controllability type, instead of approximate controllability. Such problem was first investigated for linear and semi-linear heat equations in the work [AFCS15], whose result was later improved in [AFCS17]. After this, numerous works have explored such problem for different PDEs, for instance, see [AFCdS18, AFCdS23] (for hyperbolic eq.), [AFCdS20] (for semi-linear heat eq. with followers as boundary controls), [BMO] (for heat equation with dynamic boundary conditions), [NH22] (for quasi-linear parabolic eq.), [AAFC18] (for linear and semi-linear degenerate parabolic eq.), [AFCdS24] (for Burgers eq.), [CnS19, CnS23] (for Kuramoto-Sivashinsky eq.), [AAS23] (for KdV eq.), [dCdJdSN22] (for Wave equation in a domain with moving boundary).

Control problem with Stackelberg-Nash strategy has also been explored for coupled PDE systems, see [HSdTP16, HSdT18, DKDZ23, TdTWZ23]. More precisely, in the paper [HSdTP16], the authors dealt with a Stackelberg Nash strategy for a cascade system of parabolic equations acting only in the first equation, along with a suitable weight on the follower control. The paper [HSdT18] addressed Stackelberg-Nash strategies to control a system of two coupled parabolic equations employing only one leader control. A more general multi-objective control problem for coupled 2×2 system of parabolic equations has been explored in a recent article [AS24]. Concerning to the hierarchic control for nonlinear coupled degenerate parabolic equations, we refer to the work [DKDZ23]. Lastly, let us mention the most recent work [TdTWZ23], where Stackelberg exact controllability for the Boussinesq system in two dimensions has been investigated.

It is worth mentioning that *robust Stackelberg control problem* (controllability with Stackelberg strategy in the presence of some worst disturbance signal) has been demonstrated in the works [MdT18] for incompressible Navier-Stokes equation, [HSP20] for semilinear heat equation and [BM22] for Kuramoto-Sivashinsky equation.

1.4. Outline. Throughout the paper we will use positive generic constant $C > 0$ which may change line to line. The rest of the paper is organized as follows:

- Section 2 contains all the well-posedness results relevant for this paper.
- In Section 3, we prove Theorem 1.1 dealing with the characterization of Nash equilibrium.
- Section 4 is devoted to derive a new Carleman estimate for the linear system (1.19).
- In Section 5, we demonstrate the derivation of a suitable Carleman estimate for the coupled adjoint system (1.18) using the Carleman estimate derived in the last section.
- In Section 6, the desired observability inequality has been deduced, using which we conclude the partial null controllability result for the linearized system (1.17).
- Finally, we devote Section 7 to conclude the main result of this paper as stated in Theorem 1.3 using the null controllability result obtained in the previous section and the local inversion mapping theorem.

2. WELL-POSEDNESS RESULTS

The main goal of this section is to establish the well-posedness of the optimality system (1.13), so that we can make the characterization of the Nash equilibrium f^1, f^2 as stated in Theorem 1.1 sensible. Let us start with the linear Benney-Lin equation.

2.1. Well-posedness of the linearized model. In this subsection, we establish the well-posedness of the linearized system (1.17).

2.1.1. Linear equation. Let us first consider the following linear scalar system

$$\begin{cases} \partial_t u + \partial_x^5 u + \mu_0 \partial_x^4 u + \partial_x^3 u = f \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = 0 \\ \partial_x^2 u(t, 0) = \partial_x^2 u(t, L) \\ u(0, x) = u_0(x), \end{cases} \quad (2.1)$$

where $\mu_0 > 0$. Corresponding to this system, we have the following well-posedness result.

Proposition 2.1. *For $f \in L^1(0, T; L^2(0, L))$ and $u_0 \in L^2(0, L)$, system (2.1) has a unique mild solution $u \in \mathcal{C}([0, T]; L^2(0, L)) \cap L^2((0, T); H^2(0, L))$ satisfying the continuity estimate*

$$\|u\|_{\mathcal{C}([0, T]; L^2(0, L))} + \|u\|_{L^2(0, T); H^2(0, L)} \leq C (\|u_0\|_{L^2(0, L)} + \|f\|_{L^1(0, T; L^2(0, L))}). \quad (2.2)$$

Moreover, $u, \partial_x u \in L^1(0, T; L^2(0, L))$ and satisfies the estimate

$$\|u, \partial_x u\|_{L^1(0, T; L^2(0, L))} \leq C \|u\|_{L^2(0, T; H^1(0, L))}^2.$$

Proof. The above system (2.1) can be written in an abstract infinite dimensional ODE set up as

$$u'(t) = Au(t) + f(t), \quad u(0) = u_0, \quad (2.3)$$

where the unbounded operator A is given by

$$A := -\partial_x^5 u - \mu_0 \partial_x^4 u - \partial_x^3 u$$

with the domain

$$\mathcal{D}(A) = \{u \in H^5(0, L) \cap H_0^2(0, L) : \partial_x^2 u(t, 0) = \partial_x^2 u(t, L)\}.$$

Let $u \in \mathcal{D}(A)$. Performing integration by parts along with the boundary conditions, we have

$$\begin{aligned} \langle Au, u \rangle_{L^2} &= -\frac{1}{2} \int_0^L \partial_x (\partial_x^2 u)^2 - \mu_0 \int_0^L |\partial_x^2 u|^2 + \frac{1}{2} \int_0^L \partial_x (\partial_x u)^2 \\ &= -\mu_0 \int_0^L |\partial_x^2 u|^2 \leq 0, \end{aligned}$$

which shows that the operator A is dissipative. Similarly, one can show that the adjoint operator of A given by

$$A^* = \partial_x^5 u - \mu_0 \partial_x^4 u + \partial_x^3 u$$

with domain

$$\mathcal{D}(A^*) = \mathcal{D}(A)$$

is also dissipative. Hence, by Lumer-Phillips theorem, the operator A generates a contraction semigroup $\{\mathcal{S}(t)\}_{t \geq 0}$ in $L^2(0, L)$. Thus, by classical semigroup theory result, for any $f \in L^1(0, T; L^2(0, L))$ and $u_0 \in L^2(0, L)$ the system (2.1) has a unique mild solution $u \in \mathcal{C}([0, T]; L^2(0, L))$ given by

$$u(t) = \mathcal{S}(t)u_0 + \int_0^t \mathcal{S}(t-s)f(s) ds.$$

Using this expression and the contraction property of semigroup $\{\mathcal{S}(t)\}_{t \geq 0}$, we get

$$\|u\|_{\mathcal{C}([0, T]; L^2(0, L))} \leq \|u_0\|_{L^2(0, L)} + \|f\|_{L^1(0, T; L^2(0, L))}. \quad (2.4)$$

Let us multiply (2.1) by $2u$ and integrate it over $(0, L)$. Next, we perform an integration by parts to get

$$\begin{aligned} \frac{d}{dt} \int_0^L |u(t, \cdot)|^2 + 2\mu_0 \int_0^L |\partial_x^2 u(t, \cdot)|^2 &= 2 \int_0^L f(t, \cdot) u(t, \cdot) \\ &\leq 2 \|f(t, \cdot)\|_{L^2(0, L)} \|u(t, \cdot)\|_{L^2(0, L)}. \end{aligned}$$

On integrating the above identity w.r.t t over $(0, T)$, we obtain

$$\begin{aligned} \int_0^L |u(T, \cdot)|^2 - \int_0^L |u_0(\cdot)|^2 + 2\mu_0 \int_0^T \int_0^L |\partial_x^2 u|^2 &\leq 2 \int_0^T \|f(t, \cdot)\|_{L^2(0, L)} \|u(t, \cdot)\|_{L^2(0, L)} \\ &\leq 2 \|u\|_{\mathcal{C}([0, T]; L^2(0, L))} \|f\|_{L^1(0, T; L^2(0, L))} \\ &\leq 2 (\|u_0\|_{L^2(0, L)} + \|f\|_{L^1(0, T; L^2(0, L))}) \|f\|_{L^1(0, T; L^2(0, L))}. \end{aligned}$$

Thanks to the Poincaré inequality, we have

$$\begin{aligned} C \|u\|_{L^2(0, T; H^2(0, L))}^2 &\leq \|u_0\|_{L^2(0, L)}^2 + 2 (\|u_0\|_{L^2(0, L)} + \|f\|_{L^1(0, T; L^2(0, L))}) \|f\|_{L^1(0, T; L^2(0, L))} \\ &\leq 2 (\|u_0\|_{L^2(0, L)} + \|f\|_{L^1(0, T; L^2(0, L))})^2, \end{aligned}$$

which gives

$$\|u\|_{L^2(0, T; H^2(0, L))} \leq C (\|u_0\|_{L^2(0, L)} + \|f\|_{L^1(0, T; L^2(0, L))}).$$

Combining this inequality with (2.4), we get the desired inequality.

Thanks to the fact that $H^1(0, L) \hookrightarrow L^\infty(0, L)$ (i.e., $\|u\|_{L^\infty} \leq C\|u\|_{H^1(0, L)}$), we obtain

$$\begin{aligned} \|u \partial_x u\|_{L^1(0, T; L^2(0, L))} &\leq \int_0^T \|u(t, \cdot)\|_{L^\infty(0, L)} \|\partial_x u(t, \cdot)\|_{L^2(0, L)} dt \\ &\leq C \int_0^T \|u(t, \cdot)\|_{H^1(0, L)}^2 dt \end{aligned}$$

This completes the proof. \square

Let us introduce a weak formulation for the solution of the system (2.1) when the source term f lies in the weaker space $L^2(0, T; H^{-2}(0, L))$.

Definition 2.2. Let $u_0 \in L^2(0, L)$ and $f \in L^2(0, T; H^{-2}(0, L))$. A function $u \in \mathcal{C}([0, T]; L^2(0, L))$ is called a weak solution of (2.1) if it satisfies the following identity

$$\iint_Q u g + \langle u(T), \varphi_T \rangle_{L^2(0, L)} = \int_0^T \langle f, \varphi \rangle_{H^{-2}, H_0^2} + \langle u_0, \varphi(0) \rangle_{L^2(0, L)}, \quad \forall (\varphi_T, g) \in L^2(0, L) \times L^1(0, T; L^2(0, L)), \quad (2.5)$$

where φ is the solution of the following equation

$$\begin{cases} -\partial_t \varphi - \partial_x^5 \varphi - \mu_0 \partial_x^4 \varphi - \partial_x^3 \varphi = g, & (t, x) \in Q, \\ \varphi(t, 0) = \varphi(t, L) = \partial_x \varphi(t, 0) = \partial_x \varphi(t, L) = 0, & t \in (0, T), \\ \partial_x^2 \varphi(t, 0) = \partial_x^2 \varphi(t, L), & t \in (0, T), \\ \varphi(T, x) = \varphi_T(x), & x \in (0, L). \end{cases} \quad (2.6)$$

Remark 2.3. For any $\varphi_T \in L^2(0, L)$ and $g \in L^1(0, T; L^2(0, L))$, similar argument as Proposition 2.1 ensures that the equation (2.6) possesses unique solution in the space $\mathcal{C}([0, T]; L^2(0, L)) \cap L^2(0, T; H^2(0, L))$.

Notation. For $T > 0$ and $s \in \mathbb{R}$, let us introduce the space

$$X_T^s := \mathcal{C}([0, T]; H^s(0, L)) \cap L^2(0, T; H^{s+2}(0, L)).$$

In order to prove the regularity result for the system (2.1) when $u_0 \in H^s(0, L)$ and $f \in L^2(0, T; H^{s-2}(0, L))$, one needs to define compatible condition on the initial data u_0 . We say the initial data $u_0 \in H^s(0, L)$ is compatible if it satisfies the underlying boundary conditions of the system when the associated trace makes sense, more precisely,

$$\begin{cases} u_0(0) = u_0(1) = 0, & \text{if } \frac{1}{2} < s \leq \frac{3}{2}, \\ u_0(0) = u_0(1) = u_0'(0) = u_0'(1) = 0, & \text{if } \frac{3}{2} < s \leq \frac{5}{2}, \\ u_0(0) = u_0(1) = u_0'(0) = u_0'(1) = 0, u_0''(0) = u_0''(1), & \text{if } \frac{5}{2} < s \leq 5. \end{cases} \quad (2.7)$$

Proposition 2.4. Let $s \in [0, 5]$. Then for any $f \in L^2(0, T; H_0^{s-2}(0, L))$ and $u_0 \in H^s(0, L)$ satisfying the compatibility condition (2.7), the system (2.1) exhibits a unique solution $u \in X_T^s$ in the sense of Definition 2.2, which satisfies

$$\|u\|_{X_T^s} \leq C (\|u_0\|_{H^s(0, L)} + \|f\|_{L^2(0, T; H^{s-2}(0, L))}).$$

Proof. **s = 0 :** Let us first assume $f \in L^2(0, T; L^2(0, L)) \subset L^1(0, T; L^2(0, L))$. Then, due to Proposition 2.1, system (2.1) admits a unique solution $u \in \mathcal{C}([0, T]; L^2(0, L)) \cap L^2(0, T; H^2(0, L))$. On multiplying the equation (2.1) with $2u$, and performing integration by parts, we get

$$\frac{d}{dt} \int_0^L |u|^2 + 2\mu_0 \int_0^L |\partial_x^2 u|^2 = 2 \int_0^L f(t, \cdot) u(t, \cdot) dx \leq 2 \|f\|_{H^{-2}} \|u\|_{H_0^2},$$

which on integrating w.r.t t , and using Poincaré inequality and Young's inequality gives

$$\|u\|_{\mathcal{C}([0, T]; L^2(0, L))} + \|u\|_{L^2(0, T; H^2(0, L))} \leq C (\|u_0\|_{L^2(0, L)} + \|f\|_{L^2(0, T; H^{-2}(0, L))}).$$

Finally, we use density of $L^2(0, T; L^2(0, L))$ in $L^2(0, T; H^{-2}(0, L))$, to conclude the result for $s = 0$.

s = 5 : For this case, let us denote $\tilde{u} = Au$. Note that \tilde{u} satisfies (2.1) for initial data $\tilde{u}_0 = Au_0 \in L^2(0, L)$ and non-homogeneous term $\tilde{f} = Af \in L^2(0, T; H^{-2}(0, L))$. Then by the case $s = 0$, we have:

$$\|\tilde{u}\|_{\mathcal{C}([0, T]; L^2(0, L))} + \|\tilde{u}\|_{L^2(0, T; H^2(0, L))} \leq C (\|\tilde{u}_0\|_{L^2(0, L)} + \|\tilde{f}\|_{L^2(0, T; H^{-2}(0, L))}),$$

which gives

$$\|u\|_{\mathcal{C}([0, T]; H^5(0, L))} + \|u\|_{L^2(0, T; H^7(0, L))} \leq C (\|u_0\|_{H^5(0, L)} + \|f\|_{L^2(0, T; H^3(0, L))}).$$

This proves the case $s = 5$.

For the case $s \in (0, 5)$, one can obtain the result using the standard interpolation arguments as done in [GG08]. \square

2.1.2. **Linearized scalar equation.** Concerning the well-posedness result for the linearized system

$$\begin{cases} \partial_t u + \partial_x^3 u + \mu_0 \partial_x^4 u + \partial_x^5 u + \partial_x(\bar{u}u) = f, & (t, x) \in (0, T) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = 0, & t \in (0, T), \\ \partial_x^2 u(t, L) = \partial_x^2 u(t, 0), & t \in (0, T), \\ u(0, x) = u_0(x), & x \in (0, L), \end{cases} \quad (2.8)$$

we obtain the following result.

Proposition 2.5. *Let $T > 0$ and $s \in [0, 5]$. Assume $\bar{u} \in X_T^s$. Then for any $u_0 \in X_T^s$ satisfying the compatibility condition (2.7) and for any $f \in L^2(0, T; H_0^{s-2}(0, L))$, the linearized system (2.8) admits a unique solution $u \in X_T^s$ satisfying*

$$\|u\|_{X_T^s} \leq C \left(\|u_0\|_{H^s(0, L)} + \|f\|_{L^2(0, T; H^{s-2})} \right). \quad (2.9)$$

Proof. We prove the above theorem for $s = 0$, using fixed point argument. One can similarly handle the other cases.

Let $\tau \in (0, T]$ be any real number. Consider the map $\Gamma : X_\tau^s \rightarrow X_\tau^s$ given by

$$\Gamma(\tilde{u}) = u,$$

where u solves

$$\begin{cases} \partial_t u + \partial_x^3 u + \mu_0 \partial_x^4 u + \partial_x^5 u = -\partial_x(\bar{u}\tilde{u}) + f, & (t, x) \in (0, \tau) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = 0, & t \in (0, \tau), \\ \partial_x^2 u(t, L) = \partial_x^2 u(t, 0), & t \in (0, \tau), \\ u(0, x) = u_0(x), & x \in (0, L). \end{cases}$$

For the non-homogeneous term $\partial_x(\bar{u}\tilde{u})$, we have

$$\begin{aligned} \left\| \partial_x(\bar{u}\tilde{u}) \right\|_{L^2(0, \tau; H^{-1}(0, L))}^2 &\leq C \int_0^\tau \|\bar{u}\tilde{u}\|_{L^2(0, L)}^2 \\ &\leq C \int_0^\tau \|\bar{u}\|_{L^\infty(0, L)}^2 \|\tilde{u}\|_{L^2(0, L)}^2 \leq C \int_0^\tau \|\bar{u}\|_{L^2(0, L)} \|\bar{u}\|_{H^1(0, L)} \|\tilde{u}\|_{L^2(0, L)}^2 \\ &\leq C \|\bar{u}\|_{C([0, T]; L^2(0, L))} \|\tilde{u}\|_{C([0, \tau]; L^2(0, L))}^2 \sqrt{\tau} \|\bar{u}\|_{L^2(0, T; H^1(0, L))} \\ &\leq C \sqrt{\tau} \|\bar{u}\|_{X_\tau^0}^2 \|\tilde{u}\|_{X_\tau^0}^2, \end{aligned}$$

which proves $\partial_x(\bar{u}\tilde{u}) \in L^2(0, \tau; H^{-1}(0, L)) \subset L^2(0, \tau; H^{-2}(0, L))$. This guarantees the existence of a unique $u \in X_\tau^0$ which solves the above system due to Proposition 2.4. Moreover, the solution u satisfies

$$\|\Gamma(\tilde{u})\|_{X_\tau^0} = \|u\|_{X_\tau^0} \leq C \left(\|u_0\|_{H^s(0, L)} + \sqrt{\tau} \|\bar{u}\|_{X_\tau^0} \|\tilde{u}\|_{X_\tau^0} + \|f\|_{L^2(0, T; H^{-2}(0, L))} \right). \quad (2.10)$$

This proves that the map Γ is well defined.

Next, we show the map Γ to be a contraction. Note that for $\tilde{u}_1, \tilde{u}_2 \in X_\tau^0$, the corresponding solution $u_1, u_2 \in X_\tau^0$ satisfy

$$\|\Gamma(\tilde{u}_1) - \Gamma(\tilde{u}_2)\|_{X_\tau^0} = \|u_1 - u_2\|_{X_\tau^0} \leq C \sqrt{\tau} \|\bar{u}\|_{X_\tau^0} \|\tilde{u}_1 - \tilde{u}_2\|_{X_\tau^0}.$$

Next we choose $\tau = \tau^* > 0$ small enough so that

$$\begin{cases} C \sqrt{\tau^*} \|\bar{u}\|_{X_{\tau^*}^0} < 1, \\ \text{and } N\tau^* = T, \text{ for some } N \in \mathbb{N}. \end{cases}$$

Such choice of τ^* proves that the map Γ is a contraction, and hence it has a fixed point $u \in X_{\tau^*}^0$ which solves the linearized system (2.8) in time interval $[0, \tau^*]$.

Observe that the proof showing the map Γ to be contraction is independent on the choice of initial data, and so we can mimic the above argument in the time intervals $[n\tau^*, (n+1)\tau^*]$ for $1 \leq n \leq N-1$. This essentially proves the existence and uniqueness of solution $u \in X_T^0$ for the system (2.8). \square

Using arguments similar to the one used in proving [Proposition 2.5](#), we can prove the well-posedness result for the following linear system as well.

$$\begin{cases} \partial_t u + \partial_x^3 u + \mu_0 \partial_x^4 u + \partial_x^5 u + \bar{u} \partial_x u = f, & (t, x) \in (0, T) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = 0, & t \in (0, T), \\ \partial_x^2 u(t, L) = \partial_x^2 u(t, 0), & t \in (0, T), \\ u(0, x) = u_0(x), & x \in (0, L). \end{cases} \quad (2.11)$$

More precisely, we obtain the following existence-uniqueness result.

Proposition 2.6. *Let $T > 0$ and $s \in [0, 5]$. Assume $\bar{u} \in X_T^s$. Then for any $u_0 \in X_T^s$ and $f \in L^2(0, T; H_0^{s-2}(0, L))$, the linear system (2.11) admits a unique solution $u \in X_T^s$ satisfying*

$$\|u\|_{X_T^s} \leq C \left(\|u_0\|_{H^s(0, L)} + \|f\|_{L^2(0, T; H^{s-2})} \right). \quad (2.12)$$

2.1.3. Linearized coupled system. We can finally prove the following well-posedness result for the linearized coupled system (1.17).

Proposition 2.7. *Let $v_0 \in H^s(0, L)$ be any compatible initial data, see (2.7). Suppose $\bar{u} \in X_T^s$, $f \in L^2(0, T; L^2(0, L))$, and $(g^0, g^1, g^2) \in L^2(0, T; H_0^{s-2}(0, L))$ are given. Then for sufficiently large μ_1 and μ_2 , the system (1.17) admits a unique solution $(v, \varphi^1, \varphi^2) \in [X_T^s]^3$ satisfying the estimate*

$$\|(v, \varphi^1, \varphi^2)\|_{[X_T^s]^3} \leq C \left(\|v_0\|_{H^s(0, L)} + \|f\|_{L^2(0, T; L^2(0, L))} + \sum_{i=0}^2 \|g^i\|_{L^2(0, T; H^{s-2}(0, L))} \right), \quad (2.13)$$

where $C > 0$ is a constant independent of μ_1 and μ_2 .

Proof. Let us define a map $\tilde{\Gamma} : X_T^s \times X_T^s \rightarrow X_T^s \times X_T^s$ given by $\tilde{\Gamma}(\hat{\varphi}^1, \hat{\varphi}^2) = (\check{\varphi}^1, \check{\varphi}^2)$. where the pairs $(\hat{\varphi}^1, \hat{\varphi}^2), (\check{\varphi}^1, \check{\varphi}^2)$ are co-related through the following systems

$$\begin{cases} \partial_t \hat{v} + \partial_x^5 \hat{v} + \mu_0 \partial_x^4 \hat{v} + \partial_x^3 \hat{v} + \partial_x(\bar{u} \hat{v}) = f^0 \chi_{\mathcal{O}_0} - \sum_{i=1}^2 \frac{1}{\mu_i} \hat{\varphi}^i \chi_{\mathcal{O}_i} + g^0, & (t, x) \in (0, T) \times (0, L), \\ \hat{v}(t, 0) = \hat{v}(t, L) = \partial_x \hat{v}(t, 0) = \partial_x \hat{v}(t, L) = 0, \quad \partial_x^2 \hat{v}(t, L) = \partial_x^2 \hat{v}(t, 0), & t \in (0, T), \\ \hat{v}(0, x) = v_0(x) & x \in (0, L), \end{cases} \quad (2.14)$$

and

$$\begin{cases} -\partial_t \check{\varphi}^i - \partial_x^5 \check{\varphi}^i + \mu_0 \partial_x^4 \check{\varphi}^i - \partial_x^3 \check{\varphi}^i - \bar{u} \partial_x \check{\varphi}^i = \nu_i \hat{v} \chi_{\mathcal{O}_{i,d}} + g^i, & (t, x) \in (0, T) \times (0, L), \\ \check{\varphi}^i(t, 0) = \check{\varphi}^i(t, L) = \partial_x \check{\varphi}^i(t, 0) = \partial_x \check{\varphi}^i(t, L) = 0, & t \in (0, T), \\ \partial_x^2 \check{\varphi}^i(t, L) = \partial_x^2 \check{\varphi}^i(t, 0), & t \in (0, T), \\ \check{\varphi}^i(T, x) = 0, & x \in (0, L). \end{cases} \quad (2.15)$$

Thanks to [Proposition 2.5](#), we get the existence of a unique solution $\hat{v} \in X_T^s$ which satisfies

$$\|\hat{v}\|_{X_T^s} \leq C \left(\|v_0\|_{H^s(0, L)} + \|f\|_{L^2(0, T; L^2(0, L))} + \sum_{i=1}^2 \frac{1}{\mu_i} \|\hat{\varphi}^i\|_{X_T^s} + \|g^0\|_{L^2(0, T; H^{s-2}(0, L))} \right), \quad (2.16)$$

where $C > 0$ is a constant independent of μ_i .

Next, using [Proposition 2.6](#) and the above estimate (2.16), we can show that the system (2.15) admits a unique solution $\check{\varphi}^i \in X_T^s$ for $i \in \{1, 2\}$, which satisfies the following estimate

$$\|\check{\varphi}^i\|_{X_T^s} \leq C \left[\|v_0\|_{H^s(0, L)} + \|f\|_{L^2(0, T; L^2(0, L))} + \sum_{i=1}^2 \frac{1}{\mu_i} \|\hat{\varphi}^i\|_{X_T^s} + \|g^0\|_{L^2(0, T; H^{s-2}(0, L))} + \|g^i\|_{L^2(0, T; H^{s-2}(0, L))} \right],$$

where $C > 0$ is a constant independent of μ_1 and μ_2 . This ensures that the map $\tilde{\Gamma}$ is well-defined.

Due to the last estimate and linearity of the system, we have

$$\begin{aligned} \|\tilde{\Gamma}(\hat{\varphi}_\alpha^1, \hat{\varphi}_\alpha^2) - \tilde{\Gamma}(\hat{\varphi}_\beta^1, \hat{\varphi}_\beta^2)\| &\leq C \sum_{i=1}^2 \frac{1}{\mu_i} \|\hat{\varphi}_\alpha^i - \hat{\varphi}_\beta^i\|_{X_T^s} \\ &\leq C \frac{1}{\min\{\mu_i\}} \left\| (\hat{\varphi}_\alpha^1, \hat{\varphi}_\alpha^2) - (\hat{\varphi}_\beta^1, \hat{\varphi}_\beta^2) \right\|_{X_T^s \times X_T^s}, \quad \forall (\hat{\varphi}_\alpha^1, \hat{\varphi}_\alpha^2), (\hat{\varphi}_\beta^1, \hat{\varphi}_\beta^2) \in X_T^s \times X_T^s, \end{aligned}$$

where $C > 0$ is independent of μ_1 and μ_2 . Hence, for sufficiently large μ_1 and μ_2 , the map $\tilde{\Gamma}$ is a contraction and hence possesses a unique fixed point $(\varphi^1, \varphi^2) \in [X_T^s]^2$.

Let $v \in X_T^s$ be the solution of (2.14) with this fixed point (φ^1, φ^2) , instead of $(\hat{\varphi}^1, \hat{\varphi}^2)$. Then the tuple $(v, \varphi^1, \varphi^2) \in [X_T^s]^3$ solves the coupled system (1.17), for sufficiently large μ_1 and μ_2 , and satisfies the estimate (2.13). This completes the proof. \square

2.2. Well-posedness of the nonlinear model. The main goal of this subsection is to establish the well-posedness of the main non-linear optimality system (1.13).

2.2.1. Nonlinear scalar equation. We will begin by proving the global well-posedness of the following non-linear system:

$$\begin{cases} \partial_t u + \partial_x^5 u + \mu_0 \partial_x^4 u + \partial_x^3 u + u \partial_x u = f, & (t, x) \in (0, T) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = 0, & t \in (0, T), \\ \partial_x^2 u(t, L) = \partial_x^2 u(t, 0), & t \in (0, T), \\ u(0, x) = u_0(x), & x \in (0, L). \end{cases} \quad (2.17)$$

In fact, we will establish the following result.

Proposition 2.8. *Let $T > 0$ be given. Then for $u_0 \in L^2(0, L)$, and $f \in L^2(0, T; L^2(0, L))$, the system (2.17) admits a unique solution $u \in X_T^0$ satisfying*

$$\|u\|_{X_T^0} \leq C \left(\|u_0\|_{L^2(0, L)} + \|f\|_{L^2(0, T; L^2(0, L))} \right).$$

To prove this global result, we first establish the local well-posedness of the nonlinear problem (2.17) in time, as stated below.

Proposition 2.9. *Let $T > 0$ and $f \in L^2(0, T; L^2(0, L))$ be given. Then for $u_0 \in L^2(0, L)$, there exists $T^* > 0$, depending on $\|u_0\|_{L^2(0, L)}$ such that the system (2.17) admits a unique solution in $X_{T^*}^0$.*

Proof. Let $\tau \in (0, T)$. Let us define a map $\hat{\Gamma}$ on

$$B_\tau(0, R) := \{u \in X_\tau^0 : \|u\|_{X_\tau^0} \leq R\} \subset X_\tau^0,$$

given by

$$\hat{\Gamma}(\tilde{u}) = u, \quad \text{for } \tilde{u} \in B_\tau(0, R) \subset X_\tau^0,$$

where u solves the linear system

$$\begin{cases} \partial_t u + \partial_x^5 u + \mu_0 \partial_x^4 u + \partial_x^3 u = -\tilde{u} \partial_x \tilde{u} + f, & (t, x) \in (0, \tau) \times (0, L), \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = 0, & t \in (0, \tau), \\ \partial_x^2 u(t, L) = \partial_x^2 u(t, 0), & t \in (0, \tau), \\ u(0, x) = u_0(x), & x \in (0, L). \end{cases} \quad (2.18)$$

Note that using Hölder's inequality, Young's inequality and the inclusion $H^1(0, L) \hookrightarrow L^\infty(0, L)$, we have

$$\begin{aligned} \|\tilde{u} \partial_x \tilde{u}\|_{L^1(0, \tau; L^2(0, L))} &\leq \int_0^\tau \|\partial_x \tilde{u}\|_{L^\infty(0, L)} \|\tilde{u}\|_{L^2(0, L)} \\ &\leq \|\tilde{u}\|_{C([0, \tau]; L^2(0, L))} \int_0^\tau \|\partial_x \tilde{u}\|_{H^1(0, L)} \\ &\leq \sqrt{\tau} \|\tilde{u}\|_{C([0, \tau]; L^2(0, L))} \|\tilde{u}\|_{L^2(0, \tau; H^2(0, L))} \\ &\leq \sqrt{\tau} \|\tilde{u}\|_{X_\tau^0}^2. \end{aligned}$$

Thus, the non-homogeneous term $\tilde{u} \partial_x \tilde{u} + f \in L^1(0, \tau; L^2(0, L))$, and so the Proposition 2.4 guarantees the existence of a unique solution $u \in X_\tau^0$ of (2.18) satisfying

$$\|u\|_{X_\tau^0} \leq C \left(\|u_0\|_{L^2(0, L)} + \sqrt{\tau} \|\tilde{u}\|_{X_\tau^0}^2 + \|f\|_{L^2(0, \tau; L^2(0, L))} \right).$$

Set $R := 2C \left(\|u_0\|_{L^2(0, L)} + \|f\|_{L^2(0, \tau; L^2(0, L))} \right)$, and choose $\tau = T^*$ such that $RC\sqrt{T^*} < \frac{1}{4}$. This choice of τ and R proves

$$\hat{\Gamma} : B_{T^*}(0, R) \rightarrow B_{T^*}(0, R)$$

is well defined map.

Next, for $\tilde{u}_1, \tilde{u}_2 \in B_{T^*}(0, R)$, we have:

$$\begin{aligned}
\|\widehat{\Gamma}(\tilde{u}_1) - \widehat{\Gamma}(\tilde{u}_2)\|_{X_{T^*}^0} &\leq C\|\tilde{u}_1\partial_x\tilde{u}_1 - \tilde{u}_2\partial_x\tilde{u}_2\|_{L^1(0,T^*;L^2(0,L))} \\
&\leq C\|\tilde{u}_1\partial_x(\tilde{u}_1 - \tilde{u}_2) + \partial_x\tilde{u}_2(\tilde{u}_1 - \tilde{u}_2)\|_{L^1(0,T^*;L^2(0,L))} \\
&\leq C\sqrt{T^*}\left(\|\tilde{u}_1\|_{C([0,T^*];L^2(0,L))}\|\tilde{u}_1 - \tilde{u}_2\|_{L^2(0,T^*;H^2(0,L))}\right. \\
&\quad \left. + \|\tilde{u}_1 - \tilde{u}_2\|_{C([0,T^*];L^2(0,L))}\|\tilde{u}_2\|_{L^2(0,T^*;H^2(0,L))}\right) \\
&\leq C\sqrt{T^*}\left(\|\tilde{u}_1\|_{X_{T^*}^0} + \|\tilde{u}_2\|_{X_{T^*}^0}\right)\|\tilde{u}_1 - \tilde{u}_2\|_{X_{T^*}^0} \\
&< \frac{1}{2}\|\tilde{u}_1 - \tilde{u}_2\|_{X_{T^*}^0}.
\end{aligned}$$

This shows that $\widehat{\Gamma}$ is a contraction, and hence it has a fixed point $u \in B_{T^*}(0, R)$, which is the solution of the non-linear system (2.17) for $t \in [0, T^*]$. \square

Proof of Proposition 2.8. Since, we have already proved the local well-posedness result in time, it is sufficient to prove an a priori estimate as mentioned in the Proposition 2.8 to get the global well-posedness of (2.17). On multiplying the equation (2.17) by u , we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\int_0^L|u|^2\right) + \mu_0\int_0^L|\partial_x^2u|^2 = \int_0^Lfu \leq \frac{1}{2}\int_0^L|f|^2 + \frac{1}{2}\int_0^L|u|^2.$$

Next, multiply the above equation by e^{-t} and then integrate it w.r.t t over $[0, \tau] \subset [0, T]$ to get

$$e^{-\tau}\int_0^L|u(\tau)|^2dt - \int_0^L|u_0|^2 + 2\mu_0e^{-\tau}\int_0^\tau\int_0^L|\partial_x^2u|^2 \leq e^{-\tau}\int_0^\tau\int_0^L|f|^2.$$

Taking supremum over $\tau \in [0, T]$ gives

$$\|u\|_{C([0,T];L^2(0,L))} + \|u\|_{L^2(0,T;H^2(0,L))} \leq C\left(\|u_0\|_{L^2(0,L)} + \|f\|_{L^2(0,T;L^2(0,L))}\right),$$

and hence completes the proof Proposition 2.8. \square

2.2.2. Nonlinear coupled system. Let us finally give the well-posedness result for the main non-linear optimality system (1.13).

Proposition 2.10. *Let $\epsilon_0 > 0$ be sufficiently small and μ_i be sufficiently large for $i \in \{1, 2\}$. Then for $v_0 \in L^2(0, L)$, $f \in L^2(0, T; L^2(0, L))$, and $g^i \in L^2(0, T; H^{-2}(0, L))$ such that*

$$\|v_0\|_{L^2(0,L)} + \|f\|_{L^2(0,T;L^2(0,L))} + \sum_{i=0}^2\|g^i\|_{L^2(0,T;H^{-2}(0,L))} \leq \epsilon_0, \quad (2.19)$$

the system (1.13) admits a unique solution $(v, \varphi^1, \varphi^2) \in [X_T^0]^3$ satisfying

$$\|(v, \varphi^1, \varphi^2)\|_{[X_T^0]^3} \leq C\left(\|u_0\|_{L^2(0,L)} + \|f\|_{L^2(0,T;L^2(0,L))} + \sum_{i=0}^2\|g^i\|_{L^2(0,T;H^{-2}(0,L))}\right).$$

Lemma 2.11. *For $u, v \in X_T^0 = C([0, T]; L^2(0, L)) \cap L^2(0, T; H^2(0, L))$, we have*

$$\|u\partial_xv\|_{L^2(0,T;H^{-1}(0,L))}^2 \leq C_1\|u\|_{X_T^0}^2\|v\|_{X_T^0}^2. \quad (2.20)$$

Proof. Using the definition of H^{-1} norm, we obtain

$$\|u\partial_xv\|_{H^{-1}(0,L)} \leq C\|u\|_{L^\infty(0,L)}\|v\|_{H^1(0,L)}.$$

And for $u \in H^1(0, 1)$, we have

$$\|u\|_{L^\infty(0,L)} \leq C\left(\|u\|_{L^2(0,L)} + \|u\|_{L^2(0,L)}^{1/2}\|\partial_xu\|_{L^2(0,L)}^{1/2}\right),$$

Using these two estimates and Young's inequality, we get

$$\begin{aligned}
\|u\partial_xv\|_{L^2(0,T;H^{-1})}^2 &\leq \int_0^T\|u\|_{L^\infty(0,L)}^2\|v\|_{H^1(0,L)}^2 \leq C\int_0^T\left(\|u\|_{L^2(0,L)}^2 + \|u\|_{L^2(0,L)}\|u\|_{H^1(0,L)}\right)\|v\|_{H^1(0,L)}^2 \\
&\leq C\|u\|_{C([0,T];L^2(0,L))}^2\|v\|_{L^2(0,T;H^1(0,L))}^2 + \|u\|_{C([0,T];L^2(0,L))}\|v\|_{L^4(0,T;H^1(0,L))}^2\|u\|_{L^2(0,T;H^1(0,L))} \\
&\quad (2.21)
\end{aligned}$$

Note that using the interpolation estimate

$$\|u(t, \cdot)\|_{H^1(0,L)} \leq C \|u(t, \cdot)\|_{H^2(0,L)}^{1/2} \|u(t, \cdot)\|_{L^2(0,L)}^{1/2},$$

and the Young's inequality, we have

$$\|u\|_{L^4(0,T;H^1(0,L))} \leq C \|u\|_{C([0,T];L^2(0,L))}^{1/2} \|u\|_{L^2(0,T;H^2(0,L))}^{1/2}.$$

Substituting the above estimate in (2.21), we obtain the desired estimate (2.20). \square

Proof of Proposition 2.10. For any $R > 0$, let us define $B(0, R) := \{z \in X_T^0 : \|z\|_{X_T^0} \leq R\}$. We define a map $\underline{\Gamma} : X_T^0 \rightarrow X_T^0$ given by $\underline{\Gamma}(\tilde{v}) = v$, where $(v, \varphi_1, \varphi_2)$ solves the following system

$$\begin{cases} \partial_t v + \partial_x^5 v + \mu_0 \partial_x^4 v + \partial_x^3 v + \tilde{v} \partial_x \tilde{v} + \partial_x(\tilde{u}v) = f^0 \chi_{\mathcal{O}_0} - \sum_{i=1}^2 \frac{1}{\mu_i} \varphi^i \chi_{\mathcal{O}_i} + g^0, & (t, x) \in Q, \\ -\partial_t \varphi^i - \partial_x^5 \varphi^i + \mu_0 \partial_x^4 \varphi^i - \partial_x^3 \varphi^i - (\tilde{v} + \tilde{u}) \partial_x \varphi^i = \chi_{\mathcal{O}_{i,d}} \nu_i (v - v_{i,d}) + g^i, & (t, x) \in Q, \\ v(t, 0) = v(t, L) = \partial_x v(t, 0) = \partial_x v(t, L) = 0, \partial_x^2 v(t, L) = \partial_x^2 v(t, 0), & t \in (0, T), \\ \varphi^i(t, 0) = \varphi^i(t, L) = \partial_x \varphi^i(t, 0) = \partial_x \varphi^i(t, L) = 0, \partial_x^2 \varphi^i(t, L) = \partial_x^2 \varphi^i(t, 0), & t \in (0, T), \\ v(0, x) = v_0(x), \varphi^i(T, x) = 0, & x \in (0, L). \end{cases} \quad (2.22)$$

Thanks to the Lemma 2.11, we have the following estimate for the source term of the above equation

$$\|\tilde{v} \partial_x \tilde{v}\|_{L^2(0,T;H^{-2})}^2 \leq C \|\tilde{v} \partial_x \tilde{v}\|_{L^2(0,T;H^{-1})}^2 \leq C_1 \|\tilde{v}\|_{C([0,T];L^2(0,L))}^2 \|\tilde{v}\|_{L^2(0,T;H^1(0,L))}^2,$$

So due to Proposition 2.7, the above system admits a unique solution $(v, \varphi^1, \varphi^2) \in [X_T^0]^3$ satisfying

$$\|(v, \varphi^1, \varphi^2)\|_{[X_T^0]^3} \leq C \left(C_1 \|\tilde{v}\|_{X_T^0}^2 + \|v_0\|_{L^2(0,L)} + \|f\|_{L^2(0,T;L^2(0,L))} + \sum_{i=0}^2 \|g^i\|_{L^2(0,T;H^{-2}(0,L))} \right). \quad (2.23)$$

Set

$$R := 2C \left(\|v_0\|_{L^2(0,L)} + \|f\|_{L^2(Q)} + \sum_{i=0}^2 \|g^i\|_{L^2(0,T;H^{-2}(0,L))} \right) \leq 2C\epsilon_0. \quad (2.24)$$

Choosing ϵ_0 sufficiently small, we can get a R satisfying

$$CC_1 R < \frac{1}{2}. \quad (2.25)$$

Then, from the inequality (2.23), we have

$$\|(v, \varphi^1, \varphi^2)\|_{[X_T^0]^3} < \frac{R}{2} + \frac{R}{2} = R.$$

This proves that the map $\underline{\Gamma} : B(0, R) \rightarrow B(0, R)$ is well defined.

Next, let us take $\tilde{v}_a, \tilde{v}_b \in B(0, R)$, and let $(v_a, \varphi_a^1, \varphi_a^2)$ and $(v_b, \varphi_b^1, \varphi_b^2)$ be the associated solutions of (2.22). Then, the tuple $(v, \varphi^1, \varphi^2) = (v_a - v_b, \varphi_a^1 - \varphi_b^1, \varphi_a^2 - \varphi_b^2)$ solves the system

$$\begin{cases} \partial_t v + \partial_x^5 v + \mu_0 \partial_x^4 v + \partial_x^3 v + \partial_x(\tilde{u}v) = -\sum_{i=1}^2 \frac{1}{\mu_i} \varphi^i \chi_{\mathcal{O}_i} - \tilde{v}_a \partial_x \tilde{v}_a - \tilde{v}_b \partial_x \tilde{v}_b, & (t, x) \in Q, \\ -\partial_t \varphi^i - \partial_x^5 \varphi^i + \mu_0 \partial_x^4 \varphi^i - \partial_x^3 \varphi^i - \tilde{u} \partial_x \varphi^i = \chi_{\mathcal{O}_{i,d}} \nu_i v + \tilde{v}_a \partial_x \varphi_a^i - \tilde{v}_b \partial_x \varphi_b^i, & (t, x) \in Q, \\ v(t, 0) = v(t, L) = \partial_x v(t, 0) = \partial_x v(t, L) = 0, \partial_x^2 v(t, L) = \partial_x^2 v(t, 0), & t \in (0, T), \\ \varphi^i(t, 0) = \varphi^i(t, L) = \partial_x \varphi^i(t, 0) = \partial_x \varphi^i(t, L) = 0, \partial_x^2 \varphi^i(t, L) = \partial_x^2 \varphi^i(t, 0), & t \in (0, T), \\ v(0, x) = \varphi^i(T, x) = 0, & x \in (0, L), \end{cases}$$

and so we have

$$\begin{aligned} & \sum_{i=1}^2 \|\varphi_a^i - \varphi_b^i\|_{X_T^0} + \|\underline{\Gamma}(\tilde{v}_a) - \underline{\Gamma}(\tilde{v}_b)\|_{X_T^0} \\ & \leq C \left(\|\tilde{v}_a \partial_x \tilde{v}_a - \tilde{v}_b \partial_x \tilde{v}_b\|_{L^2(0,T;H^{-2}(0,L))} + \sum_{i=1}^2 \left\| \tilde{v}_a \partial_x \varphi_a^i - \tilde{v}_b \partial_x \varphi_b^i \right\|_{L^2(0,T;H^{-2}(0,L))} \right). \end{aligned} \quad (2.26)$$

Note that

$$\begin{aligned} \left\| \tilde{v}_a \partial_x \tilde{v}_a - \tilde{v}_b \partial_x \tilde{v}_b \right\|_{L^2(0,T;H^{-2}(0,L))} & \leq C_2 \left\| \tilde{v}_a \partial_x (\tilde{v}_a - \tilde{v}_b) \right\|_{L^2(H^{-1})} + C_2 \left\| \partial_x \tilde{v}_b (\tilde{v}_a - \tilde{v}_b) \right\|_{L^2(H^{-1})} \\ & \leq C_2 \left(\|\tilde{v}_a\|_{X_T^0} + \|\tilde{v}_b\|_{X_T^0} \right) \|\tilde{v}_a - \tilde{v}_b\|_{X_T^0}. \end{aligned} \quad (2.27)$$

The last estimate has been obtained by the following argument

$$\begin{aligned} \left\| \tilde{v}_a \partial_x (\tilde{v}_a - \tilde{v}_b) \right\|_{L^2(H^{-1})}^2 &\leq C \int_0^T \|\tilde{v}_a\|_{L^\infty(0,L)}^2 \|\tilde{v}_a - \tilde{v}_b\|_{H^1(0,L)}^2 \\ &\leq C \int_0^T \|\tilde{v}_a\|_{L^2} \|\tilde{v}_a\|_{H^1(0,L)} \|\tilde{v}_a - \tilde{v}_b\|_{L^2(0,L)} \|\tilde{v}_a - \tilde{v}_b\|_{H^2(0,L)} \\ &\leq C \|\tilde{v}_a\|_{C(L^2)} \|\tilde{v}_a - \tilde{v}_b\|_{C(L^2)} \|\tilde{v}_a\|_{L^2(H^1)} \|\tilde{v}_a - \tilde{v}_b\|_{L^2(H^2)}. \end{aligned}$$

Similarly,

$$\begin{aligned} \left\| \tilde{v}_a \partial_x \varphi_a^i - \tilde{v}_b \partial_x \varphi_b^i \right\|_{L^2(0,T;H^{-2}(0,L))} &\leq C_2 \|\tilde{v}_a - \tilde{v}_b\|_{X_T^0} \|\varphi_a^i\|_{X_T^0} + C_2 \|\tilde{v}_b\|_{X_T^0} \|\varphi_a^i - \varphi_b^i\|_{X_T^0} \\ &\leq C_2 \|\tilde{v}_a - \tilde{v}_b\|_{X_T^0} \|\varphi_a^i\|_{X_T^0} + C_2 R \|\varphi^i\|_{X_T^0} \end{aligned} \quad (2.28)$$

Assume R satisfies $CC_2R < 1$. Then substituting the last two estimates in (2.26), we get

$$\|\underline{\Gamma}(\tilde{v}_a) - \underline{\Gamma}(\tilde{v}_b)\|_{X_T^0} \leq CC_2 \left(\|\tilde{v}_a\|_{X_T^0} + \|\tilde{v}_b\|_{X_T^0} + \sum_{i=1}^2 \|\varphi_a^i\|_{X_T^0} \right) \|\tilde{v}_a - \tilde{v}_b\|_{X_T^0}.$$

Further using (2.23) in the last estimate, we have

$$\begin{aligned} \|\underline{\Gamma}(\tilde{v}_a) - \underline{\Gamma}(\tilde{v}_b)\|_{X_T^0} &\leq \|\tilde{v}_a - \tilde{v}_b\|_{X_T^0} \left[CC_2 \left(\|\tilde{v}_a\|_{X_T^0} + \|\tilde{v}_b\|_{X_T^0} \right) + C^2 C_1 C_2 \|\tilde{v}_a\|_{X_T^0}^2 \right. \\ &\quad \left. + C^2 C_2 \left(\|v_0\|_{L^2(0,L)} + \|f\|_{L^2(Q)} + \sum_{i=0}^2 \|g^i\|_{L^2(0,T;H^{-2}(0,L))} \right) \right] \\ &\leq C \|\tilde{v}_1 - \tilde{v}_2\|_{X_T^0} (2R + R^2 + \epsilon_0). \end{aligned}$$

Finally, we can choose R and ϵ_0 small enough such that the map $\underline{\Gamma}$ is a contraction map. Thus, using the Banach fixed point theorem one can conclude that the map $\underline{\Gamma}$ has a fixed point, which gives the existence of solution $(v, \varphi^1, \varphi^2)$ of (1.13).

3. EXISTENCE OF NASH EQUILIBRIUM

In this section, we validate the characterization of Nash equilibrium mentioned in Theorem 1.1. To prove this, we follow the idea presented in the work [AAS23].

Proof of Theorem 1.1. Let us first assume that the control pair $(f^1, f^2) \in L^2((0, T) \times \mathcal{O}_1) \times L^2((0, T) \times \mathcal{O}_2)$ is Nash equilibrium for the functionals J_1, J_2 , i.e., (f^1, f^2) satisfies the property (1.8). Note that for $i \in \{1, 2\}$, the functionals J_i given by (1.11) are differentiable in the direction of f^i , and so the property (1.8) implies

$$\begin{aligned} &\begin{cases} J_1'(f; f^1, f^2)(\hat{f}^1, 0) = 0, & \forall \hat{f}^1 \in L^2((0, T) \times \mathcal{O}_1), \\ J_2'(f; f^1, f^2)(0, \hat{f}^2) = 0, & \forall \hat{f}^2 \in L^2((0, T) \times \mathcal{O}_2). \end{cases} \\ \iff &\begin{cases} \nu_1 \iint_{\mathcal{O}_{1,d} \times (0,T)} (v - v_{1,d}) w^1 + \mu_1 \iint_{\mathcal{O}_1 \times (0,T)} f^1 \hat{f}^1 = 0, & \forall \hat{f}^1 \in L^2((0, T) \times \mathcal{O}_1), \\ \nu_2 \iint_{\mathcal{O}_{2,d} \times (0,T)} (v - v_{2,d}) w^2 + \mu_2 \iint_{\mathcal{O}_2 \times (0,T)} f^2 \hat{f}^2 = 0, & \forall \hat{f}^2 \in L^2((0, T) \times \mathcal{O}_2). \end{cases} \end{aligned} \quad (3.1)$$

where, $\{w^i\}_{i=1,2}$ solves

$$\begin{cases} \partial_t w^i + \partial_x^5 w^i + \mu_0 \partial_x^4 w^i + \partial_x^3 w^i + \partial_x ((v + \bar{u}) w^i) = \chi_{\mathcal{O}_i} \hat{f}^i, & (t, x) \in Q, \\ w^i(t, 0) = w^i(t, L) = \partial_x w^i(t, 0) = \partial_x w^i(t, L) = 0, \partial_x^2 w^i(t, L) = \partial_x^2 w^i(t, 0), & t \in (0, T), \\ w^i(0, x) = 0, & x \in (0, L). \end{cases} \quad (3.2)$$

Let us consider the non-homogeneous adjoint system

$$\begin{cases} -\partial_t \varphi^i - \partial_x^5 \varphi^i + \mu_0 \partial_x^4 \varphi^i - \partial_x^3 \varphi^i - (v + \bar{u}) \partial_x \varphi^i = \chi_{\mathcal{O}_{i,d}} \nu_i (v - v_{i,d}), & (t, x) \in Q, \\ \varphi^i(t, 0) = \varphi^i(t, L) = \partial_x \varphi^i(t, 0) = \partial_x \varphi^i(t, L) = 0, \partial_x^2 \varphi^i(t, L) = \partial_x^2 \varphi^i(t, 0), & t \in (0, T), \\ \varphi^i(T, x) = 0, & x \in (0, L). \end{cases} \quad (3.3)$$

Then, using the systems (3.2) and (3.3) in the identity (3.1), we have

$$\begin{aligned} \iint_{\mathcal{O}_i \times (0, T)} \left(\widehat{f}^i \varphi^i + \mu_i f^i \widehat{f}^i \right) &= 0, \quad \forall \widehat{f}^i \in L^2((0, T) \times \mathcal{O}_i), \text{ for } i \in \{1, 2\}, \\ \iff f^i &= -\frac{1}{\mu_i} \varphi^i \chi_{\mathcal{O}_i} \text{ for } i \in \{1, 2\}. \end{aligned}$$

Substituting this expression of f^1 and f^2 in the control system gives the optimality system (1.13), which is well-posed due to Proposition 2.10, under the assumption that the constants $\{\mu_i\}_{i=1}^2$ are sufficiently large, and the initial data $v_0 \in L^2(0, L)$ with a sufficiently small norm. This proves the forward part.

Conversely, we prove the pair of functions (f^1, f^2) given by (1.12) is indeed a Nash equilibrium for the functionals $\{J_i\}_{i=1}^2$. To establish it, we shall use the following lemma.

Lemma 3.1. (a) For each $f^2 \in L^2(0, T; L^2(\mathcal{O}_2))$, the functional $J_1(\cdot, v^2)$ attains its infimum at some point $f^1 \in L^2(0, T; L^2(\mathcal{O}_1))$.
 (b) For each $f^1 \in L^2(0, T; L^2(\mathcal{O}_1))$, the functional $J_2(v^1, \cdot)$ attains its infimum at some point $f^2 \in L^2(0, T; L^2(\mathcal{O}_2))$.

Proof. To prove (a), it is enough to show that the map $J_1(\cdot, v^2)$ is sequentially lower semi-continuous (see Theorem 2.D of [Zei95]). For $f \in L^2(0, T; L^2(\mathcal{O}_1))$, let us consider a sequence $f_n \in L^2(0, T; L^2(\mathcal{O}_1))$ such that

$$f_n \rightharpoonup f \text{ in } L^2(0, T; L^2(\mathcal{O}_1)).$$

Let $u^n \in X_T^0$ be the solution of (1.4) corresponding to the $f^1 = f_n$, which is guaranteed by the Proposition 2.8. From the equation, we conclude $u_n \in L^2(0, T; H^{-3}(0, L))$ and so by Aubin-Lion's lemma, there exists a subsequence $\{u_{n_k}\}$ in $L^2(Q)$ such that $u_{n_k} \rightarrow u$ strongly in $L^2(Q)$. Also due to uniform boundedness of $\|u_{n_k}\|_{L^2(0, T; H^2(0, L))}$, $u_n \rightharpoonup u$ weakly in $L^2(0, T; H^1(0, L))$ up to a subsequence, and hence $u_{n_k} \partial_x u_{n_k} \rightharpoonup u \partial_x u$ weakly in $L^2(Q)$, up to a subsequence. This proves that the limit function u is a solution of (1.4) with $f^2 = f$, and so we have

$$J_1(f, f^2) \leq \liminf J_1(f_{n_k}, f^2).$$

Finally, to get the lower semi-continuity, we just need to choose a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\liminf J_1(f_{n_k}, f^2) = \liminf J_1(f_n, f^2)$. This completes the proof of the lemma. \square

Let us first observe that the functionals $J_1\left(\cdot, -\frac{1}{\mu_2} \varphi^2 \chi_{\mathcal{O}_2}\right)$, $J_2\left(-\frac{1}{\mu_1} \varphi^1 \chi_{\mathcal{O}_1}, \cdot\right)$ are well defined due to the global well posedness result Proposition 2.8. The above lemma guarantees the existence of $f^{1*} \in L^2(0, T; \mathcal{O}_1)$, $f^{2*} \in L^2(0, T; \mathcal{O}_2)$ such that

$$\begin{cases} J_1\left(f^{1*}, -\frac{1}{\mu_2} \varphi^2 \chi_{\mathcal{O}_2}\right) = \inf_{f^1} J_1\left(f^1, -\frac{1}{\mu_2} \varphi^2 \chi_{\mathcal{O}_2}\right), \\ J_2\left(-\frac{1}{\mu_1} \varphi^1 \chi_{\mathcal{O}_1}, f^{2*}\right) = \inf_{f^2} J_2\left(-\frac{1}{\mu_1} \varphi^1 \chi_{\mathcal{O}_1}, f^2\right), \end{cases}$$

which implies

$$\begin{cases} J_1'\left(f^{1*}, -\frac{1}{\mu_2} \varphi^2 \chi_{\mathcal{O}_2}\right)(\widehat{f}^1, 0) = 0, \quad \forall \widehat{f}^1 \in L^2(0, T; L^2(\mathcal{O}_1)), \\ J_2'\left(-\frac{1}{\mu_1} \varphi^1 \chi_{\mathcal{O}_1}, f^{2*}\right)(0, \widehat{f}^2) = 0, \quad \forall \widehat{f}^2 \in L^2(0, T; L^2(\mathcal{O}_2)). \end{cases}$$

Proceeding with this condition, we have seen that

$$f^{i*} = -\frac{1}{\mu_i} \varphi^i \chi_{\mathcal{O}_i}, \quad \text{for } i \in \{1, 2\},$$

where $(v, \varphi^1, \varphi^2)$ is the unique solution of (1.13), and hence (f^1, f^2) given by (1.12) is the unique Nash equilibrium point for the functionals $\{J_i\}_{i=1}^2$. \square

4. CARLEMAN ESTIMATE FOR LINEARIZED BENNEY-LIN EQUATION

This section is devoted to the derivation of Carleman estimate for the following adjoint system

$$\begin{cases} -\partial_t w - \partial_x^5 w + \mu_0 \partial_x^4 w - \partial_x^3 w - \bar{u} \partial_x w = g, & (t, x) \in Q, \\ w(t, 0) = w(t, L) = \partial_x w(t, 0) = \partial_x w(t, L) = 0, \quad \partial_x^2 w(t, L) = \partial_x^2 w(t, 0), & t \in (0, T), \\ w(T, x) = w_T, & x \in (0, L). \end{cases} \quad (4.1)$$

Such estimate would play a very important role in deriving the controllability results for the linearized system (1.17). In order to describe the Carleman inequality, we first need to introduce the relevant weight

function which serves as a crucial component in the inequality. The weight function considered here is motivated from the one introduced in [ACMS16].

Weight function. Let $\omega \subset (0, L)$ be any nonempty open subset. Consider an open interval $\tilde{\omega}$ such that $\tilde{\omega} = (a, b) \subset \subset \omega$, and set $a_0 = \frac{(a+b)}{2}$. Then for $K_1, K_2 > 0$ (to be specified later), consider the smooth functions

$$\varphi(x) = 1 + K_1 \left(1 - e^{-K_2(x-a_0)^2}\right), \quad \forall x \in [0, L], \quad (4.2)$$

$$\xi(t) = \frac{1}{t(T-t)}, \quad \forall t \in (0, T), \quad (4.3)$$

and define the function α as

$$\alpha(t, x) = \xi(t)\varphi(x), \quad \forall (t, x) \in (0, T) \times [0, L]. \quad (4.4)$$

We further introduce

$$\check{\alpha}(t) := \min_{x \in [0, L]} \alpha(t, x), \quad \hat{\alpha}(t) := \max_{x \in [0, L]} \alpha(t, x), \quad \forall t \in (0, T). \quad (4.5)$$

We now make suitable choices for the constants K_1, K_2 so that the functions defined above have some desired properties, which are crucial for the Carleman inequality.

- Note that $\alpha > 0$ in $(0, T) \times [0, L]$, for any $K_1, K_2 > 0$.
- The derivative of φ is given by

$$\varphi'(x) = 2K_1K_2(x - a_0)e^{-K_2(x-a_0)^2}, \quad x \in (0, L),$$

and so we have

$$\begin{cases} \varphi'(0) < 0, & \varphi'(L) > 0 \\ \exists \text{ some } c > 0 \text{ such that } |\varphi'| \geq c > 0 & \text{in } [0, L] \setminus \tilde{\omega} \end{cases}$$

- The second order derivative of φ is given by

$$\varphi''(x) = 2K_1K_2 \left(1 - 2K_2(x - a_0)^2\right) e^{-K_2(x-a_0)^2}.$$

We set $K_2 = \frac{4}{(b-a)^2}$, so that

$$\varphi'' < 0 \text{ in } [0, L] \setminus \tilde{\omega}.$$

- Using the sign of φ' , one can easily conclude that

$$\begin{cases} \check{\varphi} := \min_{x \in [0, L]} \varphi(x) = \varphi(a_0) = 1, \text{ and} \\ \hat{\varphi} := \max_{x \in [0, L]} \varphi(x) = \max\{\varphi(0), \varphi(L)\}, \end{cases}$$

Further, note that

$$10\check{\varphi} - 9\hat{\varphi} = 1 - 9K_1 \max\left\{\left(1 - e^{-K_2a_0^2}\right), \left(1 - e^{-K_2(L-a_0)^2}\right)\right\},$$

and so we choose $K_1 = \left[18 \max\left\{\left(1 - e^{-K_2a_0^2}\right), \left(1 - e^{-K_2(L-a_0)^2}\right)\right\}\right]^{-1}$ to get $10\check{\varphi} - 9\hat{\varphi} = \frac{1}{2}$.

Finally, we get the following relation between $\check{\alpha}$ and $\hat{\alpha}$

$$10s\check{\alpha}(t) - 9s\hat{\alpha}(t) = \frac{1}{2}s\xi(t). \quad (4.6)$$

Let us proceed to obtain a suitable Carleman estimate as mentioned in the theorem below.

Theorem 4.1 (Carleman inequality). *Let α, ξ be the weight functions defined above. Then, for any nonempty open $\omega \subset (0, L)$, there exist constants $C, s_0 > 0$ such that for any $w_T \in H^2(0, L)$, $\bar{u} \in X_T^2$, and $g \in L^2(0, T; L^2(0, L))$, the solution w of (4.1) satisfies the estimate*

$$\begin{aligned} & \iint_Q (s^9 \xi^9 |w|^2 + s^7 \xi^7 |\partial_x w|^2 + s^5 \xi^5 |\partial_x^2 w|^2 + s^3 \xi^3 |\partial_x^3 w|^2 + s \xi |\partial_x^4 w|^2) e^{-2s\alpha} \\ & \leq C \left(\iint_Q e^{-2s\alpha} |g|^2 + \iint_{Q_\omega} (s^9 \xi^9 |w|^2 + s^7 \xi^7 |\partial_x w|^2 + s^5 \xi^5 |\partial_x^2 w|^2 + s^3 \xi^3 |\partial_x^3 w|^2 + s \xi |\partial_x^4 w|^2) e^{-2s\alpha} \right). \end{aligned} \quad (4.7)$$

for every $s \geq s_0$.

Proof. The proof of this theorem has been divided into several steps for the ease of reading. Let us first introduce the operator \mathcal{K} defined as

$$\mathcal{K} := \partial_t - \mu_0 \partial_x^4 + \partial_x^5.$$

Step 1: Change of variable. Let $q = e^{-s\alpha}w$. On expanding the term $e^{-s\alpha}\mathcal{K}w = e^{-s\alpha}\mathcal{K}(e^{s\alpha}q)$, we get

$$\mathcal{G} = \mathcal{K}_1q + \mathcal{K}_2q + \mathcal{R}q, \quad (4.8)$$

where,

$$\begin{aligned} \mathcal{G} &= e^{-s\alpha}\mathcal{K}w = e^{-s\alpha}(-\partial_x^3w - \bar{u}\partial_xw - g). \\ \mathcal{K}_1q &= \partial_tq + \partial_x^5q + 10s^2(\partial_x\alpha)^2\partial_x^3q + 25s^2(\partial_x\alpha)(\partial_x^2\alpha)\partial_x^2q + 5s^4(\partial_x\alpha)^4\partial_xq + 20s^4(\partial_x\alpha)^3(\partial_x^2\alpha)q \\ &:= \sum_{i=1}^6 \mathcal{K}_1^i q. \\ \mathcal{K}_2q &= 5s\partial_x\alpha\partial_x^4q + 10s\partial_x^2\alpha\partial_x^3q + 10s^3(\partial_x\alpha)^3\partial_x^2q + 30s^3(\partial_x\alpha)^2\partial_x^2\alpha\partial_xq - 5s\partial_x^4\alpha\partial_xq + s^5(\partial_x\alpha)^5q \\ &:= \sum_{i=1}^6 \mathcal{K}_2^i q. \\ \mathcal{R}q &= \left(10s\partial_x^3\alpha + 5s^2\partial_x\alpha\partial_x^2\alpha\right)\partial_x^2q + \left(15s^2(\partial_x^2\alpha)^2 + 20s^2\partial_x\alpha\partial_x^3\alpha + 10s\partial_x^4\alpha\right)\partial_xq \\ &\quad + \left(s(\partial_t\alpha + \partial_x^5\alpha) + s^2(10\partial_x^2\alpha\partial_x^3\alpha + 5\partial_x\alpha\partial_x^4\alpha) + s^3(15\partial_x\alpha(\partial_x^2\alpha)^2 + 10(\partial_x\alpha)^2\partial_x^3\alpha) - 10s^4(\partial_x\alpha)^3\partial_x^2\alpha\right)q \\ &\quad - \mu_0\left(\partial_x^4q + 4s\partial_x\alpha\partial_x^3q + (6s\partial_x^2\alpha + 6s^2(\partial_x\alpha)^2)\partial_x^2q + (4s\partial_x^3\alpha + 12s^2\partial_x\alpha\partial_x^2\alpha + 4s^3(\partial_x\alpha)^3)\partial_xq \right. \\ &\quad \left. + (s\partial_x^4\alpha + s^2(3(\partial_x^2\alpha)^2 + 4\partial_x\alpha\partial_x^3\alpha) + 6s^3(\partial_x\alpha)^2\partial_x^2\alpha + s^4(\partial_x\alpha)^4)q\right). \end{aligned}$$

Using the identity (4.8) and the property of inner product in $L^2(Q)$, we obtain

$$\|\mathcal{G} - \mathcal{R}q\|_{L^2(Q)}^2 = \|\mathcal{K}_1q\|_{L^2(Q)}^2 + \|\mathcal{K}_2q\|_{L^2(Q)}^2 + 2\langle \mathcal{K}_1q, \mathcal{K}_2q \rangle_{L^2(Q)}. \quad (4.9)$$

To obtain the Carleman estimate (4.14), we need to find suitable lower bound for the inner product $\langle \mathcal{K}_1q, \mathcal{K}_2q \rangle_{L^2(Q)}$, and upper bound for the left-hand side term $\|\mathcal{G} - \mathcal{R}q\|_{L^2(Q)}$.

Step 2: Lower bound on $\langle \mathcal{K}_1q, \mathcal{K}_2q \rangle_{L^2(Q)}$. This step is the most technical one, where the aforementioned properties of weight functions are used. Note that the inner product $\langle \mathcal{K}_1q, \mathcal{K}_2q \rangle_{L^2(Q)}$ can be expanded as

$$\langle \mathcal{K}_1q, \mathcal{K}_2q \rangle_{L^2(Q)} = \sum_{i,j=1}^6 \mathcal{I}_{ij},$$

where $\{\mathcal{I}_{ij}\}_{i,j=1}^6$ are the cross terms given by $\mathcal{I}_{ij} := \langle \mathcal{K}_1^i q, \mathcal{K}_2^j q \rangle_{L^2(Q)}$. Note that using $q = e^{-s\alpha}w$ and boundary conditions of (4.1), we have $q(t, 0) = q(t, L) = \partial_x q(t, 0) = \partial_x q(t, L) = 0$.

The cross product terms \mathcal{I}_{ij} can be computed as listed below:

- $\mathcal{I}_{11} + \mathcal{I}_{12} + \mathcal{I}_{15} = -\frac{5s}{2} \iint_Q \partial_t \partial_x \alpha |\partial_x^2 q|^2 - \frac{5s}{2} \iint_Q \partial_x^3 \partial_t \alpha |\partial_x q|^2.$
- $\mathcal{I}_{13} + \mathcal{I}_{14} = 15s^3 \iint_Q (\partial_x \alpha)^2 \partial_t \partial_x \alpha |\partial_x q|^2.$
- $\mathcal{I}_{16} = -\frac{5}{2} s^5 \iint_Q (\partial_x \alpha)^4 \partial_t \partial_x \alpha |q|^2.$
- $\mathcal{I}_{21} = -\frac{5s}{2} \iint_Q \partial_x^2 \alpha |\partial_x^4 q|^2 + \frac{5s}{2} \int_0^T \partial_x \alpha |\partial_x^4 q|^2|_{x=0}.$
- $\mathcal{I}_{22} = 5s \iint_Q \partial_x^4 \alpha |\partial_x^3 q|^2 - 10s \iint_Q \partial_x^2 \alpha |\partial_x^4 q|^2 + s \int_0^T [10\partial_x^2 \alpha \partial_x^3 q \partial_x^4 q - 5\partial_x^3 \alpha |\partial_x^3 q|^2]_{x=0}^L.$

- $\mathcal{I}_{23} = 45s^3 \iint_Q (\partial_x \alpha)^2 \partial_x^2 \alpha |\partial_x^3 q|^2 - 15s^3 \iint_Q \partial_x^2 ((\partial_x \alpha)^2 \partial_x^2 \alpha) |\partial_x^2 q|^2$
 $+ \int_0^T \left[10s^3 (\partial_x \alpha)^3 \partial_x^2 q \partial_x^4 q - 5s^3 (\partial_x \alpha)^3 |\partial_x^3 q|^2 - 30s^3 (\partial_x \alpha)^2 \partial_x^2 \alpha \partial_x^2 q \partial_x^3 q \right. \\ \left. + 15s^3 \partial_x ((\partial_x \alpha)^2 \partial_x^2 \alpha) |\partial_x^2 q|^2 \right]_{x=0}^L.$
- $\mathcal{I}_{24} = 15s^3 \iint_Q \partial_x^4 ((\partial_x \alpha)^2 \partial_x^2 \alpha) |\partial_x q|^2 - 60s^3 \iint_Q \partial_x^2 ((\partial_x \alpha)^2 \partial_x^2 \alpha) |\partial_x^2 q|^2 + 30s^3 \iint_Q (\partial_x \alpha)^2 \partial_x^2 \alpha |\partial_x^3 q|^2$
 $- 30s^3 \int_0^T \left[(\partial_x \alpha)^2 \partial_x^2 \alpha \partial_x^2 q \partial_x^3 q - \partial_x ((\partial_x \alpha)^2 \partial_x^2 \alpha) |\partial_x^2 q|^2 \right]_{x=0}^L.$
- $\mathcal{I}_{25} = -\frac{5s}{2} \iint_Q \partial_x^8 \alpha |\partial_x q|^2 + 10s \iint_Q \partial_x^6 \alpha |\partial_x^2 q|^2 - 5s \iint_Q \partial_x^4 \alpha |\partial_x^3 q|^2$
 $+ 5s \int_0^T \left[\partial_x^4 \alpha \partial_x^2 q \partial_x^3 q - \partial_x^5 \alpha |\partial_x^2 q|^2 \right]_{x=0}^L.$
- $\mathcal{I}_{26} = -\frac{5}{2} s^5 \iint_Q \partial_x^4 ((\partial_x \alpha)^4 \partial_x^2 \alpha) |q|^2 + \frac{25}{2} s^5 \iint_Q \partial_x^2 ((\partial_x \alpha)^4 \partial_x^2 \alpha) |\partial_x q|^2 - \frac{25}{2} s^5 \iint_Q (\partial_x \alpha)^4 \partial_x^2 \alpha |\partial_x^2 q|^2$
 $+ \frac{1}{2} s^5 \int_0^T (\partial_x \alpha)^5 |\partial_x^2 q|^2 \Big|_{x=0}^L.$
- $\mathcal{I}_{31} = -75s^3 \iint_Q (\partial_x \alpha)^2 \partial_x^2 \alpha |\partial_x^3 q|^2 + 25s^3 \int_0^T (\partial_x \alpha)^3 |\partial_x^3 q|^2 \Big|_{x=0}^L.$
- $\mathcal{I}_{32} = 100s^3 \iint_Q (\partial_x \alpha)^2 \partial_x^2 \alpha |\partial_x^3 q|^2.$
- $\mathcal{I}_{33} = 100s^5 \iint_Q (\partial_x \alpha)^5 \partial_x^2 q \partial_x^3 q = -250s^5 \iint_Q (\partial_x \alpha)^4 \partial_x^2 \alpha |\partial_x^2 q|^2 + 50s^5 \int_0^T (\partial_x \alpha)^5 |\partial_x^2 q|^2 \Big|_{x=0}^L.$
- $\mathcal{I}_{34} = 150s^5 \iint_Q ((\partial_x \alpha)^4 \partial_x^2 \alpha)_{xx} |\partial_x q|^2 - 300s^5 \iint_Q (\partial_x \alpha)^4 \partial_x^2 \alpha |\partial_x^2 q|^2.$
- $\mathcal{I}_{35} = -25s^3 \iint_Q ((\partial_x \alpha)^2 \partial_x^4 \alpha)_{xx} |\partial_x q|^2 + 50s^3 \iint_Q ((\partial_x \alpha)^2 \partial_x^4 \alpha) |\partial_x^2 q|^2.$
- $\mathcal{I}_{36} = -35s^7 \iint_Q ((\partial_x \alpha)^6 \partial_x^2 \alpha)_{xx} |q|^2 + 105s^7 \iint_Q ((\partial_x \alpha)^6 \partial_x^2 \alpha) |\partial_x q|^2.$
- $\mathcal{I}_{41} = \frac{125}{2} s^3 \iint_Q ((\partial_x \alpha)^2 \partial_x^2 \alpha)_{xx} |\partial_x^2 q|^2 - 125s^3 \iint_Q (\partial_x \alpha)^2 \partial_x^2 \alpha |\partial_x^3 q|^2$
 $- \frac{125}{2} s^3 \int_0^T ((\partial_x \alpha)^2 \partial_x^2 \alpha)_x |\partial_x^2 q|^2 \Big|_{x=0}^L + 125s^3 \int_0^T ((\partial_x \alpha)^2 \partial_x^2 \alpha) \partial_x^2 q \partial_x^3 q \Big|_{x=0}^L.$
- $\mathcal{I}_{42} = -125s^3 \iint_Q (\partial_x \alpha (\partial_x^2 \alpha)^2)_x |\partial_x^2 q|^2 + 125s^3 \int_0^T \partial_x \alpha (\partial_x^2 \alpha)^2 |\partial_x^2 q|^2 \Big|_{x=0}^L.$
- $\mathcal{I}_{43} = 250s^5 \iint_Q (\partial_x \alpha)^4 \partial_x^2 \alpha |\partial_x^2 q|^2.$
- $\mathcal{I}_{44} = -375s^5 \iint_Q ((\partial_x \alpha)^3 (\partial_x^2 \alpha)^2)_x |\partial_x q|^2.$
- $\mathcal{I}_{45} = \frac{125}{2} s^3 \iint_Q (\partial_x \alpha \partial_x^2 \alpha \partial_x^4 \alpha)_x |\partial_x q|^2.$
- $\mathcal{I}_{46} = \frac{25}{2} s^7 \iint_Q ((\partial_x \alpha)^6 \partial_x^2 \alpha)_{xx} |q|^2 - 25s^7 \iint_Q (\partial_x \alpha)^6 \partial_x^2 \alpha |\partial_x q|^2.$
- $\mathcal{I}_{51} = \frac{375}{2} s^5 \iint_Q (\partial_x \alpha)^4 \partial_x^2 \alpha |\partial_x^2 q|^2 - \frac{125}{2} s^5 \iint_Q ((\partial_x \alpha)^4 \partial_x^2 \alpha)_{xx} |\partial_x q|^2 - \frac{25}{2} s^5 \int_0^T (\partial_x \alpha)^5 |\partial_x^2 q|^2 \Big|_{x=0}^L.$
- $\mathcal{I}_{52} = 25s^5 \iint_Q ((\partial_x \alpha)^4 \partial_x^2 \alpha)_{xx} |\partial_x q|^2 - 50s^5 \iint_Q ((\partial_x \alpha)^4 \partial_x^2 \alpha) |\partial_x^2 q|^2.$
- $\mathcal{I}_{53} = -175s^7 \iint_Q (\partial_x \alpha)^6 \partial_x^2 \alpha |\partial_x q|^2.$
- $\mathcal{I}_{54} = 150s^7 \iint_Q (\partial_x \alpha)^6 \partial_x^2 \alpha |\partial_x q|^2.$

- $\mathcal{I}_{55} = -25s^5 \iint_Q (\partial_x \alpha)^4 \partial_x^4 \alpha |\partial_x q|^2$
- $\mathcal{I}_{56} = -\frac{45}{2}s^9 \iint_Q (\partial_x \alpha)^8 \partial_x^2 \alpha |q|^2$
- $\mathcal{I}_{61} = 50s^5 \iint_Q \partial_x^4 ((\partial_x \alpha)^4 \partial_x^2 \alpha) |q|^2 - 200s^5 \iint_Q \partial_x^2 ((\partial_x \alpha)^4 \partial_x^2 \alpha) |\partial_x q|^2 + 100s^5 \iint_Q (\partial_x \alpha)^4 \partial_x^2 \alpha |\partial_x^2 q|^2$
- $\mathcal{I}_{62} = -100s^5 \iint_Q \partial_x^3 ((\partial_x \alpha)^3 (\partial_x^2 \alpha)^2) |q|^2 + 300s^5 \iint_Q \partial_x ((\partial_x \alpha)^3 (\partial_x^2 \alpha)^2) |\partial_x q|^2$
- $\mathcal{I}_{63} = 100s^7 \iint_Q ((\partial_x \alpha)^6 \partial_x^2 \alpha)_{xx} |q|^2 - 200s^7 \iint_Q (\partial_x \alpha)^6 \partial_x^2 \alpha |\partial_x q|^2$
- $\mathcal{I}_{64} = -300s^7 \iint_Q ((\partial_x \alpha)^5 \partial_x^2 \alpha)_x |q|^2$
- $\mathcal{I}_{65} = 50s^5 \iint_Q ((\partial_x \alpha)^3 \partial_x^2 \alpha \partial_x^4 \alpha)_x |q|^2$
- $\mathcal{I}_{66} = 20s^9 \iint_Q (\partial_x \alpha)^8 \partial_x^2 \alpha |q|^2.$

Clubbing these expansions together, we get the following identity

$$\begin{aligned}
\langle \mathcal{K}_1 q, \mathcal{K}_2 q \rangle_{L^2(Q)} &= -\frac{25}{2}s \iint_Q (\partial_x^2 \alpha) |\partial_x^4 q|^2 - 25s^3 \iint_Q (\partial_x \alpha)^2 \partial_x^2 \alpha |\partial_x^3 q|^2 + \iint_Q \left[-\frac{5}{2}s \partial_t \partial_x \alpha + 10s \partial_x^6 \alpha \right. \\
&\quad - \frac{25}{2}s^3 \partial_x^2 ((\partial_x \alpha)^2 \partial_x^2 \alpha) + 50s^3 (\partial_x \alpha)^2 \partial_x^4 \alpha - 125s^3 \partial_x (\partial_x \alpha (\partial_x^2 \alpha)^2) - 125s^3 \partial_x (\partial_x \alpha (\partial_x^2 \alpha)^2) \\
&\quad - 125s^3 \partial_x (\partial_x \alpha (\partial_x^2 \alpha)^2) - \left. \frac{150}{2}s^5 (\partial_x \alpha)^4 \partial_x^2 \alpha \right] |\partial_x^2 q|^2 \\
&\quad + \iint_Q \left[-\frac{5}{2}s (\partial_x^3 \partial_t \alpha) - \frac{5}{2}s \partial_x^8 \alpha + 15s^3 (\partial_x \alpha)^2 (\partial_x \partial_t \alpha) + 15s^3 \partial_x^4 ((\partial_x \alpha)^2 \partial_x^2 \alpha) \right. \\
&\quad + \frac{125}{2}s^3 \partial_x (\partial_x \alpha \partial_x^2 \alpha \partial_x^4 \alpha) - 25s^3 \partial_x^2 ((\partial_x \alpha)^2 \partial_x^4 \alpha) - \left. \frac{150}{2}s^5 \partial_x^2 ((\partial_x \alpha)^4 \partial_x^2 \alpha) \right. \\
&\quad - 75s^5 \partial_x ((\partial_x \alpha)^3 (\partial_x^2 \alpha)^2) - 25s^5 (\partial_x \alpha)^4 \partial_x^4 \alpha - 295s^7 (\partial_x \alpha)^6 \partial_x^2 \alpha \left. \right] |\partial_x q|^2 \\
&\quad + \iint_Q \left[-\frac{5}{2}s^5 (\partial_x \alpha)^4 (\partial_t \partial_x \alpha) + 50s^5 \partial_x^4 ((\partial_x \alpha)^4 \partial_x^2 \alpha) + 50s^5 \partial_x ((\partial_x \alpha)^3 \partial_x^2 \alpha \partial_x^4 \alpha) \right. \\
&\quad - 100s^5 \partial_x^3 ((\partial_x \alpha)^3 (\partial_x^2 \alpha)^2) + \frac{225}{2}s^7 \partial_x^2 ((\partial_x \alpha)^6 \partial_x^2 \alpha) \\
&\quad \left. - 300s^7 \partial_x ((\partial_x \alpha)^5 \partial_x^2 \alpha) - \frac{5}{2}s^9 (\partial_x \alpha)^8 \partial_x^2 \alpha \right] |q|^2 + \mathcal{B}, \tag{4.10}
\end{aligned}$$

where \mathcal{B} denotes the boundary terms given as

$$\begin{aligned}
\mathcal{B} &= \frac{5s}{2} \int_0^T \partial_x \alpha |\partial_x^4 q|^2 \Big|_{x=0}^L + s \int_0^T [10 \partial_x^2 \alpha \partial_x^3 q \partial_x^4 q - 5 \partial_x^3 \alpha |\partial_x^3 q|^2]_{x=0}^L \\
&\quad + \int_0^T [10s^3 (\partial_x \alpha)^3 \partial_x^2 q \partial_x^4 q - 5s^3 (\partial_x \alpha)^3 |\partial_x^3 q|^2 - 30s^3 (\partial_x \alpha)^2 \partial_x^2 \alpha \partial_x^2 q \partial_x^3 q + 15s^3 \partial_x ((\partial_x \alpha)^2 \partial_x^2 \alpha) |\partial_x^2 q|^2]_0^L \\
&\quad - 30s^3 \int_0^T [(\partial_x \alpha)^2 \partial_x^2 \alpha \partial_x^2 q \partial_x^3 q - \partial_x ((\partial_x \alpha)^2 \partial_x^2 \alpha) |\partial_x^2 q|^2]_{x=0}^L + 5s \int_0^T [\partial_x^4 \alpha \partial_x^2 q \partial_x^3 q - \partial_x^5 \alpha |\partial_x^2 q|^2]_{x=0}^L \\
&\quad + \frac{1}{2}s^5 \int_0^T (\partial_x \alpha)^5 |\partial_x^2 q|^2 \Big|_{x=0}^L + 25s^3 \int_0^T (\partial_x \alpha)^3 |\partial_x^3 q|^2 \Big|_{x=0}^L + 50s^5 \int_0^T (\partial_x \alpha)^5 |\partial_x^2 q|^2 \Big|_{x=0}^L \\
&\quad - \frac{125}{2}s^3 \int_0^T ((\partial_x \alpha)^2 \partial_x^2 \alpha)_x |\partial_x^2 q|^2 \Big|_{x=0}^L + 125s^3 \int_0^T ((\partial_x \alpha)^2 \partial_x^2 \alpha) \partial_x^2 q \partial_x^3 q \Big|_{x=0}^L \\
&\quad + 125s^3 \int_0^T \partial_x \alpha (\partial_x^2 \alpha)^2 |\partial_x^2 q|^2 \Big|_0^L - \frac{25}{2}s^5 \int_0^T (\partial_x \alpha)^5 |\partial_x^2 q|^2 \Big|_{x=0}^L \\
&:= \sum_{i=1}^{18} \mathcal{B}_i.
\end{aligned}$$

Estimates of the boundary terms: Using smoothness of the weight function α along with Hölder's and Young's inequality, one can easily estimate the boundary terms \mathcal{B}_i to get the following:

- $\mathcal{B}_1 = \frac{5}{2}s \int_0^T \partial_x \alpha |\partial_x^4 q|^2 \Big|_{x=L} - \frac{5}{2}s \int_0^T \partial_x \alpha |\partial_x^4 q|^2 \Big|_{x=0}$.
- $\mathcal{B}_2 \geq -Cs \int_0^T (|\partial_x^3 q(L)|^2 + |\partial_x^3 q(0)|^2) - C \int_0^T (|\partial_x^4 q(L)|^2 + |\partial_x^4 q(0)|^2)$.
- $\mathcal{B}_3 \geq -Cs \int_0^T |\partial_x^3 q(L)|^2 - Cs \int_0^T |\partial_x^3 q(0)|^2$.
- $\mathcal{B}_4 \geq -5s^5 \int_0^1 (\partial_x \alpha)^5 |\partial_x^2 q|^2 \Big|_{x=L} - 5s^5 \int_0^1 (\partial_x \alpha)^5 |\partial_x^2 q|^2 \Big|_{x=0} - 5s \int_0^1 \partial_x \alpha |\partial_x^4 q|^2 \Big|_{x=L} - 5s \int_0^1 \partial_x \alpha |\partial_x^4 q|^2 \Big|_{x=0}$.
- $\mathcal{B}_5 + \mathcal{B}_{13} = 20s^3 \int_0^T (\partial_x \alpha)^3 |\partial_x^3 q|^2 \Big|_{x=L} - 20s^3 \int_0^T (\partial_x \alpha)^3 |\partial_x^3 q|^2 \Big|_{x=0}$.
- $\mathcal{B}_6 + \mathcal{B}_8 + \mathcal{B}_{16} \geq -Cs^2 \int_0^T (|\partial_x^3 q(L)|^2 + |\partial_x^3 q(0)|^2) - Cs^4 \int_0^T (|\partial_x^2 q(L)|^2 + |\partial_x^2 q(0)|^2)$.
- $\mathcal{B}_7 + \mathcal{B}_9 + \mathcal{B}_{15} \geq -Cs^3 \int_0^T (|\partial_x^2 q(L)|^2 + |\partial_x^2 q(0)|^2)$.
- $\mathcal{B}_{10} \geq -Cs \int_0^T (|\partial_x^2 q(L)|^2 + |\partial_x^2 q(0)|^2) - Cs \int_0^T (|\partial_x^3 q(L)|^2 + |\partial_x^3 q(0)|^2)$.
- $\mathcal{B}_{11} \geq -Cs \int_0^T (|\partial_x^2 q(L)|^2 + |\partial_x^2 q(0)|^2)$.
- $\mathcal{B}_{12} + \mathcal{B}_{14} + \mathcal{B}_{18} = 36s^5 \int_0^T (\partial_x \alpha)^5 |\partial_x^2 q|^2 \Big|_{x=L} - 36s^5 \int_0^T (\partial_x \alpha)^5 |\partial_x^2 q|^2 \Big|_{x=0}$.
- $\mathcal{B}_{17} \geq Cs^3 \int_0^T |\partial_x^2 q(L)|^2 + Cs^3 \int_0^T |\partial_x^2 q(0)|^2$.

Thanks to the boundary conditions $\alpha'(0) < 0$, $\alpha'(L) > 0$ of the weight function α , the above estimates can be summed up for sufficiently large s to get the following estimate:

$$\mathcal{B} \geq C \left[s \int_0^T (|\partial_x^4 q(L)|^2 + |\partial_x^4 q(0)|^2) + s^3 \int_0^T (|\partial_x^3 q(L)|^2 + |\partial_x^3 q(0)|^2) + s^5 \int_0^T (|\partial_x^2 q(L)|^2 + |\partial_x^2 q(0)|^2) \right] \geq 0.$$

Estimates of the double integrals: Recall the fact that $\varphi'' < 0$ on $(a, b)^c$. Using this property of the weight function, the first term from (4.10) can be estimated as

$$\begin{aligned} -\frac{25}{2}s \iint_Q (\partial_x^2 \alpha) |\partial_x^4 q|^2 &= -\frac{25}{2}s \int_0^T \int_{(a,b)^c} (\partial_x^2 \alpha) |\partial_x^4 q|^2 - \frac{25}{2}s \int_0^T \int_{(a,b)} (\partial_x^2 \alpha) |\partial_x^4 q|^2 \\ &\geq C_1 s \int_0^T \int_{(a,b)^c} \xi |\partial_x^4 q|^2 - C_2 s \int_0^T \int_{(a,b)} \xi |\partial_x^4 q|^2 \\ &= C_1 s \iint_Q \xi |\partial_x^4 q|^2 - (C_1 + C_2) s \int_0^T \int_{(a,b)} \xi (\partial_x^2 \alpha) |\partial_x^4 q|^2. \end{aligned}$$

Next for the leading terms (with the highest power of s) of $|\partial_x^3 q|$, $|\partial_x^2 q|$, $|\partial_x q|$ and $|q|$, we first use the fact that $|\varphi'| > 0$ on $(a, b)^c$, and then use the same argument as above to get the following:

$$\begin{aligned} -25s^3 \iint_Q (\partial_x \alpha)^2 \partial_x^2 \alpha |\partial_x^3 q|^2 &\geq Cs^3 \iint_Q \xi^3 |\partial_x^3 q|^2 - Cs^3 \int_0^T \int_a^b \xi^3 |\partial_x^3 q|^2 \\ -\frac{150}{2}s^5 \iint_Q (\partial_x \alpha)^4 \partial_x^2 \alpha |\partial_x^2 q|^2 &\geq Cs^5 \iint_Q \xi^5 |\partial_x^2 q|^2 - Cs^5 \int_0^T \int_a^b \xi^5 |\partial_x^2 q|^2 \\ -295s^7 \iint_Q (\partial_x \alpha)^6 \partial_x^2 \alpha |\partial_x q|^2 &\geq Cs^7 \iint_Q \xi^7 |\partial_x q|^2 - Cs^7 \int_0^T \int_a^b \xi^7 |\partial_x q|^2 \\ -\frac{5}{2}s^9 \iint_Q (\partial_x \alpha)^8 \partial_x^2 \alpha |q|^2 &\geq Cs^9 \iint_Q \xi^9 |q|^2 - Cs^9 \int_0^T \int_a^b \xi^9 |q|^2. \end{aligned}$$

Using the smoothness of the function α , the rest of the terms of (4.10) can also be estimated, and we finally obtain

$$\begin{aligned} \langle \mathcal{K}_1 q, \mathcal{K}_2 q \rangle_{L^2(Q)} &\geq C \iint_Q \left(s^9 \xi^9 |q|^2 + s^7 \xi^7 |\partial_x q|^2 + s^5 \xi^5 |\partial_x^2 q|^2 + s^3 \xi^3 |\partial_x^3 q|^2 + s \xi |\partial_x^4 q|^2 \right) \\ &\quad - C \int_0^T \int_a^b \left(s^9 \xi^9 |q|^2 + s^7 \xi^7 |\partial_x q|^2 + s^5 \xi^5 |\partial_x^2 q|^2 + s^3 \xi^3 |\partial_x^3 q|^2 + s \xi |\partial_x^4 q|^2 \right), \end{aligned} \quad (4.11)$$

for sufficiently large positive parameter s .

Step 3: Upper bound for $\|\mathcal{G} - \mathcal{R}q\|_{L^2(Q)}$. In order to get the desired bound on the L^2 norm of $\mathcal{G} - \mathcal{R}q$, we simply use the smoothness property of φ . Indeed we obtain a positive constant $C(L, T, \omega, \|\bar{u}\|_{X_0^T})$ such that the following estimate hold

$$\begin{aligned} \|\mathcal{G} - \mathcal{R}q\|_{L^2(Q)}^2 &\leq C \left(\iint_Q |\partial_x^4 q|^2 + s^2 \iint_Q \xi^2 |\partial_x^3 q|^2 + s^4 \iint_Q \xi^4 |\partial_x^2 q|^2 + s^6 \iint_Q \xi^6 |\partial_x q|^2 + s^8 \iint_Q \xi^8 |q|^2 \right) \\ &\quad + \iint_Q e^{-2s\alpha} |\partial_x^3 w|^2 + \iint_Q e^{-2s\alpha} |\partial_x w|^2 + \iint_Q e^{-2s\alpha} |g|^2. \end{aligned} \quad (4.12)$$

Step 4: Combining all the estimates. Substituting the estimates (4.11) and (4.12) in the identity (4.9), we get

$$\begin{aligned} &\|M_1 q\|_{L^2(Q)}^2 + \|M_2 q\|_{L^2(Q)}^2 + C \iint_Q \left(s^9 \xi^9 |q|^2 + s^7 \xi^7 |\partial_x q|^2 + s^5 \xi^5 |\partial_x^2 q|^2 + s^3 \xi^3 |\partial_x^3 q|^2 + s \xi |\partial_x^4 q|^2 \right) \\ &\quad - C \int_0^T \int_a^b \left(s^9 \xi^9 |q|^2 + s^7 \xi^7 |\partial_x q|^2 + s^5 \xi^5 |\partial_x^2 q|^2 + s^3 \xi^3 |\partial_x^3 q|^2 + s \xi |\partial_x^4 q|^2 \right) \\ &\leq C \left(\iint_Q |\partial_x^4 q|^2 + s^2 \iint_Q \xi^2 |\partial_x^3 q|^2 + s^4 \iint_Q \xi^4 |\partial_x^2 q|^2 + s^6 \iint_Q \xi^6 |\partial_x q|^2 + s^8 \iint_Q \xi^8 |q|^2 \right) \\ &\quad + \iint_Q e^{-2s\alpha} |\partial_x^3 w|^2 + \iint_Q e^{-2s\alpha} |\partial_x w|^2 + \iint_Q e^{-2s\alpha} |g|^2. \end{aligned}$$

For sufficiently large s , we can further reduce the above estimate to

$$\begin{aligned} &\iint_Q \left(s^9 \xi^9 |q|^2 + s^7 \xi^7 |\partial_x q|^2 + s^5 \xi^5 |\partial_x^2 q|^2 + s^3 \xi^3 |\partial_x^3 q|^2 + s \xi |\partial_x^4 q|^2 \right) \\ &\leq C \int_0^T \int_a^b \left(s^9 \xi^9 |q|^2 + s^7 \xi^7 |\partial_x q|^2 + s^5 \xi^5 |\partial_x^2 q|^2 + s^3 \xi^3 |\partial_x^3 q|^2 + s \xi |\partial_x^4 q|^2 \right) \\ &\quad + \iint_Q e^{-2s\alpha} |\partial_x^3 w|^2 + \iint_Q e^{-2s\alpha} |\partial_x w|^2 + \iint_Q e^{-2s\alpha} |g|^2. \end{aligned} \quad (4.13)$$

Step 5: Returning to the original variable w

Recall that $q = e^{-s\alpha} w$, and so we have:

$$\begin{aligned} |\partial_x w|^2 e^{-2s\alpha} &\leq C \left(s^2 \xi^2 |q|^2 + |\partial_x q|^2 \right) \\ |\partial_x^2 w|^2 e^{-2s\alpha} &\leq C \left(s^4 \xi^4 |q|^2 + s^2 \xi^2 |\partial_x q|^2 + |\partial_x^2 q|^2 \right) \\ |\partial_x^3 w|^2 e^{-2s\alpha} &\leq C \left(s^6 \xi^6 |q|^2 + s^4 \xi^4 |\partial_x q|^2 + s^2 \xi^2 |\partial_x^2 q|^2 + |\partial_x^3 q|^2 \right) \\ |\partial_x^4 w|^2 e^{-2s\alpha} &\leq C \left(s^8 \xi^8 |q|^2 + s^6 \xi^6 |\partial_x q|^2 + s^4 \xi^4 |\partial_x^2 q|^2 + s^2 \xi^2 |\partial_x^3 q|^2 + |\partial_x^4 q|^2 \right) \\ |\partial_x q|^2 &\leq C e^{-2s\alpha} \left(s^2 \xi^2 |w|^2 + |\partial_x w|^2 \right) \\ |\partial_x^2 q|^2 &\leq C e^{-2s\alpha} \left(s^4 \xi^4 |w|^2 + s^2 \xi^2 |\partial_x w|^2 + |\partial_x^2 w|^2 \right) \\ |\partial_x^3 q|^2 &\leq C e^{-2s\alpha} \left(s^6 \xi^6 |w|^2 + s^4 \xi^4 |\partial_x w|^2 + s^2 \xi^2 |\partial_x^2 w|^2 + |\partial_x^3 w|^2 \right) \\ |\partial_x^4 q|^2 &\leq C e^{-2s\alpha} \left(s^8 \xi^8 |w|^2 + s^6 \xi^6 |\partial_x w|^2 + s^4 \xi^4 |\partial_x^2 w|^2 + s^2 \xi^2 |\partial_x^3 w|^2 + |\partial_x^4 w|^2 \right). \end{aligned}$$

Using these estimates in (4.13) along with the fact that $(a, b) = \tilde{\omega} \subset \omega$, we get the desired Carleman estimate (4.7) for sufficiently large s . This completes the proof. \square

The estimate (4.7) can be further modified to obtain a simplified version as mentioned below.

Proposition 4.2 (Modified Carleman inequality). *Assume $\bar{u} \in X_T^3$, and let $\omega \subset (0, L)$ be any open subset of $(0, L)$. Let α, ξ be the weight functions as defined above. Then there exist constants $C, s_0 > 0$ such that for any $w_T \in H^3(0, L), g \in L^2(0, T; H^1(0, L))$ and for every $s \geq s_0$, the solution w of (4.1) satisfies the estimate*

$$\begin{aligned} & \iint_Q \left(s^9 \xi^9 |w|^2 + s^7 \xi^7 |\partial_x w|^2 + s^5 \xi^5 |\partial_x^2 w|^2 + s^3 \xi^3 |\partial_x^3 w|^2 + s \xi |\partial_x^4 w|^2 \right) e^{-2s\alpha} \\ & \leq C \left(\iint_Q e^{-2s\alpha} |g|^2 + \int_0^T e^{-2s\hat{\alpha}} \xi \|g\|_{H^1(0, L)}^2 + s^9 \int_0^T \xi^9 e^{-2s(5\hat{\alpha}-4\hat{\alpha})} \|w\|_{L^2(\omega)}^2 \right). \end{aligned} \quad (4.14)$$

Proof. To prove this theorem, we mainly need to handle the higher order observations terms appearing in the right hand side of the Carleman estimate (4.7).

Initially, using $\alpha \geq \hat{\alpha}$ and the definition of the Sobolev norm in (4.7), we get

$$\begin{aligned} & \iint_Q \left(s^9 \xi^9 w^2 + s^7 \xi^7 |\partial_x w|^2 + s^5 \xi^5 |\partial_x^2 w|^2 + s^3 \xi^3 |\partial_x^3 w|^2 + s \xi |\partial_x^4 w|^2 \right) e^{-2s\alpha} dx dt \\ & \leq C \iint_Q e^{-2s\alpha} |g|^2 + C \int_0^T e^{-2s\hat{\alpha}} \left(s^9 \xi^9 \|w\|_{L^2(\omega)}^2 + s^7 \xi^7 \|w\|_{H^1(\omega)}^2 + s^5 \xi^5 \|w\|_{H^2(\omega)}^2 \right. \\ & \quad \left. + s^3 \xi^3 \|w\|_{H^3(\omega)}^2 + s \xi \|w\|_{H^4(\omega)}^2 \right) \\ & =: C \iint_Q e^{-2s\alpha} |g|^2 + C s^9 \int_0^T \xi^9 \|w\|_{L^2(\omega)}^2 e^{-2s\hat{\alpha}} dt + C \sum_{i=1}^4 \mathcal{Q}_i. \end{aligned} \quad (4.15)$$

Next, thanks to the Sobolev interpolation $H^1(\omega) = (H^5(\omega), L^2(\omega))_{\frac{1}{5}, \frac{4}{5}}$, \mathcal{Q}_1 can be estimated as

$$\mathcal{Q}_1 \leq \int_0^T e^{-2s\hat{\alpha}} s^7 \xi^7 \|w\|_{H^5}^{\frac{2}{5}} \|w\|_{L^2}^{\frac{8}{5}}.$$

Further we use Young's inequality with $p = 5$ and $q = 5/4$ to get

$$\mathcal{Q}_1 \leq C \int_0^T \xi e^{-2s\hat{\alpha}} \|w(t, \cdot)\|_{H^5(\omega)}^2 + C s^{\frac{35}{4}} \int_0^T \xi^{\frac{17}{2}} e^{s(-\frac{5}{2}\hat{\alpha} + \frac{6}{2})} \|w(t, \cdot)\|_{L^2(\omega)}^2.$$

Other \mathcal{Q}_i 's can be estimated in a similar fashion with the following Sobolev interpolation

$$H^2(\omega) = (H^5(\omega), L^2(\omega))_{\frac{2}{5}, \frac{3}{5}}, H^3(\omega) = (H^5(\omega), L^2(\omega))_{\frac{3}{5}, \frac{2}{5}}, H^4(\omega) = (H^5(\omega), L^2(\omega))_{\frac{4}{5}, \frac{1}{5}}.$$

Indeed, we have

$$\begin{aligned} \mathcal{Q}_2 & \leq C \int_0^T \xi e^{-2s\hat{\alpha}} \|w(t, \cdot)\|_{H^5(\omega)}^2 + C s^{\frac{25}{3}} \int_0^T \xi^{\frac{23}{3}} e^{s(-\frac{10}{3}\hat{\alpha} + \frac{4}{3}\hat{\alpha})} \|w(t, \cdot)\|_{L^2(\omega)}^2. \\ \mathcal{Q}_3 & \leq C \int_0^T \xi e^{-2s\hat{\alpha}} \|w(t, \cdot)\|_{H^5(\omega)}^2 + C s^{\frac{15}{2}} \int_0^T \xi^6 e^{s(-5\hat{\alpha} + 3\hat{\alpha})} \|w(t, \cdot)\|_{L^2(\omega)}^2. \\ \mathcal{Q}_4 & \leq C \int_0^T \xi e^{-2s\hat{\alpha}} \|w(t, \cdot)\|_{H^5(\omega)}^2 + C s^5 \int_0^T \xi e^{s(-10\hat{\alpha} + 8\hat{\alpha})} \|w(t, \cdot)\|_{L^2(\omega)}^2. \end{aligned}$$

Therefore, amalgamating these above estimates and noting the fact

$$e^{s(-\frac{5}{2}\hat{\alpha} + \frac{6}{2})}, e^{s(-\frac{10}{3}\hat{\alpha} + \frac{4}{3}\hat{\alpha})}, e^{s(-5\hat{\alpha} + 3\hat{\alpha})} \leq e^{s(-10\hat{\alpha} + 8\hat{\alpha})},$$

we conclude

$$s^9 \int_0^T \xi^9 \|w\|_{L^2(\omega)}^2 e^{-2s\hat{\alpha}} dt + \sum_{i=1}^4 \mathcal{Q}_i \leq C \int_0^T \xi e^{-2s\hat{\alpha}} \|w(t, \cdot)\|_{H^5(\omega)}^2 + C s^9 \int_0^T \xi^9 e^{8s\hat{\alpha} - 10s\hat{\alpha}} \|w\|_{L^2(\omega)}^2.$$

Substituting this in (4.15), we deduce

$$\begin{aligned} & \iint_Q \left(s^9 \xi^9 |w|^2 + s^7 \xi^7 |\partial_x w|^2 + s^5 \xi^5 |\partial_x^2 w|^2 + s^3 \xi^3 |\partial_x^3 w|^2 + s \xi |\partial_x^4 w|^2 \right) e^{-2s\alpha} \\ & \leq C \left(\iint_Q e^{-2s\alpha} |g|^2 + s^9 \int_0^T \xi^9 e^{-2s(5\hat{\alpha}-4\hat{\alpha})} \|w\|_{L^2(\omega)}^2 + \int_0^T \xi e^{-2s\hat{\alpha}} \|w(t, \cdot)\|_{H^5(\omega)}^2 \right). \end{aligned} \quad (4.16)$$

In order to get the final Carleman estimate (4.14), we need to absorb the last term from the right hand side of (4.16), which can be done by utilizing the standard bootstrap argument, generally used for dispersive equations (see [CFPR15], [Che19], [CMZ20]).

Let us consider $\tilde{\rho}(t) = e^{-s\hat{\alpha}(t)}\xi^{\frac{1}{2}}(t)$, and define $\tilde{w} = \tilde{\rho}w$. Then, \tilde{w} solves the following system

$$\begin{cases} -\partial_t \tilde{w} - \partial_x^5 \tilde{w} + \mu_0 \partial_x^4 \tilde{w} - \partial_x^3 \tilde{w} - \bar{u} \partial_x \tilde{w} = \tilde{g}, & (t, x) \in Q, \\ \tilde{w}(t, 0) = \tilde{w}(t, L) = \partial_x \tilde{w}(t, 0) = \partial_x \tilde{w}(t, L) = 0, \quad \partial_x^2 \tilde{w}(t, L) = \partial_x^2 \tilde{w}(t, 0), & t \in (0, T), \\ \tilde{w}(T, x) = 0, & x \in (0, L), \end{cases}$$

where, $\tilde{g} = \tilde{\rho}g - \partial_t \tilde{\rho}w$.

Note that $|\partial_t \tilde{\rho}| \leq C(\xi^{\frac{3}{2}} + s\xi^{\frac{5}{2}})e^{-s\hat{\alpha}} \leq Cs\xi^{\frac{5}{2}}e^{-s\hat{\alpha}}$, and so

$$\|\tilde{g}\|_{L^2(0,T;H^1(0,L))}^2 \leq C \left(\int_0^T e^{-2s\hat{\alpha}} \xi \|g\|_{H^1(0,L)}^2 + \int_0^T e^{-2s\hat{\alpha}} s^2 \xi^5 \|w\|_{H^1(0,L)}^2 \right).$$

Thanks to Proposition 2.6, the solution $\tilde{w} \in L^2(0, T; H^5(0, L))$ satisfies

$$\int_0^T e^{-2s\hat{\alpha}} \xi \|w\|_{H^5(0,L)}^2 \leq C \left(\int_0^T e^{-2s\hat{\alpha}} \xi \|g\|_{H^1(0,L)}^2 + \int_0^T e^{-2s\hat{\alpha}} s^2 \xi^5 \|w\|_{H^1(0,L)}^2 \right).$$

Further substituting this estimate in the inequality (4.16), we get

$$\begin{aligned} & \iint_Q \left(s^9 \xi^9 |w|^2 + s^7 \xi^7 |\partial_x w|^2 + s^5 \xi^5 |\partial_x^2 w|^2 + s^3 \xi^3 |\partial_x^3 w|^2 + s \xi |\partial_x^4 w|^2 \right) e^{-2s\alpha} \\ & \leq C \left(\iint_Q e^{-2s\alpha} |g|^2 + s^9 \int_0^T \xi^9 e^{-2s(5\hat{\alpha}-4\hat{\alpha})} \|w\|_{L^2(\omega)}^2 + \int_0^T e^{-2s\hat{\alpha}} \xi \|g\|_{H^1(0,L)}^2 \right. \\ & \quad \left. + \int_0^T e^{-2s\hat{\alpha}} s^2 \xi^5 \|w\|_{H^1(0,L)}^2 \right). \end{aligned}$$

Clearly, the last term of right hand side can be absorbed by the second term of left for sufficiently large s to finally obtain the desired estimate (4.14). \square

5. CARLEMAN ESTIMATE FOR THE LINEARIZED COUPLED SYSTEM (1.18)

In this section, we derive suitable Carleman estimates for the adjoint system (1.18) under the assumption (1.14), for the cases (1.15) and (1.16) separately.

Case-1.15: $\mathcal{O}_{1,d} = \mathcal{O}_{2,d}$

For this case, we introduce a new variable p given by

$$p = \nu_1 \psi^1 + \nu_2 \psi^2,$$

so that the concerned adjoint system (1.18) converts into the following system involving just two variables, namely η and p .

$$\begin{cases} -\partial_t \eta - \partial_x^5 \eta + \mu_0 \partial_x^4 \eta - \partial_x^3 \eta - \bar{u} \eta_x = p \chi_{\mathcal{O}_d} + h^0, & (t, x) \in Q, \\ \partial_t p + \partial_x^5 p + \mu_0 \partial_x^4 p + \partial_x^3 p + \partial_x(\bar{u}p) = -\sum_{i=1}^2 \frac{\nu_i}{\mu_i} \eta \chi_{\mathcal{O}_i} + h, & (t, x) \in Q, \\ \eta(t, 0) = \eta(t, L) = \partial_x \eta(t, 0) = \partial_x \eta(t, L) = 0, \quad \partial_x^2 \eta(t, L) = \partial_x^2 \eta(t, 0), & t \in (0, T), \\ p(t, 0) = p(t, L) = \partial_x p(t, 0) = \partial_x p(t, L) = 0, \quad \partial_x^2 p(t, L) = \partial_x^2 p(t, 0), & t \in (0, T), \\ \eta(T, x) = \eta_T(x), p(0, x) = 0, & x \in (0, L), \end{cases} \quad (5.1)$$

where $\mathcal{O}_d = \mathcal{O}_{1,d} = \mathcal{O}_{2,d}$ and $h = \nu_1 h^1 + \nu_2 h^2$.

Let us introduce the following shorthand notation

$$\mathcal{I}(s, w) := \iint_Q \left(s^9 \xi^9 |w|^2 + s^7 \xi^7 |\partial_x w|^2 + s^5 \xi^5 |\partial_x^2 w|^2 + s^3 \xi^3 |\partial_x^3 w|^2 + s \xi |\partial_x^4 w|^2 \right) e^{-2s\alpha},$$

where the weight functions α and ξ are given by (4.2) and (4.3), respectively. Then we obtain the following Carleman estimate for the system (5.1).

Proposition 5.1. *Let $\bar{u} \in X_T^3$ and assume the constants μ_i to be large enough. Then, there exist constants $s_0, C > 0$ such that for every $s \geq s_0$, and for any $(\eta_T, h^0, h) \in H^3(0, L) \times [L^2(0, T; H^1(0, L))]^2$, the solution (η, p) of system (5.1) satisfies*

$$\begin{aligned} \mathcal{I}(s, \eta) + \mathcal{I}(s, p) &\leq Cs^{17} \int_0^T \xi^{17} e^{-2s(10\hat{\alpha}-9\hat{\alpha})} \|\eta\|_{L^2(\omega)}^2 \\ &\quad + Cs^9 \int_0^T \xi^9 e^{-2s(10\hat{\alpha}-9\hat{\alpha})} \left(\|h\|_{H^1(0, L)}^2 + \|h^0\|_{H^1(0, L)}^2 \right). \end{aligned} \quad (5.2)$$

Proof. Since $\mathcal{O}_d \cap \mathcal{O}_0 \neq \emptyset$, let us consider a nonempty, open and connected subset $\omega \subset \mathcal{O}_d \cap \mathcal{O}_0$. Further, let $\tilde{\omega} \subset \subset \omega$ any open set. Thanks to the Carleman estimate (4.14) for the states η and p corresponding to $\tilde{\omega}$, we have

$$\begin{aligned} &\mathcal{I}(s, \eta) + \mathcal{I}(s, p) \\ &\leq C \left(\iint_Q e^{-2s\alpha} \underbrace{|p\chi_{\mathcal{O}_d} + h^0|^2}_{\kappa_1} + \iint_Q e^{-2s\alpha} \left| -\sum_{i=1}^2 \frac{\nu_i}{\mu_i} \eta \chi_{\mathcal{O}_i} + h \right|^2_{\kappa_2} + \int_0^T e^{-2s\hat{\alpha}} \xi \underbrace{\|p\chi_{\mathcal{O}_d} + h^0\|_{H^1(0, L)}^2}_{\kappa_3} \right. \\ &\quad \left. + \int_0^T e^{-2s\hat{\alpha}} \xi \left\| -\sum_{i=1}^2 \frac{\nu_i}{\mu_i} \eta \chi_{\mathcal{O}_i} + h \right\|_{H^1(0, L)}^2_{\kappa_4} + s^9 \int_0^T \xi^9 e^{-2s(5\hat{\alpha}-4\hat{\alpha})} \left(\|\eta\|_{L^2(\tilde{\omega})}^2 + \|p\|_{L^2(\tilde{\omega})}^2 \right) \right). \end{aligned}$$

Next, we use triangle inequality in the terms κ_i for $i \in \{1, 2, 3, 4\}$, so that the terms involving p and η can be absorbed by $\mathcal{I}(s, p)$ and $\mathcal{I}(s, \eta)$, respectively for sufficiently large s to get

$$\begin{aligned} &\mathcal{I}(s, \eta) + \mathcal{I}(s, p) \\ &\leq C \left(\int_0^T e^{-2s\hat{\alpha}} \left(\|h^0\|_{L^2(0, L)}^2 + \|h\|_{L^2(0, L)}^2 \right) + \int_0^T e^{-2s\hat{\alpha}} \xi \left(\|h^0\|_{H^1(0, L)}^2 + \|h\|_{H^1(0, L)}^2 \right) \right. \\ &\quad \left. + Cs^9 \int_0^T \xi^9 e^{-2s(5\hat{\alpha}-4\hat{\alpha})} \left(\|\eta\|_{L^2(\tilde{\omega})}^2 + \|p\|_{L^2(\tilde{\omega})}^2 \right) \right). \end{aligned} \quad (5.3)$$

Our next aim is to remove the local term of p from right-hand side of (5.3). In order to do that, let us first introduce a smooth cut off function $\zeta \in C_c^\infty(\omega)$ with $\zeta(x) = 1$ for $x \in \tilde{\omega}$ and $0 \leq \zeta \leq 1$. Further, thanks to the first equation of (5.1) and the fact that $\omega \subset \mathcal{O}_d \cap \mathcal{O}_0$, p can be written as

$$p = -\partial_t \eta - \partial_x^5 \eta + \mu_0 \partial_x^4 \eta - \partial_x^3 \eta - \bar{u} \eta_x - h^0 \quad \text{in } (0, T) \times \omega.$$

Therefore, the last term of (5.3) can be estimated as

$$\begin{aligned} &s^9 \int_0^T \xi^9 e^{-2s(5\hat{\alpha}-4\hat{\alpha})} \|p\|_{L^2(\tilde{\omega})}^2 \\ &\leq s^9 \int_0^T \int_\omega e^{-2s(5\hat{\alpha}-4\hat{\alpha})} \xi^9 \zeta p \left(-\partial_t \eta - \partial_x^5 \eta + \mu_0 \partial_x^4 \eta - \partial_x^3 \eta - \bar{u} \partial_x \eta - h^0 \right) \\ &= s^9 \int_0^T \int_\omega e^{-2s(5\hat{\alpha}-4\hat{\alpha})} \xi^9 \zeta \left(\partial_t p + \partial_x^5 p + \mu_0 \partial_x^4 p + \partial_x^3 p + \partial_x(\bar{u} p) \right) \eta \\ &\quad + s^9 \int_0^T \int_\omega \partial_t \left(e^{-2s(5\hat{\alpha}-4\hat{\alpha})} \xi^9 \right) \eta \zeta p - s^9 \int_0^T \int_\omega e^{-2s(5\hat{\alpha}-4\hat{\alpha})} \xi^9 \zeta p h^0 \\ &\quad + s^9 \int_0^T \int_\omega e^{-2s(5\hat{\alpha}-4\hat{\alpha})} \xi^9 \left(\partial_x^5 \zeta p + \partial_x^4 \zeta \partial_x p + \partial_x^3 \zeta \partial_x^2 p + \partial_x^2 \zeta \partial_x^3 p + \partial_x \zeta \partial_x^4 p \right) \eta \\ &\quad + \mu_0 s^9 \int_0^T \int_\omega e^{-2s(5\hat{\alpha}-4\hat{\alpha})} \xi^9 \left(\partial_x^4 \zeta p + \partial_x^3 \zeta \partial_x p + \partial_x^2 \zeta \partial_x^2 p + \partial_x \zeta \partial_x^3 p \right) \eta \\ &\quad + s^9 \int_0^T \int_\omega e^{-2s(5\hat{\alpha}-4\hat{\alpha})} \xi^9 \left(\partial_x^3 \zeta p + \partial_x^2 \zeta \partial_x p + \partial_x \zeta \partial_x^2 p \right) \eta \\ &\quad + s^9 \int_0^T \int_\omega e^{-2s(5\hat{\alpha}-4\hat{\alpha})} \xi^9 \partial_x \zeta \bar{u} p \eta =: \sum_{i=1}^7 \mathcal{J}_i. \end{aligned}$$

Using the smoothness of the function ζ and Young's inequality, one can easily estimate the terms \mathcal{J}_i as listed below

- $|\mathcal{J}_1| = \left| \int_0^T \int_\omega e^{-2s(5\hat{\alpha}-4\hat{\alpha})} s^9 \xi^9 \zeta \left(-\sum_{i=1}^2 \frac{\nu_i}{\mu_i} \eta \chi_{\mathcal{O}_i} + h \right) \eta \right|$
 $\leq C \left(\int_0^T \int_\omega e^{-2s(5\hat{\alpha}-4\hat{\alpha})} s^9 \xi^9 |\eta|^2 + \int_0^T \int_\omega e^{-2s(5\hat{\alpha}-4\hat{\alpha})} s^9 \xi^9 |h|^2 \right)$
 $\leq C \left(\int_0^T \int_\omega e^{-2s(10\hat{\alpha}-9\hat{\alpha})} s^9 \xi^9 |\eta|^2 + \int_0^T \int_\omega e^{-2s(10\hat{\alpha}-9\hat{\alpha})} s^9 \xi^9 |h|^2 \right).$
- $|\mathcal{J}_2| \leq C \int_0^T \int_\omega e^{-2s(5\hat{\alpha}-4\hat{\alpha})} s^{10} \xi^{11} |\eta| |p|$
 $\leq \epsilon \int_0^T \int_\omega e^{-2s\hat{\alpha}} s^9 \xi^9 |p|^2 + C_\epsilon \int_0^T \int_\omega e^{-2s(10\hat{\alpha}-9\hat{\alpha})} s^{11} \xi^{13} |\eta|^2.$
- $|\mathcal{J}_3| \leq C \int_0^T \int_\omega e^{-2s(5\hat{\alpha}-4\hat{\alpha})} s^9 \xi^9 |h^0| |p|$
 $\leq \epsilon \int_0^T \int_\omega e^{-2s\hat{\alpha}} s^9 \xi^9 |p|^2 + C_\epsilon \int_0^T \int_\omega e^{-2s(10\hat{\alpha}-9\hat{\alpha})} s^9 \xi^9 |h^0|^2.$
- $|\mathcal{J}_4| \leq C \int_0^T \int_\omega e^{-2s(5\hat{\alpha}-4\hat{\alpha})} s^9 \xi^9 \left(|p| + |\partial_x p| + |\partial_x^2 p| + |\partial_x^3 p| + |\partial_x^4 p| \right) |\eta|$
 $\leq \epsilon \mathcal{I}(s, p) + C_\epsilon \int_0^T \int_\omega e^{-2s(10\hat{\alpha}-9\hat{\alpha})} s^{17} \xi^{17} |\eta|^2.$
- $|\mathcal{J}_5| \leq \epsilon \mathcal{I}(s, p) + C_\epsilon \int_0^T \int_\omega e^{-2s(10\hat{\alpha}-9\hat{\alpha})} s^{15} \xi^{15} |\eta|^2.$
- $|\mathcal{J}_6| \leq \epsilon \mathcal{I}(s, p) + C_\epsilon \int_0^T \int_\omega e^{-2s(10\hat{\alpha}-9\hat{\alpha})} s^{13} \xi^{13} |\eta|^2.$
- $|\mathcal{J}_7| \leq C \|\bar{u}\|_\infty \int_0^T \int_\omega e^{-2s(5\hat{\alpha}-4\hat{\alpha})} s^9 \xi^9 |\psi^i| |\eta|$
 $\leq \epsilon \mathcal{I}(s, p) + C_\epsilon \int_0^T \int_\omega e^{-2s(10\hat{\alpha}-9\hat{\alpha})} s^9 \xi^9 |\eta|^2.$

On combining these estimates, one can conclude the following estimate

$$s^9 \int_0^T \xi^9 e^{-2s(5\hat{\alpha}-4\hat{\alpha})} \|p\|_{L^2(\omega)}^2 \leq C \left(\epsilon \mathcal{I}(s, p) + \int_0^T \int_\omega e^{-2s(10\hat{\alpha}-9\hat{\alpha})} s^{17} \xi^{17} |\eta|^2 + \int_0^T \int_\omega e^{-2s(10\hat{\alpha}-9\hat{\alpha})} s^9 \xi^9 \left(|h^0|^2 + |h|^2 \right) \right). \quad (5.4)$$

Also, note that

$$e^{-2s\hat{\alpha}} = e^{-2s(10\hat{\alpha}-9\hat{\alpha})} e^{2s(10\hat{\alpha}-9\hat{\alpha})} e^{-2s\hat{\alpha}} \leq e^{-2s(10\hat{\alpha}-9\hat{\alpha})},$$

$$e^{-2s\hat{\alpha}} = e^{-2s(10\hat{\alpha}-9\hat{\alpha})} e^{2s(10\hat{\alpha}-9\hat{\alpha})} e^{-2s\hat{\alpha}} \leq e^{-2s(10\hat{\alpha}-9\hat{\alpha})}.$$

Finally, substituting these estimate in right hand side of (5.3) and choosing ϵ sufficiently small, we obtain

$$\mathcal{I}(s, \eta) + \mathcal{I}(s, p) \leq C s^{17} \int_0^T \xi^{17} e^{-2s(10\hat{\alpha}-9\hat{\alpha})} \|\eta\|_{L^2(\omega)}^2 + C s^9 \int_0^T \xi^9 e^{-2s(10\hat{\alpha}-9\hat{\alpha})} \left(\|h\|_{H^1(0,L)}^2 + \|h^0\|_{H^1(0,L)}^2 \right).$$

This completes the proof of [Proposition 5.1](#). □

Case-1.16: $\mathcal{O}_{1,d} \cap \mathcal{O}_0 \neq \mathcal{O}_{2,d} \cap \mathcal{O}_0$

To deal with this case, we further split it into the following two subcases:

$$\mathcal{O}_{i,d} \cap \mathcal{O}_0 \not\subseteq \mathcal{O}_{j,d} \cap \mathcal{O}_0, \forall (i, j) \in \{(1, 2), (2, 1)\}. \quad (5.5)$$

$$\exists (i, j) \in \{(1, 2), (2, 1)\} : \mathcal{O}_{i,d} \cap \mathcal{O}_0 \subset \mathcal{O}_{j,d} \cap \mathcal{O}_0. \quad (5.6)$$

Let us mention a lemma from [AAS23] concerning weight functions, which is crucial for clubbing the individual Carleman estimates of the states η, ψ^1 and ψ^2 .

Lemma 5.2 (see Lemma 4.3, [AAS23]). *Let $\omega_1 = (a_1, b_1)$ and $\omega_2 = (a_2, b_2)$ be open subset of $[0, L]$ such that $b_1 < a_2$ and let \mathcal{O} be an open interval such that $\omega_1 \cup \omega_2 \subset \mathcal{O}$. Then there exists two non-negative function φ_1 and φ_2 in $\mathcal{C}^\infty(\mathbb{R})$, satisfying conditions of the weight function of the Carleman for ω_1 and ω_2 , respectively as mentioned in Section 4 such that $\varphi_1 = \varphi_2$ in $[0, L] \setminus \mathcal{O}$ and $\|\varphi_1\|_\infty = \|\varphi_2\|_\infty$.*

Since the non-empty sets $\mathcal{O}_{i,d} \cap \mathcal{O}_0$ for $i \in \{1, 2\}$ satisfy $\mathcal{O}_{1,d} \cap \mathcal{O}_0 \neq \mathcal{O}_{2,d} \cap \mathcal{O}_0$, so we can choose non-empty open sets $\omega_i \subset \mathcal{O}_{i,d} \cap \mathcal{O}_0$ for $i \in \{1, 2\}$ such that $\omega_1 \cap \omega_2 = \emptyset$. In addition, depending on the whether the sets $\mathcal{O}_{i,d} \cap \mathcal{O}_0$ satisfy (5.5) or (5.6), ω_i can be chosen to have the following properties:

- For the subcase (5.5),

$$\omega_i \cap \mathcal{O}_{j,d} = \emptyset \quad \forall (i, j) \in \{(1, 2), (2, 1)\}. \quad (5.7)$$

- For the subcase (5.6),

$$\omega_j \cap \mathcal{O}_{i,d} = \emptyset \quad \text{for } (i, j) = (1, 2) \text{ or } (i, j) = (2, 1). \quad (5.8)$$

Let $\tilde{\mathcal{O}}_0$ be a nonempty open subsets of the control region \mathcal{O}_0 such that

$$\omega_1 \cup \omega_2 \subset \subset \tilde{\mathcal{O}}_0 \subset \subset \mathcal{O}_0.$$

Thanks to Lemma 5.2, we consider the functions φ_1, φ_2 associated to the sets ω_1, ω_2 and for $\mathcal{O} = \tilde{\mathcal{O}}_0$, and thus define the weight functions α_i as

$$\alpha_i(t, x) = \varphi_i(x) \xi(t), \text{ for } i \in \{1, 2\},$$

where ξ is given by (4.3). We denote the maximum and minimum of α_i over $[0, L]$ as

$$\alpha^*(t) = \max_{x \in [0, L]} \alpha_1(t, x) = \max_{x \in [0, L]} \alpha_2(t, x), \quad \text{and} \quad \alpha_*(t) = \min_{x \in [0, L]} \alpha_1(t, x) = \min_{x \in [0, L]} \alpha_2(t, x).$$

For $i \in \{1, 2\}$, we introduce the shorthand notation

$$\mathcal{I}_i(s, w) := \iint_Q \left(s^9 \xi^9 |w|^2 + s^7 \xi^7 |\partial_x w|^2 + s^5 \xi^5 |\partial_x^2 w|^2 + s^3 \xi^3 |\partial_x^3 w|^2 + s \xi |\partial_x^4 w|^2 \right) e^{-2s\alpha_i}. \quad (5.9)$$

Then, we obtain the following Carleman estimates for the system (1.18) corresponding to the subcases (5.5) and (5.6).

Proposition 5.3. *Let $\bar{u} \in X_T^3$, and assume the constants μ_i to be large enough and $\mathcal{O}_{i,d} \cap \mathcal{O}_0 \neq \emptyset$. Then, there exists $C, s_0 > 0$ such that for $s \geq s_0$, the solution (η, ψ_1, ψ_2) of (1.18) with $(\eta_T, h^0, h^1, h^2) \in H^3(0, L) \times [L^2(0, T; H^1(0, L))]^3$ satisfies the following:*

- For the subcase (5.5), we have

$$\begin{aligned} & \mathcal{I}_1(s, \eta) + \mathcal{I}_1(s, \psi^1) + \mathcal{I}_2(s, \psi^2) \\ & \leq C \left(s^{17} \int_0^T \int_{\mathcal{O}_0} e^{-2s(10\alpha_* - 9\alpha^*)} \xi^{17} |\eta|^2 + s^9 \int_0^T e^{-2s(10\alpha_* - 9\alpha^*)} \xi^9 (\|h^0\|_{H^1}^2 + \|h^1\|_{H^1}^2 + \|h^2\|_{H^1}^2) \right). \end{aligned} \quad (5.10)$$

- For the subcase (5.6), we have

$$\begin{aligned} & \mathcal{I}_i(s, \eta) + \mathcal{I}_j(s, \psi^j) + \mathcal{I}_i(s, p) \\ & \leq C \left(s^{17} \int_0^T \int_{\mathcal{O}_0} e^{-2s(10\alpha_* - 9\alpha^*)} \xi^{17} |\eta|^2 + s^9 \int_0^T e^{-2s(10\alpha_* - 9\alpha^*)} \xi^9 (\|h^0\|_{H^1}^2 + \|h^1\|_{H^1}^2 + \|h^2\|_{H^1}^2) \right), \end{aligned} \quad (5.11)$$

where $p = \nu_1 \psi^1 + \nu_2 \psi^2$.

Proof. Subcase (5.5): Consider a function $\zeta^0 \in \mathcal{C}^5([0, L])$ such that

$$\zeta^0(x) = \begin{cases} 0, & \text{for } x \in \tilde{\mathcal{O}}_0, \\ 1, & \text{for } x \in \mathcal{O}_0^c. \end{cases}$$

Note that $\eta_0 = \zeta^0 \eta$ satisfies the following system:

$$\begin{cases} -\partial_t \eta_0 - \partial_x^5 \eta_0 + \mu_0 \partial_x^4 \eta_0 - \partial_x^3 \eta_0 - \bar{u} \partial_x \eta_0 = \sum_{i=1}^2 \zeta^0 \nu_i \psi^i \chi_{\mathcal{O}_{i,d}} + \zeta^0 h^0 + R(\eta), & (t, x) \in Q, \\ \eta_0(t, 0) = \eta_0(t, L) = \partial_x \eta_0(t, 0) = \partial_x \eta_0(t, L) = 0, \quad \partial_x^2 \eta_0(t, L) = \partial_x^2 \eta_0(t, 0), & t \in (0, T), \\ \eta_0(T, x) = \zeta^0(x) \eta_T(x), & x \in (0, L), \end{cases}$$

where

$$\begin{aligned} R(\eta) = & -(\partial_x^5 \zeta^0 \eta + \partial_x^4 \zeta^0 \partial_x \eta + \partial_x^3 \zeta^0 \partial_x^2 \eta + \partial_x^2 \zeta^0 \partial_x^3 \eta + \partial_x \zeta^0 \partial_x^4 \eta) - \bar{u} \partial_x \zeta^0 \eta \\ & + \mu_0 (\partial_x^4 \zeta^0 \eta + \partial_x^3 \zeta^0 \partial_x \eta + \partial_x^2 \zeta^0 \partial_x^2 \eta + \partial_x \zeta^0 \partial_x^3 \eta) - (\partial_x^3 \zeta^0 \eta + \partial_x^2 \zeta^0 \partial_x \eta + \partial_x \zeta^0 \partial_x^2 \eta). \end{aligned}$$

For $i \in \{1, 2\}$, let $\tilde{\omega}_i \subset\subset \omega_i$ be any nonempty open set. Then using the Carleman estimate (4.14) for η_0 with weight function $\alpha = \alpha_1$ and $\omega = \tilde{\omega}_1$, we have

$$\begin{aligned} \mathcal{I}_1(s, \eta_0) \leq & C s^9 \int_0^T \xi^9 e^{-2s(5\alpha_* - 4\alpha^*)} \|\eta_0\|_{L^2(\tilde{\omega}_1)}^2 + C \iint_Q e^{-2s\alpha_1} \left(\sum_{i=1}^2 |\zeta^0 \nu_i \chi_{\mathcal{O}_{i,d}} \psi^i|^2 + |\zeta^0 h^0|^2 + |R(\eta)|^2 \right) \\ & + C \int_0^T e^{-2s\alpha^*} \xi \left(\sum_{i=1}^2 \|\zeta^0 \nu_i \chi_{\mathcal{O}_{i,d}} \psi^i\|_{H^1(0,L)}^2 + \|\zeta^0 h^0\|_{H^1(0,L)}^2 + \|R(\eta)\|_{H^1(0,L)}^2 \right). \end{aligned}$$

Next, note that

$$\begin{aligned} \mathcal{I}_1(s, \eta_0) &= \int_0^T \int_0^L (s^9 \xi^9 |\eta_0|^2 + s^7 \xi^7 |\partial_x \eta_0|^2 + s^5 \xi^5 |\partial_x^2 \eta_0|^2 + s^3 \xi^3 |\partial_x^3 \eta_0|^2 + s \xi |\partial_x^4 \eta_0|^2) e^{-2s\alpha_1} \\ &= \int_0^T \int_{\mathcal{O}_0} (s^9 \xi^9 |\eta_0|^2 + s^7 \xi^7 |\partial_x \eta_0|^2 + s^5 \xi^5 |\partial_x^2 \eta_0|^2 + s^3 \xi^3 |\partial_x^3 \eta_0|^2 + s \xi |\partial_x^4 \eta_0|^2) e^{-2s\alpha_1} \\ &\quad + \int_0^T \int_{\mathcal{O}_0^c} (s^9 \xi^9 |\eta|^2 + s^7 \xi^7 |\partial_x \eta|^2 + s^5 \xi^5 |\partial_x^2 \eta|^2 + s^3 \xi^3 |\partial_x^3 \eta|^2 + s \xi |\partial_x^4 \eta|^2) e^{-2s\alpha_1} \\ &\geq \mathcal{I}_1(s, \eta) - \int_0^T \int_{\mathcal{O}_0} (s^9 \xi^9 |\eta|^2 + s^7 \xi^7 |\partial_x \eta|^2 + s^5 \xi^5 |\partial_x^2 \eta|^2 + s^3 \xi^3 |\partial_x^3 \eta|^2 + s \xi |\partial_x^4 \eta|^2) e^{-2s\alpha_1}. \end{aligned}$$

Thus combining the last two inequalities, we obtain

$$\begin{aligned} \mathcal{I}_1(s, \eta) \leq & C \iint_Q e^{-2s\alpha_1} \left(\sum_{i=1}^2 |\zeta^0 \nu_i \chi_{\mathcal{O}_{i,d}} \psi^i|^2 + |\zeta^0 h^0|^2 + |R(\eta)|^2 \right) \\ & + C \int_0^T e^{-2s\alpha^*} \xi \left(\sum_{i=1}^2 \|\zeta^0 \nu_i \chi_{\mathcal{O}_{i,d}} \psi^i\|_{H^1(0,L)}^2 + \|\zeta^0 h^0\|_{H^1(0,L)}^2 + \|R(\eta)\|_{H^1(0,L)}^2 \right) \\ & + \int_0^T \int_{\mathcal{O}_0} (s^9 \xi^9 |\eta|^2 + s^7 \xi^7 |\partial_x \eta|^2 + s^5 \xi^5 |\partial_x^2 \eta|^2 + s^3 \xi^3 |\partial_x^3 \eta|^2 + s \xi |\partial_x^4 \eta|^2) e^{-2s\alpha_1}. \quad (5.12) \end{aligned}$$

Following the same line of argument as used in the proof of Proposition 4.2, one can handle the local terms involving $\partial_x^k \eta$ for $k \in \{1, 2, 3, 4\}$ from the right hand side of (5.12) as sketched below:

- First we use Sobolev interpolation estimates and Young's inequality to get

$$\int_0^T \int_{\mathcal{O}_0} s^7 \xi^7 e^{-2s\alpha_1} |\partial_x \eta|^2 \leq C \int_0^T \xi e^{-2s\alpha^*} \|\eta\|_{H^5(\mathcal{O}_0)}^2 + C s^{\frac{35}{4}} \int_0^T \xi^{\frac{17}{2}} e^{s(-\frac{5}{2}\alpha_* + \frac{\alpha^*}{2})} \|\eta\|_{L^2(\mathcal{O}_0)}^2, \quad (5.13)$$

$$\int_0^T \int_{\mathcal{O}_0} s^5 \xi^5 e^{-2s\alpha_1} |\partial_x^2 \eta|^2 \leq C \int_0^T \xi e^{-2s\alpha^*} \|\eta\|_{H^5(\mathcal{O}_0)}^2 + C s^{\frac{25}{3}} \int_0^T \xi^{\frac{23}{3}} e^{s(-\frac{10}{3}\alpha_* + \frac{4}{3}\alpha^*)} \|\eta\|_{L^2(\mathcal{O}_0)}^2, \quad (5.14)$$

$$\int_0^T \int_{\mathcal{O}_0} s^3 \xi^3 e^{-2s\alpha_1} |\partial_x^3 \eta|^2 \leq C \int_0^T \xi e^{-2s\alpha^*} \|\eta\|_{H^5(\mathcal{O}_0)}^2 + C s^{\frac{15}{2}} \int_0^T \xi^6 e^{s(-5\alpha_* + 3\alpha^*)} \|\eta\|_{L^2(\mathcal{O}_0)}^2, \quad (5.15)$$

$$\int_0^T \int_{\mathcal{O}_0} s \xi e^{-2s\alpha_1} |\partial_x^4 \eta|^2 \leq C \int_0^T \xi e^{-2s\alpha^*} \|\eta\|_{H^5(\mathcal{O}_0)}^2 + C s^5 \int_0^T \xi e^{s(-10\alpha_* + 8\alpha^*)} \|\eta\|_{L^2(\mathcal{O}_0)}^2. \quad (5.16)$$

- Next, let $\tilde{\rho}(t) = e^{-s\alpha^*}(t)\xi^{\frac{1}{2}}(t)$, and define $\tilde{\eta} = \tilde{\rho}\eta$. Then, $\tilde{\eta}$ solves the following system

$$\begin{cases} -\partial_t \tilde{\eta} - \partial_x^5 \tilde{\eta} + \mu_0 \partial_x^4 \tilde{w} - \partial_x^3 \tilde{\eta} - \bar{u} \partial_x \tilde{\eta} = \tilde{h}^0, & (t, x) \in Q, \\ \tilde{\eta}(t, 0) = \tilde{\eta}(t, L) = \partial_x \tilde{\eta}(t, 0) = \partial_x \tilde{\eta}(t, L) = 0, \quad \partial_x^2 \tilde{\eta}(t, L) = \partial_x^2 \tilde{\eta}(t, 0), & t \in (0, T), \\ \tilde{\eta}(T, x) = 0, & x \in (0, L), \end{cases}$$

where $\tilde{h}^0 = \tilde{\rho} \left(\sum_{i=1}^2 \nu_i \psi^i \chi_{\mathcal{O}_i, d} + h^0 \right) - \partial_t \tilde{\rho} \eta$. Using $|\partial_t \tilde{\rho}| \leq C s \xi^{\frac{5}{2}} e^{-s\alpha^*}$, we have

$$\begin{aligned} \|\tilde{h}^0\|_{L^2(0, T; H^1(0, L))}^2 &\leq C \left(\int_0^T e^{-2s\alpha^*} \xi \|h^0\|_{H^1(0, L)}^2 + \sum_{i=1}^2 \int_0^T e^{-2s\alpha^*} \xi \|\nu_i \psi^i \chi_{\mathcal{O}_i, d}\|_{H^1(0, L)}^2 \right. \\ &\quad \left. + s^2 \int_0^T e^{-2s\alpha^*} \xi^5 \|\eta\|_{H^1(0, L)}^2 \right). \end{aligned}$$

Thus, due to the regularity result [Proposition 2.6](#), the solution $\tilde{\eta} \in L^2(0, T; H^5(0, L))$ satisfies

$$\begin{aligned} \int_0^T e^{-2s\alpha^*} \xi \|\eta\|_{H^5(0, L)}^2 &\leq C \left(\int_0^T e^{-2s\alpha^*} \xi \|h^0\|_{H^1(0, L)}^2 + \sum_{i=1}^2 \int_0^T e^{-2s\alpha^*} \xi \|\nu_i \chi_{\mathcal{O}_i, d} \psi^i\|_{H^1(0, L)}^2 \right. \\ &\quad \left. + s^2 \int_0^T e^{-2s\alpha^*} \xi^5 \|\eta\|_{H^1(0, L)}^2 \right). \quad (5.17) \end{aligned}$$

Using the definition of ζ^0 , we have

$$\iint_Q e^{-2s\alpha_i} |\zeta^0 \nu_i \chi_{\mathcal{O}_i, d} \psi^i|^2 = \int_0^T \int_{\tilde{\mathcal{O}}_i^c} e^{-2s\alpha_i} |\zeta^0 \nu_i \chi_{\mathcal{O}_i, d} \psi^i|^2, \quad \text{for } i \in \{1, 2\}. \quad (5.18)$$

Let $\epsilon > 0$ be any number sufficiently small. Then, since $\zeta^0 \in C^5([0, L])$, one can choose s large enough such that

$$\iint_Q e^{-2s\alpha_i} |\zeta^0 \nu_i \chi_{\mathcal{O}_i, d} \psi^i|^2 + \int_0^T e^{-2s\alpha^*} \xi \|\zeta^0 \nu_i \chi_{\mathcal{O}_i, d} \psi^i\|_{H^1(0, L)}^2 \leq \epsilon \mathcal{I}_i(s, \psi^i), \quad \text{for } i \in \{1, 2\}, \quad (5.19)$$

$$\iint_Q e^{-2s\alpha_i} |R(\eta)|^2 + \int_0^T e^{-2s\alpha^*} \xi \|R(\eta)\|_{H^1(0, L)}^2 \leq \epsilon \mathcal{I}_1(s, \eta) + \int_0^T e^{-2s\alpha^*} \xi \|\eta\|_{H^5(0, L)}^2. \quad (5.20)$$

Thus, using the estimates (5.13)-(5.20) and continuity of ζ^0 in (5.12), we get

$$\begin{aligned} \mathcal{I}_1(s, \eta) &\leq C \left(\iint_Q e^{-2s\alpha_i} |h^0|^2 + \int_0^T e^{-2s\alpha^*} \xi \|h^0\|_{H^1(0, L)}^2 + s^9 \int_0^T \xi^9 e^{-2s(5\alpha_* - 4\alpha^*)} \|\eta\|_{L^2(\mathcal{O}_0)}^2 \right. \\ &\quad \left. + \int_0^T e^{-2s\alpha^*} s^2 \xi^5 \|\eta\|_{H^1(0, L)}^2 + \epsilon \mathcal{I}_1(s, \psi^1) + \epsilon \mathcal{I}_2(s, \psi^2) + \epsilon \mathcal{I}_1(s, \eta) \right). \end{aligned}$$

Finally, for sufficiently large s one can absorb the term containing $\|\eta\|_{H^1(0, L)}$ from the right side to get

$$\begin{aligned} \mathcal{I}_1(s, \eta) &\leq C \left(\iint_Q e^{-2s\alpha_i} |h^0|^2 + \int_0^T e^{-2s\alpha^*} \xi \|h^0\|_{H^1(0, L)}^2 + s^9 \int_0^T \xi^9 e^{-2s(5\alpha_* - 4\alpha^*)} \|\eta\|_{L^2(\mathcal{O}_0)}^2 \right. \\ &\quad \left. + \epsilon \mathcal{I}_1(s, \psi^1) + \epsilon \mathcal{I}_2(s, \psi^2) + \epsilon \mathcal{I}_1(s, \eta) \right). \quad (5.21) \end{aligned}$$

Next, the Carleman estimate (4.14) for ψ^i with weight function $\alpha = \alpha_i$ and $\omega = \tilde{\omega}_i$ gives

$$\begin{aligned} \mathcal{I}_i(s, \psi^i) &\leq C \iint_Q e^{-2s\alpha_i} \left(\left| \frac{1}{\mu_i} \eta \chi_{\mathcal{O}_i} \right|^2 + |h^i|^2 \right) + C \int_0^T e^{-2s\alpha^*} \xi \left(\left\| \frac{1}{\mu_i} \eta \chi_{\mathcal{O}_i} \right\|_{H^1(0, L)}^2 + \|h^i\|_{H^1(0, L)}^2 \right) \\ &\quad + C \iint_{Q_{\tilde{\omega}_i}} e^{-2s\alpha_i} s^9 \xi^9 |\psi^i|^2. \quad (5.22) \end{aligned}$$

Note that as $\mathcal{O}_i \cap \mathcal{O}_0 = \emptyset$ for $i \in \{1, 2\}$, so $\varphi_1 = \varphi_2$ in $\mathcal{O}_1 \cup \mathcal{O}_2$, and hence we have

$$\iint_Q e^{-2s\alpha_i} \left| \frac{1}{\mu_i} \eta \chi_{\mathcal{O}_i} \right|^2 + \int_0^T e^{-2s\alpha^*} \xi \left\| \frac{1}{\mu_i} \eta \chi_{\mathcal{O}_i} \right\|_{H^1(0, L)}^2 \leq C \iint_Q e^{-2s\alpha_i} \xi \left(|\eta|^2 + |\partial_x \eta|^2 \right), \quad \text{for } i \in \{1, 2\}.$$

Thus for sufficiently large s , we get

$$\iint_Q e^{-2s\alpha_i} \left| \frac{1}{\mu_i} \eta \chi_{\mathcal{O}_i} \right|^2 + \int_0^T e^{-2s\alpha^*} \xi \left\| \frac{1}{\mu_i} \eta \chi_{\mathcal{O}_i} \right\|_{H^1(0,L)}^2 \leq \epsilon \mathcal{I}(s, \eta),$$

and hence the inequality (5.22) becomes

$$\mathcal{I}_i(s, \psi^i) \leq C \left(\iint_Q e^{-2s\alpha_i} |h^i|^2 + \int_0^T e^{-2s\alpha^*} \xi \|h^i\|_{H^1(0,L)}^2 + \iint_{Q_{\tilde{\omega}_i}} e^{-2s\alpha_i} s^9 \xi^9 |\psi^i|^2 + \epsilon \mathcal{I}(s, \eta) \right). \quad (5.23)$$

Amalgamating the estimates (5.21) and (5.23) for sufficiently small ϵ , we get

$$\begin{aligned} \mathcal{I}(s, \eta) + \mathcal{I}_1(s, \psi^1) + \mathcal{I}_2(s, \psi^2) &\leq C s^9 \int_0^T e^{-2s(5\alpha_* - 4\alpha^*)} \xi^9 \left(\|\eta\|_{L^2(\mathcal{O}_0)}^2 + \sum_{i=1}^2 \|\psi^i\|_{L^2(\tilde{\omega}_i)}^2 \right) \\ &\quad + C \sum_{i=0}^2 \left(\int_0^T e^{-2s\alpha_i} |h^i|^2 + \int_0^T e^{-2s\alpha^*} \xi \|h^i\|_{H^1(0,L)}^2 \right), \end{aligned} \quad (5.24)$$

where $s \geq s_0$ for sufficiently large $s_0 > 0$.

Our next goal is to remove the local term of ψ^i , which can be done using arguments similar to the one used while proving [Proposition 5.1](#). So for $i \in \{1, 2\}$, let $\zeta^i \in \mathcal{C}_c^\infty(\omega_i)$ such that $\zeta^i(x) = 1$ for $x \in \tilde{\omega}_i$ and $0 \leq \zeta^i \leq 1$. Using the differential equation and the property (5.7) of ω_i , we have

$$-\partial_t \eta - \partial_x^5 \eta + \mu_0 \partial_x^4 \eta - \partial_x^3 \eta - \bar{u} \eta_x - h^0 = \nu_i \psi^i \quad \text{in } (0, T) \times \omega_i, \quad i \in \{1, 2\}.$$

Thus, using the above expression of ψ^i , we have

$$\begin{aligned} s^9 \int_0^T \int_{\tilde{\omega}_i} e^{-2s(5\alpha_* - 4\alpha^*)} \xi^9 |\psi^i|^2 &\leq s^9 \int_0^T \int_{\omega_i} e^{-2s(5\alpha_* - 4\alpha^*)} \xi^9 \zeta^i |\psi^i|^2 \\ &= \frac{1}{\nu_i} s^9 \int_0^T \int_{\omega_i} e^{-2s(5\alpha_* - 4\alpha^*)} \xi^9 \zeta^i (-\partial_t \eta - \partial_x^5 \eta + \mu_0 \partial_x^4 \eta - \partial_x^3 \eta - \bar{u} \eta_x - h^0) \psi^i \\ &= \frac{1}{\nu_i} s^9 \int_0^T \int_{\omega_i} e^{-2s(5\alpha_* - 4\alpha^*)} \xi^9 \zeta^i (\partial_t \psi^i + \partial_x^5 \psi^i + \mu_0 \partial_x^4 \psi^i + \partial_x^3 \psi^i + \partial_x (\bar{u} \psi^i)) \eta \\ &\quad + \frac{1}{\nu_i} s^9 \int_0^T \int_{\omega_i} \partial_t \left(e^{-2s(5\alpha_* - 4\alpha^*)} \xi^9 \right) \eta \psi^i - s^9 \frac{1}{\nu_i} \int_0^T \int_{\omega_i} e^{-2s(5\alpha_* - 4\alpha^*)} \xi^9 h^0 \psi^i \\ &\quad + \frac{1}{\nu_i} s^9 \int_0^T \int_{\omega_i} e^{-2s(5\alpha_* - 4\alpha^*)} \xi^9 (\partial_x^5 \zeta^i \psi^i + \partial_x^4 \zeta^i \partial_x \psi^i + \partial_x^3 \zeta^i \partial_x^2 \psi^i + \partial_x^2 \zeta^i \partial_x^3 \psi^i + \partial_x \zeta^i \partial_x^4 \psi^i) \eta \\ &\quad + \frac{\mu_0}{\nu_i} s^9 \int_0^T \int_{\omega_i} e^{-2s(5\alpha_* - 4\alpha^*)} \xi^9 (\partial_x^4 \zeta^i \psi^i + \partial_x^3 \zeta^i \partial_x \psi^i + \partial_x^2 \zeta^i \partial_x^2 \psi^i + \partial_x \zeta^i \partial_x^3 \psi^i) \eta \\ &\quad + \frac{1}{\nu_i} s^9 \int_0^T \int_{\omega_i} e^{-2s(5\alpha_* - 4\alpha^*)} \xi^9 (\partial_x^3 \zeta^i \psi^i + \partial_x^2 \zeta^i \partial_x \psi^i + \partial_x \zeta^i \partial_x^2 \psi^i) \eta \\ &\quad + \frac{1}{\nu_i} s^9 \int_0^T \int_{\omega_i} e^{-2s(5\alpha_* - 4\alpha^*)} \xi^9 (\partial_x \zeta^i) \bar{u} \psi^i \eta \\ &=: \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4 + \mathcal{T}_5 + \mathcal{T}_6 + \mathcal{T}_7. \end{aligned}$$

Following the arguments used to estimate \mathcal{J}_i 's in the proof of [Proposition 5.1](#), one can estimate these \mathcal{T}_i 's similarly to finally deduce the following estimate for sufficiently large s

$$\begin{aligned} \int_0^T \int_{\tilde{\omega}_i} e^{-2s(5\alpha_* - 4\alpha^*)} s^9 \xi^9 |\psi^i|^2 &\leq \epsilon \mathcal{I}_i(s, \psi^i) + C \int_0^T \int_{\omega_i} e^{-2s(10\alpha_* - 9\alpha^*)} s^{17} \xi^{17} |\eta|^2 \\ &\quad + C \int_0^T \int_{\omega_i} e^{-2s(10\alpha_* - 9\alpha^*)} s^9 \xi^9 (|h^0|^2 + |h^i|^2). \end{aligned}$$

Lastly, substituting this in the estimate (5.24), we deduce the desired Carleman estimate (5.10).

Subcase (5.6): In this case we have $\mathcal{O}_i \cap \mathcal{O}_0 \subset \mathcal{O}_j \cap \mathcal{O}_0$, for any one $i \in \{1, 2\}$ and $j \neq i$. On adding and subtracting $\nu_j \psi^j \chi_{\mathcal{O}_{i,d}}$ in the right-hand side of the first differential equation of the system (1.18), we get

$$-\partial_t \eta - \partial_x^5 \eta + \mu_0 \partial_x^4 \eta - \partial_x^3 \eta - \bar{u} \eta_x = \chi_{\mathcal{O}_{i,d}} (\nu_i \psi^i + \nu_j \psi^j) + (\chi_{\mathcal{O}_{j,d}} - \chi_{\mathcal{O}_{i,d}}) \nu_j \psi^j + h^0.$$

Set $p = \nu_1 \psi^1 + \nu_2 \psi^2$. Then, the adjoint system (1.18) can be rewritten as

$$\begin{cases} -\partial_t \eta - \partial_x^5 \eta + \mu_0 \partial_x^4 \eta - \partial_x^3 \eta - \bar{u} \partial_x \eta = p \chi_{\mathcal{O}_{i,d}} + \nu_j \psi^j (\chi_{\mathcal{O}_{j,d}} - \chi_{\mathcal{O}_{i,d}}) + h^0, & (t, x) \in Q, \\ \partial_t \psi^j + \partial_x^5 \psi^j + \mu_0 \partial_x^4 \psi^j + \partial_x^3 \psi^j + \partial_x (\bar{u} \psi^j) = -\frac{1}{\mu_j} \eta \chi_{\mathcal{O}_j} + h^j, & (t, x) \in Q, \\ \partial_t p + \partial_x^5 p + \mu_0 \partial_x^4 p + \partial_x^3 p + \partial_x (\bar{u} p) = -\sum_{k=1}^2 \frac{\nu_k}{\mu_k} \eta \chi_{\mathcal{O}_k} + \sum_{k=1}^2 \nu_k h^k, & (t, x) \in Q, \\ \eta(t, 0) = \eta(t, L) = \partial_x \eta(t, 0) = \partial_x \eta(t, L) = 0, \quad \partial_x^2 \eta(t, L) = \partial_x^2 \eta(t, 0), & t \in (0, T), \\ \psi^j(t, 0) = \psi^j(t, L) = \partial_x \psi^j(t, 0) = \partial_x \psi^j(t, L) = 0, \quad \partial_x^2 \psi^j(t, L) = \partial_x^2 \psi^j(t, 0), & t \in (0, T), \\ p(t, 0) = p(t, L) = \partial_x p(t, 0) = \partial_x p(t, L) = 0, \quad \partial_x^2 p(t, L) = \partial_x^2 p(t, 0), & t \in (0, T), \\ \eta(T, x) = \eta_T(x), \psi^j(0, x) = 0, p(0, x) = 0, & x \in (0, L). \end{cases} \quad (5.25)$$

Let $\epsilon > 0$ be any number, and let $\tilde{\omega}_k \subset \subset \omega_k$, for $k \in \{1, 2\}$. Then, one can follow the same argument as used for the subcase (5.5) to conclude the following inequality, which is analogue to (5.21):

$$\begin{aligned} \mathcal{I}_i(s, \eta) \leq C & \left(\iint_Q e^{-2s\alpha_i} |h^0|^2 + \int_0^T e^{-2s\alpha^*} \xi \|h^0\|_{H^1(0,L)}^2 + s^9 \int_0^T \xi^9 e^{-2s(5\alpha^* - 4\alpha^*)} \|\eta\|_{L^2(\mathcal{O}_0)}^2 \right. \\ & \left. + \epsilon \mathcal{I}_j(s, \psi^j) + \epsilon \mathcal{I}_i(s, p) + \epsilon \mathcal{I}_i(s, \eta) \right), \end{aligned} \quad (5.26)$$

for sufficiently large value of s . Next, we use the Carleman estimate (4.14) for ψ^j and p with the weight function α_j and α_i , and the local domain $\tilde{\omega}_j$ and $\tilde{\omega}_i$, respectively to obtain

$$\begin{aligned} \mathcal{I}_j(s, \psi^j) \leq C & \iint_{Q_{\tilde{\omega}_j}} e^{-2s\alpha_j} s^9 \xi^9 |\psi^j|^2 + C \iint_Q e^{-2s\alpha_j} \left(\left| \frac{1}{\mu_j} \eta \chi_{\mathcal{O}_j} \right|^2 + |h^j|^2 \right) \\ & + C \int_0^T e^{-2s\alpha^*} \xi \left(\left\| \frac{1}{\mu_j} \eta \chi_{\mathcal{O}_j} \right\|_{H^1(0,L)}^2 + \|h^j\|_{H^1(0,L)}^2 \right), \end{aligned} \quad (5.27)$$

$$\begin{aligned} \mathcal{I}_i(s, p) \leq C & \iint_{Q_{\tilde{\omega}_i}} e^{-2s\alpha_i} s^9 \xi^9 |p|^2 + C \iint_Q e^{-2s\alpha_i} \sum_{k=1}^2 \left(\left| \frac{\nu_k}{\mu_k} \eta \chi_{\mathcal{O}_k} \right|^2 + |h^k|^2 \right) \\ & + C \int_0^T e^{-2s\alpha^*} \xi \sum_{k=1}^2 \left(\left\| \frac{\nu_k}{\mu_k} \eta \chi_{\mathcal{O}_k} \right\|_{H^1(0,L)}^2 + \|h^k\|_{H^1(0,L)}^2 \right). \end{aligned} \quad (5.28)$$

Since $\mathcal{O}_j \cap \mathcal{O}_0 = \emptyset$ and $\varphi_1 = \varphi_2$ in $[0, L] \setminus \tilde{\mathcal{O}}_0 \supset [0, L] \setminus \mathcal{O}_0$, so $\varphi_1 = \varphi_2$ in $\mathcal{O}_1 \cup \mathcal{O}_2$. Thus, for sufficiently large s , we have the following estimate:

$$\begin{aligned} & \iint_Q e^{-2s\alpha_j} \left| \frac{1}{\mu_j} \eta \chi_{\mathcal{O}_j} \right|^2 + \int_0^T e^{-2s\alpha^*} \xi \left\| \frac{1}{\mu_j} \eta \chi_{\mathcal{O}_j} \right\|_{H^1(0,L)}^2 \leq \epsilon \mathcal{I}_i(s, \eta) \\ \text{and } & \iint_Q e^{-2s\alpha_i} \sum_{k=1}^2 \left| \frac{1}{\mu_k} \eta \chi_{\mathcal{O}_k} \right|^2 + \int_0^T e^{-2s\alpha^*} \xi \sum_{k=1}^2 \left\| \frac{1}{\mu_k} \eta \chi_{\mathcal{O}_k} \right\|_{H^1(0,L)}^2 \leq \epsilon \mathcal{I}_i(s, \eta). \end{aligned}$$

Due to these estimates, the inequalities (5.27) and (5.28) can be summed together to finally get

$$\begin{aligned} \mathcal{I}_j(s, \psi^j) + \mathcal{I}_i(s, p) \leq C & \left(\iint_{Q_{\tilde{\omega}_j}} e^{-2s\alpha_j} s^9 \xi^9 |\psi^j|^2 + \iint_{Q_{\tilde{\omega}_i}} e^{-2s\alpha_i} s^9 \xi^9 |p|^2 + \sum_{k=1}^2 \iint_Q e^{-2s\alpha^*} |h^k|^2 \right. \\ & \left. + \sum_{k=1}^2 \int_0^T e^{-2s\alpha^*} \xi \|h^k\|_{H^1(0,L)}^2 + \epsilon \mathcal{I}_i(s, \eta) \right), \end{aligned} \quad (5.29)$$

for sufficiently large s .

Upon adding the inequalities (5.26) and (5.29), and choosing ϵ sufficiently small, we deduce

$$\begin{aligned} \mathcal{I}_i(s, \eta) + \mathcal{I}_j(s, \psi^j) + \mathcal{I}_i(s, p) \leq C s^9 & \int_0^T \xi^9 e^{-2s(5\alpha^* - 4\alpha^*)} \left(\|\eta\|_{L^2(\mathcal{O}_0)}^2 + \|\psi^j\|_{L^2(\tilde{\omega}_j)}^2 + \|p\|_{L^2(\tilde{\omega}_i)}^2 \right) \\ & + C \sum_{k=0}^2 \left(\iint_Q e^{-2s\alpha^*} |h^k|^2 + \int_0^T e^{-2s\alpha^*} \xi \|h^k\|_{H^1(0,L)}^2 \right). \end{aligned} \quad (5.30)$$

In order to remove the local terms of ψ^j and p , we will use the equation of η along with the conditions (5.8) and (5.8). More precisely, note that restricting the differential equation of η from (5.25) over $(0, T) \times \omega_i$ and $(0, T) \times \omega_j$, we get

$$\begin{aligned} -\partial_t \eta - \partial_x^5 \eta + \mu_0 \partial_x^4 \eta - \partial_x^3 \eta - \bar{u} \eta_x &= p + h^0, & (t, x) \in (0, T) \times \omega_i, \\ -\partial_t \eta - \partial_x^5 \eta + \mu_0 \partial_x^4 \eta - \partial_x^3 \eta - \bar{u} \eta_x &= \nu_j \psi^j + h^0, & (t, x) \in (0, T) \times \omega_j. \end{aligned}$$

Using these equations, the local terms of ψ^j and p can be removed from right hand side of (5.30) as demonstrated in the proof for the subcase (5.5), and hence for sufficiently large s we obtain the desired estimate (5.11). This finishes the proof. \square

6. NULL CONTROLLABILITY OF THE LINEARIZED SYSTEM WITH SOURCE TERM

This section is dedicated for proving the partial null controllability of the linearized system (1.17) with source terms $\{g^i\}_{i=0}^2$ in some suitable weighted space, which decay exponentially to zero as t goes to T . This controllability result would be useful in the next section to conclude the main result, i.e., to obtain the partial null controllability result for the main non-linear system (1.13).

6.1. Observability inequality. In this subsection, we derive a suitable observability inequality which will be used to deduce the desired null controllability result for non-homogeneous linearized system (1.17).

Let us first derive a new Carleman estimate with slightly tweaked weight functions. Consider the function $l \in \mathcal{C}^1([0, T])$ given by

$$l(t) = \begin{cases} \frac{T^2}{4}, & t \in [0, \frac{T}{2}], \\ t(T-t), & t \in [\frac{T}{2}, T], \end{cases}$$

and define

$$\alpha(t, x) = \varphi(x)\beta(t), \quad \alpha_i(t, x) = \varphi_i(x)\beta(t),$$

where $\beta(t) = \frac{1}{l(t)}$, and $\varphi, \{\varphi_i\}_{i=1,2}$ are the weight functions used in the derivation of the Carleman estimates mentioned in Proposition 5.1, Proposition 5.3, respectively. Denote the minimum and maximum of these functions as

$$\begin{aligned} \hat{\alpha}(t) &:= \max_{[0, L]} \alpha(t, x), \quad \check{\alpha}(t) := \min_{[0, L]} \alpha(t, x), \quad \forall t \in (0, T), \\ \alpha^*(t) &:= \max_{[0, L]} \alpha_1(t, x) = \max_{[0, L]} \alpha_2(t, x), \quad \forall t \in (0, T), \\ \alpha_*(t) &:= \min_{[0, L]} \alpha_1(t, x) = \min_{[0, L]} \alpha_2(t, x), \quad \forall t \in (0, T). \end{aligned} \tag{6.1}$$

With this slightly modified weight functions, we obtain the following Carleman estimates.

Lemma 6.1. *Let $\bar{u} \in X_T^0$, and assume the constants μ_i to be large enough and $\mathcal{O}_{i,d} \cap \mathcal{O}_0 \neq \emptyset$. Then, there exists a constant $C > 0$ such that the solution (η, ψ_1, ψ_2) of (1.18) with $(\eta_T, h^0, h^1, h^2) \in L^2(0, L) \times [L^2(0, T; H^1(0, L))]^3$ satisfies the following for fixed $s \geq s_0$:*

- For the case (1.15), we have

$$\begin{aligned} &\int_0^L |\eta(0, x)|^2 + \iint_Q e^{-2s\alpha} \beta^9 (|\eta|^2 + |\nu_1 \psi^1 + \nu_2 \psi^2|^2) + \iint_Q e^{-2s\alpha} \beta^7 (|\partial_x \eta|^2 + |\nu_1 \partial_x \psi^1 + \nu_2 \partial_x \psi^2|^2) \\ &\leq C \int_0^T \int_{\mathcal{O}_0} \beta^{17} e^{-2s(10\check{\alpha} - 9\hat{\alpha})} |\eta|^2 + C \int_0^T \beta^9 e^{-2s(10\check{\alpha} - 9\hat{\alpha})} (\|\nu_1 h^1 + \nu_2 h^2\|_{H^1(0, L)}^2 + \|h^0\|_{H^1(0, L)}^2). \end{aligned} \tag{6.2}$$

- For the case (1.16)-(5.5), we have

$$\begin{aligned} &\int_0^L |\eta(0, x)|^2 + \iint_Q e^{-2s\alpha_1} (\beta^9 |\eta|^2 + \beta^7 |\partial_x \eta|^2) + \sum_{i=1}^2 \iint_Q e^{-2s\alpha_i} (\beta^9 |\psi^i|^2 + \beta^7 |\partial_x \psi^i|^2) \\ &\leq C \left(\int_0^T \int_{\mathcal{O}_0} e^{-2s(10\alpha_* - 9\alpha^*)} \beta^{17} |\eta|^2 + \sum_{i=0}^2 \int_0^T e^{-2s(10\alpha_* - 9\alpha^*)} \beta^9 \|h^i\|_{H^1}^2 \right). \end{aligned} \tag{6.3}$$

• For the case (1.16)-(5.6), we have

$$\begin{aligned} & \int_0^L |\eta(0, x)|^2 + \iint_Q e^{-2s\alpha_1} (\beta^9 |\eta|^2 + \beta^7 |\partial_x \eta|^2) + \iint_Q e^{-2s\alpha_j} (\beta^9 |\psi^j|^2 + \beta^7 |\partial_x \psi^j|^2) \\ & \quad + \iint_Q e^{-2s\alpha_j} (\beta^9 |\nu_1 \psi^1 + \nu_2 \psi^2|^2 + \beta^7 |\nu_1 \partial_x \psi^1 + \nu_2 \partial_x \psi^2|^2) \\ & \leq C \left(\int_0^T \int_{\mathcal{O}_0} e^{-2s(10\alpha_* - 9\alpha^*)} \beta^{17} |\eta|^2 + \sum_{i=0}^2 \int_0^T e^{-2s(10\alpha_* - 9\alpha^*)} \beta^9 \|h^i\|_{H^1}^2 \right). \end{aligned} \quad (6.4)$$

Proof. The proof of all the above estimates are similar, and can be easily deduced from the Carleman estimates (5.2), (5.10) and (5.11), using the properties of new weight functions. The proof is done by first assuming η_T and \bar{u} to be sufficiently smooth, and then using standard density argument one can conclude the desired result for $\eta_T \in L^2(0, L)$ and $\bar{u} \in X_T^0$. Here, we demonstrate the proof of (6.2).

Recall $p = \nu_1 \psi^1 + \nu_2 \psi^2$ and $h = \nu_1 h^1 + \nu_2 h^2$. As $\beta = \xi$ in $[\frac{T}{2}, T]$, we have

$$\begin{aligned} & \int_{\frac{T}{2}}^T \int_0^L e^{-2s\alpha} \xi^9 |\eta|^2 + \int_{\frac{T}{2}}^T \int_0^L e^{-2s\alpha} \xi^7 |\partial_x \eta|^2 + \int_{\frac{T}{2}}^T \int_0^L e^{-2s\alpha} \xi^9 |p|^2 + \int_{\frac{T}{2}}^T \int_0^L e^{-2s\alpha} \xi^7 |\partial_x p|^2 \\ & = \int_{\frac{T}{2}}^T \int_0^L e^{-2s\alpha} \beta^9 |\eta|^2 + \int_{\frac{T}{2}}^T \int_0^L e^{-2s\alpha} \beta^7 |\partial_x \eta|^2 + \int_{\frac{T}{2}}^T \int_0^L e^{-2s\alpha} \beta^9 |p|^2 + \int_{\frac{T}{2}}^T \int_0^L e^{-2s\alpha} \beta^7 |\partial_x p|^2. \end{aligned} \quad (6.5)$$

Thanks to the estimate (5.2), we have:

$$\begin{aligned} & \int_{\frac{T}{2}}^T \int_0^L e^{-2s\alpha} (\beta^9 |\eta|^2 + \beta^7 |\partial_x \eta|^2) + \int_{\frac{T}{2}}^T \int_0^L e^{-2s\alpha} (\beta^9 |p|^2 + \beta^7 |\partial_x p|^2) \\ & \leq C \left(\int_0^T \int_{\mathcal{O}_0} e^{-2s(10\hat{\alpha} - 9\hat{\alpha})} \xi^{17} |\eta|^2 + \int_0^T e^{-2s(10\hat{\alpha} - 9\hat{\alpha})} \xi^9 (\|h^0\|_{H^1(0, L)}^2 + \|h\|_{H^1(0, L)}^2) \right). \end{aligned} \quad (6.6)$$

Since $\beta = \xi$ in $[\frac{T}{2}, T]$ and

$$e^{-2s(10\hat{\alpha} - 9\hat{\alpha})} \beta^k \geq C \text{ in } \left[0, \frac{T}{2}\right], \quad \text{for } k > 0,$$

the r.h.s. of (6.6) can be further estimated to get

$$\begin{aligned} & \int_{\frac{T}{2}}^T \int_0^L e^{-2s\alpha} (\beta^9 |\eta|^2 + \beta^7 |\partial_x \eta|^2) + \int_{\frac{T}{2}}^T \int_0^L e^{-2s\alpha} (\beta^9 |p|^2 + \beta^7 |\partial_x p|^2) \\ & \leq C \left(\int_0^T \int_{\mathcal{O}_0} e^{-2s(10\hat{\alpha} - 9\hat{\alpha})} \beta^{17} |\eta|^2 + \int_0^T e^{-2s(10\hat{\alpha} - 9\hat{\alpha})} \beta^9 (\|h^0\|_{H^1(0, L)}^2 + \|h\|_{H^1(0, L)}^2) \right). \end{aligned} \quad (6.7)$$

Next, let us introduce a function $\hat{\zeta} \in \mathcal{C}^2[0, T]$ such that

$$\hat{\zeta}(t) = \begin{cases} 1, & t \in [0, \frac{T}{2}] \\ 0, & t \in [\frac{3T}{4}, T]. \end{cases}$$

If (η, p) is a solution of (5.1), then $(\hat{\eta}, \hat{p}) = (\hat{\zeta}\eta, \hat{\zeta}p)$ solves the following system

$$\begin{cases} -\partial_t \hat{\eta} - \partial_x^5 \hat{\eta} + \mu_0 \partial_x^4 \hat{\eta} - \partial_x^3 \hat{\eta} - \bar{u} \hat{\eta}_x = \hat{p} \chi_{\mathcal{O}_d} + \hat{\zeta} h^0 - \partial_t \hat{\zeta} \eta, & (t, x) \in Q, \\ \partial_t \hat{p} + \partial_x^5 \hat{p} + \mu_0 \partial_x^4 \hat{p} + \partial_x^3 \hat{p} + \partial_x (\bar{u} \hat{p}) = -\sum_{i=1}^2 \frac{\nu_i}{\mu_i} \hat{\eta} \chi_{\mathcal{O}_i} + \hat{\zeta} h + \partial_t \hat{\zeta} p, & (t, x) \in Q, \\ \hat{\eta}(t, 0) = \hat{\eta}(t, L) = \partial_x \hat{\eta}(t, 0) = \partial_x \hat{\eta}(t, L) = 0, \partial_x^2 \hat{\eta}(t, L) = \partial_x^2 \hat{\eta}(t, 0), & t \in (0, T), \\ \hat{p}(t, 0) = p(t, L) = \partial_x \hat{p}(t, 0) = \partial_x \hat{p}(t, L) = 0, \partial_x^2 \hat{p}(t, L) = \partial_x^2 \hat{p}(t, 0), & t \in (0, T), \\ \hat{\eta}(T, x) = 0, \hat{p}(0, x) = 0, & x \in (0, L). \end{cases} \quad (6.8)$$

Using the energy estimates for the above system (analogue to (2.13)), we obtain

$$\|\hat{\eta}\|_{X_T^0} + \|\hat{p}\|_{X_T^0} \leq C \left(\|\hat{\zeta} h^0\|_{L^2(Q)} + \|\hat{\zeta} h\|_{L^2(Q)} + \|\partial_t \hat{\zeta} \eta\|_{L^2(Q)} + \|\partial_t \hat{\zeta} p\|_{L^2(Q)} \right).$$

Let us use the definition of $\widehat{\zeta}$ in the above estimate to deduce for $t \in [0, T/2]$

$$\begin{aligned} & \|\eta(t)\|_{L^2(0,L)} + \|\eta\|_{L^2(0,\frac{T}{2};L^2(0,L))} + \|\partial_x \eta\|_{L^2(0,\frac{T}{2};L^2(0,L))} + \|p\|_{L^2(0,\frac{T}{2};L^2(0,L))} + \|\partial_x p\|_{L^2(0,\frac{T}{2};L^2(0,L))} \\ & \leq C \left(\|h^0\|_{L^2(0,\frac{T}{2};L^2(0,L))} + \|h\|_{L^2(0,\frac{T}{2};L^2(0,L))} + \|\eta\|_{L^2(\frac{T}{2},\frac{3T}{4};L^2(0,L))} + \|p\|_{L^2(\frac{T}{2},\frac{3T}{4};L^2(0,L))} \right). \end{aligned} \quad (6.9)$$

Next, using the estimate (6.9) and the properties of weight functions, we readily have $\forall t \in [0, \frac{T}{2}]$

$$\begin{aligned} & \|\eta(t)\|_{L^2(0,L)} + \int_0^{\frac{T}{2}} \int_0^L e^{-2s\alpha} (\beta^9 |\eta|^2 + \beta^7 |\partial_x \eta|^2) + \int_0^{\frac{T}{2}} \int_0^L e^{-2s\alpha} (\beta^9 |p|^2 + \beta^7 |\partial_x p|^2) \\ & \leq C \left(\int_0^{\frac{T}{2}} \int_0^L |h^0|^2 + |h|^2 + \int_{\frac{T}{2}}^{\frac{3T}{4}} \int_0^L \beta^9 e^{-2s\alpha} |\eta|^2 + \int_{\frac{T}{2}}^{\frac{3T}{4}} \int_0^L \beta^9 e^{-2s\alpha} |p|^2 \right). \end{aligned}$$

Thanks to the estimate (6.7) and the fact that $e^{-2s(10\bar{\alpha}-9\hat{\alpha})}\beta^m \geq C$ in $[0, T/2]$, the above identity becomes

$$\begin{aligned} & \|\eta(0)\|_{L^2(0,L)} + \int_0^{\frac{T}{2}} \int_0^L e^{-2s\alpha} (\beta^9 |\eta|^2 + \beta^7 |\partial_x \eta|^2) + \int_0^{\frac{T}{2}} \int_0^L e^{-2s\alpha} (\beta^9 |p|^2 + \beta^7 |\partial_x p|^2) \\ & \leq C \left(\int_0^T \int_{\mathcal{O}_0} e^{-2s(10\bar{\alpha}-9\hat{\alpha})}\beta^{17} |\eta|^2 + \int_0^T e^{-2s(10\bar{\alpha}-9\hat{\alpha})}\beta^9 \left(\|h^0\|_{H^1(0,L)}^2 + \|h\|_{H^1(0,L)}^2 \right) \right). \end{aligned} \quad (6.10)$$

Finally, the inequalities (6.7) and (6.10) together imply the desired Carleman inequality (6.2). \square

Using the estimates obtained in Lemma 6.1, we derive a suitable observability inequality which is applicable for all the cases mentioned above. More precisely, we have the following result.

Proposition 6.2 (Observability inequality). *Let $\bar{u} \in X_T^0$, and assume the constants μ_i to be large enough and $\mathcal{O}_{i,d} \cap \mathcal{O}_0 \neq \emptyset$. Then, there exists a constant $C > 0$ such that the solution (η, ψ_1, ψ_2) of (1.18) with $(\eta_T, h^0, h^1, h^2) \in L^2(0, L) \times [L^2(0, T; H^1(0, L))]^3$ satisfies the following estimate*

$$\begin{aligned} & \int_0^L |\eta(0, x)|^2 + \iint_Q e^{-2s\alpha^*} (\beta^9 |\eta|^2 + \beta^7 |\partial_x \eta|^2) + \sum_{i=1}^2 \iint_Q e^{-2s\alpha^*} (|\psi^i|^2 + |\partial_x \psi^i|^2) \\ & \leq C \left(\int_0^T \int_{\mathcal{O}_0} \beta^{17} e^{-2s(10\alpha_*-9\alpha^*)} |\eta|^2 + \sum_{i=0}^2 \int_0^T \beta^9 e^{-2s(10\alpha_*-9\alpha^*)} \|h^i\|_{H^1(0,L)}^2 \right). \end{aligned} \quad (6.11)$$

Proof. • **For the Case (1.15):**

To deal with this case, we first replace the the function α by α_1 in (6.2) to get

$$\begin{aligned} & \int_0^L |\eta(0, x)|^2 + \iint_Q e^{-2s\alpha_1} \beta^9 (|\eta|^2 + |\nu_1 \psi^1 + \nu_2 \psi^2|^2) + \iint_Q e^{-2s\alpha_1} \beta^7 (|\partial_x \eta|^2 + |\nu_1 \partial_x \psi^1 + \nu_2 \partial_x \psi^2|^2) \\ & \leq C \left(\int_0^T \int_{\mathcal{O}_0} \beta^{17} e^{-2s(10\alpha_*-9\alpha^*)} |\eta|^2 + \sum_{i=0}^2 \int_0^T \beta^9 e^{-2s(10\alpha_*-9\alpha^*)} \|h^i\|_{H^1(0,L)}^2 \right). \end{aligned}$$

Let us ignore the terms associated to $\nu_1 \psi^1 + \nu_2 \psi^2$ from the left hand side and use the estimate $\alpha^* \geq \alpha_1$ in the above estimate to obtain

$$\begin{aligned} & \int_0^L |\eta(0, x)|^2 + \iint_Q e^{-2s\alpha^*} \beta^9 |\eta|^2 + \iint_Q e^{-2s\alpha^*} \beta^7 |\partial_x \eta|^2 \\ & \leq C \left(\int_0^T \int_{\mathcal{O}_0} \beta^{17} e^{-2s(10\alpha_*-9\alpha^*)} |\eta|^2 + \sum_{i=0}^2 \int_0^T \beta^9 e^{-2s(10\alpha_*-9\alpha^*)} \|h^i\|_{H^1(0,L)}^2 \right). \end{aligned} \quad (6.12)$$

To add the terms of ψ^i on the left, let us introduce $\tilde{\psi}^i = e^{-s\alpha^*} \psi^i$. Note that $\tilde{\psi}^i$ satisfies the differential equation

$$\partial_t \tilde{\psi}^i + \partial_x^5 \tilde{\psi}^i + \mu_0 \partial_x^4 \tilde{\psi}^i + \partial_x^3 \tilde{\psi}^i + \partial_x (\bar{u} \tilde{\psi}^i) = -\frac{e^{-s\alpha^*}}{\mu_i} \eta \chi_{\mathcal{O}_i} + e^{-s\alpha^*} h^i - s \partial_t \alpha^* e^{-s\alpha^*} \tilde{\psi}^i.$$

On multiplying the equation with $2\tilde{\psi}^i$ and integrating it w.r.t. x over $(0, L)$, we get

$$\begin{aligned} \frac{d}{dt} \int_0^L |\tilde{\psi}^i|^2 + 2\mu_0 \int_0^L |\partial_x^2 \tilde{\psi}^i|^2 &= - \int_0^L \partial_x (\bar{u} \tilde{\psi}^i) \tilde{\psi}^i - \int_0^L \frac{e^{-s\alpha^*}}{\mu_i} \eta \chi_{\mathcal{O}_i} \tilde{\psi}^i + \int_0^L e^{-s\alpha^*} h^i \tilde{\psi}^i \\ &\quad - s \int_0^L \partial_t \alpha^* e^{-s\alpha^*} |\tilde{\psi}^i|^2 \end{aligned}$$

Note that $\partial_t \alpha^* \geq 0$, and so we can ignore the last term of the right side. Further, performing integration by parts and using the Hölder's inequality and Young's inequality, we have

$$\begin{aligned} \frac{d}{dt} \int_0^L |\tilde{\psi}^i|^2 + 2\mu_0 \int_0^L |\partial_x^2 \tilde{\psi}^i|^2 &\leq C \left(\int_0^L e^{-2s\alpha^*} |\psi^i|^2 + \int_0^L e^{-2s\alpha^*} |\eta|^2 + \int_0^L e^{-2s\alpha^*} |h^i|^2 \right) \\ &\quad + \epsilon \int_0^L e^{-2s\alpha^*} |\partial_x \psi^i|^2, \end{aligned}$$

for any $\epsilon > 0$.

Next, using Poincaré inequality in the second term of left hand side and choosing ϵ sufficiently small, one can absorb the last term of the right side of the above inequality in left. Next, we use the Gronwall's inequality and the fact that $\tilde{\psi}^i(0, \cdot) = 0$ on $(0, L)$ to obtain

$$e^{-C\tau} \int_0^L |\tilde{\psi}^i(\tau, \cdot)|^2 + 2\mu_0 \int_0^\tau \int_0^L e^{-Ct} |\partial_x \tilde{\psi}^i|^2 \leq C \int_0^\tau e^{-Ct} \left(\int_0^L e^{-2s\alpha^*} |\eta|^2 + \int_0^L e^{-2s\alpha^*} |h^i|^2 \right)$$

for any $\tau \in (0, T)$. Using the bound $e^{-C\tau} \leq e^{-Ct} \leq 1$, and then taking supremum with respect to τ over $(0, T)$, we get

$$\int_0^L |\tilde{\psi}^i(\tau, \cdot)|^2 + \int_0^\tau \int_0^L |\partial_x \tilde{\psi}^i|^2 \leq C \int_0^T \int_0^L e^{-2s\alpha^*} |\eta|^2 + C \int_0^T \int_0^L e^{-2s\alpha^*} |h^i|^2, \text{ for } \tau \in (0, T).$$

Integrating w.r.t. τ over $(0, T)$ and using the bound

$$\beta(\tau) \geq \frac{4}{T^2} \text{ for } \tau \in (0, T) \text{ and } e^{-2s\alpha^*} \leq e^{-2s(10\alpha_* - 9\alpha^*)},$$

we deduce

$$\iint_Q e^{-2s\alpha^*} |\psi^i|^2 + \iint_Q e^{-2s\alpha^*} |\partial_x \psi^i|^2 \leq C \left(\iint_Q e^{-2s\alpha^*} \beta^9 |\eta|^2 + \iint_Q e^{-2s(10\alpha_* - 9\alpha^*)} \beta^9 |h^i|^2 \right). \quad (6.13)$$

Summing the estimates (6.12), and (6.13) for $i \in \{1, 2\}$, we finally get the desired estimate (6.11).

• **For the Case (1.16)-(5.5):**

Just using $\alpha_i \leq \alpha^*$ and the bound $\beta \geq 4/T^2$ in the left of (6.3), we get (6.11).

• **For the Case (1.16)-(5.6):**

Again using the estimates $\alpha_i \leq \alpha^*$, $\beta \geq 4/T^2$, and ignoring the $\nu_1 \psi^1 + \nu_2 \psi^2$ terms from the left side of (6.4), we get

$$\begin{aligned} \int_0^L |\eta(0, x)|^2 + \iint_Q e^{-2s\alpha_*} (\beta^9 |\eta|^2 + \beta^7 |\partial_x \eta|^2) + \iint_Q e^{-2s\alpha_*} (|\psi^j|^2 + |\partial_x \psi^j|^2) \\ \leq C \left(\int_0^T \int_{\mathcal{O}_0} e^{-2s(10\alpha_* - 9\alpha^*)} \beta^{17} |\eta|^2 + \sum_{i=0}^2 \int_0^T e^{-2s(10\alpha_* - 9\alpha^*)} \beta^9 \|h^i\|_{H^1(0, L)}^2 \right). \end{aligned}$$

Finally, we add the ψ^i term in the left of the above estimate due to (6.13) to obtain (6.11). \square

6.2. Null controllability. In this subsection, we prove the global null-controllability result for the v -component of the extended linearized system (1.17) with source terms g^i in proper weighted space for $i \in \{0, 1, 2\}$.

Let us introduce the operator \mathcal{L} and its formal adjoint \mathcal{L}^* as

$$\begin{aligned} \mathcal{L}v &= \partial_t v + \partial_x^5 v + \mu_0 \partial_x^4 v + \partial_x^3 v + \partial_x(\bar{u}v), \\ \mathcal{L}^*v &= -\partial_t v - \partial_x^5 v + \mu_0 \partial_x^4 v - \partial_x^3 v - \bar{u} \partial_x v. \end{aligned}$$

Next, we consider the Banach space

$$\begin{aligned} \mathcal{E} := & \left\{ (v, \varphi^1, \varphi^2, f) \mid e^{s(10\alpha_* - 9\alpha^*)} \beta^{-\frac{9}{2}} (v, \varphi^1, \varphi^2) \in [L^2(0, T; H^{-1}(0, L))]^3, \right. \\ & e^{s(10\alpha_* - 9\alpha^*)} \beta^{-\frac{17}{2}} f \in L^2(Q), \\ & e^{s(10\alpha_* - 9\alpha^*)} \beta^{-\frac{17}{2}} (v, \varphi^1, \varphi^2) \in [\mathcal{C}([0, T]; L^2(0, L)) \cap L^2(0, T; H^2(0, L))]^3, \\ & e^{s\alpha^*} \beta^{-\frac{7}{2}} \left(\mathcal{L}v - f^0 \chi_{\mathcal{O}_0} + \sum_{i=1}^2 \frac{1}{\mu_i} \varphi^i \chi_{\mathcal{O}_i} \right) \in L^2(0, T; H^{-1}(0, L)), \\ & \left. e^{s\alpha^*} \left(\mathcal{L}^* \varphi^i - \nu_i v \chi_{\mathcal{O}_{i,d}} \right) \in L^2(0, T; H^{-1}(0, L)) \right\}. \end{aligned} \quad (6.14)$$

We will prove the following null-controllability result for the linearized system (1.17).

Proposition 6.3. *For $v_0 \in L^2(0, L)$, and $e^{s(10\alpha_* - 9\alpha^*)} \beta^{-\frac{17}{2}} (g^0, g^1, g^2) \in [L^2(0, T; H^{-1}(0, L))]^3$, there exist a control $f \in L^2(0, T; L^2(\mathcal{O}_0))$ such that the associated solution tuple $(v, \varphi^1, \varphi^2, f)$ of (1.17) belongs to the space \mathcal{E} , in particular $v(T) = 0$. Moreover, the control f satisfies the following estimate*

$$\|f\|_{L^2(0, T; L^2(\mathcal{O}_0))} \leq C \left(\|v_0\|_{L^2(0, L)} + \sum_{i=0}^2 \|g^i\|_{L^2(0, T; H^{-1}(0, L))} \right).$$

Proof. Consider the space

$$\begin{aligned} \mathcal{H}_0 = & \left\{ (\eta, \psi^1, \psi^2) \in [\mathcal{C}^6(\bar{Q})]^3 \mid \eta(t, 0) = \eta(t, L) = \partial_x \eta(t, 0) = \partial_x \eta(t, L) = 0, \partial_x^2 \eta(t, 0) = \partial_x^2 \eta(t, L), \right. \\ & \psi^i(t, 0) = \psi^i(t, L) = \partial_x \psi^i(t, 0) = \partial_x \psi^i(t, L) = 0, \partial_x^2 \psi^i(t, 0) = \partial_x^2 \psi^i(t, L), \\ & \mathcal{L}^*(\eta)(t, 0) = \mathcal{L}^*(\eta)(t, L) = 0, \mathcal{L}(\psi^i)(t, 0) = \mathcal{L}(\psi^i)(t, L) = 0, \\ & \left. \psi^i(0, x) = 0, \text{ for } i \in \{1, 2\} \right\}. \end{aligned}$$

Let us define a bilinear form $\mathcal{B} : \mathcal{H}_0 \times \mathcal{H}_0 \rightarrow \mathbb{R}$ given by

$$\begin{aligned} \mathcal{B}((\hat{\eta}, \hat{\psi}^1, \hat{\psi}^2), (\eta, \psi^1, \psi^2)) := & \iint_Q e^{-2s(10\alpha_* - 9\alpha^*)} \beta^9 \left(\mathcal{L}^* \hat{\eta} - \sum_{i=1}^2 \nu_i \hat{\psi}^i \chi_{\mathcal{O}_{i,d}} \right)_x \left(\mathcal{L}^* \eta - \sum_{i=1}^2 \nu_i \psi^i \chi_{\mathcal{O}_{i,d}} \right)_x \\ & + \sum_{i=1}^2 \iint_Q e^{-2s(10\alpha_* - 9\alpha^*)} \beta^9 \left(\mathcal{L} \hat{\psi}^i + \frac{1}{\mu_i} \hat{\eta} \chi_{\mathcal{O}_i} \right)_x \left(\mathcal{L} \psi^i + \frac{1}{\mu_i} \eta \chi_{\mathcal{O}_i} \right)_x \\ & + \int_0^T \int_{\mathcal{O}} e^{-2s(10\alpha_* - 9\alpha^*)} \beta^{17} \hat{\eta} \eta. \end{aligned}$$

Note that the map \mathcal{B} is continuous. Thanks to the observability estimate (6.11), we have

$$\begin{aligned} & \mathcal{B}((\eta, \psi^1, \psi^2), (\eta, \psi^1, \psi^2)) \\ &= \iint_Q e^{-2s(10\alpha_* - 9\alpha^*)} \beta^9 \left| \left(\mathcal{L}^* \eta - \sum_{i=1}^2 \nu_i \psi^i \chi_{\mathcal{O}_{i,d}} \right)_x \right|^2 \\ & \quad + \sum_{i=1}^2 \iint_Q e^{-2s(10\alpha_* - 9\alpha^*)} \beta^9 \left| \left(\mathcal{L} \psi^i + \frac{1}{\mu_i} \eta \chi_{\mathcal{O}_i} \right)_x \right|^2 + \int_0^T \int_{\mathcal{O}} e^{-2s(10\alpha_* - 9\alpha^*)} \beta^{17} |\eta|^2 \\ & \geq C \left(\iint_Q e^{-2s\alpha^*} \beta^9 |\eta|^2 + \sum_{i=1}^2 \iint_Q e^{-2s\alpha^*} |\psi^i|^2 \right), \quad \forall (\eta, \psi^1, \psi^2) \in \mathcal{H}_0. \end{aligned} \quad (6.15)$$

This shows that the bilinear map \mathcal{B} is positive definite and coercive on \mathcal{H}_0 . Moreover, the map $(\eta, \psi^1, \psi^2) \mapsto \mathcal{B}((\eta, \psi^1, \psi^2), (\eta, \psi^1, \psi^2))^{1/2}$ defines a norm, denoted by $\|\cdot\|_{\mathcal{H}_0}$ on the space \mathcal{H}_0 due to the estimate (6.15). Let \mathcal{H} be the completion of the space \mathcal{H}_0 with this norm. Then \mathcal{B} is a well-defined, positive definite, continuous, and coercive bilinear map on \mathcal{H} .

Next, for $e^{s\alpha^*} (g^0, g^1, g^2) \in [L^2(0, T; H^{-1}(0, L))]^3$ and $v_0 \in L^2(0, L)$, we consider the linear map $\mathcal{F} : \mathcal{H}_0 \rightarrow \mathbb{R}$ given by

$$\mathcal{F}(\eta, \psi^1, \psi^2) := \int_0^T \langle g^0, \eta \rangle_{H^{-1}, H_0^1} + \sum_{i=1}^2 \int_0^T \langle g^i, \psi^i \rangle_{H^{-1}, H_0^1} + \int_0^L v_0 \eta(0).$$

Using the observability inequality (6.11), the map \mathcal{F} can be estimated as

$$\begin{aligned} |\mathcal{F}(\eta, \psi^1, \psi^2)| &\leq \left(\|e^{s\alpha^*} \beta^{-7/2} g^0\|_{L^2(0,T;H^{-1}(0,L))} + \sum_{i=1}^2 \|e^{s\alpha^*} g^i\|_{L^2(0,T;H^{-1}(0,L))} + \|v_0\|_{L^2(0,L)} \right) \\ &\quad \left(\|e^{-s\alpha^*} \beta^{7/2} \eta\|_{L^2(0,T;H^1(0,L))} + \sum_{i=1}^2 \|e^{-s\alpha^*} \psi^i\|_{L^2(0,T;H^1(0,L))} + \|\eta(0)\|_{L^2(0,L)} \right). \end{aligned}$$

This proves that the linear map \mathcal{F} is continuous on \mathcal{H}_0 , and hence it can be extended as a continuous linear functional on \mathcal{H} due to Hahn-Banach theorem.

Because of the properties of the maps \mathcal{B} and \mathcal{F} mentioned above, we can apply the Lax-Milgram theorem. More precisely, for any given $v_0 \in L^2(0,L)$ and any $\{g^i\}_{i=0}^2$ such that $e^{s\alpha^*} (g^0, g^1, g^2) \in [L^2(0,T;H^{-1}(0,L))]^3$, there exists a unique tuple $(\hat{\eta}, \hat{\psi}^1, \hat{\psi}^2) \in \mathcal{H}$ satisfying

$$\mathcal{B}((\hat{\eta}, \hat{\psi}^1, \hat{\psi}^2), (\eta, \psi^1, \psi^2)) = \mathcal{F}(\eta, \psi^1, \psi^2), \quad \forall (\eta, \psi^1, \psi^2) \in \mathcal{H}. \quad (6.16)$$

Let us define

$$\begin{cases} \hat{v} := -e^{-2s(10\alpha_* - 9\alpha^*)} \beta^9 \partial_x^2 \left(\mathcal{L}^* \hat{\eta} - \sum_{i=1}^2 \nu_i \hat{\psi}^i \chi_{\mathcal{O}_{i,d}} \right) & \text{in } Q, \\ \hat{\varphi}^i := -e^{-2s(10\alpha_* - 9\alpha^*)} \beta^9 \partial_x^2 \left(\mathcal{L} \hat{\psi}^i + \frac{1}{\mu_i} \hat{\eta} \chi_{\mathcal{O}_i} \right) & \text{in } Q, \\ \hat{f} := -e^{-2s(10\alpha_* - 9\alpha^*)} \beta^{17} \hat{\eta} & \text{in } (0,T) \times \mathcal{O}_0. \end{cases}$$

Performing integration by parts with respect to x , the identity (6.16) can be rewritten as

$$\begin{aligned} \int_0^T \left\langle \hat{v}, \mathcal{L}^* \eta - \sum_{i=1}^2 \nu_i \psi^i \chi_{\mathcal{O}_{i,d}} \right\rangle_{H^{-1}, H_0^1} + \sum_{i=1}^2 \int_0^T \left\langle \hat{\varphi}^i, \mathcal{L} \psi^i + \frac{1}{\mu_i} \eta \chi_{\mathcal{O}_i} \right\rangle_{H^{-1}, H_0^1} - \int_0^T \int_{\mathcal{O}_0} \hat{f} \eta - \int_0^L v_0 \eta(0) \\ = \int_0^T \langle g^0, \eta \rangle_{H^{-1}, H_0^1} + \sum_{i=1}^2 \int_0^T \langle g^i, \psi^i \rangle_{H^{-1}, H_0^1}, \quad \forall (\eta, \psi^1, \psi^2) \in \mathcal{H}. \end{aligned}$$

Using the adjoint equation (1.18), the above identity can be written as

$$\begin{aligned} \int_0^T \langle \hat{v}, h^0 \rangle_{H^{-1}, H_0^1} + \sum_{i=1}^2 \int_0^T \langle \hat{\varphi}^i, h^i \rangle_{H^{-1}, H_0^1} - \int_0^T \int_{\mathcal{O}_0} \hat{f} \eta - \int_0^L v_0 \eta(0) \\ = \int_0^T \langle g^0, \eta \rangle_{H^{-1}, H_0^1} + \sum_{i=1}^2 \int_0^T \langle g^i, \psi^i \rangle_{H^{-1}, H_0^1}, \quad \forall (h^0, h^1, h^2) \in [L^2(0,T;H_0^1(0,L))]^3, \quad (6.17) \end{aligned}$$

where (η, ψ^1, ψ^2) is the solution of (1.18) corresponding to (h^0, h^1, h^2) .

Note that for $(v_0, g^0, g^1, g^2) \in L^2(0,L) \times [L^2(0,T;H^{-1}(0,L))]^3$ and $f = \hat{f} \in L^2(0,T;L^2(\mathcal{O}_0))$ the system (1.17) admits a unique solution, say $(\tilde{v}, \tilde{\varphi}^1, \tilde{\varphi}^2) \in L^2(Q)$ in the sense of transposition, i.e. $(\tilde{v}, \tilde{\varphi}^1, \tilde{\varphi}^2)$ satisfies

$$\begin{aligned} \iint_Q \tilde{v} h^0 + \sum_{i=1}^2 \iint_Q \tilde{\varphi}^i h^i - \int_0^T \int_{\mathcal{O}_0} \hat{f} \eta - \int_0^L v_0 \eta(0) \\ = \int_0^T \langle g^0, \eta \rangle_{H^{-1}, H_0^1} + \sum_{i=1}^2 \int_0^T \langle g^i, \psi^i \rangle_{H^{-1}, H_0^1}, \quad \forall (h^0, h^1, h^2) \in [L^2(0,T;H_0^1(0,L))]^3, \quad (6.18) \end{aligned}$$

where (η, ψ^1, ψ^2) is the solution of (1.18) corresponding to this (h^0, h^1, h^2) and $\eta_T = 0$. Comparing the identities (6.17) and (6.18), we get $(\tilde{v}, \tilde{\varphi}^1, \tilde{\varphi}^2) = (\hat{v}, \hat{\varphi}^1, \hat{\varphi}^2)$. Thus, $(\hat{v}, \hat{\varphi}^1, \hat{\varphi}^2) \in L^2(Q)$ solves the system (1.18).

Finally, in order to end the proof, we just need to show that $(\hat{v}, \hat{\varphi}^1, \hat{\varphi}^2, \hat{f}) \in \mathcal{E}$.

• Let us first start with $L^2(0,T;H^{-1}(0,L))$ estimate associated to \hat{v} and $\hat{\varphi}^i$ for $i \in \{1, 2\}$.

$$\begin{aligned} \|e^{s(10\alpha_* - 9\alpha^*)} \beta^{-9/2} \hat{v}\|_{L^2(0,T;H^{-1}(0,L))}^2 &= \int_0^T e^{-2s(10\alpha_* - 9\alpha^*)} \beta^9 \left\| \partial_x^2 \left(\mathcal{L}^* \hat{\eta} - \sum_{i=1}^2 \nu_i \hat{\psi}^i \chi_{\mathcal{O}_{i,d}} \right) \right\|_{H^{-1}(0,L)}^2 \\ &\leq C \int_0^T e^{-2s(10\alpha_* - 9\alpha^*)} \beta^9 \left\| \partial_x \left(\mathcal{L}^* \hat{\eta} - \sum_{i=1}^2 \nu_i \hat{\psi}^i \chi_{\mathcal{O}_{i,d}} \right) \right\|_{L^2(0,L)}^2. \end{aligned}$$

Similarly, we have

$$\|e^{s(10\alpha_* - 9\alpha^*)} \beta^{-9/2} \hat{\varphi}^i\|_{L^2(0,T;H^{-1}(0,L))}^2 \leq C \int_0^T e^{-2s(10\alpha_* - 9\alpha^*)} \beta^9 \left\| \partial_x \left(\mathcal{L} \hat{\psi}^i + \frac{1}{\mu_i} \hat{\eta} \chi_{\mathcal{O}_i} \right) \right\|_{L^2(0,L)}^2.$$

Next, we look for the estimates of the control \hat{f} . Note that

$$\|e^{s(10\alpha_* - 9\alpha^*)} \beta^{-17/2} \hat{f}\|_{L^2(Q)} = \iint_Q e^{-2s(10\alpha_* - 9\alpha^*)} \beta^{17} |\hat{\eta}|^2.$$

Summing the last three estimates, and using the expression (6.15), we obtain

$$\begin{aligned} & \|e^{s(10\alpha_* - 9\alpha^*)} \beta^{-9/2} \hat{v}\|_{L^2(0,T;H^{-1}(0,L))}^2 + \sum_{i=1}^2 \|e^{s(10\alpha_* - 9\alpha^*)} \beta^{-9/2} \hat{\varphi}^i\|_{L^2(0,T;H^{-1}(0,L))}^2 \\ & + \|e^{s(10\alpha_* - 9\alpha^*)} \beta^{-17/2} \hat{f}\|_{L^2(Q)} \leq C \mathcal{B}((\hat{\eta}, \hat{\psi}^1, \hat{\psi}^2), (\hat{\eta}, \hat{\psi}^1, \hat{\psi}^2)) < \infty. \end{aligned}$$

• In order to derive X_T^0 estimate associated to $(\hat{v}, \hat{\varphi}^1, \hat{\varphi}^2)$, let us define

$$\begin{cases} v^* := e^{s(10\alpha_* - 9\alpha^*)} \beta^{-\frac{17}{2}} \hat{v}, \\ \varphi^{i*} := e^{s(10\alpha_* - 9\alpha^*)} \beta^{-\frac{17}{2}} \hat{\varphi}^i, \\ g^{0*} := e^{s(10\alpha_* - 9\alpha^*)} \beta^{-\frac{17}{2}} (\chi_{\mathcal{O}_0} \hat{f} + g^0), \\ g^{i*} := e^{s(10\alpha_* - 9\alpha^*)} \beta^{-\frac{17}{2}} g^i. \end{cases}$$

Note that these functions solves the following system

$$\begin{cases} \mathcal{L}v^* = g^{0*} - \sum_{i=1}^2 \frac{1}{\mu_i} \partial_t \left(e^{s(10\alpha_* - 9\alpha^*)} \beta^{-\frac{17}{2}} \right) \hat{\varphi}^i \chi_{\mathcal{O}_i}, & (t, x) \in Q, \\ \mathcal{L}^* \varphi^{i*} = g^{i*} + \partial_t \left(e^{s(10\alpha_* - 9\alpha^*)} \beta^{-\frac{17}{2}} \right) \hat{v} \chi_{\mathcal{O}_{i,d}}, & (t, x) \in Q, \\ v^*(t, 0) = v^*(t, L) = \partial_x v^*(t, 0) = \partial_x v^*(t, L) = 0, \quad \partial_x^2 v^*(t, L) = \partial_x^2 v^*(t, 0), & t \in (0, T), \\ \varphi^{i*}(t, 0) = \varphi^{i*}(t, L) = \partial_x \varphi^{i*}(t, 0) = \partial_x \varphi^{i*}(t, L) = 0, \quad \partial_x^2 \varphi^{i*}(t, L) = \partial_x^2 \varphi^{i*}(t, 0), & t \in (0, T), \\ v^*(0, x) = e^{s(10\alpha_* - 9\alpha^*)} \beta^{-\frac{17}{2}} \Big|_{t=0} v_0(x), \quad \varphi^{i*}(T, x) = 0, & x \in (0, L). \end{cases} \quad (6.19)$$

For the right hand side term, we have

$$\left| \partial_t \left(e^{s(10\alpha_* - 9\alpha^*)} \beta^{-\frac{17}{2}} \right) \right| \leq C e^{s(10\alpha_* - 9\alpha^*)} \beta^{-\frac{13}{2}},$$

and so

$$\begin{aligned} \left\| \partial_t \left(e^{s(10\alpha_* - 9\alpha^*)} \beta^{-\frac{17}{2}} \right) \hat{\varphi}^i \right\|_{L^2(Q)}^2 & \leq C \iint_Q e^{2s(10\alpha_* - 9\alpha^*)} \beta^{-13} |\hat{\varphi}^i|^2 \\ & \leq C \iint_Q e^{2s(10\alpha_* - 9\alpha^*)} \beta^{-9} |\hat{\varphi}^i|^2 < \infty. \end{aligned}$$

Also, g^{0*} satisfies the following estimate

$$\begin{aligned} \|g^{0*}\|_{L^2(0,T;H^{-1}(0,L))} & \leq C \|e^{s(10\alpha_* - 9\alpha^*)} \beta^{-\frac{17}{2}} \hat{f}\|_{L^2(Q)} + \|e^{s(10\alpha_* - 9\alpha^*)} \beta^{-\frac{17}{2}} g^0\|_{L^2(0,T;H^{-1}(0,L))} \\ & \leq C \|e^{s(10\alpha_* - 9\alpha^*)} \beta^{-\frac{17}{2}} \hat{f}\|_{L^2(Q)} + \|e^{s\alpha_*} \beta^{-\frac{7}{2}} g^0\|_{L^2(0,T;H^{-1}(0,L))} < \infty. \end{aligned}$$

Similarly, one can show that

$$g^{i*} + \partial_t \left(e^{s(10\alpha_* - 9\alpha^*)} \beta^{-\frac{17}{2}} \right) \hat{v} \chi_{\mathcal{O}_{i,d}} \in L^2(0, T; H^{-1}(0, L)).$$

Thus, the system (6.19) admits a unique solution $(v^*, \varphi^{1*}, \varphi^{2*}) \in \mathcal{C}([0, T; L^2(0, L)]) \cap L^2(0, T; H^2(0, L))$ due to the well-posedness result Proposition 2.7. This completes the proof of the fact that $(\hat{v}, \hat{\varphi}^1, \hat{\varphi}^2, \hat{f}) \in \mathcal{E}$, and thus ends the proof of controllability result Proposition 6.3. \square

7. NULL CONTROLLABILITY OF THE NON-LINEAR SYSTEM

The proof of the controllability result for the non-linear case relies on the local inversion mapping theorem as mentioned below.

Theorem 7.1 (Inverse mapping theorem). *Consider any two Banach spaces $\mathcal{B}_1, \mathcal{B}_2$, and let $\mathcal{A} : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ be a C^1 map. Suppose $\mathcal{A}(b_1^*) = b_2^*$ for some $b_1^* \in \mathcal{B}_1, b_2^* \in \mathcal{B}_2$, and assume that the derivative map $\mathcal{A}'(b_1^*) : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is surjective. Then, there exists $\delta > 0$ such that for any $b_2 \in \mathcal{B}_2$ satisfying $\|b_2^* - b_2\|_{\mathcal{B}_2} < \delta$ there exists $b_1 \in \mathcal{B}_1$ satisfying $\mathcal{A}(b_1) = b_2$.*

Proposition 7.2 (Local controllability). *Let $T > 0$ and assume $\bar{u} \in X_T^0$ solves the control free system (1.5). Then there exists a constant $\delta > 0$ such that for any $v_0 \in L^2(0, L)$ with $\|v_0\|_{L^2(0, L)} < \delta$, there exists a control $f \in L^2(0, T; L^2(\mathcal{O}_0))$ such that the solution $(v, \varphi^1, \varphi^2)$ of (1.13) satisfies $v(T, \cdot) = 0$.*

Proof. To apply the local inversion theorem, we set

$$\mathcal{B}_1 := \mathcal{E} \text{ and } \mathcal{B}_2 := \left[L^2\left(e^{s(10\alpha_* - 9\alpha^*)} \beta^{-\frac{17}{2}}(0, T); H^{-1}(0, L)\right) \right]^3 \times L^2(0, L),$$

and define the map \mathcal{A} as

$$\mathcal{A} : \mathcal{B}_1 \rightarrow \mathcal{B}_2 : \mathcal{A}(v, \varphi^1, \varphi^2, f) = \left(\mathcal{L}v + v \partial_x v + \sum_{i=1}^2 \frac{1}{\mu_i} \varphi^i \chi_{\mathcal{O}_i} - f^0 \chi_{\mathcal{O}_0}, \mathcal{L}^* \varphi^1 - v \partial_x \varphi^1 - \nu_1 v \chi_{\mathcal{O}_{1,d}}, \right. \\ \left. \mathcal{L}^* \varphi^2 - v \partial_x \varphi^2 - \nu_2 v \chi_{\mathcal{O}_{2,d}}, v(0, \cdot) \right).$$

Since the derivative of the linear term is identity which is continuous as well, so one has to only focus on the non-linear term while calculating the derivative. In fact the derivative map $\mathcal{A}' : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is given by

$$\mathcal{A}'(\tilde{v}, \tilde{\varphi}^1, \tilde{\varphi}^2, \tilde{f})(v, \varphi^1, \varphi^2, f) = \left(\mathcal{L}v + v \partial_x \tilde{v} + \tilde{v} \partial_x v + \sum_{i=1}^2 \frac{1}{\mu_i} \varphi^i \chi_{\mathcal{O}_i} - \tilde{f}, \mathcal{L}^* \varphi^1 - \tilde{v} \partial_x \varphi^1 - v \partial_x \tilde{\varphi}^1 - \nu_1 v \chi_{\mathcal{O}_{1,d}}, \right. \\ \left. \mathcal{L}^* \varphi^2 - \tilde{v} \partial_x \varphi^2 - v \partial_x \tilde{\varphi}^2 - \nu_2 v \chi_{\mathcal{O}_{2,d}}, v(0, \cdot) \right).$$

In order to prove the continuity of \mathcal{A}' , note that

$$\left\| e^{s(10\alpha_* - 9\alpha^*)} \beta^{-\frac{17}{2}} \partial_x (v_1 v_2) \right\|_{L^2(0, T; H^{-1}(0, L))}^2 \\ \leq \int_0^T e^{2s(10\alpha_* - 9\alpha^*)} \beta^{-17} \|v_1 v_2\|_{L^2(0, L)} \\ \leq C \int_0^T e^{2s(10\alpha_* - 9\alpha^*)} \beta^{-17} \|v_1\|_{H^1(0, L)} \|v_2\|_{L^2(0, L)} \\ \leq C \left(\int_0^T e^{2s(10\alpha_* - 9\alpha^*)} \beta^{-17} \|v_1\|_{H^1(0, L)}^2 \right)^{1/2} \left(\int_0^T e^{2s(10\alpha_* - 9\alpha^*)} \beta^{-17} \|v_2\|_{L^2(0, L)}^2 \right)^{1/2}.$$

This shows that the bilinear map

$$\mathfrak{B} : \left[L^2\left(e^{s(10\alpha_* - 9\alpha^*)} \beta^{-\frac{17}{2}}(0, T); H^2(0, L)\right) \right]^2 \rightarrow \left[L^2\left(e^{s(10\alpha_* - 9\alpha^*)} \beta^{-\frac{17}{2}}(0, T); H^{-1}(0, L)\right) \right]^2$$

given by $\mathfrak{B}(v_1, v_2) = \partial_x(v_1 v_2)$ is continuous, and hence $\mathcal{A} \in \mathcal{C}^1(\mathcal{B}_1, \mathcal{B}_2)$.

Finally to conclude the controllability result, note that $\mathcal{A}(0, 0, 0, 0) = (0, 0, 0, 0)$, and the map

$$\mathcal{A}'(0, 0, 0, 0) : \mathcal{B}_1 \rightarrow \mathcal{B}_2$$

given by

$$\mathcal{A}'(0, 0, 0, 0)(v, \varphi^1, \varphi^2, f) = \left(\mathcal{L}v + \sum_{i=1}^2 \frac{1}{\mu_i} \varphi^i \chi_{\mathcal{O}_i} - f, \mathcal{L}^* \varphi^1 - \nu_1 v \chi_{\mathcal{O}_{1,d}}, \mathcal{L}^* \varphi^2 - \nu_2 v \chi_{\mathcal{O}_{2,d}}, v(0, \cdot) \right)$$

is surjectivity due to [Proposition 6.3](#). Hence, [Theorem 7.1](#) guarantees the existence of $\delta > 0$ such that for $(g^0, g^1, g^2) = (0, 0, 0)$ and $v_0 \in L^2(0, L)$ with $\|v_0\|_{L^2} < \delta$, there exists a control $f \in L^2(0, T; L^2(\mathcal{O}_0))$ such that the solution $(v, \varphi^1, \varphi^2)$ of (1.13) satisfies $v(T, x) = 0$ for $x \in (0, L)$. This completes the proof. \square

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