Maximal Line Digraphs

Quentin Japhet, Dimitri Watel, Dominique Barth, Marc-Antoine Weisser

Maximal Line Digraphs

Quentin Japhet[1], Dimitri Watel[2,3], Dominique Barth[1], and Marc-Antoine Weisser[4]

[1]DAVID, University of Versailles-St Quentin, Paris-Saclay University, 45 avenue des États-Unis, 78035, Versailles, France
[2]ENSIIE, 1 square de la Résistance, 91000, Evry, France
[3]SAMOVAR, 9 Rue Charles Fourier, 91000, Evry, France
[4]LISN, CentraleSupelec, Paris-Saclay University, 1 Rue Raimond Castaing, 91190, Gif-sur-Yvette, France

quentin.japhet@uvsq.fr (Corresponding author), dimitri.watel@ensiie.fr, dominique.barth@uvsq.fr, marc-antoine.weisser@centralesupelec.fr

Abstract

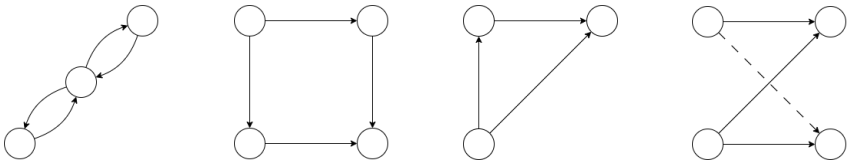
A line digraph $L(G) = (A, E)$ is the digraph constructed from the digraph $G = (V, A)$ such that there is an arc (a, b) in $L(G)$ if the terminal node of a in G is the initial node of b. The maximum number of arcs in a line digraph with m nodes is $(m/2)^2 + (m/2)$ if m is even, and $((m-1)/2)^2 + m - 1$ otherwise. For $m \geq 7$, there is only one line digraph with as many arcs if m is even, and if m is odd, there are two line digraphs, each being the transpose of the other.

Keywords: Line Digraph, Graph theory, Combinatorial, Linegraph

1 Introduction

Introduced in [4], the ***line digraph*** transformation is, given a simple digraph $G = (V, A)$ with m arcs, there is an arc (a, b) in the line digraph $L(G) = (A, E)$ if the terminal node of a in G is the initial node of b. We say G is a ***root digraph*** of $L(G)$. The following caracterization of line digraphs is provided in [4].

Theorem 1 ([4], Theorem 7)**.** *A digraph is a line digraph if and only if none of the **Shortcut**, **Eight** and **Deviation** digraphs shown in Figure 1 is a subgraph, and every **Z** digraph is in a $K_{2,2}$.*



(a) Eight (b) Deviation (c) Shortcut (d) Z

Figure 1: Forbidden subdigraphs.
The subgraph Z is allowed if the dotted arc is present, which forms $K_{2,2}$.

Given a fixed value of m, we want to caracterize the line digraphs of m nodes with maximum number of arcs. Equivalently, we want to caracterize the root digraphs with m arcs maximizing $\Phi(G) = \sum_{v \in V} d_v^+ \cdot d_v^-$ (the outgoing and incoming degrees of v) as it is the number of arcs in its line digraph.

Motivations The undirected version of this question is trivial, the graph with the most edges, the clique, being a line graph. Previous works have been devoted to the same question for root graph with a fixed number of nodes and edges [1, 2]. The result is not obvious on the directed version of the problem, as forbidden digraphs are not induced. Similar work has been done on other graph classes[3, 5, 6].

Results In this paper, we first show in Section 2 that the maximum number of arcs in the line digraphs with m nodes is $\left(\frac{m}{2}\right)^2 + \frac{m}{2}$ arcs if m is even, and $\left(\frac{m-1}{2}\right)^2 + m - 1$ arcs otherwise. In Section 3, we then show that those bounds are tight by providing all the root digraphs achieving them.

Notations Like used previously, $\boldsymbol{m}$ denotes the number of arcs in the root digraphs and thus the number of node in the line digraphs while $\boldsymbol{n}$ denotes the number of nodes in the root digraphs. We use d_u^- and d_u^+ for the incoming and outgoing degrees of u. If necessary, we specify the digraph G with $d_{G,u}$.
A ***circuit*** is a sequence of consecutive arcs whose two extremity nodes are identical. An arc is ***incident*** on a node if one of its extremities is that node.
Given $G = (V, A)$, $\boldsymbol{G - u}$ is the digraph $(V \setminus u, A \setminus \{(u, x), (x, u)\}_{\forall x \in V})$ and $\boldsymbol{G - (u, v)}$ is the digraph $(V, A \setminus (u, v))$. An ***optimal digraph*** G with m arcs is such that, for all digraphs F with m arcs, $\Phi(F) \leq \Phi(G)$. The ***transpose digraph*** $G^T = (V, A^T)$ of the digraph $G = (V, A)$ is such $(a, b) \in A \Leftrightarrow (b, a) \in A^T$.

2 Maximum number of arcs in a line digraph

In this section we provide a close formula for the maximum number of arcs in a line digraph of some given order.

Lemma 1. *Let H be a line digraph of odd order $m \geq 7$, then H contains at least one node with $d_v \leq \frac{m-1}{2}$.*

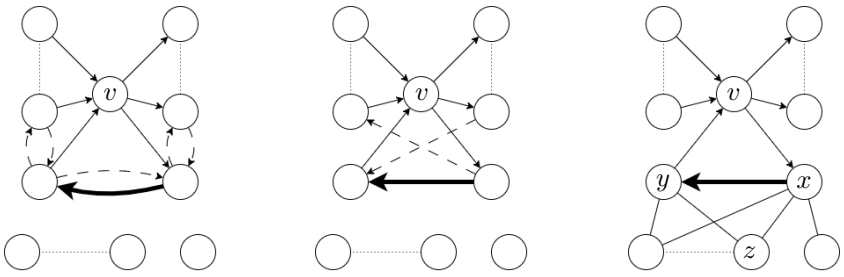
Proof. Let $m = 2p + 1$ and $H = (A, E)$ be a line digraph with order m. We assume that, for every node v_i, $d_{v_i} \geq p + 1$.

As m is odd, some node $v \in A$ does not belong to a 2-circuit otherwise there exists an Eight (figure 1) digraph in H. Consequently there are at least $p + 1$ distinct neighbors to v. With Γ_v^-, the set of predecessor nodes of v, Γ_v^+ the set of successor nodes of v and Γ_v^0 the set of nodes not adjacent to v, we have $|\Gamma_v^-| + |\Gamma_v^+| \geq p + 1$, $|\Gamma_v^0| \leq p - 1$ and $|\Gamma_v^-| + |\Gamma_v^+| + |\Gamma_v^0| = 2p$.

As shown in Figure 2a, if two nodes of Γ_v^- or two nodes of Γ_v^+ are linked, H contains a Shortcut. Similarly, a node of Γ_v^- cannot be the predecessor of a node in Γ_v^+. Consequently, if a node of $\Gamma_v^- \cup \Gamma_v^+$ belongs to a 2-circuit, then the other node of the circuit is in Γ_v^0. As $|\Gamma_v^- \cup \Gamma_v^+| - |\Gamma_v^0| \geq 2$, there exist two nodes in $\Gamma_v^- \cup \Gamma_v^+$ that are not contained in a 2-circuit. Let x be one of those nodes, there are at least $p + 1$ distinct neighbors to it.

As shown in Figure 2b, there is at most one successor in Γ_v^- to a node in Γ_v^+ otherwise H contains a Deviation. Similarly, there is at most one predecessor in Γ_v^+ to a node in Γ_v^-. Thus there is at most one neighbor y in $\Gamma_v^- \cup \Gamma_v^+$ to x.

As there are $p+1$ distinct neighbors to x and $|\Gamma_v^0| \leq p-1$, every node of Γ_v^0 is neighbor of x. As y may belong to at most one 2-circuit, then at least $p - 2$ nodes of Γ_v^0 are neighbors of y (and thus common neighbors of x and y). Note that $p - 2 \geq 1$ as $m \geq 7$. Let z be any of those nodes.



(a) Adding a dotted arc creates a Shortcut; adding a bold arc is possible.

(b) If the bold arc is added, dotted arcs create a Deviation with the node v

(c) Arcs between x or y and Γ_v^0 nodes are not currently oriented.

Figure 2: Arcs between Γ_v^-, Γ_v^+ and Γ_v^0

Figure 2c illustrates those links when $x \in \Gamma_v^+$. The case where $x \in \Gamma_v^-$ is identical (and consists in swapping x and y on the figure). Any orientation of the links (y, z) and (x, z) leads either to a Shortcut or a Deviation. This last contradiction invalidates the hypothesis and prove the lemma. □

Theorem 2. *In a line digraph H of order m, there are at most $\left(\frac{m}{2}\right)^2 + \frac{m}{2}$ arcs if m is even, and at most $\left(\frac{m-1}{2}\right)^2 + m - 1$ arcs otherwise. Those bounds are tight.*

Proof. We prove Theorem 2 by induction on m, initialization for digraphs of order 6 or less is available in the annexe.

If m is odd, by Lemma 1, there exists one node v of degree at most $\frac{m-1}{2}$. Let $H' = H - v$. By induction on the digraph order, one can show that there are at most $\left(\frac{m-1}{2}\right)^2 + \frac{m-1}{2}$ arcs in H'. As none of those arcs are incident to v, then there are at most $\left(\frac{m-1}{2}\right)^2 + \frac{m-1}{2} + \frac{m-1}{2}$ arcs in H. So the theorem is true in the odd case.

If m is even. By induction on the digraph order, for every node v of H, There are at most $B_m = \left(\frac{m-2}{2}\right)^2 + m - 2$ arcs in the digraph $H - v$. We consider the following linear program where x_{ij} is a binary variable representing the presence of the arc (i, j) in the digraph. Note that no constraint is given on the fact that H is a line digraph. Consequently, the program gives an upper bound on the number of arcs. We also provide the dual of the program on the right.

$$\max \sum_{i=1}^{m} \sum_{\substack{j=1 \\ j \neq i}}^{m} x_{ij} \qquad\qquad \min \sum_{i=1}^{m} B_m R_i$$

$$\sum_{\substack{j=1 \\ j \neq i}}^{m} \sum_{\substack{k=1 \\ k \neq i \\ k \neq j}}^{m} X_{jk} \leq B_m \qquad \forall i \in [\![1; m]\!] \qquad \sum_{\substack{k \neq i \\ k \neq j}} R_k \leq 1 \qquad \forall i \neq j \in [\![1; m]\!]$$

$$x_{ij} \in \{0, 1\} \quad \forall i \neq j \in [\![1; m]\!] \qquad R_i \geq 0 \qquad \forall i \in [\![1; m]\!]$$

A feasible primal solution for the linear relaxation consists in setting x_{ij} to $\frac{\left(\frac{m-2}{2}\right)^2 + m - 2}{(m-1)(m-2)}$ for all i and j. A feasible dual solution consists in setting $R_i = 1/(m-2)$ for all i. In the two cases, we get the following objective value

$$\left(\left(\frac{m-2}{2}\right)^2 + m - 2\right) \cdot \frac{m}{m-2} = \left(\frac{(m-2) \cdot m}{4}\right) + m = \left(\frac{m}{2}\right)^2 + \frac{m}{2}$$

We also get the desired upper bound. The two upper bounds are tight as we can achieve them with the digraphs of Figure 3.

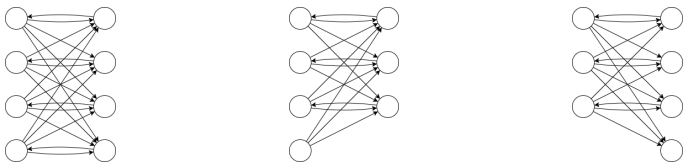


Figure 3: Example of a line digraph with maximum number of arcs. This is a complete oriented bipartite with a maximum number of return arcs, and an additional node in the complete bipartite for the odd case.

Let see that these digraphs are line-digraphs. Since being bipartite, there is no triangle in the undirected graph underlying, and so there is no Shortcut. Each node v has $d_v^- = 1$ or $d_v^+ = 1$, so there are no Eights either. In a Deviation,

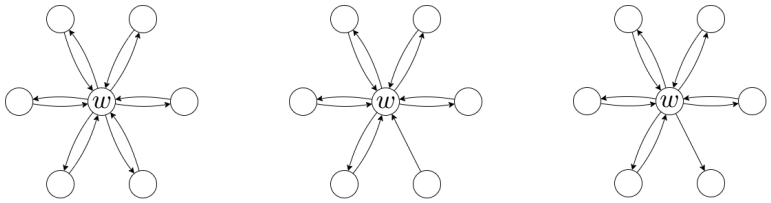
there is a node u with $d_u^+ > 1$, a node v with $d_v^- > 1$, and two paths of size two from u to v. Since this is bipartite, the paths from a node u with an $d_u^+ > 1$ to v with $d_v^- > 1$ are all odd, so there is no Deviation. All nodes with an outgoing degree strictly greater than 1 have the same successors, and all nodes with an ingoing degree strictly greater than 1 have the same predecessors, so there is no Z outside a $K_{2,2}$. This concludes the proof of Theorem 2. □

3 Unicity of maximum line digraphs

Now that we know the maximum number of arcs, let us show that the only digraphs reaching this value are those shown in Figure 3. Since $\Phi(G)$ is the number of arcs in $L(G)$, as a consequence of Theorem 2

Corollary 1. *Let G a digraph with m arcs, $\Phi(G) = \sum_{v \in V} d_v^+ \cdot d_v^- \leq \left(\frac{m}{2}\right)^2 + \frac{m}{2}$ arcs if m is even, and $\Phi(G) \leq \left(\frac{m-1}{2}\right)^2 + m - 1$ arcs otherwise.*

Let O_m be the digraph with a central node w, $\lfloor m/2 \rfloor$ 2-length circuit from the central node and, if m is odd, a final incoming arc on w as shown in Figure 4.



(a) O_m m even (b) O_m m odd (c) O_m^T m odd

Figure 4: Optimal digraphs for $m = 12$ and $m = 11$. The one on the left verifies $\Phi(G) = 6 * 6 + (1 * 1) \cdot 6 = 42$. Those on the right verify $\Phi(G) = 5 * 6 + (1 * 1) \cdot 5 + 1 * 0 = 35$

These are the root digraphs of the line digraphs shown in Figure 3. For theses digraphs, by Theorem 2, $\Phi(O_m) = \frac{(m-1)^2}{4} + m - 1$ if m is odd and $\Phi(O_m) = \Phi(O_m^T) = \frac{m^2}{4} + \frac{m}{2}$ if it's even. Let us show that every optimal digraph with m arcs is isomorphic to O_m or O_m^T. Note that $O_m = O_m^T$ if m is even.

Lemma 2. *If G is optimal, then, for any arc $(u, v) \in A$,*

$$d_u^- + d_v^+ \geq \begin{cases} \frac{m}{2} + 1 & \text{if } m \text{ is even} \\ \frac{m-1}{2} & \text{if } m \text{ is odd} \end{cases}$$

Proof. Consider the digraph $F = G - (u, v)$. Then $\Phi(F) = \Phi(G) - (d_u^- + d_v^+)$.

There are $m-1$ arcs in the digraph F. Suppose m is even. By Theorem 2:

$$\Phi(G) = \frac{m^2}{4} + \frac{m}{2} \quad \text{and} \quad \Phi(F) \leq \frac{(m-2)^2}{4} + m - 2$$
$$\text{So} \qquad d_u^- + d_v^+ \geq \frac{m^2}{4} + \frac{m}{2} - \left(\frac{(m-2)^2}{4} + m - 2\right) \geq \frac{m}{2} + 1$$

The same applies if m is odd, by Theorem 2:

$$\Phi(G) = \frac{(m-1)^2}{4} + m - 1 \quad \text{and} \quad \Phi(F) \leq \frac{(m-1)^2}{4} + \frac{m-1}{2}$$
$$\text{So} \qquad d_u^- + d_v^+ \geq \frac{(m-1)^2}{4} + m - 1 - \left(\frac{(m-1)^2}{4} + \frac{m-1}{2}\right) \geq \frac{m-1}{2} \qquad \square$$

Corollary 2. *If G is optimal, for any pair of arcs (u,v) and (x,y) then, if $F = G - (u,v)$, $d_{G,u}^- + d_{G,v}^+ + d_{F,x}^- + d_{F,y}^+ \geq m$*

Proof. This is an application of Lemma 2 first with arc (u,v) in digraph G and then with arc (x,y) in digraph F. $\square$

Lemmas 3 to 5 show that if G is optimal, it must contain a 2-length circuit.

Lemma 3. *If G is optimal, with no 2-length circuit, then, for any pair of consecutive arcs (u,v) and (v,x) we have $(x,u) \in A$.*

Proof. Let $F = G - (u,v)$. Then $d_{F,v}^- = d_{G,v}^- - 1$. Since there is no 2-length circuit in G, then $x \neq u$ and $d_{F,x}^+ = d_{G,x}^+$. By Corollary 2, $\beta = d_{G,u}^- + d_{G,v}^+ + d_{G,v}^- + d_{G,x}^+ - 1 \geq m$. For all $(s,t) \in A$, if $\alpha^{s,t} = \mathbb{1}_{u=t} + \mathbb{1}_{v=s} + \mathbb{1}_{v=t} + \mathbb{1}_{x=s} \leq 1$ then $\beta = \sum_{(s,t)\in A} \alpha^{s,t} - 1 < m$, contradiction. Note that $\alpha^{s,t} \leq 2$ because $u \neq v$ and $v \neq x$. So $\exists (s,t) \in A$, $\alpha^{s,t} = 2$. There are three cases for arc (s,t) : the arc (v,u) or the arc (x,v), which are excluded since there is no 2-length circuit in G, the last option being the arc (x,u), which is the desired result. $\square$

Lemma 4. *If G is optimal with no 2-length circuit, then every pair of arcs are incident on a same node.*

Proof. Suppose there are two arcs in G with no common extremity, (u,v) and (x,y). Note that the degree of x and y in $G-(u,v)$ is unchanged. By Corollary 2, $\beta = d_u^- + d_v^+ + d_x^- + d_y^+ \geq m$. Let $\alpha^{s,t} = \mathbb{1}_{u=t} + \mathbb{1}_{v=s} + \mathbb{1}_{x=t} + \mathbb{1}_{y=s} \leq 2$. Since $\alpha^{u,v} = \alpha^{x,y} = 0$ and $\beta = \sum_{(s,t)\in A} \alpha^{s,t}$ then $\exists (p,q), (s,t) \in A$, $\alpha^{p,q} = \alpha^{s,t} = 2$. There are four cases, the arcs (v,u), (y,x), (v,x) and (y,u). The first two cases are excluded because G contains no 2-length circuit. So there is the circuit (u,v,x,y,u) in G. According to Lemma 3, the arcs (x,u) and (u,x) are in G, which is excluded by hypothesis. $\square$

Lemma 5. *If G is optimal, then there is at least one 2-length circuit in G.*

Proof. It is assumed that $m \geq 4$. The cases $m = 2$ and $m = 3$ can be handled by exhaustive enumeration. Assume that there is no symmetric arcs in G. By Lemma 4, two arcs of G have a common node. If there is a triangle (u, v, w) in the undirected graph underlying G, any other arc must be incident on at least two nodes of the triangle to have a node in common with each arc. Since $m \geq 4$ then there is a 2-length circuit in G.

If there is no triangle in the undirected graph underlying G, and all pairs of arcs have a node in common, then all arcs are incident to the same node w. If there is no 2-length circuit in G, by Lemma 3, w is either a ***source*** ($d_w^- = 0$) or a ***sink*** ($d_w^+ = 0$), but then $\Phi(G) = 0$ and G is not optimal. □

Lemmas 6 to 9 show that if G is optimal with $m \geq 7$ arcs and there is a 2-length circuit, then G is isomorphic to O_m or O_m^T (Figure 4).

Lemma 6. *If G is optimal and there is a 2-length circuit (u, v, u) then every arc is incident to u or v.*

Proof. Let $F = G - (u, v)$. Then $d_{F,v}^- = d_{G,v}^- - 1$ and $d_{F,u}^+ = d_{G,v}^+ - 1$. By Corollary 2, $d_{G,u}^- + d_{G,v}^+ + d_{G,v}^- + d_{G,u}^+ - 2 \geq m$. Let $\alpha^{s,t} = \mathbb{1}_{u=t} + \mathbb{1}_{v=s} + \mathbb{1}_{v=t} + \mathbb{1}_{u=s}$. We have $\alpha^{u,v} = \alpha^{v,u} = 2$ and $\forall (s, t) \in A \setminus \{(u, v), (v, u)\}$, $\alpha^{s,t} \leq 1$ (otherwise it would be a loop or G would be a multigraph). We have $\sum_{(s,t)\in A} \alpha^{s,t} \geq m + 2$, so $\forall (s, t) \in A \setminus \{(u, v), (v, u)\}$, $\alpha^{s,t} = 1$ and every arc is incident on u or v. □

Lemma 7. *If G is optimal with $m \geq 7$*

- *and there is a 2-length circuit $C_1 = (u, v, u)$*
- *and there is another 2-length circuit C_2*
- *and all nodes of G are neighbors of u or all nodes of G are neighbors of v*

Then all arcs are incident to u or all arcs are incident to v.

Proof. C_2 arcs are incident to u or v by Lemma 6. Without loss of generality, let assume that $C_2 = (u, x, u)$. Then by Lemma 6, every arc is incident to u or v and every arc is incident to u or x so an arc that is not incident to u is necessarily (v, x) or (x, v). There can therefore be no more than 2 arcs that are not incident to u by Lemma 6. If $m \geq 7$, there is at least one other arc incident to u. So if we consider the digraph F where (v, x) and (x, v) are replaced by (u, y) and (y, u) where y is a new node, we verify that $\Phi(F) > \Phi(G)$, which is excluded by optimality of G. □

Lemma 8. *If G is optimal, with a unique 2-length circuit (u, v, u) and all nodes are neighbors of u or all nodes are neighbors of v, then all arcs are incident to u or all arcs are incident to v.*

Proof. Let assume that all nodes are neighbors of u (case of v is symmetrical).

Let separate the nodes of $G - (u, v)$ into six sets:

- X_{uv} successor nodes of u and predecessors of v.
- X_{vu} successor nodes of v and predecessors of u.
- Γ^-_{uv} predecessor nodes of u and v.
- Γ^-_u predecessor nodes of u but not adjacent to v.
- Γ^+_u successor nodes of u but not adjacent to v.
- Γ^+_{uv} successor nodes of u and v.

Since there is no 2-length circuit in G except (u, v, u), there are no other categories. Note that all the nodes in these 6 categories are of degree 1 (if they are adjacent only to u) or 2 (if they are adjacent to u and v).

Let now consider the digraph F where

- for $x \in X_{uv}$, we replace the arc (x, v) by (x, u)
- for $x \in X_{vu}$, we replace the arc (v, x) by (u, x)
- for $x \in \Gamma^-_{uv}$, we replace the arc (x, v) by arc (y, u) where y is a new node.
- for $x \in \Gamma^+_{uv}$, we replace the arc (v, x) by arc (u, z) where z is a new node.

Let $\Gamma = \Gamma^-_{uv} \cup \Gamma^+_{uv} \cup \Gamma^-_u \cup \Gamma^+_u$. So we have

$$\Phi(G) = d^-_u \cdot d^+_u + d^-_v \cdot d^+_v + \sum_{x \in X_{uv}} d^-_x \cdot d^+_x + \sum_{x \in X_{vu}} d^-_x \cdot d^+_x + \sum_{x \in \Gamma} d^-_x \cdot d^+_x$$

$$\Phi(G) = d^-_u \cdot d^+_u + d^-_v \cdot d^+_v + \sum_{x \in X_{uv}} 1 + \sum_{x \in X_{vu}} 1 + 0$$

In F, v is now adjacent only to u with the 2-length circuit. All other arcs incident to v are now incident to u.

$$\Phi(F) = (d^-_u + d^-_v - 1) \cdot (d^+_u + d^+_v - 1) + 1 \cdot 1 + \sum_{x \in X_{uv}} 1 + \sum_{x \in X_{vu}} 1$$

$$\Phi(F) = \Phi(G) + d^-_u \cdot d^+_v + d^-_v \cdot d^+_u - d^-_v - d^+_v - d^-_u - d^+_u + 2$$

$$\Phi(F) = \Phi(G) + (d^-_u \cdot d^+_v + 1 - d^-_u - d^+_v) + (d^-_v \cdot d^+_u + 1 - d^-_v - d^+_u)$$

So F is optimal and since $\Phi(G) = \Phi(F)$ then

$$(d^-_u \cdot d^+_v + 1 - d^-_u - d^+_v) + (d^-_v \cdot d^+_u + 1 - d^-_v - d^+_u) = 0$$

Note that for any pair of positive non-zero integers, we have $xy + 1 \geq x + y$. Equality exists only if $x = 1$ or $y = 1$. We deduce from this inequality that $d^-_u = 1$ or $d^+_v = 1$ and that $d^-_v = 1$ or $d^+_u = 1$. If $d^-_v = d^+_v = 1$ then the lemma is proved. If $d^-_u = d^+_u = 1$ then u has only v as a neighbor, the digraph contains only these two nodes and the lemma is proved. If $d^-_u = d^-_v = 1$ then among the neighbors of u subsist the sets Γ^+_{uv} and Γ^+_u. If $\Gamma^+_{uv} = \emptyset$ then the lemma is proved. So let consider a node $x \in \Gamma^+_{uv}$ and G" the digraph where the arc (v, x) is replaced by (x, u). Let $\Sigma = \sum_{y \in V \setminus \{x,u,v\}} d^+_y d^-_y$.

$$\Phi(G) = d^-_u \cdot d^+_u + d^-_v \cdot d^+_v + d^-_x \cdot d^+_x + \Sigma = 1 \cdot d^+_u + 1 \cdot d^+_v + 2 \cdot 0 + \Sigma$$

$$\Phi(G") = (d^-_u + 1) \cdot d^+_u + d^-_v \cdot (d^+_v - 1) + (d^-_x - 1) \cdot (d^+_x + 1) + \Sigma$$

$$\Phi(G") = \Phi(G) + d^+_u - d^-_v + (d^-_x - 1) \cdot (d^+_x + 1) = \Phi(G) + d^+_u - 1 + 1$$

This contradicts the optimality of G. The case $d_u^+ = 1$ and $d_v^+ = 1$ is symmetrical. The lemma has been proved. □

Lemma 9. *If G is optimal with $m \geq 7$ and a 2-length circuit (u, v, u) then every arc is incident to u or every arc is incident to v.*

Proof. By Lemma 6, every arc is incident to u or v. If any node is neighbor of u or any node is neighbor of v then according to Lemma 7 and 8, the lemma is proved. Else consider $X = \{x_1, x_2, \ldots, x_p\}$ the set of neighbors of u which are not neighbors of v and $Y = \{y_1, y_2, \ldots, y_q\}$ the set of v which are not neighbors of u. Suppose $q \leq p$. Let construct the digraph F where the arcs (y_i, v) (respectively (v, y_i)) are replaced by (x_i, v) (respectively (v, x_i)). By optimality of G, for all $i \leq q$, x_i and y_i are sources or x_i and y_i are sinks. Otherwise, $d^-_{F,x_i} \cdot d^+_{F,x_i} = 1$ and so $\Phi(F) > \Phi(G)$. Without loss of generality, let assume that x_1 and y_1 are sources. So there are the arcs (x_1, u) and (x_1, v) in F. Since G is optimal, so is F. Note that in F every node is neighbor of u or every node is neighbor of v so by Lemma 7 and 8, every arc of F is incident to u or every arc of F is incident to v. This contradicts the existence of (x_1, u) and (x_1, v). □

Theorem 3. *If G is optimal and $m \geq 7$ then G is isomorphic to O_m or O_m^T.*

Proof. By Lemma 5, there are two symmetric arcs (u, v) and (v, u) in G. By Lemma 9, every arc is incident to u or every arc is incident to v. Let assume without loss of generality that they are incident to u. So G is a star where every node is adjacent to u and is connected to u by one arc or two symmetrical arcs. Note that if there is a source and a sink in G connected to u then we can merge these nodes to increase $\Phi(G)$. So there are only sources or only sinks in G. If there are no source and no sink in G, then we are in the case of the first digraph in Figure 4. If there are only sources in G, we can deduce that $d_u^+ \leq d_u^-$. If there is only one source in G, then we are in the case of the last digraph in Figure 4. And if there are two sources x and y in G, then by replacing (y, u) by (u, x), we obtain a digraph F such that

$$\Phi(F) = \Phi(G) - d_u^+ + d_u^- + 1$$

Since $d_u^+ \leq d_u^-$, we have a contradiction with the optimality of G. Similarly, if there are only sinks in G, we are in the case of the second digraph in Figure 4. □

4 Conclusion

In this paper, we have shown that the maximum number of arcs in a line digraph with n nodes is $\left(\frac{n}{2}\right)^2 + \frac{n}{2}$ arcs if n is even, and $\left(\frac{n-1}{2}\right)^2 + n - 1$ arcs otherwise. We have also shown that , for $n \geq 7$, the only line digraphs with so many arcs are those shown in Figure 3.

References

[1] Ábrego, B. M., Fernández-Merchant, S., Neubauer, M. G., and Watkins, W. Sum of squares of degrees in a graph. *JIPAM 10*, 3 (2008).

[2] Ahlswede, R., and Katona, G. O. Graphs with maximal number of adjacent pairs of edges. *Acta Mathematica Hungarica 32*, 1 (1978), 97–120.

[3] Babiński, S., and Grzesik, A. Maximal edge colorings of graphs. *European Journal of Combinatorics 117* (2024), 103824.

[4] Beineke, L. W. Derived graphs and digraphs. *Beiträge zur graphentheorie* (1968), 17–33.

[5] Hoffmann, M., and Reddy, M. M. The number of edges in maximal 2-planar graphs, 2023.

[6] Paul, C., and Telle, J. A. Edge-maximal graphs of branchwidth k: the k-branches. *Discrete mathematics 309*, 6 (2009), 1467–1475.

5 Annexe

Here we present the initialization of the proof of Theorem 2, for line digraphs of order ≤ 6. We show this using the root digraph and we want to maximize $\Phi(G) = \sum_{v \in V} d_v^+ \cdot d_v^-$. All digraphs are assumed to be connected. We call u a ***best node***, if $\forall v \in V, d_v^+ \cdot d_v^- \leq d_u^+ \cdot d_u^-$.

Lemma 10. *If all arcs of a digraph G are incident to the best node b, with x incoming arcs and y outgoing arcs, then $\Phi(G) \leq x * y + min(x, y)$, with equality if there are $max(x, y) + 1$ nodes in G.*

Proof. Let assume that $x \geq y$, the other case is symmetrical. There are at least x neighbors at b. If there are $x + 1$ nodes in G then there is an outgoing arc from all the neighbors of b, and there is an incoming arc to y of them. Thus $\Phi(G) = x * y + min(x, y)$. If there are $x + 1 + k$ nodes in G $(0 \leq k \leq y)$, then there is an incoming arc to k nodes with $d^+ = 0$, an outgoing arc from $x - y + k$ neighbors of b with $d^- = 0$, and there is a 2-length circuit with $y - k$ of them. Thus $\Phi(G) = x * y + y - k \leq x * y + y$. □

We will treat each digraph according to its number of arcs and the value of the best node.
If there are two arcs in G. $\forall v \in V, d_v^+ \cdot d_v^- \leq 1$. If there are two nodes $\{u, v\}$ in G, then there must be the arcs (u, v) and (v, u), thus $\Phi(G) = 2$ (figure 5). If there are three nodes in G, then $\Phi(G) \leq 1$.

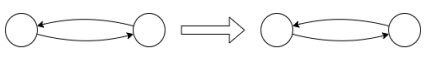


Figure 5: root digraph with four arcs and their line digraph

if there are three arcs in G**.** $\forall v \in V, d_v^+ \cdot d_v^- \leq 2$.

If a best node u is such that $d_u^+ \cdot d_u^- = 2$, then $d_u^+ = 2$ and $d_u^- = 1$ (resp $d_u^+ = 1$ and $d_u^- = 2$). In this case by Lemma 10, $\Phi(G) = 3$ if there are three nodes in G (figure 6, resp figure 7) and $\Phi(G) < 3$ if there are more nodes.

If a best node u is such that $d_u^+ \cdot d_u^- = 1$, and $\Phi(G) \geq 3$, then there are at least three nodes with $d^+ \cdot d^- = 1$ in G, this only happens with the 3-circuit (figure 8). In this case, it requires more than three arcs to have $\Phi(G) > 3$.

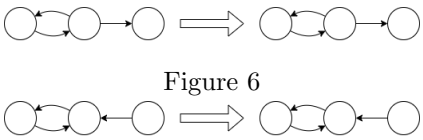


Figure 6

Figure 7

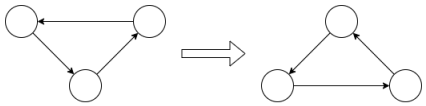


Figure 8

root digraph with four arcs and their line digraph

if there are four arcs in G**.** $\forall v \in V, d_v^+ \cdot d_v^- \leq 4$.

If a best node u is such that $d_u^+ \cdot d_u^- = 4$, then $d_u^+ = 2$ and $d_u^- = 2$. In this case by Lemma 10, $\Phi(G) = 6$ if there are three nodes in G (figure 9) and $\Phi(G) < 6$ if there are more nodes.

If a best node u is such that $d_u^+ \cdot d_u^- = 3$, then $d_u^+ = 3$ and $d_u^- = 1$ (resp $d_u^+ = 1$ and $d_u^- = 3$). In this case by Lemma 10, $\Phi(G) \leq 4 < 6$

If a best node u is such that $d_u^+ \cdot d_u^- = 2$, then $d_u^+ = 2$ and $d_u^- = 1$ (resp $d_u^+ = 1$ and $d_u^- = 2$). Let assume that $\Phi(G) \geq 6$. If there is only one best node in G, then there are at least four nodes with $d^+ \cdot d^- = 1$, then $\sum_{v \in V} d_v^+ > 4$, impossible. If there are two best nodes in G, then there are at least two nodes with $d^+ \cdot d^- = 1$, then $\sum_{v \in V} (d_v^+ + d_v^-) > 2 * 4$, impossible. If there are at least three best nodes in G, then $\sum_{v \in V} (d_v^+ + d_v^-) > 2 * 4$, impossible.

If a best node u is such that $d_u^+ \cdot d_u^- = 1$ and $\Phi(G) \geq 6$, then there are at least six nodes with $d^+ \cdot d^- = 1$ in G, then $\sum_{v \in V} d_v^+ > 4$, impossible.

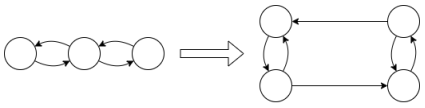


Figure 9: root digraph with four arcs and their line digraph

if there are five arcs in G**.** $\forall v \in V, d_v^+ \cdot d_v^- \leq 6$.

If a best node u is such that $d_u^+ \cdot d_u^- = 6$, then $d_u^+ = 3$ and $d_u^- = 2$ (resp $d_u^+ = 2$ and $d_u^- = 3$). In this case by Lemma 10, $\Phi(G) = 8$ if there are four nodes in G (figure 10a, resp figure 10b) and $\Phi(G) < 8$ if there are more nodes.

If a best node u is such that $d_u^+ \cdot d_u^- = 4$, then $d_u^+ = 4$ and $d_u^- = 1$ (resp $d_u^+ = 1$ and $d_u^- = 4$) or $d_u^+ = 2$ and $d_u^- = 2$. If $d_u^+ = 4$ (resp $d_u^- = 4$), then by Lemma 10, $\Phi(G) \leq 5 < 8$. If $d_u^+ = 2$ and $d_u^- = 2$, then there is only one best node in G, otherwise it would have at least six arcs since two nodes can only share two arcs. For the same reason, there is not a node v such that $d_v^+ \cdot d_v^- = 3$

in G. If there is a node v such that $d_v^+ \cdot d_v^- = 2$ in G, then it must share two arcs with u. Then there are two arcs between u and v, an incoming arc on u, an outgoing arc from u, and an incident arc on v. If there are three nodes in G, then $\Phi(G) = 4 + 2 + 2 = 8$ (figure 10c). If there are four nodes in G, then $\Phi(G) \leq 4 + 2 + 1 + 0 < 8$. If there are five nodes in G, then $\Phi(G) = 6 < 8$. If there is no node such that $d^+ \cdot d^- = 2$ in G and $\Phi(G) \geq 8$, then there are at least four nodes with $d^+ \cdot d^- = 1$, and $\sum_{v \in V} d_v^+ > 5$, impossible.

If a best node u is such that $d_u^+ \cdot d_u^- = 3$, then $d_u^+ = 3$ and $d_u^- = 1$ (resp $d_u^+ = 1$ and $d_u^- = 3$) and there is only one best node in G, otherwise it would have at least six arcs since two nodes can only share two arcs. Let assume $\Phi(G) \geq 8$. If there is one node such that $d^+ \cdot d^- = 2$ in G, then there are at least three nodes with $d^+ \cdot d^- = 1$, and $\sum_{v \in V}(d_v^+ + d_v^-) > 2 * 5$. If there are two nodes such that $d^+ \cdot d^- = 2$ in G, then there is at least one node with $d^+ \cdot d^- = 1$, and $\sum_{v \in V}(d_v^+ + d_v^-) > 2 * 5$. If there are at least three nodes such that $d^+ \cdot d^- = 2$ in G, then $\sum_{v \in V}(d_v^+ + d_v^-) > 2 * 5$. If there is no node such that $d^+ \cdot d^- = 2$ in G and $\Phi(G) \geq 8$, then there is at least five nodes with $d^+ \cdot d^- = 1$, and $\sum_{v \in V} d_v^+ > 5$, impossible.

If a best node u is such that $d_u^+ \cdot d_u^- = 2$, then $d_u^+ = 2$ and $d_u^- = 1$ (resp $d_u^+ = 1$ and $d_u^- = 2$). Let assume that $\Phi(G) \geq 8$. If there are at most two best nodes in G, then there are at least four nodes with $d^+ \cdot d^- = 1$, and $\sum_{v \in V}(d_v^+ + d_v^-) > 2 * 5$, impossible. If there are three best nodes in G, then there are at least two nodes with $d^+ \cdot d^- = 1$, and $\sum_{v \in V}(d_v^+ + d_v^-) > 2 * 5$, impossible. If there are at least four best nodes in G, then $\sum_{v \in V}(d_v^+ + d_v^-) > 2 * 5$, impossible.

If a best node u is such that $d_u^+ \cdot d_u^- = 1$ and $\Phi(G) \geq 8$, then there are at least eight nodes with $d^+ \cdot d^- = 1$ in G, and $\sum_{v \in V} d_v^+ > 5$, impossible.

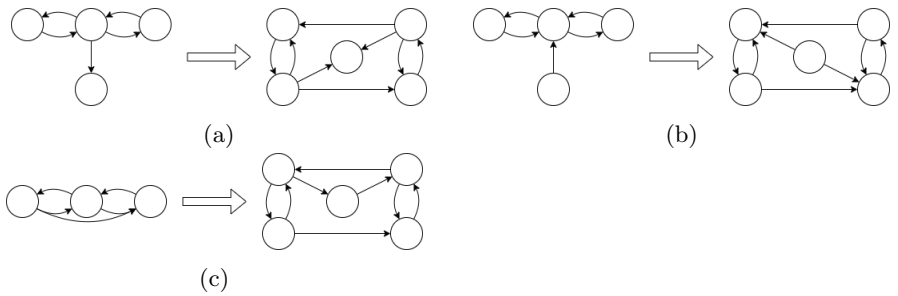


Figure 10: root digraph with five arcs and their line digraph

if there are six arcs in G. $\forall v \in V, d_v^+ \cdot d_v^- \leq 9$.

If a best node u is such that $d_u^+ \cdot d_u^- = 9$, then $d_u^+ = 3$ and $d_u^- = 3$. In this case by Lemma 10, $\Phi(G) = 12$ if there are four nodes in G (figure 11a) and $\Phi(G) < 12$ if there are more nodes.

If a best node u is such that $d_u^+ \cdot d_u^- = 8$, then $d_u^+ = 4$ and $d_u^- = 2$ (resp $d_u^+ = 2$ and $d_u^- = 4$). In this case by Lemma 10, $\Phi(G) \leq 10 < 12$.

If a best node u is such that $d_u^+ \cdot d_u^- = 6$, then $d_u^+ = 3$ and $d_u^- = 2$ (resp $d_u^+ = 2$ and $d_u^- = 3$) and there is no other node such that $d^+ \cdot d^- \geq 3$ in G,

otherwise it would have at least seven arcs since two nodes can only share two arcs. Let assume $\Phi(G) \geq 12$. If there is one node such that $d^+ \cdot d^- = 2$ in G, then there are at least four nodes with $d^+ \cdot d^- = 1$, and $\sum_{v \in V}(d_v^+ + d_v^-) > 2 * 6$. If there are two nodes such that $d^+ \cdot d^- = 2$ in G, then there are at least two nodes with $d^+ \cdot d^- = 1$, and $\sum_{v \in V}(d_v^+ + d_v^-) > 2 * 6$. If there are at least three nodes such that $d^+ \cdot d^- = 2$, then $\sum_{v \in V}(d_v^+ + d_v^-) > 2 * 6$. If there is no node such that $d^+ \cdot d^- = 2$ in G and $\Phi(G) \geq 12$, then there are at least six nodes with $d^+ \cdot d^- = 1$, and $\sum_{v \in V} d_v^+ > 6$, impossible.

If a best node u is such that $d_u^+ \cdot d_u^- = 5$, then $d_u^+ = 5$ and $d_u^- = 1$ (resp $d_u^+ = 1$ and $d_u^- = 5$). In this case by Lemma 10, $\Phi(G) \leq 10 < 12$.

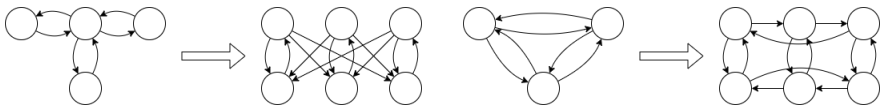
If a best node u is such that $d_u^+ \cdot d_u^- = 4$, then $d_u^+ = 4$ and $d_u^- = 1$ (resp $d_u^+ = 1$ and $d_u^- = 4$) or $d_u^+ = 2$ and $d_u^- = 2$. If $d_u^+ = 4$ (resp $d_u^- = 4$), then there is no other node such that $d^+ \cdot d^- \geq 2$ in G, otherwise it would have at least seven arcs since two nodes can only share two arcs. If $\Phi(G) \geq 12$, then there are at least eight nodes with $d^+ \cdot d^- = 1$ in G, and $\sum_{v \in V} d_v^+ > 6$, impossible. If $d_u^+ = 2$ and $d_u^- = 2$, then there are at most three nodes such that $d^+ \cdot d^- \geq 3$ in G, otherwise it would have at least seven arcs since two nodes can only share two arcs. If there are three best nodes in G, then $\Phi(G) = 12$ (figure 11a). Else, if there is a node v such that $d_v^+ \cdot d_v^- = 3$ in G, then it must share two arcs with u. So there are two arcs between u and v, an incoming arc on u, an outgoing arc from u, and two arcs with the same orientation on v. If there are four nodes in G, then $\Phi(G) \leq 4 + 3 + 2 + 0 < 12$. If there are five nodes in G, then $\Phi(G) \leq 4 + 3 + 1 + 0 + 0 < 12$. If there are six nodes in G, then $\Phi(G) \leq 4 + 3 + 0 + 0 + 0 < 12$. If there are exactly two best nodes in G, then they must share two arcs and there must be at least four nodes in G. If there are four nodes in G, then $\Phi(G) = 4 + 4 + 1 + 1 < 12$. If there are five nodes in G, then $\Phi(G) = 4 + 4 + 1 + 0 + 0 < 12$. If there are six nodes in G, then $\Phi(G) = 4+4+0+0+0+0 < 12$. If there is only one best node in G, let assume $\Phi(G) \geq 12$. If there are one or two nodes such that $d^+ \cdot d^- = 2$ in G, then there are at least four nodes with $d^+ \cdot d^- = 1$, and $\sum_{v \in V}(d_v^+ + d_v^-) > 2 * 6$. If there are at least three nodes such that $d^+ \cdot d^- = 2$ in G, then $\sum_{v \in V}(d_v^+ + d_v^-) > 2 * 6$. If there is no node such that $d^+ \cdot d^- = 2$ in G and $\Phi(G) \geq 12$, then there are at least eight nodes with $d^+ \cdot d^- = 1$, and $\sum_{v \in V} d_v^+ > 6$, impossible.

If a best node u is such that $d_u^+ \cdot d_u^- = 3$, then $d_u^+ = 3$ and $d_u^- = 1$ (resp $d_u^+ = 1$ and $d_u^- = 3$) and there are at most three nodes such that $d^+ \cdot d^- = 3$ in G, otherwise it would have at least seven arcs since two nodes can only share two arcs. Let assume $\Phi(G) = 12$, if there are at least two best nodes $\{u, v\}$ in G, then they must share two arcs. So there are two arcs between u and v. If $d_u^+ = d_v^+$, then there are four, five or six nodes in G and $\Phi(G) = 3+3+0 < 12$. If $d_u^+ \neq d_v^+$, then there are four nodes in G, $\Phi(G) < 3+3+1+1 < 12$. If there are five nodes in G, then $\Phi(G) = 3+3+1+0+0 < 12$. If there are six nodes in G, then $\Phi(G) = 3+3+0+0+0+0 < 12$. Let assume that there is only one best node in G. If there are one or two nodes such that $d^+ \cdot d^- = 2$ in G, then there are at least five nodes with $d^+ \cdot d^- = 1$, and $\sum_{v \in V}(d_v^+ + d_v^-) > 2 * 6$. If there are at least three nodes such that $d^+ \cdot d^- = 2$ in G, then $\sum_{v \in V}(d_v^+ + d_v^-) > 2 * 6$. If there is no node such that $d^+ \cdot d^- = 2$ in G and $\Phi(G) \geq 12$, then there are at

least nine nodes with $d^+ \cdot d^- = 1$, and $\sum_{v \in V} d_v^+ > 6$, impossible.

If a best node u is such that $d_u^+ \cdot d_u^- = 2$, then $d_u^+ = 2$ and $d_u^- = 1$ (resp $d_u^+ = 1$ and $d_u^- = 2$). Let assume that $\Phi(G) \geq 12$. If there are at most three best nodes in G, then there are at least six nodes with $d^+ \cdot d^- = 1$, and $\sum_{v \in V} d_v^+ > 6$, impossible. If there are four best nodes in G, then there are at least four nodes with $d^+ \cdot d^- = 1$, and $\sum_{v \in V} d_v^+ > 6$, impossible. If there are at least five best nodes in G, then $\sum_{v \in V} (d_v^+ + d_v^-) > 2 * 6$, impossible.

If a best node u is such that $d_u^+ \cdot d_u^- = 1$ and $\Phi(G) \geq 12$, then there are at least twelve nodes with $d^+ \cdot d^- = 1$ in G, then $\sum_{v \in V} d_v^+ > 6$, impossible.



(a) (b)

Figure 11: root digraph with six arcs and their line digraph