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# Mean-field limit of particle systems with absorption

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## Abstract

We consider a particle system of singular interaction where particles are removed from the system once they hit some barrier. We show the wellposedness of the particle system and its mean-field limit and prove the propagation of chaos.

**Keywords:**    particle system, mean-field games, Fokker-Planck equation, propagation of chaos

## 1 Introduction

We consider interacting particle systems in 1- $d$  whose components are absorbed as soon as they hit some barrier, saying zero without loss of generality. Denote by  $X^{N,1}, \dots, X^{N,N}$  the  $N$  particles whose dynamics are given as follows: for  $i = 1, \dots, N$ ,

$$X_t^{N,i} = Z^{N,i} + \int_0^t \iota(X_s^{N,i}) \frac{1}{N} \sum_{j=1}^N b(s, X_s^{N,i}, X_s^{N,j}) \iota(X_s^{N,j}) ds + \int_0^t \sqrt{2} \iota(X_s^{N,i}) dW_s^i, \quad \forall t \geq 0, \quad (1)$$

where  $\iota(x) := \mathbb{1}_{\{x > 0\}}$  denotes the indicator function,  $b : \mathbb{R}_+ \times \mathbb{R}^2 \rightarrow \mathbb{R}$  is a measurable drift and

- $(Z^{N,1}, \dots, Z^{N,N})$  is a family of exchangeable real valued random variables;
- $W^1, \dots, W^N$  are independent Brownian motions that are independent of  $(Z^{N,1}, \dots, Z^{N,N})$ .

Provided that (1) is well posed, every process  $X^{N,i}$  is absorbed as soon as it hits zero, i.e.

$$X_t^{N,i} = X_{\tau_i^{N, \wedge t}}^{N,i}, \quad \forall t \geq 0,$$

where  $\tau_i^N := \inf\{t \geq 0 : X_t^{N,i} \leq 0\}$  stands for the first hitting time of  $X^{N,i}$  at zero. An equivalent formulation to (1) is

$$dX_t^{N,i} = \mathbb{1}_{\{\tau_i^N > t\}} \left[ \frac{1}{N} \sum_{j=1}^N b(t, X_t^{N,i}, X_t^{N,j}) \mathbb{1}_{\{\tau_j^N > t\}} dt + \sqrt{2} dW_t^i \right], \quad \forall t \geq 0.$$

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We aim to show the propagation of chaos, as  $N \rightarrow \infty$ , of the particle system (1) towards the following mean-field (non-linear) stochastic differential equation (SDE):

$$X_t = Z + \int_0^t \iota(X_s) \left( \int_{(0,\infty)} b(s, X_t, y) \mu_t(dy) \right) ds + \int_0^t \iota(X_s) \sqrt{2} dW_s, \quad \forall t \geq 0, \quad (2)$$

where  $\mu_t := \mathcal{L}(X_t)$  stands for the law of  $X_t$ .

It is sometimes mathematically more convenient to study the mean-field limit of an interacting system to avoid, for example, the curse of dimensionality. This is also usually done in mean-field games literature. However, justifying rigorously the propagation of chaos is central to this reasoning. Therefore, the goal of this paper is to show the wellposedness of (1) and (2), and more importantly to establish the propagation of chaos while  $N \rightarrow \infty$  under suitable hypothesis on the coefficient  $b$  (essentially continuity and boundedness).

The difficulty when dealing with (1) and (2) is that the diffusion coefficient is not uniformly elliptic due to the presence of the hitting time. In addition, we do not have a priori any regularity on the law of the hitting time(s). This in some sens means that the cross-interaction of (1) is singular, and the strength of this interaction is highly related to the number of active particles. Hence, we are not in the classical framework of [24], but we will see that under boundedness and continuity assumption on  $b$  we can obtain a tightness-consistency result when  $N \rightarrow \infty$ . Enforcing more regularity on  $b$ , and working on the mean-field limit, we will obtain the strong well-posedness of (2) that will ensure the propagation of chaos result.

In general, interacting particle systems are studied in many different contexts of applications, including the dynamics of granular media [6, 7], mathematical biology [15, 9], economics and social networks [19, 17], and deep neural networks [21, 20]. A more detailed exposition can be found, for example, in [2, 3, 4, 5].

The motivations for studying (1) comes from systemic risk arising in banking systems. Particle systems of the form (1) that interact through hitting times are largely used to model the evolution of complex banking networks with mutual exposures, see [18]. Under this framework, a bank defaults when its capital level drops below zero, and its default causes instantaneous variation of the other active banks' drift. In contrast to [18], an alternative modelling framework is that defaults do not change capital values but affect their dynamic coefficients, see e.g. [16] for the study of the mean-field limit model with mutual holding. Our model enters this second framework.

We mention here other models of interacting systems and their mean field limits involving hitting times. In the context of neurosciences and integrate and fire models see [10, 11]. In finance, besides already mentionned [16, 18], see [1] and the references therein.

We finish this section with a future perspective for studying systems of type (1) in ecological or biophysical context. In such setting, consider individuals (population in a habitat/group of cells in a domain) that move in their environment interacting with each other, but when an individual exits the habitat/domain it never comes back and it no longer influences the mutual interaction. Contrary to the current setting, this would be more relevant in  $\mathbb{R}^2$  and with more singularity on the interaction

b. One example of a limit model would be in the vein of Keller-Segel systems (see e.g. [14]), but with homogeneous boundary conditions. There, a population of cells interacts in an attractive way by following a chemical gradient that it generates and the homogeneous Dirichlet boundary condition is seen as the disappearance of cells and chemicals near the boundary.

**Plan of the paper.** The rest of the paper is structured as follows. In Section 2 we give our main hypothesis and results and we discuss the proof strategies. In Section 3 we deal with the existence and convergence of the particle system (1). Finally, strong well-posedness for (2) is obtained in Section 4.

**Notations** For a generic Polish space  $E$ , denote by  $C(E)$  (resp.  $\mathcal{P}(E)$ ) the set of continuous functions (resp. probability measures) on  $E$ . Set  $\Omega := C(\mathbb{R}_+)$  and  $\Omega_T := C([0, T])$  for every  $T > 0$ . Note that  $\Omega_T$  can be identified as a subset of  $\Omega$  since every  $f \in \Omega_T$  can be seen as an element  $f \in \Omega$  by setting  $f(t) := f(t \wedge T)$ . Therefore, we say simply  $\Omega_T \subset \Omega$  without any danger of confusion. Let  $F$  be the coordinate process on  $\Omega$ , i.e.  $F_t(f) := f(t)$  for all  $f \in \Omega$  and  $t \geq 0$ . Define further the canonical filtration  $\mathbb{F} \equiv (\mathcal{F}_t := \sigma(F_s, s \leq t))_{t \geq 0}$ . In what follows, we need the empirical measures

$$\mu^N := \frac{1}{N} \sum_{i=1}^N \delta_{X^{N,i}} \quad \text{and} \quad \mu_t^N := \frac{1}{N} \sum_{i=1}^N \delta_{X_t^{N,i}}, \quad \forall t \geq 0, \quad (3)$$

that take values respectively in  $\mathcal{P}(\Omega)$  and  $\mathcal{P}(\mathbb{R}_+)$ .

## 2 Main results

Let us start with the following set of assumptions. In every statement, it will be made precise which of the assumptions below is necessary.

**Assumption 1.** (i) *The measurable function  $b : \mathbb{R}_+ \times \mathbb{R}^2 \rightarrow \mathbb{R}$  is bounded.*

(ii) *For every  $t \geq 0$ ,  $b(t, \cdot)$  is continuous almost everywhere on  $\mathbb{R}^2$ , and for every  $x \geq 0$  the limits  $\lim_{y \rightarrow 0} b(s, x, y)$ ,  $\lim_{y \rightarrow 0} b(s, y, x)$  exist.*

(iii)  *$b, \partial_x b$  are bounded and Hölder continuous.*

Now, we give the definition of strong and weak solution to the non linear process (2).

**Definition 2.** (i) *A process  $X$  is said to be a (strong) solution to (2) if it is adapted with respect to the filtration generated by  $W$  and (2) holds for all  $t \geq 0$  almost surely.*

(ii)  $\mathbb{Q} \in \mathcal{P}(\Omega)$  *is called a weak solution to (2) if, with  $\mathbb{Q}_t := \mathbb{Q} \circ F_t^{-1}$*

1.  $\mathbb{Q}_0 = \rho$ ;

2. *For every  $\varphi \in C_c^2(\mathbb{R})$ , the process  $M^\varphi$  defined by*

$$M_t^\varphi := \varphi(F_t) - \varphi(F_0) - \int_0^t \iota(F_s) \left( \varphi'(F_s) \left( \int_{(0, \infty)} b(s, F_s, y) \mathbb{Q}_s(dy) \right) + \varphi''(F_s) \right) ds, \quad \forall t \geq 0$$

is an  $\mathbb{F}$ -martingale under  $\mathbb{Q}$ .

3.  $\mathbb{Q}_t[\mathbb{R}_+] = 1$ , for all  $t \geq 0$ .

Our first main result is the well-posedness of the limit SDE.

**Theorem 3.** *Let  $\rho \in \mathcal{P}(\mathbb{R}_+)$ . Under Assumption 1-(iii), the mean-field SDE (2) admits a unique (strong) solution.*

For the  $N$ -particle system (1), we first prove that there exists a unique weak solution. Then, we prove tightness for the empirical measure in (3) and that any limit point of the empirical measures must be a weak solution to (2). This yields the desired propagation of chaos by the uniqueness in law of the solution to (2).

**Theorem 4.** *1) Let Assumption 1-(i) hold. Then, for each  $N \geq 1$ , there exists a unique weak solution  $(X^{N,1}, \dots, X^{N,N})$  to the particle system (1).*

*2) Suppose in addition that Assumption 1-(ii) holds and that  $\mu_0^N = \frac{1}{N} \sum_{i=1}^N \delta_{Z^{N,i}}$  converges in probability to some deterministic measure  $\rho \in \mathcal{P}(\mathbb{R}_+)$ . Then, any (possibly random) limit point  $\mu$  of  $(\mu^N)_{N \geq 2}$  a.s. solves in the weak sense (2) with initial law  $\rho$ .*

*3) Suppose in addition that Assumption 1-(iii) holds. Then,  $\mu^N$  converges in probability to the law of  $X$ .*

**Strategy of the proof.** For Theorem 3, the proof of the existence for the McKean-Vlasov SDE relies on Schauder's fixed-point theorem. One delicate point here is that the measure that solves it accumulates dynamically mass in zero, hence the functional on the space of flows of probability measures for which we would naturally exhibit a fixed point contains an indicator function. This is not continuous for Wasserstein distance for probability measures, so the space on which we exhibit a fixed point is enlarged to take into account this dynamical accumulation of mass in zero. It becomes the product space of flows of probability measures and  $\Omega_T$ .

As for the uniqueness, we study the marginal distributions of any solution  $X$  to (2). For  $t > 0$ , we first show that the law of  $X_t$  is of the form  $p(t, x)dx + (1 - \int p(t, x)dx)\delta_0(dx)$ , where  $p(t, \cdot)$  is a sub-probability density satisfying a parametric Fokker-Planck equation whose coefficients depend on  $p$  itself. We show the one-to-one relation between  $X$  and such parametric equation, and the uniqueness of  $X$  is equivalent to that of  $p$  satisfying this PDE. Due to the non-linearity, we consider PDEs of parametric coefficients and estimate the explicit dependency of their solutions on the parameters. By introducing a suitably defined operator, we prove that the operator is a contraction and thus the uniqueness result follows.

For Theorem 4, the existence follows through an application of a Girsanov-Cameron-Martin transform. The boundedness of the coefficient  $b$  easily gives the tightness of the sequence of the empirical measures. The identification of the limit is done through the martingale approach and is more delicate due to the singularity the stopping time introduces in the drift. We will thus regularize  $\iota(\cdot)$  and show that regularized and non-regularized problems are close. In the latter, it will be crucial to use the fact that individual particles behave like stopped Brownian motions modulo a change of measure. To make such a change uniform in  $N$ , we adopt the strategy of [22] and introduce the *Partial Girsanov transforms* (see the discussion after Lemma 5).

We finish this part with a following remark. We believe that the first two parts of Theorem 4 could be adapted to the so called  $L^q - L^p$  assumption on  $b$  with more involved computations as in [22]. We leave possible generalizations of assumptions on  $b$  for a future work.

### 3 Proof of Theorem 4

We split this section into three parts. In the first part we show Theorem 4-1). Then, in the second part we show the tightness of the empirical measures. Finally, in the last part, we show Theorem 4-2). The last claim in Theorem 4 simply follows from uniqueness of solutions to (2) that is, under more regularity on  $b$ , guaranteed by Theorem 4.

#### 3.1 Wellposedness of (1)

We fix an arbitrary  $N \geq 1$  and show the wellposedness of (1) under only Assumption 1-(i). It suffices to prove that the particle system (1) has a weak solution on any interval  $[0, T]$ . Pick some filtered probability space  $(E, \mathcal{E}, \mathbb{G} = (\mathcal{G}_t)_{t \geq 0}, \mathbb{W})$  that supports random variables  $\bar{Z}^{N,1}, \dots, \bar{Z}^{N,N}$  and independent Brownian motions  $\bar{W}^1, \dots, \bar{W}^N$  such that  $\mathcal{L}(\bar{Z}^{N,1}, \dots, \bar{Z}^{N,N}) = \mathcal{L}(Z^{N,1}, \dots, Z^{N,N})$ . Define processes  $Y^{N,i}$  and  $\bar{X}^{N,i}$  by

$$Y_t^{N,i} := \bar{Z}^{N,i} + \sqrt{2}\bar{W}_t^i \quad \text{and} \quad \bar{X}_t^{N,i} := Y_{\sigma_i^N \wedge t}^{N,i}, \quad \forall t \geq 0,$$

where  $\sigma_i^N := \inf\{s \geq 0 : Y_s^{N,i} \leq 0\}$ . Then a straightforward verification yields

$$\bar{X}_t^{N,i} = \bar{Z}^{N,i} + \int_0^t \sqrt{2}\iota(\bar{X}_s^{N,i})d\bar{W}_s^i, \quad \forall t \geq 0. \quad (4)$$

Set  $\bar{X}_t^N := (\bar{X}_t^{N,1}, \dots, \bar{X}_t^{N,N})$  and  $\bar{W}_t := (\bar{W}_t^1, \dots, \bar{W}_t^N)$ . For  $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ , denote

$$b_t^{N,i}(x) := \frac{1}{N} \sum_{j=1}^N b(t, x_i, x_j) \iota(x_j).$$

and  $B_t^N(x) := (b_t^{N,1}(x), \dots, b_t^{N,N}(x))$ . As  $\sup_{t \leq T} \|B_t^N\|_\infty \leq \sqrt{N}\|b\|_{T,\infty} < \infty$ , we introduce the probability measure  $\mathbb{Q}^N \in \mathcal{P}(\Omega_T)$  via the Radon-Nikodym derivative

$$\left. \frac{d\mathbb{Q}^N}{d\mathbb{W}} \right|_T := \exp \left( \int_0^T B_u^N(\bar{X}_u^N) \cdot d\bar{W}_u - \frac{1}{2} \int_0^T |B_u^N(\bar{X}_u^N)|^2 du \right) =: Z_T^N.$$

Applying the Cameron-Martin-Girsanov theorem, see e.g. [13, Thm 4.1 p. 191] or [23, Thm. 6.4.2 p. 154], we deduce that

$$\left( V_t := \overline{W}_t - \int_0^t B_s^N(\overline{X}_s^N) ds \right)_{0 \leq t \leq T}$$

is a vector consisting of  $N$  independent Brownian motions under  $\mathbb{Q}^N$ . In addition, we have in view of [13, Thm 4.1 p. 191]

$$\overline{X}_t^{N,i} = \overline{Z}^{N,i} + \int_0^t \iota(\overline{X}_s^{N,i}) \frac{1}{N} \sum_{j=1}^N b(s, \overline{X}_s^{N,i}, \overline{X}_s^{N,j}) \iota(\overline{X}_s^{N,j}) ds + \int_0^t \sqrt{2} \iota(\overline{X}_s^{N,i}) dV_s^i, \quad \forall t \in [0, T],$$

which implies that  $\mathbb{Q} := \mathbb{Q}^N \circ (\overline{X}^N)^{-1} \in \mathcal{P}(\Omega_T)$  is a weak solution to (1). The uniqueness in law follows by [13, Thm 4.2 p. 194]. This concludes the first part of Theorem 4.

### 3.2 Tightness of $\{\mu^N\}_{N \geq 1}$ and Partial Girsanov transforms

Let us first show the tightness of  $\{\mu^N\}_{N \geq 1}$ .

**Lemma 5.** *Suppose that  $\mu_0^N = \frac{1}{N} \sum_{i=1}^N \delta_{Z_{N,i}}$  converges in probability to some deterministic measure  $\rho \in \mathcal{P}(\mathbb{R}_+)$ . In addition, let Assumption 1-(i) hold. Then, the sequence of random measures  $\{\mu^N\}_{N \geq 1}$  is tight in  $\mathcal{P}(\Omega_T)$  under  $\mathbb{Q}^N$ .*

**Proof.** By [24, Prop. 2.2-ii], its tightness results from the tightness of the intensity measure  $\{\mathbb{E}_{\mathbb{Q}^N} \mu^N(\cdot)\}_{N \geq 1}$ . By symmetry, it suffices to check the tightness of  $\{\mathcal{L}(X_t^{N,1} : 0 \leq t \leq T)\}_{N \geq 1}$ . The boundedness of  $b$  yields the existence of some constant  $C > 0$  such that

$$\mathbb{E}_{\mathbb{Q}^N} [|X_t^{N,1} - X_s^{N,1}|^4] \leq C(t-s)^4 + C \mathbb{E}_{\mathbb{Q}^N} \left[ \left| \int_s^t \iota(X_u^{N,1}) dW_u^1 \right|^4 \right] \leq C(t-s)^4 + C(t-s)^2,$$

which fulfills the proof.  $\square$

We notice here that the fact that  $b$  is bounded does not only play an important role when showing tightness, but it will also be crucial for the identification of the limit as it enables us to use the Girsanov transform. The fact that particles behave as stopped Brownian motions (up to a Girsanov transform) will be used in what follows. However, passing from the interacting system to the system of independent stopped Brownian motions has a cost that was not uniform in  $N$  (see the existence section) and as such a full Girsanov transform is not useful when we pass to the limit as  $N \rightarrow \infty$ . Nevertheless we can use that the individual behaviours are the ones of stopped Brownian motions by introducing *Partial transforms* related to (1).

For a fixed  $r < N$  that corresponds to  $r$  particles that are transformed to independent stopped Brownian motions and whose influence is removed from the other  $N - r$  particles. The cost of such transforms will be uniform in  $N$ . Although for the identification of the limit we will only use the case  $r = 1$ , we give them below for an arbitrary  $r < N$ .

Let us choose, for simplicity, the first  $r$  particles to be transformed, and consider the following reference system defined on the probability space  $(E, \mathcal{E}, \mathbb{G} = (\mathcal{G}_t)_{t \geq 0}, \mathbb{W}^{(r, N)})$ :

$$\begin{cases} \hat{X}_t^{N, i} = Z^{N, i} + \int_0^t \iota(\hat{X}_s^{N, i}) d\hat{W}_s^i, & i \leq r, t \leq T \\ \hat{X}_t^{N, i} = Z^{N, i} + \int_0^t \iota(\hat{X}_s^{N, i}) d\hat{W}_s^i + \iota(\hat{X}_s^{N, i}) \frac{1}{N} \sum_{j=r+1}^N \int_0^t b(s, \hat{X}_s^{N, i}, \hat{X}_s^{N, j}) \iota(\hat{X}_s^{N, j}) ds, & r+1 \leq i \leq N, t \leq T. \end{cases}$$

Obviously, it is well-posed as it can be obtained, as in the previous section, from  $\bar{X}$ . We now study the change of measure between  $\hat{X}$  and  $X$ . The corresponding drift vector is

$$\beta_t^{(r)}(x) := \left( b_t^{N, 1}(x), \dots, b_t^{N, r}(x), \frac{1}{N} \sum_{i=1}^r b(t, x^{r+1}, x^i) \iota(x^i), \dots, \frac{1}{N} \sum_{i=1}^r b(t, x^N, x^i) \iota(x^i) \right).$$

In the sequel we will need uniform w.r.t  $N$  bounds for moments of

$$Z_T^{(r)} := \exp \left\{ \int_0^T \beta_t^{(r)}(\hat{X}_t) \cdot dW_t - \frac{1}{2} \int_0^T |\beta_t^{(r)}(\hat{X}_t)|^2 dt \right\}. \quad (5)$$

**Proposition 6.** *For any  $T > 0$ ,  $\gamma > 0$  and  $r \geq 1$  there exists  $C(T, \gamma, r)$  s.t.*

$$\forall N \geq 1, \quad \mathbb{E}_{\mathbb{W}^{(r, N)}} \exp \left\{ \gamma \int_0^T |\beta_t^{(r)}(\hat{X}_t)|^2 dt \right\} \leq C(T, \gamma, r).$$

**Proof.** Observe that

$$|\beta_t^{(r)}(x)|^2 = \sum_{i=1}^r |b_t^{N, i}(x)|^2 + \frac{1}{N^2} \sum_{j=r+1}^N \left( \sum_{i=1}^r b(t, x^{j+1}, x^i) \iota(x^i) \right)^2 \leq r \|b\|_\infty^2 + \frac{(N-r)r^2}{N^2} \|b\|_\infty^2$$

This yields the desired result with  $C(T, \gamma, r) = e^{2\gamma T r \|b\|_\infty^2}$ .  $\square$

### 3.3 Identification of the limit

Now we are ready to identify the limit. Having showed that  $\{\mu^N\}_{N \geq 1}$  is tight, we may extract a convergent subsequence, which is still denoted by  $\{\mu^N\}_{N \geq 1}$ , of limit  $\mu$ . Since  $\mu^N \rightarrow \mu$  in law in  $\mathcal{P}(\Omega)$ , we have that  $\mu^N \times \mu^N \rightarrow \mu \times \mu$ .

It remains to prove that  $\mu$  is a weak solution to (2). For any  $\varphi \in C_c^2(\mathbb{R})$ , the process  $(M_t^\varphi)_{t \geq 0}$ , defined in the martingale problem, should be a  $\mu$  martingale. To this end, it suffices to show that for all  $t > s > 0$ , all continuous bounded function  $\Phi \in C(\Omega_s)$ , we have  $\Psi(\mu) = 0$  a.s., where for  $\mathbb{Q} \in \mathcal{P}(\Omega)$ ,

$$\Psi(\mathbb{Q}) := \mathbb{E}^{\mathbb{Q}} \left[ \Phi((F_r)_{r \in [0, s]}) \left( \varphi(F_t) - \varphi(F_s) - \int_s^t \iota(F_u) \left[ \varphi''(F_u) + \varphi'(F_u) \int_{\mathbb{R}} b(u, F_u, y) \iota(y) \mathbb{Q}_u(dy) \right] du \right) \right],$$

We observe that for any  $\mathbb{Q} \in \mathcal{P}(\Omega)$ , it holds that  $\Psi(\mathbb{Q}) = \Theta(\mathbb{Q} \otimes \mathbb{Q})$ , where we define for  $\Pi \in \mathcal{P}(\Omega^2)$

$$\Theta(\Pi) := \int_{\Omega^2} \Phi((x_r)_{r \in [0, s]}) \left( \varphi(x_t) - \varphi(x_s) - \int_s^t \iota(x_u) \left[ \varphi''(x_u) + \varphi'(x_u) b(u, x_u, y_u) \iota(y_u) \right] du \right) \Pi(dx, dy).$$



We separate the rest of the proof in several steps.

*Step 2.1.* Here we show that for some constant  $A$ , for all  $N \geq 2$ ,

$$\mathbb{E} \left[ [\Theta(\mu^N \times \mu^N)]^2 \right] \leq \frac{A}{N}. \quad (6)$$

Applying Itô's formula on  $\varphi(X_t^{N,i})$  and summing in  $N$  we have that

$$\begin{aligned} & \frac{1}{N} \sum_{i=1}^N \left[ \varphi(X_t^{N,i}) - \varphi(Z^{N,i}) - \int_0^t \iota(X_s^{N,i}) \left[ \varphi''(X_s^{N,i}) - \varphi'(X_s^{N,i}) \sum_{j=1}^N \frac{1}{N} b(s, X_s^{N,i}, X_s^{N,j}) \iota(X_s^{N,j}) ds \right] \right] \\ &= \frac{1}{N} \sum_{i=1}^N \int_0^t \iota(X_s^{N,i}) \varphi'(X_s^{N,i}) dW_s^i. \end{aligned}$$

We recognise in the left hand side of the above expression parts of  $\Theta(\mu^N \times \mu^N)$ . Hence

$$\mathbb{E} \left[ [\Theta(\mu^N \times \mu^N)]^2 \right] \leq \frac{T \|\varphi'\|_\infty^2 \|\Phi\|_\infty^2}{N}.$$

*Step 2.2.* for any  $\eta \in (0, 1]$ , we define a regularised version of the function  $\iota$  by  $f_\eta$ , e.g.

$$f_\eta := \min \left( 1, C_\eta \left( \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^4}{2} \right)^+ \right), \quad \text{with } C_\eta := \frac{30}{10\eta + 6\eta^5 - 15\eta^3}.$$

It is straightforward to check that  $f_\eta$  is twice continuously differentiable on  $\mathbb{R}$  and takes values in  $[0, 1]$ . Accordingly, we define  $\Theta_\eta$ . We have

$$\begin{aligned} \mathbb{E}[\|\Psi(\mu)\|] &\leq \mathbb{E}[\|\Theta(\mu \otimes \mu) - \Theta_\eta(\mu \otimes \mu)\|] + \limsup_N \|\mathbb{E}[\|\Theta_\eta(\mu \times \mu)\|] - \mathbb{E}[\|\Theta_\eta(\mu^N \otimes \mu^N)\|]\| \\ &\quad + \limsup_N \mathbb{E}[\|\Theta_\eta(\mu^N \otimes \mu^N) - \Theta(\mu^N \otimes \mu^N)\|] + \limsup_N \mathbb{E}[\|\Theta(\mu^N \otimes \mu^N)\|]. \end{aligned}$$

By *Step 2.1.* the last term converges to zero as  $N \rightarrow \infty$ . The second term converges to zero by weak convergence and the fact that  $\Theta_\eta$  is continuous and bounded (this is where the continuity in space of  $b$  plays a role). In the next step we will prove that the remaining terms are uniformly small w.r.t.  $\eta$ .

*Step 2.3.* Let us start with

$$I_\eta := \limsup_N \mathbb{E}[\|\Theta_\eta(\mu^N \otimes \mu^N) - \Theta(\mu^N \otimes \mu^N)\|].$$

Observe that

$$\begin{aligned} |\Theta_\eta(\mu^N \otimes \mu^N) - \Theta(\mu^N \otimes \mu^N)| &\leq \phi_\infty \varphi''_\infty \int_s^t \frac{1}{N} \sum |\iota(X_u^{N,i}) - f_\eta(X_u^{N,i})| du \\ &\quad + \|\phi\|_\infty \|\varphi'\|_\infty \|b\|_\infty \int_s^t \frac{1}{N^2} \sum_{i,j} |\iota(X_u^{N,i}) \iota(X_u^{N,j}) - f_\eta(X_u^{N,i}) f_\eta(X_u^{N,j})| du \\ &\leq A \int_s^t \frac{1}{N} \sum_{i=1}^N |\iota(X_u^{N,i}) - f_\eta(X_u^{N,i})| du, \end{aligned}$$

for some  $A > 0$  that may change from line to line. This leads by exchangeability to

$$\mathbb{E}[|\Theta_\eta(\mu^N \otimes \mu^N) - \Theta(\mu^N \otimes \mu^N)|] \leq A \int_s^t \mathbb{E}|l(X_u^{N,1}) - f(X_u^{N,1})| du \leq A \int_s^t \mathbb{Q}[0 < X_u^{N,1} < \eta] du.$$

Now, we use the partial transformation to transform  $X^{1,N}$  to a stopped Brownian motion. This leads to

$$\mathbb{E}[|\Theta_\eta(\mu^N \otimes \mu^N) - \Theta(\mu^N \otimes \mu^N)|] \leq A \int_s^t \mathbb{E}_{\mathbb{Q}_{(1,N)}}[Z_T^{(1)} \mathbb{1}_{\{X_u^{N,1} \in (0,\eta)\}}] du.$$

Now, using Proposition 6 for  $r = 1$ , we get

$$\mathbb{E}[|\Theta_\eta(\mu^N \otimes \mu^N) - \Theta(\mu^N \otimes \mu^N)|] \leq A \int_s^t \mathbb{P}[0 < W_{\tau \wedge u} < \eta]^{1/2} du$$

This leads us to the control of  $\mathbb{P}[W_{\tau \wedge u} \in (0, \eta)]$ . The latter converges, for every fixed  $u > 0$ , to zero as  $\eta \rightarrow 0$ . By dominated convergence we can conclude the desired result.

For the term that remains, we use Fatou's lemma and the above computation.

$$\begin{aligned} \mathbb{E}[|\Theta_\eta(\mu \otimes \mu) - \Theta(\mu \otimes \mu)|] &\leq C \int_s^t \int_{\Omega^2} \mathbb{1}_{\{f(u) \in (0,\eta)\}} du \mu \otimes \mu(df, dy) \\ &\leq C \lim_N \int_\Omega \int_s^t \mathbb{1}_{\{f(u) \in (0,\eta)\}} du \mu^N(df) \end{aligned}$$

Repeat the above Partial Girsanov transform argument to conclude the proof of 3.

Finally, note that the subset  $\{f \in \Omega : f(t) \geq 0, \forall t \in \mathbb{R}_+\}$  is closed. Applying the Portmanteau theorem, we may conclude  $\mathbb{Q}[\{f \in \Omega : f(t) \geq 0, \forall t \in \mathbb{R}_+\}] \geq \limsup_{N \rightarrow \infty} \mathbb{Q}^N[\{f \in \Omega : f(t) \geq 0, \forall t \in \mathbb{R}_+\}] = 1$  and thus  $\mathbb{Q}[F_t \geq 0, \forall t \geq 0] = 1$ .

Next, we show in Section 4 that the mean-field SDE (2) admits a unique strong solution which is clearly a weak solution. Combined with Theorem 4-2), this yields the propagation of chaos and thus proves Theorem 3-3).

## 4 Proof of Theorem 3

We consider in Section 4 the wellposedness of (2). It is worth noting that, thanks to Section 3.3, there exists at least a weak solution to (2) under the hypothesis of Theorem 4-2). Nevertheless, to obtain the uniqueness result for (2), we need more subtle properties for its solution, which requires more conditions on the coefficient  $b$ . This yields, as a by-product, the existence of a (strong) solution to (2). Therefore, we combine in the first part of Section 4 the derivation of the desired properties and the existence of solutions to (2) using a fixed-point argument. The second part is devoted to the uniqueness. In the last subsections we give proofs of some technical lemmas.

Throughout this section we fix an arbitrary time horizon  $T > 0$ .

## 4.1 Proof of the existence

We first deal with the existence. Let  $\mathcal{P} \subset \mathcal{P}(\mathbb{R}_+)$  be the set of probability measures on  $\mathbb{R}_+$  of finite first order moment. Define the set of probability flows

$$\mathcal{P}_T := \{\mu = (\mu_t)_{0 \leq t \leq T} : \mu_t \in \mathcal{P}\}$$

and recall  $\Omega_T$  is the space of continuous functions on  $[0, T]$ . We endow  $\mathcal{P}_T$  with the metric  $\mathcal{W}_T(\mu, \nu) := \sup_{0 \leq t \leq T} \mathcal{W}(\mu_t, \nu_t)$  and endow  $\Omega_T$  with the uniform norm  $\|\cdot\|_T$ . Let  $d_T$  be the distance on  $\mathcal{P}_T \times \Omega_T$  defined by  $d_T((\mu, f), (\nu, g)) := \mathcal{W}_T(\mu, \nu) + \|f - g\|_T$ . Rewriting (2) in differential form, one has

$$dX_t = \iota(X_t) \left[ \left( \int_{(0, \infty)} b(t, X_t, y) \mu_t(dy) \right) dt + \sqrt{2} dW_t \right] = \mathbf{1}_{\{\tau > t\}} \left[ B(t, X_t, \mu_t, \mu_t(\mathbb{R}_+^*)) dt + \sqrt{2} dW_t \right],$$

where  $\mu_t := \mathcal{L}(X_t)$ ,  $\tau := \inf\{t \geq 0 : X_t \leq 0\}$ ,  $\mathbb{R}_+^* := (0, \infty)$  and  $B : \mathbb{R}_+ \times \mathbb{R} \times \mathcal{P} \times [0, 1] \rightarrow \mathbb{R}$  is given as

$$B(t, x, \lambda, a) := \int_{\mathbb{R}_+} b(t, x, y) \lambda(dy) - b(t, x, 0)(1 - a).$$

Let us introduce the operator  $\Gamma$  on  $\mathcal{P}_T \times C_T$  defined by

$$\Gamma(\mu, f) := \left( (\mathcal{L}(Y_{t \wedge \tau^{\mu, f}}^{\mu, f}))_{0 \leq t \leq T}, \alpha^{\mu, f} \right),$$

where  $\alpha^{\mu, f}(t) := \mathbb{P}[\tau^{\mu, f} > t]$ ,  $\tau^{\mu, f} := \inf\{t \geq 0 : Y_t^{\mu, f} \leq 0\}$  and

$$Y_t^{\mu, f} = Z + \int_0^t B(s, Y_s^{\mu, f}, \mu_s, f(s)) ds + \sqrt{2} W_t, \quad \forall t \in [0, T]. \quad (7)$$

For any  $(\mu, f) \in \mathcal{P}_T \times C_T$ , the SDE (7) has a unique solution and thus  $\Gamma$  is well defined. Next we point out that any fixed point of  $\Gamma$  allows us to construct a solution to (2). Namely, let  $(\mu, f)$  be a fixed point of  $\Gamma$ , i.e.  $\mathcal{L}(Y_{t \wedge \tau^{\mu, f}}^{\mu, f}) = \mu_t$  and  $\alpha^{\mu, f}(t) = \mathbb{P}[\tau^{\mu, f} > t] = f(t)$ . Then a straightforward verification yields

$$\begin{aligned} dY_{t \wedge \tau^{\mu, f}}^{\mu, f} &= \mathbf{1}_{\{\tau^{\mu, f} > t\}} \left[ B(t, Y_t^{\mu, f}, \mu_t, f(t)) dt + \sqrt{2} dW_t \right] \\ &= \iota(Y_{t \wedge \tau^{\mu, f}}^{\mu, f}) \left[ \left( \int_{(0, \infty)} b(t, Y_{t \wedge \tau^{\mu, f}}^{\mu, f}, y) \mu_t(dy) \right) dt + \sqrt{2} dW_t \right], \quad \forall t \in [0, T], \end{aligned}$$

which implies that  $(Y_{t \wedge \tau^{\mu, f}}^{\mu, f})_{0 \leq t \leq T}$  solves (2) up to time  $T$ . Therefore, the proposition below ensures a fixed point of  $\Gamma$  and thus a solution to (2) on  $[0, T]$ . For each  $L > 0$ , set

$$\begin{aligned} \mathcal{P}_T^L &:= \left\{ \nu \in \mathcal{P}_T : \sup_{0 \leq s < t \leq T} \left( \int_{\mathbb{R}_+} x^2 \nu_t(dx) + \frac{\mathcal{W}(\nu_t, \nu_s)}{|t - s|^{1/6}} \right) \leq L \right\} \\ \Omega_T^L &:= \left\{ f \in \Omega_T : \sup_{0 \leq s < t \leq T} \frac{|f(t) - f(s)|}{|t - s|^{1/6}} \leq L \right\}. \end{aligned}$$

Then we have the following proposition.

**Proposition 7.** *Under suitable conditions,  $\Gamma$  has a fixed point in  $\mathcal{P}_T \times \Omega_T$ .*

**Proof.** The proof of Proposition 7 is a combination of the lemmas below. More precisely, Lemma 8 shows that  $\Gamma(\mathcal{P}_T^L \times \Omega_T^L) \subset \Gamma(\mathcal{P}_T \times \Omega_T) \subset \mathcal{P}_T^L \times \Omega_T^L$  for some  $L > 0$  large enough. As  $\mathcal{P}_T^L \times \Omega_T^L$  is compact and  $\Gamma$  is continuous with respect to  $d_T$  in view of Lemmas 8 and 10, Schauder's fixed-point theorem allows to show the existence of  $(\mu, f) \in \mathcal{P}_T^L \times \Omega_T^L$  such that  $\Gamma(\mu, f) = (\mu, f)$ .  $\square$

Let  $\Phi : \mathbb{R} \rightarrow [0, 1]$  the cumulative distribution function of a standard normal distribution, i.e.

$$\Phi(x) := \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy, \quad \forall x \in \mathbb{R}.$$

**Lemma 8.**  *$\mathcal{P}_T^L \times \Omega_T^L$  is  $d_T$ -compact for every  $L > 0$ .*

**Proof.** The compactness of  $\Omega_T^L$  is straightforward by Arzelà-Ascoli's theorem. It remains to treat  $\mathcal{P}_T^L$ . Pick an arbitrary sequence  $(\nu^n)_{n \geq 1} \subset \mathcal{P}_T^L$  and let us show that it has a convergent subsequence of limit belonging to  $\mathcal{P}_T^L$ . For each  $t \in [0, T]$ , note that there exists a weakly convergent subsequence of  $(\nu_t^n)_{n \geq 1}$  as it is uniformly integrable. Using the diagonal arguments, one may extract some subsequence, still denoted by  $(\nu^n)_{n \geq 1}$  for the sake of simplicity, such that  $(\nu_t^n)_{n \geq 1}$  converges weakly for each  $t \in [0, T] \cap \mathbb{Q}$ . Combined with the fact

$$\int_{\mathbb{R}_+} x^2 \nu_t^n(dx) \leq L, \quad \forall t \in [0, T], n \geq 1,$$

the above weak convergence is indeed the convergence under  $\mathcal{W}$ , i.e.  $\lim_{m, n \rightarrow \infty} \mathcal{W}(\nu_t^m, \nu_t^n) = 0$  for all  $t \in [0, T] \cap \mathbb{Q}$ . Next, for each  $t \in [0, T]$ , take a sequence  $(t_k)_{k \geq 1} \subset [0, T] \cap \mathbb{Q}$  converging to  $t$ . For every  $\varepsilon > 0$ , one has

$$\lim_{m, n \rightarrow \infty} \mathcal{W}(\nu_t^m, \nu_t^n) \leq \lim_{m, n \rightarrow \infty} (\mathcal{W}(\nu_{t_k}^m, \nu_{t_k}^n) + \mathcal{W}(\nu_{t_k}^m, \nu_t^m) + \mathcal{W}(\nu_{t_k}^n, \nu_t^n)) \leq 2L|t_k - t|^{1/6},$$

which shows that  $(\nu_t^n)_{n \geq 1}$  is a Cauchy sequence and thus converges under  $\mathcal{W}$ . Denote  $\nu \equiv (\nu_t := \lim_{n \rightarrow \infty} \nu_t^n)_{0 \leq t \leq T} \in \mathcal{P}_T$ . One deduces thus  $\mathcal{W}(\nu_t, \nu_s) \leq L|t - s|^{1/6}$  for all  $t, s \in [0, T]$  and further  $\lim_{n \rightarrow \infty} \mathcal{W}_T(\nu^n, \nu) = 0$ . Finally, by Fatou's lemma, one has for every  $t \in [0, T]$

$$\int_{\mathbb{R}_+} x^2 \nu_t(dx) \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}_+} x^2 \nu_t^n(dx) \leq L,$$

which fulfills the proof.  $\square$

**Lemma 9.** *There exists some constant  $L > 0$  such that  $\Gamma(\mathcal{P}_T \times \Omega_T) \subset \mathcal{P}_T^L \times \Omega_T^L$ .*

**Proof.** Take an arbitrary  $(\nu, f) \in \mathcal{P}_T \times \Omega_T$ . For simplicity we drop the superscript  $(\nu, f)$  without any danger of confusion, i.e.  $Y \equiv Y^{\nu, f}$ ,  $\tau \equiv \tau^{\nu, f}$ , etc. Denote further  $B_t \equiv B(t, Y_t, \nu_t, f(t))$ . For any  $0 \leq t < t + \Delta t \leq T$ , let us estimate respectively  $\mathcal{W}(\mathcal{L}(Y_{\tau \wedge t}), \mathcal{L}(Y_{\tau \wedge (t + \Delta t)}))$  and  $\alpha(t) - \alpha(t + \Delta t)$ . By definition, one has

$$\mathcal{W}(\mathcal{L}(Y_{\tau \wedge t}), \mathcal{L}(Y_{\tau \wedge (t + \Delta t)})) \leq \mathbb{E}[|Y_{\tau \wedge (t + \Delta t)} - Y_{\tau \wedge t}|] = \mathbb{E} \left[ \left| \int_{\tau \wedge t}^{\tau \wedge (t + \Delta t)} B_s ds + \sqrt{2} dW_s \right| \right] \leq C\sqrt{\Delta t}.$$

Hereafter,  $C > 0$  is used to denote generic constants that may change from line to line, but depends only on the designated variables. Applying Girsanov's theorem with

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_t := \exp \left( - \int_0^t \lambda_s dW_s - \frac{1}{2} \int_0^t \lambda_s^2 ds \right), \quad \text{where } \lambda_s := \frac{B_s}{\sqrt{2}},$$

$(W_t^{\mathbb{Q}} := W_t + \int_0^t \lambda_s ds)_{0 \leq t \leq T}$  is a Brownian motion under  $\mathbb{Q}$  and one has

$$\begin{aligned} \alpha(t) - \alpha(t + \Delta t) &= \mathbb{P}[\tau > t] - \mathbb{P}[\tau > t + \Delta t] \\ &= \mathbb{E}^{\mathbb{Q}} \left[ \left( \frac{d\mathbb{P}}{d\mathbb{Q}} \right)_{t+\Delta t} \left( \iota \left( \inf_{0 \leq r \leq t} (Z + \int_0^r \sqrt{2} dW_s^{\mathbb{Q}}) \right) - \iota \left( \inf_{0 \leq r \leq t+\Delta t} (Z + \int_0^r \sqrt{2} dW_s^{\mathbb{Q}}) \right) \right) \right] \\ &\leq \exp \left( \frac{3}{2} \int_0^T \lambda_s^2 ds \right) \left( \mathbb{Q} \left[ \inf_{0 \leq r \leq t} (Z + \int_0^r \sqrt{2} dW_s^{\mathbb{Q}}) > 0 \right] - \mathbb{Q} \left[ \inf_{0 \leq r \leq t+\Delta t} (Z + \int_0^r \sqrt{2} dW_s^{\mathbb{Q}}) > 0 \right] \right) \\ &\leq C \left( \int_{\mathbb{R}_+} \left( \Phi(x/\sqrt{2t}) - \Phi(x/\sqrt{2(t+\Delta t)}) \right) \rho(dx) \right)^{1/2} \leq C \Delta t^{1/6}, \end{aligned}$$

where the first inequality follows from Hölder inequality. Finally, we see the fact that

$$\mathbb{E}[|Y_{\tau \wedge t}|^2] \leq 3 \left( \mathbb{E}[|Z|^2] + \mathbb{E} \left[ \left| \int_0^T |B_s| ds \right|^2 \right] + \mathbb{E} \left[ \int_0^T 2 ds \right] \right) \leq C,$$

which allows to conclude the proof.  $\square$

**Lemma 10.**  $\Gamma : \mathcal{P}_T \times \Omega_T \rightarrow \mathcal{P}_T \times \Omega_T$  is  $d_T$ -continuous.

**Proof.** Let  $(\nu^n, f^n) \rightarrow (\nu, f)$ . Again, we denote for simplicity

$$\begin{aligned} Y^n &\equiv Y^{\nu^n, f^n} & Y &\equiv Y^{\nu, f} \\ \tau_n &\equiv \tau^{\nu^n, f^n} & \tau &\equiv \tau^{\nu, f} \\ B_t^n &\equiv B(t, Y_t^n, \nu_t^n, f^n(t)) & B_t &\equiv B(t, Y_t, \nu_t, f(t)), \text{ etc.} \end{aligned}$$

We estimate first  $\mathbb{E}[\sup_{0 \leq t \leq T} |Y_t^n - Y_t|^2]$  following the classic arguments. For each  $t \in [0, T]$ , one has

$$Y_t^n - Y_t = \int_0^t (B(u, Y_u^n, \nu_u^n, f^n(u)) - B(u, Y_u, \nu_u, f(u))) du.$$

By the Lipschitz continuity of  $B$  and the inequality  $\max(\mathcal{W}(\nu_u^n, \nu_u), |f^n(u) - f(u)|) \leq d_T((\nu^n, f^n), (\nu, f))$ , there exists  $C > 0$  large enough such that

$$\mathbb{E} \left[ \sup_{0 \leq s \leq t} |Y_s^n - Y_s|^2 \right] \leq C \left( d_T((\nu^n, f^n), (\nu, f)) + \int_0^t \mathbb{E} \left[ \sup_{0 \leq s \leq u} |Y_s^n - Y_s|^2 \right] du \right), \quad \forall t \in [0, T].$$

This, combined with Gronwall's inequality, yields  $\lim_{n \rightarrow \infty} \mathbb{E}[\sup_{0 \leq t \leq T} |Y_t^n - Y_t|^2] = 0$ . In particular,  $\sup_{0 \leq t \leq T} |Y_t^n - Y_t|$  converges to zero in probability. Let  $\varepsilon > 0$  be arbitrary. The continuity of  $Y$  implies

$$\{\tau > t\} = \bigcup_{k \in \mathbb{N}} A_k := \bigcup_{k \in \mathbb{N}} \{Y_s \geq 1/k, \forall 0 \leq s \leq t\}.$$

Hence, we have that for some  $k_0 \in \mathbb{N}$  that  $\mathbb{P}[A_{k_0}] > \mathbb{P}[\tau > t] - \varepsilon/2$ . Since  $Y^n \rightarrow Y$  uniformly in probability, there exists  $N$  such that

$$\mathbb{P}\left[\sup_{s \in [0, t]} |Y_s^n - Y_s| \geq 1/k_0\right] \leq \frac{\varepsilon}{2}$$

holds for all  $n > N$ . We obtain thus

$$\mathbb{P}[\tau_n > t] \geq \mathbb{P}[A_{k_0} \cup \{\sup_{s \in [0, t]} |Y_s^n - Y_s| > 1/k_0\}^c] \geq \mathbb{P}[\tau > t] - \varepsilon.$$

The reverse inequality can be argued similarly, where we note that  $\mathbb{P}[\tau \leq t] = \mathbb{P}[\inf_{0 \leq s \leq t} Y_s \leq 0] = \mathbb{P}[\inf_{0 \leq s \leq t} Y_s < 0]$  as  $Y$  is a drifted Brownian motion, and we can rewrite

$$\{\inf_{0 \leq s \leq t} Y_s < 0\} = \bigcup_{k \in \mathbb{N}} \{\inf_{s \in [0, t]} Y_s \leq -1/k\}.$$

Thus with a similar calculation, we obtain  $\lim_{n \rightarrow \infty} \mathbb{P}[\tau_n \leq t] \geq \mathbb{P}[\tau \leq t] - \varepsilon$  and thus  $\lim_{n \rightarrow \infty} \alpha^n(t) = \lim_{n \rightarrow \infty} \mathbb{P}[\tau_n > t] = \mathbb{P}[\tau > t] = \alpha(t)$ . Next, we compute  $\mathcal{W}(\mathcal{L}(Y_{\tau_n \wedge t}^n), \mathcal{L}(Y_{\tau \wedge t}))$ . By definition, one has  $\mathcal{W}(\mathcal{L}(Y_{\tau_n \wedge t}^n), \mathcal{L}(Y_{\tau \wedge t})) \leq \mathbb{E}[|Y_{\tau_n \wedge t}^n - Y_{\tau \wedge t}|]$ . Further,

$$\begin{aligned} \mathbb{E}[|Y_{\tau_n \wedge t}^n - Y_{\tau \wedge t}|] &= \mathbb{E}[|Y_t^n - Y_t| \mathbf{1}_{\{\tau_n > t, \tau > t\}}] + \mathbb{E}[|Y_t^n| \mathbf{1}_{\{\tau_n > t, \tau \leq t\}}] + \mathbb{E}[|Y_t| \mathbf{1}_{\{\tau_n \leq t, \tau > t\}}] \\ &\leq \mathbb{E}[|Y_t^n - Y_t|] + \mathbb{E}[|Y_t^n|^2]^{1/2} \mathbb{P}[\tau_n > t, \tau \leq t]^{1/2} + \mathbb{E}[|Y_t|^2]^{1/2} \mathbb{P}[\tau_n \leq t, \tau > t]^{1/2}. \end{aligned}$$

Repeating the above reasoning, one may show  $\lim_{n \rightarrow \infty} \mathbb{P}[\tau_n > t, \tau \leq t] = 0 = \lim_{n \rightarrow \infty} \mathbb{P}[\tau_n \leq t, \tau > t]$ , which yields  $\lim_{n \rightarrow \infty} \mathcal{W}(\mathcal{L}(Y_{\tau_n \wedge t}^n), \mathcal{L}(Y_{\tau \wedge t})) = 0$ . Having shown

$$\lim_{n \rightarrow \infty} \mathcal{W}(\mathcal{L}(Y_{\tau_n \wedge t}^n), \mathcal{L}(Y_{\tau \wedge t})) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \alpha^n(t) = \alpha(t), \quad \forall t \in [0, T],$$

the uniform continuity in Lemma 9 finally shows  $\lim_{n \rightarrow \infty} d_T(\Gamma(\nu^n, f^n), \Gamma(\nu, f)) = 0$ .  $\square$

## 4.2 Proof of the uniqueness

This section is reserved for showing the uniqueness of solution to (2) on  $[0, T]$ . We fix an arbitrary solution  $X$  to (2). We first show the one-to-one relation between  $X$  and the parametric nonlinear Fokker-Planck equation below.

**Proposition 11.** *The uniqueness of  $X$  is equivalent to the uniqueness of  $u : \mathbb{R}_+^2 \rightarrow \mathbb{R}$  satisfying*

$$\begin{aligned} \partial_t u(t, x) &= \partial_{xx}^2(u(t, x)) - \partial_x(B^{\mu, \alpha}(t, x)u(t, x)), \quad \forall t, x > 0 \\ u(0, x) &= \rho(x), \quad u(t, 0) = 0, \quad \forall t, x > 0 \\ \beta(t) &= \int_0^\infty u(t, y) dy, \quad \forall t > 0 \\ \nu_t(dx) &= (1 - \beta(t))\delta_0(dx) + u(t, x)dx, \quad \forall t > 0 \\ B^{\nu, \beta}(t, x) &= B(t, x, \nu_t, \beta(t)), \quad \forall t, x > 0. \end{aligned} \tag{8}$$

Proposition 11 is the basis to adopt the fixed-point argument, and we need the following auxiliary result to prove it. Denote  $(\mu_t)_{t \geq 0} := (\mathcal{L}(X_t))_{t \geq 0}$ ,  $\tau := \inf\{t \geq 0 : X_t \leq 0\}$  and  $\alpha(t) := \mathbb{P}[\tau > t]$ . For  $(\nu, f) \in \mathcal{P}_T \times \Omega_T$ , define the function  $B^{\nu, f} : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$  by  $B^{\nu, f}(t, x) := B(t, x, \nu_t, f(t))$ . Then we have the following proposition.

**Proposition 12.** *Under the hypothesis of Theorem 3 and writing  $B \equiv B^{\mu, \alpha}$ , we have the following results.*

1. *For each  $t > 0$ , there exists a sub-probability density  $p(t, \cdot)$  supported on  $(0, \infty)$  such that*

$$\mu_t(dx) = (1 - \alpha(t))\delta_0(dx) + p(t, x)dx \quad \text{and} \quad \alpha(t) = \int_0^\infty p(t, x)dx.$$

2. *The family  $(\mu_t)_{t \geq 0}$  is the unique weak solution to the following Fokker-Planck equation, i.e.*

$$\frac{d}{dt} \int_{\mathbb{R}} f(x)\mu_t(dx) = \int_{(0, \infty)} [B(t, x)f'(x) + f''(x)] \mu_t(dx), \quad \forall t \in (0, T], \quad (9)$$

*holds for all smooth functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  of compact support.*

3. *The function  $p$  is a solution to (8), with  $\alpha$ ,  $\mu$  and  $B^{\mu, \alpha}$  defined above.*

**Proof.** By definition, one may rewrite

$$\mu_t(dx) = (1 - \alpha(t))\delta_0(dx) + \iota(x)\mathbb{P}[X_t \in dx].$$

(i) Define the SDE for  $Y$  by

$$Y_t = Z + \int_0^t B(s, Y_s)ds + \sqrt{2}W_t, \quad \forall t \geq 0.$$

Note that  $X_t = Y_t$  on the event  $\{\tau > t\}$ . So, for any Borel set  $A \subseteq (0, \infty)$ , we have

$$\mathbb{P}[X_t \in A] = \mathbb{P}[Y_t \in A, \tau > t] \leq \mathbb{P}[Y_t \in A],$$

which implies that the distribution of  $X_t$  restricted on  $(0, \infty)$  is absolutely continuous with respect to the distribution of  $Y_t$ . The latter admits a probability density as  $B$  is bounded.

(ii) One has by definition

$$B(t, x) = \int_{\mathbb{R}_+} b(t, x, y)\mu_t(dy) - b(t, x, 0)(1 - \alpha(t)) = \mathbb{E}[b(t, x, X_t)] - b(t, x, 0)(1 - \alpha(t)),$$

which implies, by Lemma 9, that  $t \mapsto B^{\mu, \alpha}(t, \cdot)$  is Hölder continuous and  $\partial_x B^{\mu, \alpha}, \partial_{xx}^2 B^{\mu, \alpha}$  are bounded and Lipschitz in  $x$ . For the SDE

$$Z_t = Z + \int_0^t \iota(Z_s)B(s, Z_s)ds + \int_0^t \iota(Z_s)\sqrt{2}dW_s, \quad \forall t \in [0, T] \quad (10)$$

$X$  is its solution by definition, where we use the fact that (10) has a unique solution if and only if the same holds for the corresponding SDE without absorption

$$dZ_t = B(t, Z_t)dt + \sqrt{2}dW_t, \quad \forall t \in [0, T].$$

This is ensured as  $B^{\mu, \alpha}$  is Lipschitz in  $x$ . Therefore,  $X$  is the unique solution of (10) and the family  $(\mu_t)_{t \geq 0}$ , by Lemma 2.3 of [8], is the unique weak solution to the Fokker-Planck equation

$$\frac{d}{dt} \int_{\mathbb{R}} f(x) \nu_t(dx) = \int_{(0, \infty)} [B(t, x)f'(x) + f''(x)] \nu_t(dx), \quad \forall t \in (0, T], f \in C_c^\infty(\mathbb{R}). \quad (11)$$

(iii) To conclude, we apply Lemma 1.10 and Theorem 2.2 of Chapter VI in [12]. Namely, the following (linear) Fokker-Planck equation on the half space

$$\begin{aligned} \partial_t u(t, x) &= \partial_{xx}^2(u(t, x)) - \partial_x(B(t, x)u(t, x)), \quad \forall t, x > 0 \\ u(0, x) &= \rho(x), \quad u(t, 0) = 0, \quad \forall t, x > 0 \end{aligned} \quad (12)$$

has a unique classical solution  $u$  satisfying for some  $C > 0$

$$|\partial_t u(t, x)| + |\partial_x u(t, x)| + |\partial_{xx}^2 u(t, x)| \leq \frac{C}{t} \int_0^\infty \exp\left(-\frac{(x-y)^2}{Ct}\right) \rho(y) dy, \quad \forall t, x > 0. \quad (13)$$

By integration by parts, one deduces that

$$\nu_t(dx) = \left(1 - \int_0^\infty u(t, x) dx\right) \delta_0(dx) + u(t, x) dx, \quad \forall t \in [0, T] \quad (14)$$

is also a weak solution to (11), and further by uniqueness  $\nu = \mu$  and  $u = p$ . Hence the equation (8) holds for  $p$  by noting  $B \equiv B^{\mu, \alpha}$ .  $\square$

We are now ready to prove Proposition 11.

**Proof.** [of Proposition 11] It is clear that for the given solution  $X$ ,  $p$  defined in the proof of Proposition 12 is a solution to (8). On the other hand, for any solution  $u$  to (8), define  $Z$  to be the solution of

$$dZ_t = B(t, Z_t, \nu_t, \beta(t))dt + \sqrt{2}dW_t, \quad \forall t \in [0, T].$$

Then a straightforward verification yields  $(Z_t \nu(Z_t))_{t \geq 0}$  is a solution to (2).  $\square$

It remains to prove the uniqueness of solutions to (8). Introduce the SDE

$$dY_t = B^{\mu, \alpha}(t, Y_t)dt + \sqrt{2}dW_t, \quad \forall t \geq 0,$$

and denote by  $(Y_s^{t, x})_{s \geq t}$  the solution satisfying  $Y_t^{t, x} = x$ . Let  $g(t, x, s, \cdot)$  be the density function of  $Y_s^{t, x}$  for  $s > t$ , i.e.  $\mathbb{P}[Y_s^{t, x} \in dy] = g(t, x, s, y)dy$ . It is known that, for any fixed  $s > 0$  and  $y \in \mathbb{R}$ ,  $g$  satisfies the backward Kolmogorov equation

$$\begin{aligned} \partial_t g(t, x, s, y) &= -\partial_{xx}^2 g(t, x, s, y) - B^{\mu, \alpha}(t, x) \partial_x g(t, x, s, y), \quad \forall (t, x) \in [0, s) \times \mathbb{R}, \\ g(s, x, s, y) &= \delta_y(x), \quad \forall x \in \mathbb{R}. \end{aligned}$$

Then  $\alpha$  admits the following representation.



**Lemma 13.** *Under the conditions of Theorem 3, one has for all  $s > 0$*

$$\frac{\alpha(s)}{2} = \int_0^\infty \rho(x) dx \int_0^\infty g(0, x, s, y) dy - \int_0^\infty g(0, 0, s, y) dy - \int_0^s \alpha(t) dt \int_0^\infty \partial_t g(t, 0, s, y) dy. \quad (15)$$

**Proof.** Integrating the Fokker-Planck equation of (8) over  $x \in (0, \infty)$ , one obtains

$$\int_0^\infty \partial_t p(t, x) dx = \int_0^\infty \partial_{xx}^2 p(t, x) dx - \int_0^\infty \partial_x (B^{\mu, \alpha}(t, x) p(t, x)) dx,$$

which yields by Fubini's theorem and integration by parts

$$\alpha'(t) = \frac{d}{dt} \int_0^\infty p(t, x) dx = \int_0^\infty \partial_t p(t, x) dx = \int_0^\infty [\partial_{xx}^2 p(t, x) - \partial_x (B^{\mu, \alpha}(t, x) p(t, x))] dx = -\partial_x p(t, 0), \quad \forall t > 0.$$

Further, compute for all  $(t, x) \in (0, s) \times (0, \infty)$

$$\partial_t (pg) + \partial_x (B^{\mu, \alpha} pg) - \partial_x (\partial_x pg - p \partial_x g) = (\partial_t p - \partial_{xx}^2 p + \partial_x (B^{\mu, \alpha} p)) g + p (\partial_t g + \partial_{xx}^2 g + B^{\mu, \alpha} \partial_x g) = 0,$$

which yields by integrating over  $(0, s) \times (0, \infty)$

$$\int_0^s \int_0^\infty \partial_t (pg) dt dx + \int_0^s \int_0^\infty \partial_x (B^{\mu, \alpha}(t, x) pg) dt dx - \int_0^s \int_0^\infty \partial_x (\partial_x pg - p \partial_x g) dt dx = 0.$$

Again, applying Fubini's theorem combined with the initial and boundary conditions, one has

$$\begin{aligned} 0 &= \int_0^\infty (p(s, x) \delta_y(x) - \rho(x) g(0, x, s, y)) dx + \int_0^s \partial_x p(t, 0) g(t, 0, s, y) dt \\ &= p(s, y) - \int_0^\infty \rho(x) g(0, x, s, y) dx - \int_0^s \alpha'(t) g(t, 0, s, y) dt, \quad \forall (s, y) \in (0, \infty) \times (0, \infty). \end{aligned}$$

Integrating  $y$  over  $(0, \infty)$  for both sides and using integration by parts, one has

$$\begin{aligned} \alpha(s) &= \int_0^\infty \rho(x) dx \int_0^\infty g(0, x, s, y) dy + \int_0^s \alpha'(t) dt \int_0^\infty g(t, 0, s, y) dy \\ &= \int_0^\infty \rho(x) dx \int_0^\infty g(0, x, s, y) dy + \frac{\alpha(s)}{2} - \int_0^\infty g(0, 0, s, y) dy - \int_0^s \alpha(t) dt \int_0^\infty \partial_t g(t, 0, s, y) dy \end{aligned}$$

which fulfils the proof.  $\square$

We are now ready to prove the uniqueness of solution to (2). We argue by contradiction and assume that there are two distinct solutions  $X, X'$  to (2) on  $[0, T]$ . Denote  $\mu_t := \mathcal{L}(X_t), \nu_t := \mathcal{L}(X'_t)$  and  $\alpha(t) := \mathbb{P}[X_t > 0], \beta(t) := \mathbb{P}[X'_t > 0]$  for all  $t \in [0, T]$ . Define further by  $p(t, \cdot), q(t, \cdot)$  the corresponding sub-probability densities of  $\mu_t, \nu_t$ . Write

$$A(t, x) := \int_0^\infty b(t, x, y) p(t, y) dy \quad \text{and} \quad B(t, x) := \int_0^\infty b(t, x, y) q(t, y) dy$$

and the SDEs

$$Y_s^{t,x} = x + \int_t^s A(u, Y_u^{t,x}) du + \sqrt{2}(W_s - W_t) \quad \text{and} \quad Z_s^{t,x} = x + \int_t^s B(u, Z_u^{t,x}) du + \sqrt{2}(W_s - W_t), \quad \forall s \geq t.$$

Let  $f(t, x, s, \cdot), g(t, x, s, \cdot)$  be the densities of  $Y_s^{t,x}, Z_s^{t,x}$ . Hence, one has by Lemma 13

$$\begin{aligned}\alpha(s) &= 2 \int_0^\infty \rho(x) dx \int_0^\infty f(0, x, s, y) dy - 2 \int_0^\infty f(0, 0, s, y) dy - 2 \int_0^s \alpha(t) dt \int_0^\infty \partial_t f(t, 0, s, y) dy \\ &=: \Theta[\alpha, A](s) \\ \beta(s) &= 2 \int_0^\infty \rho(x) dx \int_0^\infty g(0, x, s, y) dy - 2 \int_0^\infty g(0, 0, s, y) dy - 2 \int_0^s \beta(t) dt \int_0^\infty \partial_t g(t, 0, s, y) dy \\ &=: \Theta[\beta, B](s)\end{aligned}$$

and similarly

$$\begin{aligned}A(s, z) &= \int_0^\infty \rho(x) dx \int_0^\infty b(s, z, y) f(0, x, s, y) dy + \frac{b(s, z, 0)\alpha(s)}{2} - \int_0^\infty b(s, z, y) f(0, 0, s, y) dy \\ &\quad - \int_0^s \alpha(t) dt \int_0^\infty b(s, z, y) \partial_t f(t, 0, s, y) dy =: \Lambda[\alpha, A](s, z) \\ B(s, z) &= \int_0^\infty \rho(x) dx \int_0^\infty b(s, z, y) g(0, x, s, y) dy + \frac{b(s, z, 0)\beta(s)}{2} - \int_0^\infty b(s, z, y) g(0, 0, s, y) dy \\ &\quad - \int_0^s \beta(t) dt \int_0^\infty b(s, z, y) \partial_t g(t, 0, s, y) dy =: \Lambda[\beta, B](s, z).\end{aligned}$$

**Proposition 14.** (8) has a unique solution and thus the uniqueness of solutions to (2) holds.

**Proof.** The proof relies essentially on the estimation of  $|\Theta[\alpha, A](s) - \Theta[\beta, B](s)|$  and  $|\Lambda[\alpha, A](s) - \Lambda[\beta, B](s)| := \sup_{z \in \mathbb{R}_+} |\Lambda[\alpha, A](s, z) - \Lambda[\beta, B](s, z)|$ . Write further  $\Lambda := \Lambda_0 + \Lambda_1$ , where

$$\begin{aligned}\Lambda_0[\alpha, A](s) &:= \int_0^\infty \rho(x) dx \int_0^\infty b(s, z, y) f(0, x, s, y) dy - \int_0^\infty b(s, z, y) f(0, 0, s, y) dy \\ &\quad - \int_0^s \alpha(t) dt \int_0^\infty b(s, z, y) \partial_t f(t, 0, s, y) dy \\ \Lambda_1[\alpha, A](s, z) &:= \frac{b(s, z, 0)\alpha(s)}{2}\end{aligned}$$

and  $\Lambda_0[\beta, B], \Lambda_1[\beta, B]$  are defined similarly.

*Step 1.* Recall  $\|f\|_t := \sup_{0 \leq u \leq t} |f(u)|$  for  $f \in \Omega_T$ . We claim that there exists  $C > 0$  such that

$$\|\Theta[\alpha, A] - \Theta[\beta, B]\|_s + \|\Lambda_0[\alpha, A] - \Lambda_0[\beta, B]\|_s \leq C\sqrt{s}(\|\alpha - \beta\|_s + \|A - B\|_s), \quad \forall s \in [0, T]. \quad (16)$$

Let us show that the desired uniqueness follows from the inequality (16). Namely, combining with  $\alpha = \Theta[\alpha, A], \beta = \Theta[\beta, B], A = \Lambda[\alpha, A], B = \Lambda[\beta, B]$ , (16) yields

$$\begin{aligned}\|\alpha - \beta\|_s + \|A - B\|_s &= \|\Theta[\alpha, A] - \Theta[\beta, B]\|_s + \|\Lambda[\alpha, A] - \Lambda[\beta, B]\|_s \\ &\leq \|\Theta[\alpha, A] - \Theta[\beta, B]\|_s + \|\Lambda_0[\alpha, A] - \Lambda_0[\beta, B]\|_s + \|\Lambda_1[\alpha, A] - \Lambda_1[\beta, B]\|_s \\ &\leq C\sqrt{s}(\|\alpha - \beta\|_s + \|A - B\|_s) + \|\Lambda_1[\alpha, A] - \Lambda_1[\beta, B]\|_s \\ &\leq C\sqrt{s}(\|\alpha - \beta\|_s + \|A - B\|_s) + \|b\|\|\alpha - \beta\|_s \\ &= C\sqrt{s}(\|\alpha - \beta\|_s + \|A - B\|_s) + \|b\|\|\Theta[\alpha, A] - \Theta[\beta, B]\|_s \\ &\leq C(1 + \|b\|)\sqrt{s}(\|\alpha - \beta\|_s + \|A - B\|_s) \\ &\leq \bar{C}\sqrt{s}(\|\alpha - \beta\|_s + \|A - B\|_s), \quad \forall s \in [0, T].\end{aligned}$$

This implies that  $\|\alpha - \beta\|_s + \|A - B\|_s = 0$  whenever  $s \leq \min(1/\overline{C}^2, T) =: \Delta T > 0$ . Repeating the above reasoning on the interval  $[\Delta T, T]$  with an alternative initial condition, one may deduce, after dividing a finite number of the interval  $[0, T]$ ,  $\|\alpha - \beta\|_T = \|A - B\|_T = 0$ . Hence the uniqueness is derived.

$$\begin{aligned}
\|\Lambda[c] - \Lambda[c']\|_t &= \|\Lambda_0[c] - \Lambda_0[c']\|_t + \|\Lambda_1[c] - \Lambda_1[c']\|_t \\
&\leq \|\Lambda_0[c] - \Lambda_0[c']\|_t + \|\Lambda_1^+[c] - \Lambda_1^+[c']\|_t + \|\Lambda_1^-[c] - \Lambda_1^-[c']\|_t \\
&= \|\Lambda^+[c] - \Lambda^+[c']\|_t + \|\Lambda_1^-[c] - \Lambda_1^-[c']\|_t \\
&\leq C\sqrt{(t-t^*)^+}\|c - c'\|_t + \frac{\bar{b}}{2}\|c_0 - c'_0\|_t \\
&= C\sqrt{(t-t^*)^+}\|c - c'\|_t + \frac{\bar{b}}{2}\|\Lambda_0[c] - \Lambda_0[c']\|_t \\
&\leq C\left(1 + \frac{\bar{b}}{2}\right)\sqrt{(t-t^*)^+}\|c - c'\|_t,
\end{aligned}$$

which yields a contraction when  $t$  is close enough to  $t^*$ .

*Step 2.* It remains to prove (16). Set  $\Delta_s := \|\alpha - \beta\|_s + \|A - B\|_s$ . In what follows,  $R, r > 0$  always denote the constants that may vary from line to line. By assumption, it holds

$$\|A\|_T + \|B\|_T + \|\partial_x A\|_T + \|\partial_x B\|_T \leq R. \quad (17)$$

Let us recall the parametrix expressions of  $f$  and  $g$  by Aronson as follows:

$$f(t, x, s, y) = q(t, x, s, y) + \sum_{k=1}^{\infty} q \otimes F^{(k)}(t, x, s, y) \quad \text{and} \quad g(t, x, s, y) = q(t, x, s, y) + \sum_{k=1}^{\infty} q \otimes G^{(k)}(t, x, s, y),$$

where  $q(t, x, s, y) := \phi(2(s-t), y-x)$ ,  $\phi(r, z) := e^{-z^2/2r}/\sqrt{2\pi r}$ ,  $F(t, x, s, y) := A(t, x)\partial_x q(t, x, s, y)$ ,  $G(t, x, s, y) := B(t, x)\partial_x q(t, x, s, y)$  and  $\otimes$  denotes the time-space convolution, i.e.

$$\begin{aligned}
q \otimes F(t, x, s, y) &:= \int_t^s du \int_{-\infty}^{\infty} q(t, x, u, z) F(u, z, s, y) dz \\
q \otimes F^{(k)}(t, x, s, y) &:= (q \otimes F^{(k-1)}) \otimes F(t, x, s, y)(t, x, s, y), \quad \text{for all } k \geq 2.
\end{aligned}$$

Hence, one obtains by a straightforward computation

$$\begin{aligned}
\frac{1}{2}|\Theta[\alpha, A](s) - \Theta[\beta, B](s)| &\leq \left| \int_0^{\infty} \rho(x) I_1(x) dx \right| + |I_1(0)| + \Delta_s \int_0^s I_2(t) dt + \int_0^s I_3(t) dt, \\
|\Lambda_0[\alpha, A](s, z) - \Lambda_0[\beta, B](s, z)| &\leq \left| \int_0^{\infty} \rho(x) J_1(x) dx \right| + |J_1(0)| + \Delta_s \int_0^s J_2(t) dt + \int_0^s J_3(t) dt,
\end{aligned}$$

where

$$\begin{aligned}
I_1(x) &:= \int_0^{\infty} (f(0, x, s, y) - g(0, x, s, y)) dy \quad \text{and} \quad J_1(x) := \int_0^{\infty} b(s, z, y) (f(0, x, s, y) - g(0, x, s, y)) dy \\
I_2(t) &:= \left| \int_0^{\infty} \partial_t f(t, 0, s, y) dy \right| \quad \text{and} \quad J_2(t) := \left| \int_0^{\infty} b(s, z, y) \partial_t f(t, 0, s, y) dy \right| \\
I_3(t) &:= \left| \int_0^{\infty} (\partial_t f(t, 0, s, y) - \partial_t g(t, 0, s, y)) dy \right| \quad \text{and} \quad J_3(t) := \left| \int_0^{\infty} b(s, z, y) (\partial_t f(t, 0, s, y) - \partial_t g(t, 0, s, y)) dy \right|.
\end{aligned}$$

In view of Lemmas 16 and 17 below, there exists  $R > 0$  such that

$$|\Theta[\alpha, A](s) - \Theta[\beta, B](s)| + |\Lambda_0[\alpha, A](s, z) - \Lambda_0[\beta, B](s, z)| \leq R s^{1/2} \Delta_s$$

which completes the proof.  $\square$

To establish Lemmas 16 and 17, we need the following preliminary result whose proof adopts almost the same reasoning of Lemma 3 in [25].

**Lemma 15.** *With the above notation, there exists  $R, r > 0$  such that*

$$|F^{(k)}(t, x, s, y)| + |G^{(k)}(t, x, s, y)| \leq k \frac{R^k \Gamma(1/2)^{k-1}}{\Gamma((k+1)/2)} \phi(r(s-t), y-x) (t-s)^{k/2-1} \quad (18)$$

$$|F^{(k)}(t, x, s, y) - G^{(k)}(t, x, s, y)| \leq k \Delta_s \frac{C^k \Gamma(1/2)^{k-1}}{\Gamma((k+1)/2)} \phi(r(s-t), y-x) (t-s)^{k/2-1} \quad (19)$$

hold for all  $0 \leq t < s \leq T$ ,  $x, y \in \mathbb{R}$  and  $k \geq 1$ , where  $\Gamma : \mathbb{R}_+^* \rightarrow \mathbb{R}$  is the Gamma function defined by

$$\Gamma(z) := \int_0^\infty a^{z-1} e^{-a} da.$$

### 4.3 Estimation of $I_1/J_1$

**Lemma 16.** *There exists  $R > 0$  such that*

$$\left| \int_0^\infty p_0(x) I_1(x) dx \right| + \left| \int_0^\infty p_0(x) J_1(x) dx \right| + |I_1(0)| + |J_1(0)| \leq R(s-t)^{1/2} \Delta_s$$

**Proof.** First, we make use of the following estimation for  $k \geq 1$

$$\left| \tilde{p} \otimes G^{(k)}(0, x, s, y) - \tilde{q} \otimes H^{(k)}(0, x, s, y) \right| \leq (k+1) s^{k/2} \frac{R^{k+1} \Gamma(1/2)}{\Gamma(1+k/2)} \phi(rs, y-x) \Delta_s,$$

which yields

$$\left| \int_0^\infty p_0(x) I_1(x) dx \right| \leq \sum_{k=1}^\infty \int_0^\infty p_0(x) dx \left| \int_0^\infty \tilde{p} \otimes G^{(k)}(0, x, s, y) dy - \int_0^\infty \tilde{q} \otimes H^{(k)}(0, x, s, y) dy \right| \leq R\sqrt{s} \Delta_s.$$

As for  $J_1$ , carrying out the similar reasoning for the terms with  $k \geq 1$ , one has

$$\left| \int_0^\infty p_0(x) J_1(x) dx \right| + |J_1(0)| \leq R\sqrt{s} \Delta_s.$$

$\square$

### 4.4 Estimation of $I_2/J_2$ and $I_3/J_3$

**Lemma 17.** *There exists  $R > 0$  such that*

$$|I_2(t)| + |J_2(t)| \leq R(s-t)^{1/2} \quad \text{and} \quad |I_3(t)| + |J_3(t)| \leq R(s-t)^{1/2} \Delta_s.$$

**Proof.** As the estimation of  $I_2/J_2$  can be seen as a particular case of that of  $I_3/J_3$ , we only deal with  $I_3/J_3$ . We start by estimating  $I_3$ .

$$I_3(t) \leq \sum_{k=1}^{\infty} \left| \int_0^{\infty} (\partial_t(\tilde{p} \otimes G^{(k)})(t, 0, s, y) - \partial_t(\tilde{q} \otimes H^{(k)})(t, 0, s, y)) dy \right|.$$

Next,

$$\begin{aligned} \int_0^{\infty} \partial_t(\tilde{p} \otimes G^{(k)})(t, 0, s, y) dy &= \int_0^{\infty} \partial_t \left( \int_t^s \int_{-\infty}^{\infty} \tilde{p}(t, 0, u, z) G^{(k)}(u, z, s, y) dz du \right) dy \\ &= - \int_0^{\infty} G^{(k)}(t, 0, s, y) dy + \int_0^{\infty} \int_t^s \int_{-\infty}^{\infty} \partial_t \tilde{p}(t, 0, u, z) G^{(k)}(u, z, s, y) dz du dy \end{aligned}$$

and similarly

$$\int_0^{\infty} \partial_t(\tilde{q} \otimes H^{(k)})(t, 0, s, y) dy = - \int_0^{\infty} H^{(k)}(t, 0, s, y) dy + \int_0^{\infty} \int_t^s \int_{-\infty}^{\infty} \partial_t \tilde{q}(t, 0, u, z) H^{(k)}(u, z, s, y) dz du dy.$$

Hence,

$$\begin{aligned} & \left| \int_0^{\infty} \partial_t(\tilde{p} \otimes G^{(k)})(t, 0, s, y) dy - \int_0^{\infty} \partial_t(\tilde{q} \otimes H^{(k)})(t, 0, s, y) dy \right| \\ & \leq \left| \int_0^{\infty} (G^{(k)}(t, 0, s, y) - H^{(k)}(t, 0, s, y)) dy \right| \\ & \quad + \left| \int_0^{\infty} \int_t^s \int_{-\infty}^{\infty} (\partial_t \tilde{p}(t, 0, u, z) - \partial_t \tilde{q}(t, 0, u, z)) H^{(k)}(u, 0, s, y) dz du dy \right| \\ & \quad + \left| \int_0^{\infty} \int_t^s \int_{-\infty}^{\infty} (\partial_t \tilde{p}(t, 0, u, z) - \partial_t \tilde{q}(t, 0, u, z)) (H^{(k)}(u, z, s, y) - H^{(k)}(u, 0, s, y)) dz du dy \right| \\ & \quad + \left| \int_0^{\infty} \int_t^s \int_{-\infty}^{\infty} \partial_t \tilde{q}(t, 0, u, z) (H^{(k)}(u, z, s, y) - G^{(k)}(u, z, s, y)) dz du dy \right| \\ & =: I_{31} + I_{32} + I_{33} + I_{34}. \end{aligned}$$

For  $I_{31}$ , one has by Lemma 15

$$I_{31} \leq k \Delta_s \frac{R^k \Gamma(1/2)^{k-1}}{\Gamma((1+k)/2)} (s-t)^{k/2-1} \int_0^{\infty} \phi(r(s-t), y) dy \leq k \Delta_s \frac{R^k \Gamma(1/2)^{k-1}}{\Gamma((1+k)/2)} (s-t)^{k/2-1}. \quad (20)$$

For  $I_{32}$ , one has

$$\begin{aligned} I_{32} &= \left| \int_0^{\infty} \int_t^s \int_{-\infty}^{\infty} (\partial_t \tilde{p}(t, 0, u, z) - \partial_t \tilde{q}(t, 0, u, z)) H^{(k)}(u, 0, s, y) dz du dy \right| \\ &= \left| \int_0^{\infty} \int_t^s H^{(k)}(u, 0, s, y) dy du \int_{-\infty}^{\infty} (\partial_t \tilde{p}(t, 0, u, z) - \partial_t \tilde{q}(t, 0, u, z)) dz \right| \\ &\leq \left| \int_0^{\infty} \int_t^s |H^{(k)}(u, 0, s, y)| dy du \right| \left| \int_{-\infty}^{\infty} (\partial_t \tilde{p}(t, 0, u, z) - \partial_t \tilde{q}(t, 0, u, z)) dz \right| \\ &\leq R \Delta_s \int_t^s (u-t)^{-1/2} du \int_0^{\infty} |H^{(k)}(u, 0, s, y)| dy \\ &\leq R \Delta_s \int_t^s (u-t)^{-1/2} \frac{R^{k+1} \Gamma(1/2)^k}{\Gamma(1+k/2)} (s-u)^{k/2-1} du \\ &\leq \Delta_s \frac{R^{k+2} \Gamma(1/2)^{k-1}}{\Gamma((1+k)/2)} (s-t)^{(k-1)/2}. \end{aligned} \quad (21)$$

For  $I_{33}$ , we start with  $k = 1$ . Then it follows that

$$\begin{aligned}
& |H(t, x, s, y) - H(t, 0, s, y)| = \left| \int_0^x \partial_x H(t, z, s, y) dz \right| \\
&= \left| \int_0^x \left[ \left( \frac{\sigma(t, z) \partial_x \sigma(t, z)}{(1 + \beta(t))^2} + B(t, x) \right) \partial_{xx}^2 \tilde{q}(t, z, s, y) + \frac{\sigma(t, z)^2 - \sigma(t, y)^2}{2(1 + \beta(t))^2} \partial_{xxx}^3 \tilde{q}(t, z, s, y) \right] dz \right| \\
&\leq R \int_0^x \phi(r(s-t), y-z) \left( \frac{1}{s-t} + \frac{|y-z|}{(s-t)^{3/2}} \right) dz.
\end{aligned}$$

Hence,

$$\int_0^\infty |H(t, x, s, y) - H(t, 0, s, y)| dy \leq R \int_0^\infty \int_0^x \phi(r(s-t), y-z) \left( \frac{1}{s-t} + \frac{|y-z|}{(s-t)^{3/2}} \right) dz dy \leq \frac{R|x|}{s-t}.$$

Fix any  $\gamma \in (0, 1)$ , e.g.  $\gamma = 1/2$ . We distinguish two cases. If  $|x| \leq \sqrt{s-t}$ , then

$$\int_0^\infty |H(t, x, s, y) - H(t, 0, s, y)| dy \leq \frac{R|x|}{s-t} \leq \frac{R|x|^\gamma (s-t)^{(1-\gamma)/2}}{s-t} = \frac{R|x|^\gamma}{(s-t)^{(1+\gamma)/2}}.$$

If  $|x| > \sqrt{s-t}$ , then

$$\begin{aligned}
|H(t, x, s, y) - H(t, 0, s, y)| &\leq [|H(t, x, s, y)| + |H(t, 0, s, y)|] \frac{|x|^\gamma}{(s-t)^{\gamma/2}} \\
&\leq R \frac{\phi(r(s-t), y-x) + \phi(r(s-t), y)}{\sqrt{s-t}} \frac{|x|^\gamma}{(s-t)^{\gamma/2}}
\end{aligned}$$

and thus

$$\int_0^\infty |H(t, x, s, y) - H(t, 0, s, y)| dy \leq \frac{R|x|^\gamma}{(s-t)^{(1+\gamma)/2}}.$$

We compute for all  $k \geq 1$

$$\begin{aligned}
& \int_0^\infty |H^{(k)}(t, x, s, y) - H^{(k)}(t, 0, s, y)| dy \\
&= \int_0^\infty \left| \int_t^s \int_{-\infty}^\infty (H(t, x, u, z) - H(t, 0, u, z)) H^{(k-1)}(u, z, s, y) dz du \right| dy \\
&\leq \int_t^s du \int_{-\infty}^\infty |H(t, x, u, z) - H(t, 0, u, z)| dz \int_0^\infty |H^{(k-1)}(u, z, s, y)| dy \\
&\leq \int_t^s du \int_{-\infty}^\infty |H(t, x, u, z) - H(t, 0, u, z)| dz \int_0^\infty \frac{R^{k-1} \Gamma(1/2)^{k-2}}{\Gamma((k-1)/2)} \phi(r(s-u), y-z) (s-u)^{(k-1)/2-1} dy \\
&\leq \int_t^s \frac{|x|^\gamma}{(u-t)^{(1+\gamma)/2}} \frac{R^k \Gamma(1/2)^{k-2}}{\Gamma((k-1)/2)} (s-u)^{(k-1)/2-1} du \\
&\leq \frac{R^{k+1} \Gamma(1/2)^{k-2} |x|^\gamma}{\Gamma((k-1)/2)} (s-t)^{(k-\gamma)/2-1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
I_{33} &= \left| \int_0^\infty \int_t^s \int_{-\infty}^\infty (\partial_t \tilde{p}(t, 0, u, z) - \partial_t \tilde{q}(t, 0, u, z)) (H^{(k)}(u, z, s, y) - H^{(k)}(u, 0, s, y)) dz du dy \right| \\
&\leq \int_t^s \int_{-\infty}^\infty |\partial_t \tilde{p}(t, 0, u, z) - \partial_t \tilde{q}(t, 0, u, z)| dz du \int_0^\infty |H^{(k)}(u, z, s, y) - H^{(k)}(u, 0, s, y)| dy \\
&\leq \frac{R^{k+1} \Gamma(1/2)^{k-2}}{\Gamma((k-1)/2)} (s-t)^{(k-\gamma)/2-1} \Delta_s \int_t^s (s-u)^{(k-\gamma)/2-1} du \int_{-\infty}^\infty |z|^\gamma |\partial_t \tilde{p}(t, 0, u, z) - \partial_t \tilde{q}(t, 0, u, z)| dz \\
&\leq \frac{R^{k+1} \Gamma(1/2)^{k-2}}{\Gamma((k-1)/2)} (s-t)^{(k-\gamma)/2-1} \Delta_s \int_t^s du \int_{-\infty}^\infty \frac{R|z|^\gamma}{(u-t)} \phi(r(u-t), z) dz \\
&\leq \frac{R^{k+2} \Gamma(1/2)^{k-2}}{\Gamma((k-1)/2)} (s-t)^{k/2-1} \Delta_s. \tag{22}
\end{aligned}$$

Finally, let us turn to  $I_{34}$  whose estimation is almost the same of that of  $I_{34}$ . Write

$$\begin{aligned}
I_{34} &= \left| \int_0^\infty \int_t^s \int_{-\infty}^\infty \partial_t \tilde{q}(t, 0, u, z) (H^{(k)}(u, z, s, y) - G^{(k)}(u, z, s, y)) dz du dy \right| \\
&\leq \left| \int_0^\infty \int_t^s \int_{-\infty}^\infty \partial_t \tilde{q}(t, 0, u, z) ((H^{(k)} - G^{(k)})(u, z, s, y) - (H^{(k)} - G^{(k)})(u, 0, s, y)) dz du dy \right| \\
&\quad + \left| \int_0^\infty \int_t^s \int_{-\infty}^\infty \partial_t \tilde{q}(t, 0, u, z) (H^{(k)} - G^{(k)})(u, 0, s, y) dz du dy \right|.
\end{aligned}$$

Repeating the above reasoning by replacing  $H^{(k)}$  by  $H^{(k)} - G^{(k)}$ , one obtains

$$\int_0^\infty |(H^{(k)} - G^{(k)})(t, x, s, y) - (H^{(k)} - G^{(k)})(t, 0, s, y)| dy \leq \frac{R^{k+1} \Gamma(1/2)^{k-2} |x|^\gamma}{\Gamma((k-1)/2)} (s-t)^{(k-\gamma)/2-1} \Delta_s.$$

Therefore,

$$\begin{aligned}
I_{34} &\leq \frac{R^{k+2} \Gamma(1/2)^{k-2}}{\Gamma((k-1)/2)} (s-t)^{k/2-1} \Delta_s + \left| \int_0^\infty \int_t^s \int_{-\infty}^\infty \partial_t \tilde{q}(t, 0, u, z) (H^{(k)} - G^{(k)})(u, 0, s, y) dz du dy \right| \\
&= \frac{R^{k+2} \Gamma(1/2)^{k-2}}{\Gamma((k-1)/2)} (s-t)^{k/2-1} \Delta_s + \left| \int_0^\infty dy \int_t^s (H^{(k)} - G^{(k)})(u, 0, s, y) du \int_{-\infty}^\infty \partial_t \tilde{q}(t, 0, u, z) dz \right| \\
&\leq \frac{R^{k+2} \Gamma(1/2)^{k-2}}{\Gamma((k-1)/2)} (s-t)^{k/2-1} \Delta_s + \int_0^\infty dy \int_t^s |H^{(k)} - G^{(k)}|(u, 0, s, y) du \left| \int_{-\infty}^\infty \partial_t \tilde{q}(t, 0, u, z) dz \right| \\
&\leq \frac{R^{k+2} \Gamma(1/2)^{k-2}}{\Gamma((k-1)/2)} (s-t)^{k/2-1} \Delta_s + \int_0^\infty dy \int_t^s k \Delta_s \frac{R^k \Gamma(1/2)^{k-1}}{\Gamma((k+1)/2)} \phi(r(s-u), y) (s-u)^{k/2-1} \frac{R}{\sqrt{u-t}} du \\
&\leq \frac{R^{k+2} \Gamma(1/2)^{k-2}}{\Gamma((k-1)/2)} (s-t)^{k/2-1} \Delta_s + k \frac{R^{k+1} \Gamma(1/2)^{k-1}}{\Gamma((k+1)/2)} (s-t)^{(k-1)/2} \Delta_s. \tag{23}
\end{aligned}$$

Summing up (20), (21), (22) and (23), one obtains  $I_3(t) \leq R \Delta_s (s-t)^{1/2}$ . Adopting the above reasoning we obtain similarly  $J_3(t) \leq R \Delta_s (s-t)^{1/2}$ .  $\square$

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