

# Supplementary material for: Sketched over-parametrized projected gradient descent for sparse spike estimation

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## I. SUPPLEMENTARY MATERIAL

### A. Proof of Lemma 1

*Proof.* Let  $\theta^* \in \arg \min_{\theta \in \Theta} g_{S_d}(\theta)$ . We have, for any  $\theta \in \Theta$

$$\sqrt{g_S(\theta^*)} = \|\mathcal{S}A^{id}\phi(\theta^*) - \mathcal{S}A^{id}\phi(\theta_0)\|_2 \quad (1)$$

$$= \|\mathcal{S}A^{id}\phi(\theta^*) - \mathcal{S}_dA\phi(\theta_0) + \mathcal{S}_dA\phi(\theta_0) - \mathcal{S}A^{id}\phi(\theta_0)\|_2 \quad (2)$$

With the triangle inequality, we obtain

$$\sqrt{g_S(\theta^*)} \leq \|\mathcal{S}A^{id}\phi(\theta^*) - \mathcal{S}_dA\phi(\theta_0)\|_2 + \|\mathcal{S}_dA\phi(\theta_0) - \mathcal{S}A^{id}\phi(\theta_0)\|_2 \quad (3)$$

$$= \sqrt{g_{S_d}(\theta^*)} + \|(\mathcal{S} - \mathcal{S}_dT_\Gamma)A^{id}\phi(\theta_0)\|_2 \quad (4)$$

$$\leq \sqrt{g_{S_d}(\theta^*)} + \|\mathcal{S} - \mathcal{S}_dT_\Gamma\|_{\text{op}}^{A^{id},\Sigma} \|y\|_2 \quad (5)$$

Let  $\tilde{\theta} \in \arg \min_{\theta \in \Theta} g_S(\theta)$ , we have, similarly

$$\sqrt{g_{S_d}(\tilde{\theta}^*)} \leq \sqrt{g_{S_d}(\tilde{\theta})} \quad (6)$$

$$= \|\mathcal{S}A^{id}\phi(\tilde{\theta}) - \mathcal{S}_dA\phi(\theta_0)\|_2 \quad (7)$$

$$\leq \|\mathcal{S}A^{id}\phi(\tilde{\theta}) - \mathcal{S}A^{id}\phi(\theta_0)\|_2 + \|\mathcal{S}A^{id}\phi(\theta_0) - \mathcal{S}_dA\phi(\theta_0)\|_2 \quad (8)$$

$$\leq \sqrt{g_S(\tilde{\theta})} + \|\mathcal{S} - \mathcal{S}_dT_\Gamma\|_{\text{op}}^{A^{id},\Sigma} \|y\|_2 \quad (9)$$

We deduce from the two inequalities (5) and (9) that

$$\sqrt{g_S(\theta^*)} \leq \sqrt{g_S(\tilde{\theta})} + 2\|\mathcal{S} - \mathcal{S}_dT_\Gamma\|_{\text{op}}^{A^{id},\Sigma} \|y\|_2 \quad (10)$$

□

### B. Proof of Lemma 2

*Proof.* Let

$$\alpha_{i,k}(t) = \frac{f_k(z)}{2\pi\sigma_x\sigma_y} \times e^{-\left(\frac{(x_i-x)^2}{2\sigma_x^2} + \frac{(y_i-y)^2}{2\sigma_y^2}\right)} \quad (11)$$

the linear operator defined in our paper. Let us take the Fourier transform of (11),

$$[\mathcal{F}(A_k^{id}\delta_t)](w_l) = \int_{\mathbb{R}^2} \alpha_{i,k}(t) e^{-j\langle w_l, s_i \rangle} ds_i \quad (12)$$

with  $t = (x, y, z) \in \mathbb{R}^3$ ,  $w_l = (w_{l,x}, w_{l,y}) \in \mathbb{R}^2$  and  $s_i = (x_i, y_i) \in \mathbb{R}^2$ .

$$[\mathcal{F}(A_k^{id}\delta_t)](w_l) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{f_k(z)}{2\pi\sigma_x\sigma_y} e^{-\left(\frac{(x_i-x)^2}{2\sigma_x^2} + \frac{(y_i-y)^2}{2\sigma_y^2}\right)} e^{-j\langle w_l, s_i \rangle} ds_i \quad (13)$$

$$[\mathcal{F}(A_k^{id}\delta_t)](w_l) = \frac{f_k(z)}{2\pi\sigma_x\sigma_y} \left( \int_{-\infty}^{+\infty} e^{-\frac{(x_i-x)^2}{2\sigma_x^2}} e^{-jw_{l,x}x_i} dx_i \right) \left( \int_{-\infty}^{+\infty} e^{-\frac{(y_i-y)^2}{2\sigma_y^2}} e^{-jw_{l,y}y_i} dy_i \right) \quad (14)$$

Let  $I$  be the first integral,

$$I := \int_{-\infty}^{+\infty} e^{-\frac{(x_i-x)^2}{2\sigma_x^2}} e^{-jw_{l,x}x_i} dx_i = \int_{-\infty}^{+\infty} e^{-\frac{(x_i-x)^2 + 2jw_{l,x}x_i\sigma_x^2}{2\sigma_x^2}} dx_i \quad (15)$$

We isolate  $x_i$  from the rest of the terms in the exponent,

$$(x_i - x)^2 + 2jw_{l,x}x_i\sigma_x^2 = x_i^2 + 2x_i(jw_{l,x}\sigma_x^2 - x) + x^2 = (x_i + jw_{l,x}\sigma_x^2 - x)^2 - (jw_{l,x}\sigma_x^2 - x)^2 + x^2 \quad (16)$$

$$(17)$$

We inject this result in our integral to get

$$I = \int_{-\infty}^{+\infty} e^{-\frac{(x_i + jw_{l,x}\sigma_x^2 - x)^2 - (jw_{l,x}\sigma_x^2 - x)^2 + x^2}{2\sigma_x^2}} dx_i \quad (18)$$

$$= e^{\frac{(jw_{l,x}\sigma_x^2 - x)^2 - x^2}{2\sigma_x^2}} \int_{-\infty}^{+\infty} e^{-\frac{(x_i + jw_{l,x}\sigma_x^2 - x)^2}{2\sigma_x^2}} dx_i \quad (19)$$

$$(20)$$

By proceeding by substitution in this integral and using  $x'_i := \frac{x_i + jw_{l,x}\sigma_x^2 - x}{\sqrt{2}\sigma_x}$ , we get that  $dx'_i = \frac{1}{\sqrt{2}\sigma_x} dx_i \leftrightarrow dx_i = \sqrt{2}\sigma_x dx'_i$ . We then obtain

$$I = \sqrt{2}\sigma_x e^{\frac{(jw_{l,x}\sigma_x^2 - x)^2 - x^2}{2\sigma_x^2}} \int_{-\infty}^{+\infty} e^{-(x'_i)^2} dx'_i \quad (21)$$

We recognize the integral of a one dimensional Gaussian which is equal to  $\sqrt{\pi}$ ,

$$I = \sqrt{2}\pi\sigma_x e^{\frac{(jw_{l,x}\sigma_x^2 - x)^2 - x^2}{2\sigma_x^2}} \quad (22)$$

$$= \sqrt{2}\pi\sigma_x e^{-\frac{w_{l,x}^2\sigma_x^2}{2} - jw_{l,x}x} \quad (23)$$

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We note that we obtain a similar result with the other integral in (14). By injecting our new result (23), we obtain,

$$[\mathcal{F}(A_k^{id} \delta_t)](w_l) = \frac{f_k(z)}{2\pi\sigma_x\sigma_y} \begin{pmatrix} \sqrt{2\pi}\sigma_x e^{-\frac{\omega_{l,x}^2\sigma_x^2}{2} - j\omega_{l,x}x} \\ \sqrt{2\pi}\sigma_y e^{-\frac{\omega_{l,y}^2\sigma_y^2}{2} - j\omega_{l,y}y} \end{pmatrix} \quad (24)$$

By simplifying, we get

$$[\mathcal{F}(A_k^{id} \delta_t)](w_l) = f_k(z) e^{-\frac{\omega_{l,x}^2\sigma_x^2}{2} - j\omega_{l,x}x} e^{-\frac{\omega_{l,y}^2\sigma_y^2}{2} - j\omega_{l,y}y} \quad (25)$$

$$= f_k(z) e^{-\frac{\langle \omega_l^2, \sigma^2 \rangle}{2} - j\langle \omega_l, (x,y) \rangle}. \quad (26)$$

□

### C. Gradients

We provide the gradients needed to perform OP-COMP with PGD from sketched measurements.

$$\frac{\partial}{\partial x} [\mathcal{F}(A_k^{id} \delta_t)](w_l) = -j\omega_x [\mathcal{F}(A_k^{id} \delta_t)](w_l) \quad (27)$$

$$\frac{\partial}{\partial y} [\mathcal{F}(A_k^{id} \delta_t)](w_l) = -j\omega_y [\mathcal{F}(A_k^{id} \delta_t)](w_l) \quad (28)$$

$$\frac{\partial}{\partial z} [\mathcal{F}(A_k^{id} \delta_t)](w_l) = f'_k(z) e^{-\frac{\langle \omega_l^2, \sigma^2 \rangle}{2} - j\langle \omega_l, (x,y) \rangle}. \quad (29)$$