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Study of a class of triangular starvation driven cross-diffusion systems

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Abstract

We study the existence, regularity and uniqueness for a general class of triangular reaction-cross-diffusion systems coming from the study of *starvation driven* behavior for two species in competition. This study involves an equivalent system in *non-divergence* form, for which existence can be obtained thanks to Schauder's fixed point theorem.

Keywords. Cross-diffusion, starvation-driven models, existence, uniqueness
2020 Mathematics Subject Classification. Primary 35K57, Secondary 92D25

1 Introduction

This paper is devoted to the analysis of a class of triangular cross-diffusion systems with unknowns $u = u(t, x) \geq 0$ and $v = v(t, x) \geq 0$, representing the densities of two populations. We assume that the species represented by v diffuses with a given constant rate, while the species represented by u diffuses with a rate $B(u, v)$, which depends on both u and v . In other terms, the equation satisfied by u involves a cross-diffusion term. The system is said to be triangular as no cross-diffusion term appears in the equation satisfied by v . Moreover, we include reaction terms modeling the competition between the two species.

We consider the evolution on a smooth (C^∞) bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, over a time T . Denoting $\Omega_T := (0, T) \times \Omega$, the system writes as

$$\begin{cases} \partial_t u = \Delta(u B(u, v)) + u f(u, v), & \text{in } \Omega_T, \\ \partial_t v = d_v \Delta v + v g(u, v), & \text{in } \Omega_T. \end{cases} \quad (1.1)$$

It is endowed with zero flux (homogeneous Neumann) boundary conditions

$$\nabla(u B(u, v)) \cdot \sigma = \nabla v \cdot \sigma = 0, \quad \text{on } (0, T) \times \partial\Omega, \quad (1.2)$$

and with nonnegative initial data

$$u(0, x) = u_{\text{in}}(x) \geq 0, \quad v(0, x) = v_{\text{in}}(x) \geq 0, \quad x \in \Omega. \quad (1.3)$$

In the sequel, we will consider a diffusion coefficient d_v and functions B, f, g which satisfy the following assumptions.

Assumption A. The diffusion rate associated to the species v is strictly positive, i.e.

$$d_v > 0.$$

The functions f, g are $C^1(\mathbb{R}_+ \times \mathbb{R}_+)$ and satisfy conditions which are typical of Lotka-Volterra type reaction terms for competing species, i.e. there exist constants $C_f, C'_f, C_g, C'_g > 0$ such that for all $u, v \geq 0$ (and denoting by ∂_1, ∂_2 the derivatives with respect to the first and second variable)

$$\begin{aligned} -C_f(1+u+v) &\leq f(u, v) \leq C_f, \\ -C_g(1+u+v) &\leq g(u, v) \leq C_g, \\ |\partial_1 f(u, v)|, |\partial_2 f(u, v)| &\leq C'_f, \\ |\partial_1 g(u, v)|, |\partial_2 g(u, v)| &\leq C'_g. \end{aligned} \tag{R1}$$

We denote

$$A(u, v) := uB(u, v), \tag{1.4}$$

and assume that

$$B \in C^1(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+). \tag{D1}$$

We also suppose that there exist $a_0, a_1, a_3 > 0$ such that for all $u, v \geq 0$,

$$0 < a_0 \leq B(u, v) \leq a_1, \tag{1.5}$$

and

$$|\partial_2 B(u, v)| \leq a_3. \tag{1.6}$$

We assume moreover that there exists $a_2 > 0$ such that for all $u, v \geq 0$,

$$0 < a_0 \leq \partial_1 A(u, v) \leq a_1 \quad \text{and} \quad |\partial_2 A(u, v)| \leq a_2. \tag{D2}$$

The motivation for studying such systems comes from the modeling of the effect of starvation on the movement of individuals belonging to species in competition. They are sometimes called *starvation driven* cross-diffusion systems. In [3, Chapter 3], a whole class of *starvation driven* cross-diffusion systems is obtained from a microscopic description of the interaction between individuals. The obtained systems are

$$\begin{cases} \partial_t u - \Delta(d_a u_a^*(u, v) + d_b u_b^*(u, v)) = f_a(u_a^*(u, v), u_b^*(u, v), v) + f_b(u_a^*(u, v), u_b^*(u, v), v), \\ \partial_t v - d_v \Delta v = f_v(u_a^*(u, v), u_b^*(u, v), v), \end{cases}$$

over $(0, +\infty) \times \Omega$, with diffusion coefficients $d_a, d_b, d_v > 0$, $d_a \neq d_b$, and reaction terms f_a, f_b, f_v of Lotka-Volterra type (for competing species), where $(u_a^*(u, v), u_b^*(u, v))$ are defined as the unique solution to the nonlinear system

$$\begin{cases} u = u_a^*(u, v) + u_b^*(u, v), \\ \phi(bu_b^*(u, v) + dv)u_b^*(u, v) - \psi(au_a^*(u, v) + cv)u_a^*(u, v) = 0, \end{cases}$$

where ϕ and ψ are suitable conversion rates, and $a, b, c, d > 0$ are parameters, see [3, Chapter 3] for more details. Those systems belong to the class of systems studied in this paper by setting

$$\begin{aligned} A(u, v) &:= d_a u_a^*(u, v) + d_b u_b^*(u, v), & vg(u, v) &:= f_v(u_a^*(u, v), u_b^*(u, v), v), \\ u f(u, v) &:= f_a(u_a^*(u, v), u_b^*(u, v), v) + f_b(u_a^*(u, v), u_b^*(u, v), v). \end{aligned}$$

Our main theorem shows the existence of strong solutions to system (1.1)–(1.3) under Assumption A.

Theorem 1.1. *Consider a smooth bounded domain $\Omega \subset \mathbb{R}^N$ for $N \geq 1$ and suppose that d_v, B, f, g satisfy Assumption A. Consider also nonnegative initial data $u_{in} \in (L^\infty \cap H^1)(\Omega)$, $v_{in} \in (L^\infty \cap W^{2,p}(\Omega) \cap H^3)(\Omega)$ for all $p \in [1, +\infty)$, compatible with the Neumann boundary condition (1.2).*

Then, there exists a strong nonnegative global solution (u, v) to system (1.1)–(1.3), in the sense that

- i) each term in the two equations of (1.1) is locally integrable and the equations are satisfied a.e. in Ω_T ,*
- ii) the boundary and initial conditions (1.2), (1.3) hold in the sense of traces.*

Moreover, for all $T > 0$ and for $i, j = 1, \dots, N$, it holds for all $p \in [1, +\infty)$

- i) $u \in L^\infty([0, T]; L^p(\Omega))$, $\partial_{x_i} u \in L^\infty([0, T]; L^2(\Omega))$, $\partial_t u, \partial_{x_i x_j}^2 A(u, v) \in L^2(\Omega_T)$,*
- ii) $v \in L^\infty(\Omega_T)$, $\partial_t v, \partial_{x_i x_j}^2 v \in L^p(\Omega_T)$, $\partial_{t t}^2 v, \partial_{x_i x_j}^3 v \in L^2(\Omega_T)$.*

Finally, if $N \leq 3$, it holds for $i, j = 1, \dots, N$,

$$u \in L^\infty(\Omega_T), \quad \partial_{x_i} u \in L^4(\Omega_T), \quad \partial_t v, \partial_{x_i x_j}^2 v \in L^2([0, T]; L^\infty(\Omega)),$$

and if $N = 1$ and $u_{in} \in W^{1,p}(\Omega)$ for all $p \in [1, +\infty)$, it holds

$$\partial_x u \in L^p(\Omega_T).$$

Theorem 1.1 is proved in *Section 2*. The main difficulty in constructing strong solutions to system (1.1)–(1.3) is the presence of the nonlinear cross-diffusion term in the equation satisfied by u . A key estimate is formally obtained by using suitable multipliers when considering the evolution of u in (1.1). It leads to the estimates: $\partial_{x_i} u \in L^\infty([0, T]; L^2(\Omega))$, $\partial_t u, \partial_{x_i x_j}^2 A(u, v) \in L^2(\Omega_T)$, for $i, j = 1, \dots, N$. We present in this paper an original way of finding a suitable regularization preserving these estimates (see *Subsection 2.1*). This is achieved by first proving the existence of solutions to a regularized system in a *non-divergence* form, using a convenient change of variable (see *Subsections 2.2, 2.3*). Then, we show uniform (with respect to the regularization parameter) a priori bounds, which allow to pass to the limit when the regularization parameter tends to 0 (see *Subsection 2.4*).

We conclude this introduction with the statement of two stability/uniqueness results. If the space dimension is $N \leq 2$, the regularity of the strong solutions obtained in *Theorem 1.1*, is sufficient to prove stability in a strong norm (see *Theorem 1.2*) under the following slightly stricter assumption.

Assumption B. The diffusivity function A is such that

$$\partial_{i j}^2 A \text{ is bounded for } i, j = 1, 2.$$

This regularity is also sufficient to ensure stability in a weak norm if $N \leq 3$, without the extra Assumption B. More precisely, we prove in *Theorem 1.3* a stability result in $(H^1)'$ (Ω), denoted as the dual space of $H^1(\Omega)$, with the norm

$$\|w\|_{(H^1)'\Omega}^2 := (w)_\Omega^2 + \|\nabla \phi\|_{L^2(\Omega)}^2, \quad w \in L^2(\Omega), \quad (1.7)$$

where $(w)_\Omega$ is the average value of w in Ω and ϕ is the unique solution of the Neumann problem

$$-\Delta\phi = w - (w)_\Omega, \quad \text{in } \Omega, \quad \nabla\phi \cdot \sigma = 0 \quad \text{on } \partial\Omega, \quad (\phi)_\Omega = 0.$$

Theorem 1.2. *Let $N \leq 2$ and Ω be a smooth bounded open set of \mathbb{R}^N . We suppose that Assumptions A and B hold, and we take two solutions (u_i, v_i) , $i = 1, 2$ of system (1.1), (1.2), given by Theorem 1.1, corresponding to the nonnegative initial data $(u_{i,in}, v_{i,in})$, $i = 1, 2$ with $u_{i,in} \in (L^\infty \cap H^1)(\Omega)$, $v_{i,in} \in H^3(\Omega)$.*

Then, there exists a constant $C_{stab} > 0$, depending on Ω , T , a_0 , a_1 , a_2 , a_3 , d_v , C_f , C'_f , C_g , C'_g , $\|\partial_{ij}^2 A\|_\infty$, and on $\|u_2\|_{L^\infty(\Omega_T)}$, $\|\nabla u_2\|_{L^4(\Omega_T)}$, $\|v_2\|_{L^\infty(\Omega_T)}$, $\|\nabla v_2\|_{L^2([0,T];L^\infty(\Omega))}$, such that

$$\begin{aligned} & \|u_1 - u_2\|_{L^\infty([0,T];L^2(\Omega)) \cap L^2([0,T];H^1(\Omega))}^2 + \|v_1 - v_2\|_{L^\infty([0,T];L^2(\Omega)) \cap L^2([0,T];H^1(\Omega))}^2 \\ & \leq C_{stab} (\|u_{1,in} - u_{2,in}\|_{L^2(\Omega)}^2 + \|v_{1,in} - v_{2,in}\|_{L^2(\Omega)}^2). \end{aligned} \quad (1.8)$$

Finally, if $u_{1,in} = u_{2,in}$ and $v_{1,in} = v_{2,in}$ for a.e. $x \in \Omega$, then

$$u_1(t, x) = u_2(t, x) \quad \text{and} \quad v_1(t, x) = v_2(t, x), \quad \text{a.e. } (t, x) \in \Omega_T,$$

so that uniqueness holds for system (1.1), (1.2).

Theorem 1.3. *Let $N \leq 3$ and Ω be a smooth bounded open set of \mathbb{R}^N . We suppose that Assumption A holds, and we take two solutions (u_i, v_i) , $i = 1, 2$ of system (1.1), (1.2), given by Theorem 1.1, corresponding to the nonnegative initial data $(u_{i,in}, v_{i,in})$, $i = 1, 2$ with $u_{i,in} \in (L^\infty \cap H^1)(\Omega)$, $v_{i,in} \in (W^{2,p} \cap H^3)(\Omega)$ for all $p \in [1, +\infty)$.*

Then, there exists a constant $C'_{stab} > 0$, depending on Ω , T , a_0 , a_1 , a_2 , a_3 , d_v , C_f , C'_f , C_g , C'_g , and on $\|u_1\|_{L^\infty(\Omega_T)}$, $\|u_2\|_{L^\infty(\Omega_T)}$, $\|v_1\|_{L^\infty(\Omega_T)}$, $\|v_2\|_{L^\infty(\Omega_T)}$, such that

$$\begin{aligned} & \|u_1 - u_2\|_{L^\infty([0,T];(H^1)'(\Omega))}^2 + \|v_1 - v_2\|_{L^\infty([0,T];(H^1)'(\Omega))}^2 \\ & \leq C'_{stab} (\|u_{1,in} - u_{2,in}\|_{(H^1)'(\Omega)}^2 + \|v_{1,in} - v_{2,in}\|_{(H^1)'(\Omega)}^2). \end{aligned} \quad (1.9)$$

Finally, if $u_{1,in} = u_{2,in}$ and $v_{1,in} = v_{2,in}$ for a.e. $x \in \Omega$, then

$$u_1(t, x) = u_2(t, x) \quad \text{and} \quad v_1(t, x) = v_2(t, x), \quad \text{a.e. } (t, x) \in \Omega_T,$$

so that uniqueness holds for system (1.1), (1.2).

Remark 1. *The stability results only depend on the dimension through the assumptions on the solutions (as obtained in Theorem 1.1) and can be rephrased as conditional stability results in all dimensions.*

For the weak stability result, the stability constant depends on both solutions but the dependency on the first solution (u_1, v_1) can be measured in a weaker norm, see the remark at the end of the proof of Theorem 1.3.

For the modeling of biological populations in starvation driven situations, we refer to [4], [14] and [3]. We refer to [18] for one of the first models in population dynamics involving cross-diffusion terms. Existence, uniqueness and regularity results for triangular cross-diffusion systems can be found in [16], [19], [5], [6], [7] (note here that by triangular, we mean that cross-diffusion terms appear only in one of the equations of

the system). Finally, for results on the stability of equilibria for cross-diffusion models, we refer to [13], [10], [11], [12], [2], [8], and the references therein.

The rest of the paper is organised as follows: *Section 2* aims to prove the existence result, stated in *Theorem 1.1*. More precisely, in *Subsection 2.1*, we introduce a regularized system in *non-divergence* form and in *Subsection 2.2*, we prove the existence of a strong solution to this regularized system. Then, we explain in *Subsection 2.3* how it enables to obtain existence for an equivalent regularized system in *divergence* form. In *Subsection 2.4*, we conclude the proof of *Theorem 1.1* by removing the regularization. The stability statements are then proven in *Section 3*. A classical result for parabolic equations is finally recalled in *Appendix A*.

2 Proof of the main Theorem 1.1

2.1 Introduction of a regularized system in *non-divergence* form

We first introduce a parabolic system which is formally equivalent to (1.1)–(1.3). For this, we consider the equation satisfied by $a := A(u, v)$.

By assumption (D2), we can define the reciprocal U of A with respect to the first variable, that is, for a given $v \geq 0$,

$$a = A(u, v) \iff u = U(a, v). \quad (2.1)$$

Using this change of variable, we can rewrite (1.1)–(1.3) as follows:

$$\begin{cases} \partial_t a = \mu(a, v) \Delta a + a s(a, v, \partial_t v), & \text{in } \Omega_T, \\ \partial_t v = d_v \Delta v + v g(U(a, v), v), & \text{in } \Omega_T, \\ \nabla a \cdot \sigma = \nabla v \cdot \sigma = 0, & \text{on } (0, T) \times \partial\Omega, \end{cases}$$

with $a(0, \cdot) = a_{\text{in}} := A(u_{\text{in}}, v_{\text{in}})$, $v(0, \cdot) = v_{\text{in}}$,

$$\mu(a, v) := \partial_1 A(U(a, v), v), \quad (2.2)$$

and for all $a > 0$

$$s(a, v, \partial_t v) := \frac{U(a, v)}{a} [f(U(a, v), v) \partial_1 A(U(a, v), v) + \partial_2 B(U(a, v), v) \partial_t v].$$

By (D1), (D2) we observe that from (2.2), $\mu \in C(\mathbb{R}_+ \times \mathbb{R}_+)$ and

$$\text{for all } a, v \geq 0, \quad 0 < a_0 \leq \mu(a, v) \leq a_1, \quad (2.3)$$

and thanks to estimate (1.5), we get for all $a > 0$ and $v \geq 0$

$$0 < \frac{1}{a_1} \leq \frac{U(a, v)}{a} \leq \frac{1}{a_0}. \quad (2.4)$$

Moreover by (D1), (D2), the implicit function theorem guarantees the C^1 character of U and, for all $a, v \geq 0$, the estimates

$$\frac{1}{a_1} \leq \partial_1 U(a, v) = \left(\partial_1 A(U(a, v), v) \right)^{-1} = \frac{1}{\mu(a, v)} \leq \frac{1}{a_0}, \quad (2.5)$$

using (2.2), (2.3), and thanks to (D2),

$$-\frac{a_2}{a_0} \leq \partial_2 U(a, v) = -\frac{\partial_2 A(U(a, v), v)}{\mu(a, v)} \leq \frac{a_2}{a_0}. \quad (2.6)$$

We now introduce the following truncated-regularized system in *non-divergence* form for any $\varepsilon, M > 0$,

$$\begin{cases} \partial_t a_{\varepsilon, M} = \mu(a_{\varepsilon, M}, v_{\varepsilon, M}) \Delta a_{\varepsilon, M} + a_{\varepsilon, M} s_M(a_{\varepsilon, M}, v_{\varepsilon, M}, \partial_t v_{\varepsilon, M}), & \text{in } \Omega_T, \\ \partial_t v_{\varepsilon, M} = d_v \Delta v_{\varepsilon, M} + v_{\varepsilon, M} g_{\varepsilon, M}(U(a_{\varepsilon, M}, v_{\varepsilon, M}), v_{\varepsilon, M}), & \text{in } \Omega_T, \\ \nabla a_{\varepsilon, M} \cdot \sigma = \nabla v_{\varepsilon, M} \cdot \sigma = 0, & \text{on } (0, T) \times \partial\Omega, \end{cases} \quad (2.7)$$

with

$$s_M(a, v, \partial_t v) := \frac{U(a, v)}{a} [f(\min\{U(a, v), M\}, v) \partial_1 A(U(a, v), v) + \partial_2 B(U(a, v), v) \partial_t v], \quad (2.8)$$

and where we define, for a.e. $(t, x) \in \mathbb{R}^{N+1}$ and for all $\varepsilon > 0$,

$$g_{\varepsilon, M}(u, v) := g(\min\{u, M\}, \min\{v, M\}) *_{t, x} \varphi_\varepsilon. \quad (2.9)$$

Note that, slightly abusing notations, u and v in (2.9) are identified to their respective extension defined on \mathbb{R}^{N+1} , by continuity in time outside $(0, T)$ and by zero in space outside Ω . Finally, $*_{t, x}$ stands for the convolution operation in time and space variables and $(\varphi_\varepsilon)_{\varepsilon > 0}$ is a family of standard mollifiers on \mathbb{R}^{N+1} .

Moreover, the system (2.7)–(2.9) is endowed with the initial data

$$\begin{aligned} a_{\varepsilon, M}(0, x) &= a_{\text{in}, \varepsilon}(x) = A((u_{\text{in}} * x \psi_\varepsilon)(x), (v_{\text{in}} * x \psi_\varepsilon)(x)), & x \in \Omega, \\ v_{\varepsilon, M}(0, x) &= v_{\text{in}, \varepsilon}(x) = (v_{\text{in}} * x \psi_\varepsilon)(x), & x \in \Omega, \end{aligned} \quad (2.10)$$

where again, $u_{\text{in}}, v_{\text{in}}$ are extended by zero outside Ω , and $(\psi_\varepsilon)_{\varepsilon > 0}$ is a family of standard mollifiers on \mathbb{R}^N .

It is worth noticing that the regularization and truncation only affect the reaction part in (2.7) and the initial conditions (2.10). Note also that we truncate the function f only with respect to u , while no truncation with respect to v is needed.

2.2 Existence for the regularized system in *non-divergence* form

In this subsection, we prove existence to system (2.7)–(2.10). We first state the following existence result.

Proposition 2.1. *Let $N \geq 1$ and Ω be a smooth bounded open subset of \mathbb{R}^N . We suppose that the parameters d_v, B, f, g satisfy Assumption A. We consider nonnegative initial data $(a_{\text{in}}, v_{\text{in}}) \in (L^\infty \cap H^1)(\Omega) \times L^\infty(\Omega)$.*

Then, for any fixed $\varepsilon, M > 0$, there exists a nonnegative strong (in the same sense as in Theorem 1.1) solution $(a_{\varepsilon, M}, v_{\varepsilon, M})$ to (2.7)–(2.10), such that for all $i, j = 1, \dots, N$,

- i) $a_{\varepsilon, M} \in L^\infty(\Omega_T)$, $\partial_t a_{\varepsilon, M}, \partial_{x_i x_j}^2 a_{\varepsilon, M} \in L^2(\Omega_T)$, $\partial_{x_i} a_{\varepsilon, M} \in L^4(\Omega_T)$,
- ii) $v_{\varepsilon, M}, \partial_{x_i} v_{\varepsilon, M}, \partial_t v_{\varepsilon, M} \in L^\infty(\Omega_T)$, $\partial_{x_i x_j}^2 v_{\varepsilon, M} \in L^p(\Omega_T)$ for all $p \in [1, \infty)$.

We now prove *Proposition 2.1*.

Proof. We want to use a fixed point argument. In order to be able to use Schauder's theorem, we need first to introduce yet another regularization.

Let $\varepsilon, M > 0$ be fixed. We introduce the approximating system below for all $\delta > 0$ (we only indicate the dependence of the unknowns a, v with respect to δ since ε and M are fixed),

$$\begin{cases} \partial_t a_\delta = (\mu(a_\delta, v_\delta) *_x \psi_\delta) \Delta a_\delta + a_\delta s_M(a_\delta, v_\delta, \partial_t v_\delta), & \text{in } \Omega_T, \\ \partial_t v_\delta = d_v \Delta v_\delta + v_\delta g_{\varepsilon, M}(U(a_\delta, v_\delta), v_\delta), & \text{in } \Omega_T, \\ \nabla a_\delta \cdot \sigma = \nabla v_\delta \cdot \sigma = 0, & \text{on } (0, T) \times \partial\Omega, \end{cases} \quad (2.11)$$

where, once again slightly abusing notations, $\mu(a_\delta, v_\delta)$ is identified to its extension by zero defined on $[0, T] \times \mathbb{R}^N$, while $(\psi_\delta)_\delta$ is a standard mollifier on \mathbb{R}^N . Finally, $\mu, s_M, g_{\varepsilon, M}$ are defined as in (2.2), (2.8), (2.9) respectively, and the nonnegative initial data are defined as in (2.10), i.e.

$$a_\delta(0, x) := a_{\text{in}, \varepsilon}(x) \geq 0, \quad v_\delta(0, x) := v_{\text{in}, \varepsilon}(x) \geq 0. \quad (2.12)$$

The existence to system (2.11), (2.12) is obtained by applying Schauder's fixed point theorem [9] on the Banach space

$$E := L^\infty([0, T]; L^2(\Omega)),$$

and with the map

$$\Phi_\delta : (a_\delta, v_\delta) \in E^2 \mapsto (\bar{a}_\delta, \bar{v}_\delta), \quad (2.13)$$

where \bar{v}_δ satisfies

$$\begin{cases} \partial_t \bar{v}_\delta = d_v \Delta \bar{v}_\delta + \bar{v}_\delta g_{\varepsilon, M}(U(a_\delta, v_\delta), v_\delta), & \text{in } \Omega_T, \\ \nabla \bar{v}_\delta \cdot \sigma = 0, & \text{on } (0, T) \times \partial\Omega, \\ \bar{v}_\delta(0, x) = v_{\text{in}, \varepsilon}, & \text{on } \Omega, \end{cases} \quad (2.14)$$

and \bar{a}_δ solves

$$\begin{cases} \partial_t \bar{a}_\delta = (\mu(a_\delta, \bar{v}_\delta) *_x \psi_\delta) \Delta \bar{a}_\delta + \bar{a}_\delta s_M(a_\delta, \bar{v}_\delta, \partial_t \bar{v}_\delta), & \text{in } \Omega_T, \\ \nabla \bar{a}_\delta \cdot \sigma = 0, & \text{on } (0, T) \times \partial\Omega, \\ \bar{a}_\delta(0, x) = a_{\text{in}, \varepsilon}, & \text{on } \Omega. \end{cases} \quad (2.15)$$

Note that (2.14) and (2.15) are linear parabolic problems for which the existence of a unique nonnegative strong solution is classical (see *Proposition A.1* of the Appendix). Then, we show that the map (2.13) satisfies the assumptions of Schauder's fixed point theorem.

Indeed, thanks to the maximum principle for the heat equation, we first see that since

$$g_{\varepsilon, M}(U(a_\delta, v_\delta), v_\delta) \leq C_g,$$

we get the estimate

$$\|\bar{v}_\delta\|_{L^\infty(\Omega_T)} \leq e^{TC_g} \|v_{\text{in}, \varepsilon}\|_{L^\infty(\Omega)}. \quad (2.16)$$

Then, we take the derivative in time of (2.14):

$$(\partial_t - d_v \Delta)(\partial_t \bar{v}_\delta) = \partial_t \bar{v}_\delta g(\min\{U(a_\delta, v_\delta), M\}, \min\{v_\delta, M\}) *_t, x \varphi_\varepsilon$$

$$+ \bar{v}_\delta g(\min\{U(a_\delta, v_\delta), M\}, \min\{v_\delta, M\}) *_{t,x} \partial_t \varphi_\varepsilon.$$

Observing that

$$\|g(\min\{U(a_\delta, v_\delta), M\}, \min\{v_\delta, M\}) *_{t,x} \varphi_\varepsilon\|_{L^\infty(\Omega)}$$

$$+ \|g(\min\{U(a_\delta, v_\delta), M\}, \min\{v_\delta, M\}) *_{t,x} \partial_t \varphi_\varepsilon\|_{L^\infty(\Omega)} \leq C(\varepsilon, M, T),$$

where $C(\varepsilon, M, T)$ is a generic constant depending on ε, M, T but not δ , and using estimate (2.16), we can now use the maximum principle for the heat equation, and get

$$\|\partial_t \bar{v}_\delta\|_{L^\infty(\Omega_T)} \leq C(\varepsilon, M, T), \quad \|s_M(a_\delta, \bar{v}_\delta, \partial_t \bar{v}_\delta)\|_{L^\infty(\Omega_T)} \leq C(\varepsilon, M, T), \quad (2.17)$$

with s_M is defined in (2.8) and using (D2), (R1), (1.6), (2.4). Moreover, the maximal regularity implies

$$\|\partial_{x_i x_j}^2 \bar{v}_\delta\|_{L^p(\Omega_T)} \leq C(\varepsilon, M, T), \quad (2.18)$$

for all $i, j = 1, \dots, N$, $p \in [1, +\infty)$. Then \bar{a}_δ satisfies, thanks to Proposition A.1 (and an interpolation between $L^\infty(\Omega_T)$ and $L^2([0, T]; H^2(\Omega))$) for all $i, j = 1, \dots, N$,

$$\|\bar{a}_\delta\|_{L^\infty(\Omega_T)} + \|\partial_t \bar{a}_\delta\|_{L^2(\Omega_T)} + \|\partial_{x_i x_j}^2 \bar{a}_\delta\|_{L^2(\Omega_T)} + \|\partial_{x_i} \bar{a}_\delta\|_{L^4(\Omega_T)} \leq C(\varepsilon, M, T). \quad (2.19)$$

Thanks to estimates (2.16)–(2.19), the map $\Phi : E^2 \rightarrow E^2$ is compact, using the Rellich-Kondrakov theorem [1]. Moreover, it satisfies the inclusion

$$\Phi(B_Q \times B_Q) \subset B_Q \times B_Q,$$

with $B_Q := B_{L^\infty(\Omega_T)}(0, Q)_+ := \{w \in E \text{ s.t. } 0 \leq w \leq Q\}$ and

$$Q := \max\left(e^{T C_g} \|v_{\text{in}, \varepsilon}\|_{L^\infty(\Omega)}^2, C(\varepsilon, M, T)\right) > 0.$$

Finally, it is possible to show that $\Phi : E^2 \rightarrow E^2$ is continuous (see [3, Subsection 3.2.3] for details).

Therefore, using Schauder's fixed point theorem, there exists at least one strong solution $(a_\delta \geq 0, v_\delta \geq 0)$ to (2.11), (2.12), satisfying estimates (2.16)–(2.19) (with (a_δ, v_δ) replacing $(\bar{a}_\delta, \bar{v}_\delta)$).

Using the second equation of (2.11) and the properties of the heat equation, we also observe that

$$\|\partial_{x_i} \bar{v}_\delta\|_{L^\infty(\Omega_T)} \leq C(\varepsilon, M, T), \quad \text{for } i = 1, \dots, N. \quad (2.20)$$

We now use this solution (a_δ, v_δ) and let $\delta \rightarrow 0$. Estimates (2.16)–(2.19) ensure that we can extract subsequences (still denoted by (a_δ, v_δ)) such that for some $a, v \in L^\infty(\Omega_T)$,

$$a_\delta \rightarrow a, \quad v_\delta \rightarrow v \quad \text{a.e. in } \Omega_T \text{ and strongly in } L^p(\Omega_T), \text{ for all } p \in [1, \infty), \quad (2.21)$$

$$\partial_t a_\delta \rightharpoonup \partial_t a, \quad \Delta a_\delta \rightharpoonup \Delta a, \quad \nabla a_\delta \rightharpoonup \nabla a, \quad \text{weakly in } L^2(\Omega_T), \quad (2.22)$$

and

$$\partial_t v_\delta \rightharpoonup \partial_t v, \quad \Delta v_\delta \rightharpoonup \Delta v, \quad \nabla v_\delta \rightharpoonup \nabla v, \quad \text{weakly in } L^p(\Omega_T), \text{ for all } p \in [1, \infty). \quad (2.23)$$

Recalling estimate (2.3), we end up with

$$\left(\mu(a_\delta, v_\delta) *_{x} \psi_\delta\right) \Delta a_\delta \rightharpoonup \mu(a, v) \Delta a, \quad \text{weakly in } L^1(\Omega_T), \quad \text{as } \delta \rightarrow 0. \quad (2.24)$$

We now take the weak limit in the r.h.s. of the first equation in (2.11). The convergences (2.21), (2.23) and the δ -uniform bounds (2.16)–(2.20) ensure that $s_M(a_\delta, v_\delta, \partial_t v_\delta)$ converges to $s_M(a, v, \partial_t v)$, weakly in $L^p(\Omega_T)$ for all $p \in [1, \infty)$, so that (a, v) satisfy the first equation of system (2.11).

The convergence in the second equation in system (2.11) is obtained in a similar way. We also pass to the limit in the boundary condition of (2.11), using (2.21)–(2.23) and the continuity of the trace operator.

Finally, using the weak lower semi-continuity property of the $L^p(\Omega_T)$ norm for $p \in (1, +\infty]$ and estimates (2.16)–(2.20), we conclude that $(a_{\varepsilon, M}, v_{\varepsilon, M})$ is a strong solution to the system (2.7)–(2.10), satisfying the bounds i) and ii) announced in Proposition 2.1. \square

2.3 Existence for the regularized system in *divergence* form

Hereafter, we restore the ε, M -dependency in the notation, so that we refer to the a.e. limit of a_δ, v_δ as $a_{\varepsilon, M}, v_{\varepsilon, M}$, respectively.

In this subsection, we deduce from Proposition 2.1 the existence to the following regularized system in *divergence* form, satisfied by the unknowns $(u_{\varepsilon, M}, v_{\varepsilon, M})$

$$\begin{cases} \partial_t u_{\varepsilon, M} = \Delta(A(u_{\varepsilon, M}, v_{\varepsilon, M})) + u_{\varepsilon, M} f_M(u_{\varepsilon, M}, v_{\varepsilon, M}), & \text{on } \Omega_T, \\ \partial_t v_{\varepsilon, M} = d_v \Delta v_{\varepsilon, M} + v_{\varepsilon, M} g_{\varepsilon, M}(u_{\varepsilon, M}, v_{\varepsilon, M}), & \text{on } \Omega_T, \\ \nabla(A(u_{\varepsilon, M}, v_{\varepsilon, M})) \cdot \sigma = \nabla v_{\varepsilon, M} \cdot \sigma = 0, & \text{on } (0, T) \times \partial\Omega, \end{cases} \quad (2.25)$$

with $f_M(u, v) := f(\min\{u, M\}, v)$ (and, as previously, $g_{\varepsilon, M}(u, v) := g(\min\{u, M\}, \min\{v, M\})_{*t, x} \varphi_\varepsilon$). We complete the system with the initial conditions

$$\begin{aligned} u_{\varepsilon, M}(0, x) &= u_{\text{in}, \varepsilon}(x) := (u_{\text{in}} *_x \psi_\varepsilon)(x), & \forall x \in \Omega, \\ v_{\varepsilon, M}(0, x) &= v_{\text{in}, \varepsilon}(x) := (v_{\text{in}} *_x \psi_\varepsilon)(x), & \forall x \in \Omega. \end{aligned} \quad (2.26)$$

The existence result is stated in the following corollary.

Corollary 2.2. *Let $N \geq 1$ and Ω be a smooth bounded open subset of \mathbb{R}^N . We suppose that the parameters d_v, B, f, g satisfy Assumption A. We consider the initial data $u_{\text{in}} \in (L^\infty \cap H^1)(\Omega)$, $v_{\text{in}} \in L^\infty(\Omega)$ compatible with Neumann boundary condition.*

Then, for any fixed $\varepsilon, M > 0$ there exists a nonnegative strong (in the sense of Theorem 1.1) solution $(u_{\varepsilon, M}, v_{\varepsilon, M})$ to system (2.25), (2.26), such that for all $i, j = 1, \dots, N$,

- i) $u_{\varepsilon, M} \in L^\infty(\Omega_T)$, $\partial_t u_{\varepsilon, M}, \partial_{x_i x_j}^2 A(u_{\varepsilon, M}, v_{\varepsilon, M}) \in L^2(\Omega_T)$, $\partial_{x_i} u_{\varepsilon, M} \in L^4(\Omega_T)$,
- ii) $v_{\varepsilon, M}, \partial_t v_{\varepsilon, M}, \partial_{x_i} v_{\varepsilon, M} \in L^\infty(\Omega_T)$, $\partial_{x_i x_j}^2 v_{\varepsilon, M} \in L^p(\Omega_T)$ for all $p \in [1, \infty)$.

Proof. For any ε, M fixed, let $(a_{\varepsilon, M}, v_{\varepsilon, M})$ be a solution to (2.7)–(2.10), given by Proposition 2.1. Then, by a density argument and recalling the C^1 character of U in (2.1), we can prove that $u_{\varepsilon, M} = U(a_{\varepsilon, M}, v_{\varepsilon, M})$ admits a weak time derivative, given by

$$\partial_t u_{\varepsilon, M} = \partial_1 U(a_{\varepsilon, M}, v_{\varepsilon, M}) \partial_t a_{\varepsilon, M} + \partial_2 U(a_{\varepsilon, M}, v_{\varepsilon, M}) \partial_t v_{\varepsilon, M}, \quad (2.27)$$

belonging to $L^2(\Omega_T)$, and a weak space derivative

$$\partial_{x_i} u_{\varepsilon, M} = \partial_1 U(a_{\varepsilon, M}, v_{\varepsilon, M}) \partial_{x_i} a_{\varepsilon, M} + \partial_2 U(a_{\varepsilon, M}, v_{\varepsilon, M}) \partial_{x_i} v_{\varepsilon, M}, \quad \forall i = 1, \dots, N, \quad (2.28)$$

belonging to $L^4(\Omega_T)$. Note also that $\partial_{x_i x_j}^2 A(u_{\varepsilon, M}, v_{\varepsilon, M}) = \partial_{x_i x_j}^2 a_{\varepsilon, M}$. Then, using (2.27) and the identity $\partial_1 U(a, v) \partial_1 A(U(a, v), v) = 1$, we see that $u_{\varepsilon, M}$ satisfies the first equation of (2.25) (a.e. in Ω_T). Moreover, the trace of $\nabla a_{\varepsilon, M}$ on $[0, T] \times \partial\Omega$ is the trace of $\nabla(A(u_{\varepsilon, M}, v_{\varepsilon, M}))$, so that the first Neumann boundary condition in (2.25) is satisfied. The same holds for the initial conditions. Finally, the equation, boundary condition and initial condition related to $v_{\varepsilon, M}$ are identical in the system satisfied by $(a_{\varepsilon, M}, v_{\varepsilon, M})$ and in the system satisfied by $(u_{\varepsilon, M}, v_{\varepsilon, M})$ (when $U(a_{\varepsilon, M}, v_{\varepsilon, M})$ is replaced by $u_{\varepsilon, M}$ in the equations). Therefore, $(u_{\varepsilon, M}, v_{\varepsilon, M})$ is a nonnegative strong solution to system (2.25), (2.26). The bounds satisfied by $u_{\varepsilon, M}, v_{\varepsilon, M}$ in Corollary 2.2 are a direct consequence of the bounds satisfied by $a_{\varepsilon, M}, v_{\varepsilon, M}$ in Proposition 2.1. \square

2.4 Concluding the existence result

By the previous subsections, we now have at our disposal solutions to an approximated problem. Most of the bounds obtained for those solutions are however depending on the truncation-regularization parameters ε and M . Before letting ε tend to 0 and M tend to ∞ , we need to establish bounds (still for the solutions to the considered approximated problem) which do not depend on ε and M . This is the object of the lemma below.

Lemma 2.3. *We consider the solution $(u_{\varepsilon, M}, v_{\varepsilon, M})$ to system (2.25), (2.26), given by Corollary 2.2, and we assume that $u_{in} \in (L^\infty \cap H^1)(\Omega)$ and $v_{in} \in (L^\infty \cap W^{2,p} \cap H^3)(\Omega)$ for all $p \in [1, +\infty)$.*

Then, $(u_{\varepsilon, M}, v_{\varepsilon, M})$ satisfies the following ε, M -uniform a priori estimates, for all $p \in [1, +\infty)$

$$\|v_{\varepsilon, M}\|_{L^\infty(\Omega_T)} \leq C(T), \quad \|u_{\varepsilon, M}\|_{L^\infty([0, T]; L^p(\Omega))} \leq C(T), \quad (2.29)$$

and for all $i, j = 1, \dots, N$

$$\|\partial_t v_{\varepsilon, M}\|_{L^p(\Omega_T)} + \|\partial_{x_i x_j}^2 v_{\varepsilon, M}\|_{L^p(\Omega_T)} \leq C(T), \quad (2.30)$$

and

$$\begin{aligned} \|\partial_t u_{\varepsilon, M}\|_{L^2(\Omega_T)} + \|\partial_{x_i} u_{\varepsilon, M}\|_{L^\infty([0, T]; L^2(\Omega))} \\ + \|\partial_{x_i x_j}^2 (A(u_{\varepsilon, M}, v_{\varepsilon, M}))\|_{L^2(\Omega_T)} \leq C(T). \end{aligned} \quad (2.31)$$

Moreover, for all $i, j = 1, \dots, N$ and for any $\eta > 0$

$$\|\partial_{x_i x_j}^3 v_{\varepsilon, M}\|_{L^2(\Omega_T)} + \|\partial_{tt}^2 v_{\varepsilon, M}\|_{L^2(\Omega_T)} + \|\partial_{t x_i}^2 v_{\varepsilon, M}\|_{L^{4-\eta}(\Omega_T)} \leq C(T). \quad (2.32)$$

Finally, if $N \leq 3$,

$$\begin{aligned} \|\partial_t v_{\varepsilon, M}\|_{L^2([0, T]; L^\infty(\Omega))} + \|\partial_{x_i x_j}^2 v_{\varepsilon, M}\|_{L^2([0, T]; L^\infty(\Omega))} \\ + \|u_{\varepsilon, M}\|_{L^\infty(\Omega_T)} + \|\partial_{x_i} u_{\varepsilon, M}\|_{L^4(\Omega_T)} \leq C(T), \end{aligned} \quad (2.33)$$

and if $N = 1$ and $u_{in} \in W^{1,p}(\Omega)$ for all $p \in [1, \infty)$,

$$\|\partial_x u_{\varepsilon, M}\|_{L^p(\Omega_T)} \leq C(T). \quad (2.34)$$

In the estimates above, we denote as $C(T) > 0$ constants which may depend on T, Ω , on the parameters appearing in Assumption A (that are $d_v, C_f, C'_f, C_g, C'_g, a_0, a_1, a_2, a_3$), and on the bounds of the initial data, but not on ε, M .

Remark 2. We observe that the hypothesis on the initial datum $u_{in} \in L^p(\Omega)$, for any $p \in [1, +\infty)$, is sufficient to prove the regularity of $u_{\varepsilon, M}$ the second inequality stated in (2.29), and estimate (2.31). Indeed, the assumption $u_{in} \in L^\infty(\Omega)$ is only used to obtain the $L^\infty(\Omega_T)$ boundedness of $u_{\varepsilon, M}$ if $N \leq 3$ (see (2.33)). It is also worth noticing that the hypothesis $v_{in} \in H^3(\Omega)$ is only used to get estimate (2.32).

Proof. The first estimate in (2.29) follows from Proposition A.1 (or estimate (2.16), under the assumption $v_{in} \in L^\infty(\Omega)$).

We then show the $L^2(\Omega_T)$ boundedness of $\nabla v_{\varepsilon, M}$, uniformly in ε, M , by multiplying by $v_{\varepsilon, M}$ the second equation of (2.25) and integrating on Ω_t (for any $t \in [0, T]$). Thus, we get

$$\frac{1}{2} \int_{\Omega} v_{\varepsilon, M}^2(t) dx + d_v \int_{\Omega_t} |\nabla v_{\varepsilon, M}|^2 dx ds \leq \frac{1}{2} \int_{\Omega} v_{\varepsilon, M}^2(0) dx + C_g \int_{\Omega_t} v_{\varepsilon, M}^2 dx ds \leq C(T),$$

thanks to the assumption on the initial data and estimate (2.29). Hence we end up with

$$\|\nabla v_{\varepsilon, M}\|_{L^2(\Omega_T)} \leq C(T). \quad (2.35)$$

We now show the second inequality of (2.29) and inequality (2.30), by proving the following estimate for any $q \in \mathbb{N} - \{0\}$ and $i, j = 1, \dots, N$,

$$\begin{aligned} \|u_{\varepsilon, M}\|_{L^\infty([0, T]; L^{2q}(\Omega))} + \|\partial_t v_{\varepsilon, M}\|_{L^{2q}(\Omega_T)} + \|\partial_{x_i x_j}^2 v_{\varepsilon, M}\|_{L^{2q}(\Omega_T)} \\ + \|\partial_{x_i} v_{\varepsilon, M}\|_{L^{2q+1}(\Omega_T)} \leq C(T), \end{aligned} \quad (2.36)$$

using an induction on q . More precisely, we will get the boundedness of the first term of the l.h.s. of (2.36) by multiplying the first equation of (2.25) by $u_{\varepsilon, M}^{2q-1}$. Then, the boundedness of the remaining terms in (2.36) will be obtained by maximal regularity and interpolation inequalities.

We prove (2.36) with $q = 1$. Denoting (for any fixed $\varepsilon, M > 0$)

$$A_{\varepsilon, M} := A(u_{\varepsilon, M}, v_{\varepsilon, M}), \quad f_M := f_M(u_{\varepsilon, M}, v_{\varepsilon, M}), \quad g_{\varepsilon, M} := g_{\varepsilon, M}(u_{\varepsilon, M}, v_{\varepsilon, M}),$$

we have by Young's inequality

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} u_{\varepsilon, M}^2 dx \\ = - \int_{\Omega} \partial_1 A_{\varepsilon, M} |\nabla u_{\varepsilon, M}|^2 dx - \int_{\Omega} \partial_2 A_{\varepsilon, M} \nabla u_{\varepsilon, M} \cdot \nabla v_{\varepsilon, M} dx + \int_{\Omega} u_{\varepsilon, M}^2 f_M dx \\ \leq - \frac{1}{2} \int_{\Omega} \partial_1 A_{\varepsilon, M} |\nabla u_{\varepsilon, M}|^2 dx + \frac{1}{2} \int_{\Omega} \frac{|\partial_2 A_{\varepsilon, M}|^2}{\partial_1 A_{\varepsilon, M}} |\nabla v_{\varepsilon, M}|^2 dx + C_f \int_{\Omega} u_{\varepsilon, M}^2 dx \\ \leq \frac{a_2^2}{2a_0} \int_{\Omega} |\nabla v_{\varepsilon, M}|^2 dx + C_f \int_{\Omega} u_{\varepsilon, M}^2 dx, \end{aligned}$$

by (R1), (D2). Then Gronwall's lemma and estimate (2.35) yield

$$\|u_{\varepsilon, M}\|_{L^\infty([0, T]; L^2(\Omega))}^2 \leq e^{2C_f T} \|u_{\varepsilon, M}(0)\|_{L^2(\Omega)}^2 + \frac{a_2^2}{a_0} e^{2C_f T} \|\nabla v_{\varepsilon, M}\|_{L^2(\Omega_T)}^2 \leq C(T). \quad (2.37)$$

Moreover, the first inequality in (2.29) and the obtained estimate (2.37) imply the boundedness of $g_{\varepsilon, M}$ in $L^2(\Omega_T)$ uniformly in ε, M , so that the reaction term of the equation of $v_{\varepsilon, M}$ in (2.25) is bounded in $L^2(\Omega_T)$. Remembering that by assumption

$\nabla v_{\text{in}} \in L^2(\Omega)$, maximal regularity and an interpolation inequality ensure that for all $i, j = 1, \dots, N$,

$$\|\partial_t v_{\varepsilon, M}\|_{L^2(\Omega_T)} + \|\partial_{x_i x_j}^2 v_{\varepsilon, M}\|_{L^2(\Omega_T)} + \|\partial_{x_i} v_{\varepsilon, M}\|_{L^4(\Omega_T)} \leq C(T), \quad (2.38)$$

thus (2.36) is proved for $q = 1$.

We now prove (2.36) for $q \in \mathbb{N} - \{0, 1\}$, assuming that it holds for $q - 1$. In other words, we assume that for all $i, j = 1, \dots, N$

$$\begin{aligned} \|u_{\varepsilon, M}\|_{L^\infty([0, T]; L^{2^{q-1}}(\Omega))} + \|\partial_t v_{\varepsilon, M}\|_{L^{2^{q-1}}(\Omega_T)} + \|\partial_{x_i x_j}^2 v_{\varepsilon, M}\|_{L^{2^{q-1}}(\Omega_T)} \\ + \|\partial_{x_i} v_{\varepsilon, M}\|_{L^{2^q}(\Omega_T)} \leq C(T). \end{aligned} \quad (2.39)$$

By multiplying the first equation of (2.25) by $u_{\varepsilon, M}^{2^q - 1}$ and integrating on Ω , we get

$$\begin{aligned} \frac{1}{2^q} \frac{d}{dt} \int_{\Omega} u_{\varepsilon, M}^{2^q} dx &= - \int_{\Omega} \nabla(A_{\varepsilon, M}) \nabla(u_{\varepsilon, M}^{2^q - 1}) dx + \int_{\Omega} u_{\varepsilon, M}^{2^q} f_M dx \\ &\leq -(2^q - 1) \int_{\Omega} u_{\varepsilon, M}^{2^q - 2} \left(\partial_1 A_{\varepsilon, M} \nabla u_{\varepsilon, M} + \partial_2 A_{\varepsilon, M} \nabla v_{\varepsilon, M} \right) \cdot \nabla u_{\varepsilon, M} dx + C_f \int_{\Omega} u_{\varepsilon, M}^{2^q} dx \\ &\leq -(2^q - 1) \int_{\Omega} \partial_1 A_{\varepsilon, M} (u_{\varepsilon, M}^{2^q - 2}) |\nabla u_{\varepsilon, M}|^2 dx + \frac{(2^q - 1)}{2} \int_{\Omega} \partial_1 A_{\varepsilon, M} (u_{\varepsilon, M}^{2^q - 2}) |\nabla u_{\varepsilon, M}|^2 dx \\ &\quad + \frac{(2^q - 1)}{2} \int_{\Omega} \frac{|\partial_2 A_{\varepsilon, M}|^2}{\partial_1 A_{\varepsilon, M}} u_{\varepsilon, M}^{2^q - 2} |\nabla v_{\varepsilon, M}|^2 dx + C_f \int_{\Omega} u_{\varepsilon, M}^{2^q} dx \\ &\leq C_q \int_{\Omega} u_{\varepsilon, M}^{2^q} dx + \frac{1}{2^{q-1}} \int_{\Omega} |\nabla v_{\varepsilon, M}|^{2^q} dx, \end{aligned} \quad (2.40)$$

where we used assumption (D2) and Hölder's inequality with coefficients $\left(\frac{2^q}{2^{q-2}}, \frac{2^q}{2}\right)$. Then, Gronwall's lemma, the assumption on u_{in} and estimate (2.39) yield

$$\|u_{\varepsilon, M}\|_{L^\infty([0, T]; L^{2^q}(\Omega))} \leq C(T), \quad (2.41)$$

giving the boundedness of the first term in (2.36). Moreover, the first inequality in (2.29) and the obtained estimate (2.41) imply the boundedness of $g_{\varepsilon, M}$ in $L^{2^q}(\Omega_T)$ uniformly in ε, M , so that the right hand side of (2.25) is bounded in $L^{2^q}(\Omega_T)$. Thus, maximal regularity (which holds thanks to the assumption on v_{in}) and an interpolation inequality give (2.36). This concludes the proof of the second inequality in (2.29) and inequality (2.30).

We now prove estimate (2.31). First, multiplying by $-\Delta v_{\varepsilon, M}$ the equation satisfied by $v_{\varepsilon, M}$, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla v_{\varepsilon, M}|^2 dx + d_v \int_{\Omega} |\Delta v_{\varepsilon, M}|^2 dx &= \int_{\Omega} \Delta v_{\varepsilon, M} v_{\varepsilon, M} g_{\varepsilon, M} dx \\ &\leq \frac{d_v}{2} \int_{\Omega} |\Delta v_{\varepsilon, M}|^2 dx + \frac{\|v_{\varepsilon, M}\|_{L^\infty(\Omega)}^2}{2d_v} \int_{\Omega} g_{\varepsilon, M}^2 dx, \end{aligned}$$

so that integrating w.r.t time and using the uniform $L^2(\Omega_T)$ boundedness of $g_{\varepsilon, M}$, and the assumption $v_{\text{in}} \in H^1(\Omega)$, we end up with

$$\|\nabla v_{\varepsilon, M}\|_{L^\infty([0, T]; L^2(\Omega))} \leq C(T). \quad (2.42)$$

Alternatively, one could use the properties of the heat equation to directly obtain that $\|\nabla v_{\varepsilon,M}\|_{L^\infty(\Omega_T)} \leq C_T$, but we will not use this extra information in the sequel. system (2.7). Inequality (2.4) and the second estimate in (2.29) (with $p = 4$) imply

$$\|a_{\varepsilon,M}\|_{L^\infty([0,T];L^4(\Omega))} \leq C(T). \quad (2.43)$$

Moreover, thanks to estimates (2.29), (2.30) (with $p = 4$) we find from Definition (2.8):

$$\|s_M\|_{L^4(\Omega_T)} \leq C(T). \quad (2.44)$$

Multiplying the first equation of (2.7) by $-\Delta a_{\varepsilon,M}$ and integrating on Ω_T , we see that under the assumption $a_{\text{in}} \in H^1(\Omega)$ (equivalent to $u_{\text{in}} \in H^1(\Omega)$),

$$\begin{aligned} & \|\partial_t a_{\varepsilon,M}\|_{L^2(\Omega_T)}^2 + \|\nabla a_{\varepsilon,M}\|_{L^\infty([0,T];L^2(\Omega))}^2 + \|\Delta a_{\varepsilon,M}\|_{L^2(\Omega_T)}^2 \\ & \leq C(T) \|a_{\varepsilon,M} s_M\|_{L^2(\Omega_T)}^2 \leq C(T) \|a_{\varepsilon,M}\|_{L^4(\Omega_T)}^2 \|s_M\|_{L^4(\Omega_T)}^2 \leq C(T), \end{aligned} \quad (2.45)$$

thanks to estimates (2.43) and (2.44). Using identity (2.1), we see that

$$\partial_1 A_{\varepsilon,M} \partial_t u_{\varepsilon,M} = \partial_t a_{\varepsilon,M} - \partial_2 A_{\varepsilon,M} \partial_t v_{\varepsilon,M},$$

so that using (D2), estimates (2.30) with $p = 2$ and (2.45), we end up with

$$\|\partial_t u_{\varepsilon,M}\|_{L^2(\Omega_T)}^2 \leq \frac{2}{a_0^2} \|\partial_t a_{\varepsilon,M}\|_{L^2(\Omega_T)}^2 + 2 \left(\frac{a_2}{a_0} \right)^2 \|\partial_t v_{\varepsilon,M}\|_{L^2(\Omega_T)}^2 \leq C(T). \quad (2.46)$$

Similarly,

$$\nabla(A(u_{\varepsilon,M}, v_{\varepsilon,M})) = \partial_1 A_{\varepsilon,M} \nabla u_{\varepsilon,M} + \partial_2 A_{\varepsilon,M} \nabla v_{\varepsilon,M}, \quad (2.47)$$

so that using (D2) again, (2.42) and (2.45), we get

$$\|\nabla u_{\varepsilon,M}\|_{L^\infty([0,T];L^2(\Omega))}^2 \leq C(T), \quad (2.48)$$

Finally, estimates (2.45)–(2.48) are collected in (2.31).

In order to prove estimate (2.32), we take the time derivative in the second equation of (2.25). Then, using (2.29)–(2.31) we have

$$\begin{aligned} & \|\partial_t(\partial_t v_{\varepsilon,M}) - d_v \Delta(\partial_t v_{\varepsilon,M})\|_{L^2(\Omega_T)} \\ & = \|(\partial_t v_{\varepsilon,M}) g_{\varepsilon,M} + v_{\varepsilon,M} \partial_1 g_{\varepsilon,M} \partial_t u_{\varepsilon,M} + v_{\varepsilon,M} \partial_2 g_{\varepsilon,M} \partial_t v_{\varepsilon,M}\|_{L^2(\Omega_T)} \\ & \leq \|\partial_t v_{\varepsilon,M}\|_{L^4(\Omega_T)} \|g_{\varepsilon,M}\|_{L^4(\Omega_T)} \\ & \quad + \|v_{\varepsilon,M}\|_{L^\infty(\Omega_T)} \|\partial_1 g_{\varepsilon,M}\|_{L^\infty(\Omega_T)} \|\partial_t u_{\varepsilon,M}\|_{L^2(\Omega_T)} \\ & \quad + \|v_{\varepsilon,M}\|_{L^\infty(\Omega_T)} \|\partial_2 g_{\varepsilon,M}\|_{L^\infty(\Omega_T)} \|\partial_t v_{\varepsilon,M}\|_{L^2(\Omega_T)} \\ & \leq C(T). \end{aligned} \quad (2.49)$$

Therefore, we apply the maximal regularity, using the assumption $v_{\text{in}} \in H^3(\Omega)$ (and $u_{\text{in}}, v_{\text{in}} \in L^\infty(\Omega)$) and an interpolation inequality, using estimate (2.30), to get for all $i, j = 1, \dots, N$ and for $\eta > 0$

$$\|\partial_{tt}^2 v_{\varepsilon,M}\|_{L^2(\Omega_T)} + \|\partial_{tx_i x_j}^3 v_{\varepsilon,M}\|_{L^2(\Omega_T)} + \|\partial_{tx_i}^2 v_{\varepsilon,M}\|_{L^{4-\eta}(\Omega_T)} \leq C(T), \quad (2.50)$$

that is estimate (2.32).

We now consider the case when $N \leq 3$, so that we can use the continuous injection $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$. We see that thanks to estimates (2.45) and (2.50),

$$\|A(u_{\varepsilon,M}, v_{\varepsilon,M})\|_{L^2([0,T]; L^\infty(\Omega))} + \|\partial_t v_{\varepsilon,M}\|_{L^2([0,T]; L^\infty(\Omega))} \leq C(T), \quad (2.51)$$

which implies, thanks to Definition 2.8,

$$\int_0^T \sup_{x \in \Omega} |s_M(t, x)| dt \leq C(T).$$

Thanks to estimate (A.4) in Proposition A.1 applied to the first equation of system (2.7), and the assumption $a_{\text{in}} \in (L^\infty \cap H^1)(\Omega)$ (equivalent to $u_{\text{in}} \in (L^\infty \cap H^1)(\Omega)$), we get

$$\|a_{\varepsilon,M}\|_{L^\infty(\Omega_T)} \leq C(T), \quad (2.52)$$

and finally

$$\|u_{\varepsilon,M}\|_{L^\infty(\Omega_T)} \leq C(T). \quad (2.53)$$

Using Gagliardo-Nirenberg's inequality (with the constant in the inequality denoted by C_{GN}), we see that for all $i = 1, \dots, N$, (when $N \leq 3$)

$$\|\partial_{x_i} a_{\varepsilon,M}\|_{L^4(\Omega)}^4 \leq C_{GN} \sup_{k,l=1,\dots,N} \|\partial_{x_k x_l}^2 a_{\varepsilon,M}\|_{L^2(\Omega)}^2 \|a_{\varepsilon,M}\|_{L^\infty(\Omega)}^2 + C_{GN} \|a_{\varepsilon,M}\|_{L^\infty(\Omega)}^4.$$

Then, by integrating in time over $(0, T)$, and using estimate (2.31), we obtain

$$\|\partial_{x_i} a_{\varepsilon,M}\|_{L^4(\Omega_T)} \leq C(T). \quad (2.54)$$

Using now (D2), (2.30) with $p = 4$, (2.47) and the above inequality, we end up for all $i = 1, \dots, N$ (and $N \leq 3$), with

$$\|\partial_{x_i} u_{\varepsilon,M}\|_{L^4(\Omega_T)} \leq C(T) (\|\partial_{x_i} a_{\varepsilon,M}\|_{L^4(\Omega_T)} + \|\partial_{x_i} v_{\varepsilon,M}\|_{L^4(\Omega_T)}) \leq C(T).$$

Finally, we observe that (still when $N \leq 3$)

$$\|g_{\varepsilon,M}(u_{\varepsilon,M}, v_{\varepsilon,M})\|_{L^\infty(\Omega_T)} \leq C_g \|1 + u_{\varepsilon,M} + v_{\varepsilon,M}\|_{L^\infty(\Omega_T)} \leq C(T),$$

thanks to (2.53). Then, using (2.51), we see that for all $i, j = 1, \dots, N$,

$$\|\partial_{x_i x_j}^2 v_{\varepsilon,M}\|_{L^2([0,T]; L^\infty(\Omega))} \leq C(T). \quad (2.55)$$

Collecting estimates (2.51)–(2.55), we obtain (2.33).

We finally consider the case when $N = 1$. Denoting for simplicity $\mu_{\varepsilon,M} = \mu(a_{\varepsilon,M}, v_{\varepsilon,M})$ and $s_{\varepsilon,M} = s_M(a_{\varepsilon,M}, v_{\varepsilon,M}, \partial_t v_{\varepsilon,M})$, we take $p \geq 1$ and compute (remembering that $a_{\varepsilon,M}$ satisfies Neumann's boundary condition)

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} |\partial_x a_{\varepsilon,M}|^{2p} dx &= 2p \int_{\Omega} |\partial_x a_{\varepsilon,M}|^{2p-1} \partial_x (\mu_{\varepsilon,M} \partial_{xx}^2 a_{\varepsilon,M} + a_{\varepsilon,M} s_{\varepsilon,M}) dx \\ &= -2p \int_{\Omega} \partial_x (|\partial_x a_{\varepsilon,M}|^{2p-1}) (\mu_{\varepsilon,M} \partial_{xx}^2 a_{\varepsilon,M} + a_{\varepsilon,M} s_{\varepsilon,M}) dx \\ &= -2p(2p-1) \int_{\Omega} |\partial_x a_{\varepsilon,M}|^{2p-2} \partial_{xx}^2 a_{\varepsilon,M} (\mu_{\varepsilon,M} \partial_{xx}^2 a_{\varepsilon,M} + a_{\varepsilon,M} s_{\varepsilon,M}) dx. \end{aligned}$$

As a consequence, it holds by (2.3)

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} |\partial_x a_{\varepsilon, M}|^{2p} dx + 2p(2p-1)a_0 \int_{\Omega} |\partial_x a_{\varepsilon, M}|^{2p-2} |\partial_{xx}^2 a_{\varepsilon, M}|^2 dx \\ \leq 2p(2p-1) \int_{\Omega} |\partial_x a_{\varepsilon, M}|^{2p-2} |\partial_{xx} a_{\varepsilon, M}| |a_{\varepsilon, M}| |s_{\varepsilon, M}| dx. \end{aligned}$$

Thus, by Young's inequality, we end up with

$$\frac{d}{dt} \int_{\Omega} |\partial_x a_{\varepsilon, M}|^{2p} dx \leq \frac{p(2p-1)}{a_0} \int_{\Omega} |\partial_x a_{\varepsilon, M}|^{2p-2} |a_{\varepsilon, M}|^2 |s_{\varepsilon, M}|^2 dx. \quad (2.56)$$

Recalling the uniform bound (see estimates (2.3), (2.4)), definition (2.8) and Assumption A)

$$|s_M(a_{\varepsilon, M}, v_{\varepsilon, M}, \partial_t v_{\varepsilon, M})| \leq \frac{1}{a_0} (a_1 C_f (1 + |U(a_{\varepsilon, M}, v_{\varepsilon, M})| + v_{\varepsilon, M}) + a_3 |\partial_t v_{\varepsilon, M}|),$$

we use the estimates (2.29) and (2.53) to get

$$|s_M(a_{\varepsilon, M}, v_{\varepsilon, M}, \partial_t v_{\varepsilon, M})|^2 \leq C(T) (1 + |\partial_t v_{\varepsilon, M}|^2).$$

Therefore using (2.52), the right-hand side of (2.56) is estimated as

$$\frac{d}{dt} \int_{\Omega} |\partial_x a_{\varepsilon, M}|^{2p} dx \leq p(2p-1) C(T) \|a_{\varepsilon, M}\|_{L^\infty(\Omega)}^2 \int_{\Omega} |\partial_x a_{\varepsilon, M}|^{2p-2} (1 + |\partial_t v_{\varepsilon, M}|^2) dx.$$

Finally, using Hölder's inequality with coefficients $(\frac{2p}{2p-2}, \frac{2p}{2})$ for any $p \geq 1$, we get

$$\frac{d}{dt} \|\partial_x a_{\varepsilon, M}\|_{L^{2p}(\Omega)}^{2p} dx \leq C_p(T) \|a_{\varepsilon, M}\|_{L^\infty(\Omega)}^2 (1 + \|\partial_t v_{\varepsilon, M}\|_{L^{2p}(\Omega)}^2) \|a_{\varepsilon, M}\|_{L^{2p}(\Omega)}^{2p-2}.$$

Thus, we obtain estimate (2.34) by Duhamel's formula, recalling that we assumed that $u_{in} \in W^{1,p}(\Omega)$ for all $p \in [1, \infty)$, and using estimates (2.30). \square

We now conclude the proof of *Theorem 1.1*.

By the ε, M -uniform estimates shown in *Lemma 2.3*, we can extract subsequences from $u_{\varepsilon, M}, v_{\varepsilon, M}$ (still denoted by $u_{\varepsilon, M}, v_{\varepsilon, M}$) such that for some $u \in L^\infty([0, T]; L^p(\Omega))$, for all $p \in [1, +\infty)$, and $v \in L^\infty(\Omega)$,

$$u_{\varepsilon, M} \rightharpoonup u, \quad v_{\varepsilon, M} \rightarrow v, \quad \text{a.e. in } \Omega_T \quad \text{as } \varepsilon \rightarrow 0, M \rightarrow +\infty, \quad (2.57)$$

and for all $p \in [1, \infty)$

$$\begin{aligned} \partial_t u_{\varepsilon, M} \rightharpoonup \partial_t u, & \quad \text{weakly in } L^2(\Omega_T), \\ \partial_t v_{\varepsilon, M} \rightharpoonup \partial_t v, \Delta v_{\varepsilon, M} \rightharpoonup \Delta v, \nabla v_{\varepsilon, M} \rightharpoonup \nabla v, & \quad \text{weakly in } L^p(\Omega_T). \end{aligned} \quad (2.58)$$

Moreover, by Assumption A and estimates (2.29), (2.31), the convergence (2.57) ensures that

$$\Delta A(u_{\varepsilon, M}, v_{\varepsilon, M}) \rightharpoonup \Delta A(u, v), \quad \text{weakly in } L^2(\Omega_T). \quad (2.59)$$

Now, we take the $D'(\Omega_T)$ limit as $\varepsilon \rightarrow 0, M \rightarrow +\infty$, in (2.25), (2.26). and observe that $u_{\varepsilon, M} f_M(u_{\varepsilon, M}, v_{\varepsilon, M})$ and $v_{\varepsilon, M} g_{\varepsilon, M}(u_{\varepsilon, M}, v_{\varepsilon, M})$ converge towards $u f(u, v)$ and $v g(u, v)$ strongly in $L^1(\Omega_T)$, thanks to assumption (R1) and the estimates (2.29).

Then, using the convergences obtained above, all the terms in the first two equations of (2.25) converge in $D'(\Omega_T)$. We conclude by taking the limit in the boundary conditions of (2.25), using the continuity of the trace operator and the weak convergence of $\Delta v_{\varepsilon, M}, \Delta A(u_{\varepsilon, M}, v_{\varepsilon, M})$ in (2.58), (2.59). Finally, using the lower semi-continuity property of the $L^p(\Omega_T)$ norm for $p \in (1, +\infty]$, we conclude that u, v satisfy the estimates stated in lines i) and ii) of *Theorem 1.1*.

3 Stability and uniqueness

In this section, we present the proof of the two stability results (*Theorem 1.2* and *Theorem 1.3*).

For a better readability, we first introduce for $i = 1, 2$ the notations

$$\begin{aligned} A_i &:= A(u_i, v_i), & f_i &:= f(u_i, v_i), & g_i &:= g(u_i, v_i), \\ \partial_1 A_i &:= \partial_1 A(u_i, v_i), & \partial_2 A_i &:= \partial_2 A(u_i, v_i) \end{aligned}$$

and

$$\mathcal{N}_H := \sup_{i,j=1,2} \|\partial_{ij}^2 A\|_\infty.$$

Hereafter, all constants C are strictly positive and may change from line to line. Moreover, if foreseen, the dependency of parameter to be chosen thereafter is denoted as index of the constant.

Proof of Theorem 1.2. We compute the equations satisfied by $u_1 - u_2$ and $v_1 - v_2$ and we multiply by $u_1 - u_2$ and $\lambda(v_1 - v_2)$, respectively, where the parameter $\lambda > 0$ will be chosen later. Then, we integrate over Ω and we add the obtained formulations to get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} |u_1 - u_2|^2 dx + \lambda \int_{\Omega} |v_1 - v_2|^2 dx \right) \\ &= - \int_{\Omega} (\partial_1 A_1 \nabla u_1 + \partial_2 A_1 \nabla v_1) \cdot \nabla (u_1 - u_2) dx \\ & \quad + \int_{\Omega} (\partial_1 A_2 \nabla u_2 + \partial_2 A_2 \nabla v_2) \cdot \nabla (u_1 - u_2) dx - d_v \lambda \int_{\Omega} |\nabla (v_1 - v_2)|^2 dx \\ & \quad + \int_{\Omega} (u_1 f_1 - u_2 f_2)(u_1 - u_2) dx + \lambda \int_{\Omega} (v_1 g_1 - v_2 g_2)(v_1 - v_2) dx \\ &=: I_{diff} + I_{rea}. \end{aligned} \tag{3.1}$$

The reaction part is then estimated as below

$$\begin{aligned} I_{rea} &= \int_{\Omega} f_1 |u_1 - u_2|^2 dx + \int_{\Omega} u_2 (f_1 - f_2)(u_1 - u_2) dx \\ & \quad + \lambda \int_{\Omega} g_1 |v_1 - v_2|^2 dx + \lambda \int_{\Omega} v_2 (g_1 - g_2)(v_1 - v_2) dx \\ &\leq \max(C_f, C_g) \left(\int_{\Omega} |u_1 - u_2|^2 dx + \lambda \int_{\Omega} |v_1 - v_2|^2 dx \right) \\ & \quad + C'_f \|u_2\|_{L^\infty} \int_{\Omega} \left[|u_1 - u_2|^2 + |u_1 - u_2| |v_1 - v_2| \right] dx \\ & \quad + \lambda C'_g \|v_2\|_{L^\infty} \int_{\Omega} \left[|v_1 - v_2|^2 + |u_1 - u_2| |v_1 - v_2| \right] dx, \end{aligned} \tag{3.2}$$

thanks to (R1). Using Young's inequality, we get from (3.2)

$$I_{rea} \leq C_\lambda (1 + \|u_2\|_{L^\infty} + \|v_2\|_{L^\infty}) (\|u_1 - u_2\|_{L^2(\Omega)}^2 + \lambda \|v_1 - v_2\|_{L^2(\Omega)}^2). \tag{3.3}$$

Concerning the diffusion part, it holds by Young's inequality and (D2),

$$\begin{aligned}
I_{diff} &= - \int_{\Omega} \partial_1 A_1 |\nabla(u_1 - u_2)|^2 dx - d_v \lambda \int_{\Omega} |\nabla(v_1 - v_2)|^2 dx \\
&\quad - \int_{\Omega} \partial_1(A_1 - A_2) \nabla u_2 \cdot \nabla(u_1 - u_2) dx - \int_{\Omega} \partial_2 A_1 \nabla(v_1 - v_2) \cdot \nabla(u_1 - u_2) dx \\
&\quad - \int_{\Omega} \partial_2(A_1 - A_2) \nabla v_2 \cdot \nabla(u_1 - u_2) dx \\
&\leq -\frac{a_0}{4} \int_{\Omega} |\nabla(u_1 - u_2)|^2 dx - (d_v \lambda - \frac{a_2^2}{a_0}) \int_{\Omega} |\nabla(v_1 - v_2)|^2 dx \\
&\quad + \int_{\Omega} \frac{|\nabla u_2|^2}{\partial_1 A_1} |\partial_1(A_1 - A_2)|^2 dx + \int_{\Omega} \frac{|\nabla v_2|^2}{\partial_1 A_1} |\partial_2(A_1 - A_2)|^2 dx. \tag{3.4}
\end{aligned}$$

We now focus on the last two integrals in (3.4). The second one is estimated as follows

$$\begin{aligned}
&\int_{\Omega} \frac{|\nabla v_2|^2}{\partial_1 A_1} |\partial_2(A_1 - A_2)|^2 dx \\
&\leq \frac{2}{a_0} \mathcal{N}_H \|\nabla v_2\|_{L^\infty(\Omega)}^2 \int_{\Omega} (|u_1 - u_2|^2 + |v_1 - v_2|^2) dx \\
&\leq C_\lambda \|\nabla v_2\|_{L^\infty(\Omega)}^2 \int_{\Omega} (|u_1 - u_2|^2 + \lambda |v_1 - v_2|^2) dx. \tag{3.5}
\end{aligned}$$

In order to estimate the first integral, we use (D2) and get

$$\begin{aligned}
&\int_{\Omega} \frac{|\nabla u_2|^2}{\partial_1 A_1} |\partial_1(A_1 - A_2)|^2 dx \\
&\leq \frac{2}{a_0} \|\nabla u_2\|_{L^4(\Omega)}^2 \|\partial_1(A_1 - A_2)\|_{L^4(\Omega)}^2 \\
&\leq \frac{2}{a_0} \mathcal{N}_H^2 \|\nabla u_2\|_{L^4(\Omega)}^2 \left(\|u_1 - u_2\|_{L^4(\Omega)}^2 + \|v_1 - v_2\|_{L^4(\Omega)}^2 \right). \tag{3.6}
\end{aligned}$$

Then, Gagliardo-Nirenberg's inequality in dimension 2 [17] allows us to estimate the L^4 norm of $(u_1 - u_2)$ (resp. $(v_1 - v_2)$) in terms of the L^2 norm of $(u_1 - u_2)$ (resp. $(v_1 - v_2)$) and $\nabla(u_1 - u_2)$ (respectively $\nabla(v_1 - v_2)$), as follows

$$\begin{aligned}
&\|\nabla u_2\|_{L^4(\Omega)}^2 \|u_1 - u_2\|_{L^4(\Omega)}^2 \\
&\leq C_{GN} \|\nabla u_2\|_{L^4(\Omega)}^2 (\|\nabla(u_1 - u_2)\|_{L^2(\Omega)} \|u_1 - u_2\|_{L^2(\Omega)} + \|u_1 - u_2\|_{L^2(\Omega)}^2) \\
&\leq \delta C_{GN} \|\nabla(u_1 - u_2)\|_{L^2(\Omega)}^2 + \frac{C_{GN}}{\delta} \|\nabla u_2\|_{L^4(\Omega)}^4 \|u_1 - u_2\|_{L^2(\Omega)}^2 \\
&\quad + C_{GN} \|\nabla u_2\|_{L^4(\Omega)}^2 \|u_1 - u_2\|_{L^2(\Omega)}^2 \\
&\leq \delta C_{GN} \|\nabla(u_1 - u_2)\|_{L^2(\Omega)}^2 + C_\delta (1 + \|\nabla u_2\|_{L^4}^4) \|u_1 - u_2\|_{L^2(\Omega)}^2,
\end{aligned}$$

where we denote by C_{GN} the constant involved in the Gagliardo-Nirenberg inequality, $\delta > 0$ a parameter to be chosen later, (and C_δ a constant depending on δ). Similarly, for the second term in (3.6) it holds

$$\|\nabla u_2\|_{L^4(\Omega)}^2 \|v_1 - v_2\|_{L^4(\Omega)}^2 \leq \delta C_{GN} \|\nabla(v_1 - v_2)\|_{L^2(\Omega)}^2 + C_\delta (1 + \|\nabla u_2\|_{L^4}^4) \|v_1 - v_2\|_{L^2(\Omega)}^2,$$

so that (3.6) is estimated as

$$\begin{aligned} \int_{\Omega} \frac{|\nabla u_2|^2}{\partial_1 A_1} |\partial_1(A_1 - A_2)|^2 dx &\leq \frac{2\delta}{a_0} C_{GN} \mathcal{N}_H^2 \left(\|\nabla(u_1 - u_2)\|_{L^2(\Omega)}^2 + \|\nabla(v_1 - v_2)\|_{L^2(\Omega)}^2 \right) \\ &\quad + C_{\delta,\lambda} (1 + \|\nabla u_2\|_{L^4}^4) (\|u_1 - u_2\|_{L^2(\Omega)}^2 + \lambda \|v_1 - v_2\|_{L^2(\Omega)}^2). \end{aligned} \quad (3.7)$$

Therefore, gathering (3.5) and (3.7) into (3.4), the term I_{diff} is estimated as, renaming the constants,

$$\begin{aligned} I_{diff} &\leq -\left(\frac{a_0}{4} - \frac{2\delta}{a_0} C_{GN} \mathcal{N}_H^2\right) \|\nabla(u_1 - u_2)\|_{L^2(\Omega)}^2 \\ &\quad - \left(d_v \lambda - \frac{a_2^2}{a_0} - \frac{2\delta}{a_0} C_{GN} \mathcal{N}_H^2\right) \|\nabla(v_1 - v_2)\|_{L^2(\Omega)}^2 \\ &\quad + C_{\delta,\lambda} (1 + \|\nabla u_2\|_{L^4}^4 + \|\nabla v_2\|_{L^\infty}^2) (\|u_1 - u_2\|_{L^2(\Omega)}^2 + \lambda \|v_1 - v_2\|_{L^2(\Omega)}^2). \end{aligned} \quad (3.8)$$

Finally, plugging (3.3), (3.8) into (3.1), we end up with

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left(\|u_1 - u_2\|_{L^2(\Omega)}^2 + \lambda \|v_1 - v_2\|_{L^2(\Omega)}^2 \right) \\ &\leq -\left(\frac{a_0}{4} - \frac{2\delta}{a_0} C_{GN} \mathcal{N}_H^2\right) \|\nabla(u_1 - u_2)\|_{L^2(\Omega)}^2 \\ &\quad - \left(d_v \lambda - \frac{a_2^2}{a_0} - \frac{2\delta}{a_0} C_{GN} \mathcal{N}_H^2\right) \|\nabla(v_1 - v_2)\|_{L^2(\Omega)}^2 \\ &\quad + C_{\delta,\lambda} (1 + \|\nabla u_2\|_{L^4}^4 + \|\nabla v_2\|_{L^\infty}^2 + \|u_2\|_{L^\infty}^2 + \|v_2\|_{L^\infty}^2) \\ &\quad \left(\|u_1 - u_2\|_{L^2(\Omega)}^2 + \lambda \|v_1 - v_2\|_{L^2(\Omega)}^2 \right). \end{aligned} \quad (3.9)$$

Now, we pick $\delta \in (0, \frac{a_0^2}{8C_{GN}\mathcal{N}_H^2})$ and $\lambda \in (\frac{a_2^2}{a_0 d_v} + \frac{a_0}{4d_v}, \infty)$, and obtain, for some $c > 0$,

$$\begin{aligned} \frac{d}{dt} \left(\|u_1 - u_2\|_{L^2(\Omega)}^2 + \lambda \|v_1 - v_2\|_{L^2(\Omega)}^2 \right) &+ c \left(\|\nabla(u_1 - u_2)\|_{L^2(\Omega)}^2 + \|\nabla(v_1 - v_2)\|_{L^2(\Omega)}^2 \right) \\ &\leq C \left(\|u_1 - u_2\|_{L^2(\Omega)}^2 + \lambda \|v_1 - v_2\|_{L^2(\Omega)}^2 \right), \end{aligned}$$

where C depends on $\|u_2\|_{L^\infty(\Omega_T)}$, $\|\nabla u_2\|_{L^\infty([0,T];L^4(\Omega))}$, $\|v_2\|_{L^\infty(\Omega_T)}$, $\|\nabla v_2\|_{L^\infty(\Omega_T)}$.

Finally, we get (1.9) thanks to Gronwall's lemma. \square

We now present the proof of stability stated in *Theorem 1.3* by looking at the evolution in time of the $(H^1)'(\Omega)$ norm.

Proof of Theorem 1.3. Let consider $\phi = \phi(t, x)$ as the unique solution to the Neumann problem, for all $t \in (0, T)$,

$$-\Delta \phi = (u_1 - u_2) - (u_1 - u_2)_\Omega \quad \text{in } \Omega, \quad \nabla \phi \cdot \sigma = 0 \quad \text{on } \partial\Omega, \quad (\phi)_\Omega = 0. \quad (3.10)$$

and $\psi = \psi(t, x)$ as the unique solution to the Neumann problem, for all $t \in (0, T)$,

$$-\Delta \psi = (v_1 - v_2) - (v_1 - v_2)_\Omega \quad \text{in } \Omega, \quad \nabla \psi \cdot \sigma = 0 \quad \text{on } \partial\Omega, \quad (\psi)_\Omega = 0. \quad (3.11)$$

We compute

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \phi|^2 dx &= \frac{1}{2} \frac{d}{dt} \int_{\Omega} (-\Delta \phi + (u_1 - u_2)_{\Omega}) dx = \frac{1}{2} \frac{d}{dt} \int_{\Omega} \phi(u_1 - u_2) dx \\
&= \frac{1}{2} \int_{\Omega} \frac{d}{dt} \phi (-\Delta \phi + (u_1 - u_2)_{\Omega}) dx + \frac{1}{2} \int_{\Omega} \phi \frac{d}{dt} (u_1 - u_2) dx \\
&= -\frac{1}{2} \int_{\Omega} \left(\Delta \frac{d}{dt} \phi \right) \phi dx + \frac{1}{2} \int_{\Omega} \phi \frac{d}{dt} (u_1 - u_2) dx = \int_{\Omega} \phi \frac{d}{dt} (u_1 - u_2) dx \\
&= \int_{\Omega} \phi \Delta (A_1 - A_2) dx + \int_{\Omega} \phi (u_1 f_1 - u_2 f_2) dx =: I_{diff} + I_{rea}. \quad (3.12)
\end{aligned}$$

Using the first equation in (3.10), the assumption (D2) and Young's inequality, we estimate

$$\begin{aligned}
I_{diff} &= \int_{\Omega} (A(u_1, v_1) - A(u_2, v_1)) (-u_1 - u_2) + (u_1 - u_2)_{\Omega} dx \\
&\quad + \int_{\Omega} (A(u_2, v_1) - A(u_2, v_2)) (-u_1 - u_2) + (u_1 - u_2)_{\Omega} dx \\
&\leq -a_0 \int_{\Omega} (u_1 - u_2)^2 dx + a_1 |\Omega| (u_1 - u_2)_{\Omega}^2 \\
&\quad + a_2 \int_{\Omega} |u_1 - u_2| |v_1 - v_2| dx + \frac{a_2}{2} |\Omega| ((u_1 - u_2)_{\Omega}^2 + (v_1 - v_2)_{\Omega}^2) \\
&\leq -a_0 \left(1 - \frac{1}{8} \right) \int_{\Omega} (u_1 - u_2)^2 dx + (a_1 + a_2) |\Omega| (u_1 - u_2)_{\Omega}^2 \\
&\quad + \frac{2a_2^2}{a_0} \int_{\Omega} (v_1 - v_2)^2 dx + a_2 |\Omega| (v_1 - v_2)_{\Omega}^2, \quad (3.13)
\end{aligned}$$

and by (R1), we estimate for any $M > 0$ to be determined later

$$\begin{aligned}
I_{rea} &= \int_{\Omega} \phi f_1 (u_1 - u_2) dx + \int_{\Omega} \phi u_2 (f(u_1, v_1) - f(u_1, v_2) + f(u_1, v_2) - f(u_2, v_2)) dx \\
&\leq \frac{a_0}{8} \int_{\Omega} (u_1 - u_2)^2 dx + \frac{4}{a_0} C_f^2 (1 + \|u_1\|_{L^\infty(\Omega)}^2 + \|v_1\|_{L^\infty(\Omega)}^2) \int_{\Omega} \phi^2 dx \\
&\quad + C_f' \|u_2\|_{L^\infty(\Omega)} \int_{\Omega} |\phi| (|u_1 - u_2| + |v_1 - v_2|) dx \\
&\leq \frac{a_0}{4} \int_{\Omega} (u_1 - u_2)^2 dx + \frac{d_v M}{8} \int_{\Omega} (v_1 - v_2)^2 dx + C_1 \int_{\Omega} \phi^2 dx, \quad (3.14)
\end{aligned}$$

with C_1 depending on $a_0, d_v, M, |\Omega|, C_f, C_f', \|u_i\|_{L^\infty(\Omega)}, \|v_i\|_{L^\infty(\Omega)}, i = 1, 2$. Then, we put (3.13) and (3.14) into (3.12) to get

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \phi|^2 dx &\leq -\frac{5a_0}{8} \int_{\Omega} (u_1 - u_2)^2 dx + \left(\frac{d_v M}{8} + \frac{2a_2^2}{a_0} \right) \int_{\Omega} (v_1 - v_2)^2 dx \\
&\quad + C_1 \int_{\Omega} \phi^2 dx + (a_1 + a_2) |\Omega| (u_1 - u_2)_{\Omega}^2 + a_2 |\Omega| (v_1 - v_2)_{\Omega}^2. \quad (3.15)
\end{aligned}$$

Similarly as in (3.12), using the first equation in (3.11), we have for any $M > 0$

$$\begin{aligned} & \frac{M}{2} \frac{d}{dt} \int_{\Omega} |\nabla \psi|^2 dx \\ &= d_v M \int_{\Omega} (v_1 - v_2) (-(v_1 - v_2) + (v_1 - v_2)_{\Omega}) dx + \int_{\Omega} M \psi (v_1 g_1 - v_2 g_2) dx \\ &=: II_{diff} + II_{rea}, \end{aligned}$$

with

$$II_{diff} \leq -d_v M \left(1 - \frac{1}{8}\right) \int_{\Omega} (v_1 - v_2)^2 dx + 2d_v M |\Omega| (v_1 - v_2)_{\Omega}^2,$$

and

$$II_{rea} \leq \frac{a_0}{8} \int_{\Omega} (u_1 - u_2)^2 dx + \frac{d_v M}{4} \int_{\Omega} (v_1 - v_2)^2 dx + C_2 \int_{\Omega} \psi^2 dx,$$

with C_2 depending on $a_0, d_v, M, |\Omega|, C_g, C'_g, \|u_i\|_{L^\infty(\Omega)}, \|v_i\|_{L^\infty(\Omega)}, i = 1, 2$. Therefore, we end up with

$$\begin{aligned} \frac{M}{2} \frac{d}{dt} \int_{\Omega} |\nabla \psi|^2 dx &\leq -\frac{5}{8} d_v M \int_{\Omega} (v_1 - v_2)^2 dx + \frac{a_0}{8} \int_{\Omega} (u_1 - u_2)^2 dx \\ &\quad + C_2 \int_{\Omega} \psi^2 dx + 2d_v M |\Omega| (v_1 - v_2)_{\Omega}^2. \end{aligned} \quad (3.16)$$

By gathering (3.15), (3.16), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} (|\nabla \phi|^2 + M |\nabla \psi|^2) dx \\ &\leq -\frac{a_0}{2} \int_{\Omega} (u_1 - u_2)^2 dx - \left(\frac{d_v M}{2} - \frac{2a_0^2}{a_0}\right) \int_{\Omega} (v_1 - v_2)^2 dx + (C_1 + C_2) \int_{\Omega} (\phi^2 + \psi^2) dx \\ &\quad + (a_1 + a_2) |\Omega| (u_1 - u_2)_{\Omega}^2 + (a_2 + d_v M) |\Omega| (v_1 - v_2)_{\Omega}^2. \end{aligned} \quad (3.17)$$

Now, we analyse the evolution in time of $(u_1 - u_2)_{\Omega}$ and $(v_1 - v_2)_{\Omega}$, respectively.

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (u_1 - u_2)_{\Omega}^2 &= \frac{1}{|\Omega|^2} \int_{\Omega} (u_1 - u_2) dx \int_{\Omega} (u_1 f_1 - u_2 f_2) dx \\ &= \frac{1}{|\Omega|^2} \int_{\Omega} (u_1 - u_2) dx \int_{\Omega} f_1 (u_1 - u_2) dx \\ &\quad + \frac{1}{|\Omega|^2} \int_{\Omega} (u_1 - u_2) dx \int_{\Omega} u_2 (f_1 - f_2) dx =: J_1 + J_2, \end{aligned} \quad (3.18)$$

thus using (R1), we compute by Young's inequality and Hölder's inequality

$$\begin{aligned} J_1 &\leq \frac{a_0}{8|\Omega|} \left(\int_{\Omega} (u_1 - u_2) dx \right)^2 + \frac{2}{a_0 |\Omega|^3} \left(\int_{\Omega} f_1 (u_1 - u_2) dx \right)^2 \\ &\leq \frac{a_0}{8} \|u_1 - u_2\|_{L^2(\Omega)}^2 + C' (u_1 - u_2)_{\Omega}^2, \end{aligned} \quad (3.19)$$

and similarly

$$\begin{aligned} J_2 &= \frac{1}{|\Omega|^2} \int_{\Omega} (u_1 - u_2) dx \int_{\Omega} u_2 (f(u_1, v_1) - f(u_1, v_2) + f(u_1, v_2) - f(u_2, v_2)) dx \\ &\leq \frac{a_0}{8} \|u_1 - u_2\|_{L^2(\Omega)}^2 + \frac{d_v M}{8} \|v_1 - v_2\|_{L^2(\Omega)}^2 + C'' (u_1 - u_2)_{\Omega}^2. \end{aligned} \quad (3.20)$$

By putting (3.19), (3.20) into (3.18), we obtain

$$\frac{1}{2} \frac{d}{dt} (u_1 - u_2)_\Omega^2 \leq \frac{a_0}{4} \|u_1 - u_2\|_{L^2(\Omega)}^2 + \frac{d_v M}{8} \|v_1 - v_2\|_{L^2(\Omega)}^2 + C_3 (u_1 - u_2)_\Omega^2, \quad (3.21)$$

where the constant C_3 depends on $a_0, d_v, \eta_u, \eta_v, M, \Omega$ and on $\|u_i\|_{L^\infty(\Omega)}, \|v_i\|_{L^\infty(\Omega)}$, $i = 1, 2$. Similarly, it holds

$$\frac{1}{2} \frac{d}{dt} (v_1 - v_2)_\Omega^2 \leq \frac{a_0}{8} \|u_1 - u_2\|_{L^2(\Omega)}^2 + \frac{d_v M}{4} \|v_1 - v_2\|_{L^2(\Omega)}^2 + C_4 (v_1 - v_2)_\Omega^2, \quad (3.22)$$

where the constant C_4 depends on $a_0, d_v, \eta_u, \eta_v, M, \Omega$ and on $\|u_i\|_{L^\infty(\Omega)}, \|v_i\|_{L^\infty(\Omega)}$, $i = 1, 2$.

Adding estimates (3.17), (3.21), (3.22), using Poincaré's inequality (remember that $\int_\Omega \phi \, dx = \int_\Omega \psi \, dx = 0$) and renaming the constants, we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_\Omega (|\nabla \phi|^2 + M |\nabla \psi|^2 + (u_1 - u_2)_\Omega^2 + (v_1 - v_2)_\Omega^2) \, dx \\ & \leq -\frac{a_0}{8} \|u_1 - u_2\|_{L^2(\Omega)}^2 - \left(\frac{d_v M}{8} - \frac{2a_2^2}{a_0} \right) \|v_1 - v_2\|_{L^2(\Omega)}^2 \\ & \quad + C_M \int_\Omega (|\nabla \phi|^2 + M |\nabla \psi|^2 + (u_1 - u_2)_\Omega^2 + (v_1 - v_2)_\Omega^2) \, dx. \end{aligned}$$

Thus, taking $M > 0$ large enough, this finishes the proof by Gronwall's lemma.

Finally, we remark how the dependency on the first solution can be weakened and explain it on the first term of I_{rea} . With p satisfying $1/p > 1/2 - 1/N$ and the conjugate index p^* , we can estimate the term as

$$\begin{aligned} \int_\Omega \phi f_1(u_1 - u_2) \, dx & \leq \|\phi\|_p \|f_1(u_1 - u_2)\|_{p^*} \\ & \leq \|\phi\|_p \|f_1\|_r \|u_1 - u_2\|_2, \end{aligned}$$

where $1/r = 1/p^* - 1/2$. Then the stability constant only depends on $\|u_1\|_r$ as by the choice of $1/p > 1/2 - 1/N$ we can estimate for $\alpha \in (0, 1)$

$$\|\phi\|_p \lesssim \|\phi\|_2^\alpha \|\nabla \phi\|_2^{1-\alpha} + \|\phi\|_2$$

so that the Gronwall estimate can be closed. \square

A Some classical results for linear parabolic equation

We state here the following existence result for a linear parabolic equation in *non-divergence* form, together with some standard estimates.

Proposition A.1. *We consider the following linear parabolic problem defined on a smooth bounded domain Ω of \mathbb{R}^N :*

$$\begin{cases} \partial_t b - \gamma(t, x) \Delta b = r(t, x) b, & \text{in } \Omega_T, \\ \nabla b \cdot \sigma = 0, & \text{on } (0, T) \times \partial\Omega, \\ b(0, x) = b_{in}(x) \geq 0, & \text{on } \Omega, \end{cases} \quad (\text{A.1})$$

where

i) $\gamma : \Omega_T \rightarrow \mathbb{R}_+$, γ lies in $L^\infty([0, T]; W^{1, \infty}(\Omega))$ and there exist two constants $\gamma_0, \gamma_1 > 0$ such that

$$0 < \gamma_0 \leq \gamma(t, x) \leq \gamma_1, \quad \text{a.e. in } \Omega_T, \quad (\text{A.2})$$

ii) $r : \Omega_T \rightarrow \mathbb{R}$, r lies in $L^2(\Omega_T)$ and $L^1([0, T]; L^\infty(\Omega))$,

iii) $b_{in} : \Omega \rightarrow \mathbb{R}_+$ satisfies

$$b_{in} \in (L^\infty \cap H^1)(\Omega). \quad (\text{A.3})$$

Then, there exists a nonnegative strong (in the sense of Theorem 1.1) solution b to (A.1) such that

i) for all $t \in (0, T)$, b satisfies

$$\|b\|_{L^\infty(\Omega_T)} \leq \|b_{in}\|_{L^\infty(\Omega)} e^{\int_0^t \sup_{x \in \Omega} r(t, x) dt}, \quad (\text{A.4})$$

ii) there exists a constant $C > 0$ depending on $b_{in}, \gamma_0, \gamma_1$, such that

$$\|\partial_t b\|_{L^2(\Omega_T)}^2 + \|\nabla b\|_{L^\infty([0, T]; L^2(\Omega))}^2 + \|\Delta b\|_{L^2(\Omega_T)}^2 \leq C(1 + \|rb\|_{L^2(\Omega_T)}^2). \quad (\text{A.5})$$

We briefly explain the strategy of proof of Proposition A.1:

We first consider a regularized version of (A.1), which admits a unique classical solution, thanks to classical results of parabolic PDEs theory [15].

Then, we can prove uniform (with respect to the regularizing parameter) a priori estimates corresponding to (A.4), (A.5) and pass to the limit when that parameter tends to 0.

For (A.4), this is done by looking at the equation satisfied by $b(t, x) \exp(-\int_0^t r(s, x) ds)$ and by applying the maximum principle. For (A.5), the estimate is obtained thanks to the use of the multiplier Δb .

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References

- [1] Brezis H., *Functional analysis, Sobolev spaces and partial differential equations*, Springer, Vol. 2. (2011).
- [2] Capone F., and Fiorentino L., *Turing instability for a Leslie–Gower model*, *Ricerche di Matematica* (2023).
- [3] Brocchieri, E., *Evolutionary dynamics of populations structured by dietary diversity and starvation: cross-diffusion systems*, PhD thesis, Université Paris-Saclay; Università degli studi La Sapienza (Rome). Dipartimento di matematica (2023).
- [4] Cho, E., and Kim, Y.-J., *Starvation driven diffusion as a survival strategy of biological organisms*, *Bull. Math. Biol.* **75** (2013), 845–870.
- [5] Conforto, F., Desvillettes, L., and Soresina, C., *About reaction-diffusion systems involving the Holling-type II and the Beddington-De Angelis functional responses for predator prey models*, *Nonlinear Differ. Equ. Appl.* **25** (2018).

- [6] Desvillettes, L., *About the triangular Shigesada–Kawasaki–Teramoto reaction cross diffusion system*. Ricerche di Matematica. **73** Suppl 1 (2024), 105–114.
- [7] Desvillettes, L., and Trescases, A., *New results for triangular reaction cross diffusion system*, J. Math. Anal. Appl. **430** (2015), 32–59.
- [8] Desvillettes L., and Soresina C., *Non-triangular cross-diffusion systems with predator–prey reaction terms*. Ricerche di Matematica. **68** (1) (2019), 295–314.
- [9] Evans, L. C., *Partial differential equations*. American Mathematical Society, Vol. 19 (2022).
- [10] Gambino, G., Lombardo, M. C., and Sammartino, M., *Pattern formation driven by cross-diffusion in a 2D domain*, Nonlinear Analysis: Real World Applications, **14** (3) (2013), 1755–1779.
- [11] Gambino, G., Lombardo, M. C., and Sammartino, M., *Turing instability and traveling fronts for a nonlinear reaction–diffusion system with cross-diffusion*, Mathematics and Computers in Simulation, **82** (6) (2012), 1112–1132.
- [12] Gambino, G., Lombardo, M. C., and Sammartino, M., *Cross-diffusion driven instability for a Lotka–Volterra competitive reaction–diffusion system*. In Waves and Stability in Continuous Media (2008), 297–302.
- [13] Iida, M., Mimura, M., and Ninomiya, H., *Diffusion, Cross-diffusion and Competitive Interaction*, J. Math. Biol. **53** (2006), 617–641.
- [14] Kim, Y. J., Kwon, O., and Li, F., *Global asymptotic stability and the ideal free distribution in a starvation driven diffusion*. J. Math. Biol. **68** (6) (2014), 1341–1370.
- [15] Ladyženskaja, O.A., Solonnikov, V.A., and Ural’ceva, N.N., *Linear and quasi-linear equations of parabolic type*, American Mathematical Soc., Vol 23 (1988).
- [16] Lou, Y., Ni, W.M., and Wu, Y., *On the global existence of a cross-diffusion system*. Discrete and Continuous Dynamical Systems, **4** (1998), 193–204.
- [17] Nirenberg, L. *An extended interpolation inequality*, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze **20.4** (1966), 733–737.
- [18] Shigesada, N., Kawasaki, K., and Teramoto, E., *Spatial segregation of interacting species*, J. Theor. Biol. **79** (1979), 83–99.
- [19] Yamada, Y., *Global solutions for quasilinear parabolic systems with cross-diffusion effects*, Nonlinear Analysis: Theory, Methods and Applications, **24** (9) (1995), 1395–1412.

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