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# STABLE MINIMAL HYPERSURFACES IN $\mathbb{R}^6$

LAURENT MAZET

ABSTRACT. Following the strategy developed by Chodosh, Li, Minter and Stryker, and using the volume estimate of Antonelli and Xu, we prove that, in  $\mathbb{R}^6$ , a complete, two-sided, stable minimal hypersurfaces is flat.

## 1. INTRODUCTION

A minimal hypersurface  $M^n$  of  $\mathbb{R}^{n+1}$  is a critical point of the  $n$ -volume functional. It is characterized by its vanishing mean curvature. If a unit normal vectorfield  $\nu$  is defined along  $M$  and  $\varphi$  is a function with compact support on  $M$ , one can consider a deformation of  $M$  with initial speed  $\varphi\nu$ . The computation of the second derivative of the  $n$ -volume along this deformation at initial time gives

$$\int_M |\nabla\varphi|^2 - |A_M|^2\varphi^2$$

where  $A_M$  is the second fundamental form of  $M$ . So asking that this quantity is non negative for any  $\varphi$  means that  $M$  is a minimum at order 2 of the  $n$ -volume. Such a minimal hypersurface is called stable.

The stable Bernstein problem asks whether a complete stable minimal hypersurface is a flat affine hyperplane. We give a positive answer in the case  $n = 5$ .

**Theorem 1.1.** *Let  $M^5 \looparrowright \mathbb{R}^6$  be an immersed, complete, connected, two-sided, stable minimal hypersurface. Then  $M$  is a Euclidean hyperplane.*

A particular class of stable minimal hypersurface is given by minimal graphs over  $\mathbb{R}^n$ . In [4], Bernstein proved that a minimal graph over  $\mathbb{R}^2$  has to be a plane. In the sixties, the same question for higher dimensions was studied in a series of papers by Fleming [17], De Giorgi [14], Almgren [1] and Simons [26]. They proved that minimal graphs over  $\mathbb{R}^n$  are planes if  $n \leq 7$ . For  $n \geq 8$ , Bombieri, De Giorgi and Giusti [5] were able to construct counter-examples and gave also in  $\mathbb{R}^8$  an example of a stable minimal hypersurfaces that is not a hyperplane.

Concerning the stable Bernstein problem, the question was solved positively in  $\mathbb{R}^3$  by Do Carmo and Peng [15], Fischer-Colbrie and Schoen [16] and Porogelov [22] in the early eighties. In higher dimension, Schoen, Simon and Yau [23, 24] were able to settle the stable Bernstein in  $\mathbb{R}^{n+1}$ ,  $n \leq 6$ , under a Euclidean volume growth assumption (see also the recent work of Bellettini [3]).

Recently Chodosh and Li [9] were able to answer positively the stable Bernstein problem in  $\mathbb{R}^4$ . Later two alternative proofs came out: one by Catino, Mastrolia and Roncoroni [8] and one by Chodosh and Li [10]. Actually in [10], Chodosh and Li develop a second strategy to prove the result. Then, in a joint work Minter and Stryker [12], they were able to apply this strategy in the case of  $\mathbb{R}^5$  to solve the stable Bernstein problem in this dimension as well.

As in [13, 12], it is well known that Theorem 1.1 comes with corollaries like curvature estimates for stable minimal immersions in 6-dimensional manifolds and characterization of finite Morse index minimal hypersurfaces in  $\mathbb{R}^6$ . For example, we have

**Corollary 1.1.** *Let  $(X^6, g)$  be a complete Riemannian manifold whose sectional curvature satisfies  $|sec_g| \leq K$ . Then any compact, two-sided, stable minimal immersion  $M^5 \looparrowright X$  satisfies*

$$|A_M|(q) \min(1, d_M(q, \partial M)) \leq C(K)$$

for  $q \in M$ .

The basic idea to prove Theorem 1.1 is to obtain a Euclidean growth estimate for the volume of  $M$  and then apply the work of Schoen, Simon and Yau. The strategy of Chodosh and Li is a way towards this estimate. We refer to [10, 12] for a good presentation of their ideas. Let us give some elements. Let  $M$  be a stable minimal hypersurface in  $\mathbb{R}^{n+1}$  with induced metric  $g$ . Inspired by the work of Gulliver and Lawson [18], they consider the conformal metric  $\tilde{g} = r^{-2}g$  where  $r$  is the Euclidean distance to 0 in  $\mathbb{R}^{n+1}$ . If  $M$  was a hyperplane passing through the origin  $(M \setminus \{0\}, \tilde{g})$  would be isometric to the Euclidean product  $\mathbb{S}^{n-1} \times \mathbb{R}$ . In the general case, the idea of Chodosh and Li is that the stability assumption implies that the geometry of  $(M \setminus \{0\}, \tilde{g})$  should look like  $\mathbb{S}^{n-1} \times \mathbb{R}$ . In [12], the authors consider the bi-Ricci curvature which is a certain combination of sectional curvatures. It was introduced by Shen and Ye in [25], already to study minimal surfaces (see precise definition in Section 2). Notice that on  $\mathbb{S}^{n-1} \times \mathbb{R}$ , the bi-Ricci curvature is lower bounded by  $n-2$ . In [12], the authors prove that the stability of  $M$  implies a positive spectral lower bound for the bi-Ricci curvature of  $(M \setminus \{0\}, \tilde{g})$ . More precisely they prove that, on  $(M \setminus \{0\}, \tilde{g})$ , the operator  $-a\tilde{\Delta} + (\widetilde{\text{BRic}}_- - 1)$  is non-negative where  $\widetilde{\text{BRic}}_-$  is the punctual minimum of the bi-Ricci curvature of  $\tilde{g}$ . This should be understood as a weak form of  $\widetilde{\text{BRic}} \geq 1$ .

The second step of the strategy consists in the construction of a  $\mu$ -bubble in  $(M \setminus \{0\}, \tilde{g})$  with a spectral lower bound for its Ricci curvature. In some sense, they identify in any sufficiently large part of  $(M \setminus \{0\}, \tilde{g})$  a hypersurface that play the role of  $\mathbb{S}^{n-1} \times \{t\}$  in  $\mathbb{S}^{n-1} \times \mathbb{R}$ .

The last step in [12] is to obtain a Bishop-Gromov volume estimate for the  $\mu$ -bubble under the spectral lower bound on the Ricci curvature. In their paper, the proof of this volume estimate was specific to dimension 3. Recently, Antonelli and Xu [2] have proved such a Bishop-Gromov estimate in any dimension.

Once the  $\tilde{g}$ -volume of the  $\mu$ -bubble is controlled, this gives an estimate of its volume in the original metric  $g$  and then control the growth of the volume of  $M$  tanks to an isoperimetric inequality due to Michael and Simon [20] and Brendle [6].

In the present paper, we also follow the above strategy of [12]. Here we consider a weighted bi-Ricci curvature  $\text{BRic}_\alpha$  where the parameter  $\alpha$  does not give the same weight to all sectional curvature in the combination (a similar idea appear in the recent article by Hong and Yan [19]). We prove a spectral lower bound for the weighted bi-Ricci curvature: the operator  $-a\tilde{\Delta} + (\widetilde{\text{BRic}}_{\alpha-} - \delta)$  is non-negative where  $a, \delta \in \mathbb{R}$ . At that step,  $a, \alpha$  are two parameters that should be chosen such that  $\delta > 0$ .

By imposing some new constraints on  $a$  and  $\alpha$ , we are then able to construct the  $\mu$ -bubble with a spectral lower bound on the Ricci curvature. At the last step, we apply the Bishop-Gromov estimate of Antonelli and Xu [2]. In order to do so, this imposes some new

constraints on the parameters  $a$  and  $\alpha$ . Nevertheless, the choice  $a = \frac{11}{10}$  and  $\alpha = \frac{40}{43}$  fits all the constraints. The end of the proof then follows the line of [12].

**Organization.** In Section 2, we fix some notations that we use all along the paper. Section 3 is devoted to the proof the spectral lower bound for  $\widetilde{\text{BRic}}_\alpha$  for the Gulliver-Lawson metric on a stable minimal hypersurface. In Section 4, we construct the  $\mu$ -bubble with a spectral lower Ricci bound. We end the proof of Theorem 1.1 in Section 5. Along the paper, we specify the value of  $n$ ,  $a$  and  $\alpha$  only when it is necessary, we hope this allows to understand where the constraints come from.

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## 2. PRELIMINARIES

Let  $(M^n, g)$  be a Riemannian manifold and  $(e_i)_{1 \leq i \leq n}$  be an orthonormal basis of  $T_p M$ . For  $\alpha \in \mathbb{R}$ , we recall or define

- the Ricci curvature  $\text{Ric}(e_1, e_1) = \sum_{i=2}^n R(e_1, e_i, e_i, e_1)$ ,
- the punctual minimum of the Ricci curvature  $\lambda(p) = \min_{v \in T_p M, |v|=1} \text{Ric}(e, e)$ ,
- the weighted bi-Ricci or  $\alpha$ -bi-Ricci curvature

$$\text{BRic}_\alpha(e_1, e_2) = \sum_{i=2}^n R(e_1, e_i, e_i, e_1) + \alpha \sum_{j=3}^n R(e_2, e_j, e_j, e_2)$$

- the minimum of the  $\alpha$ -bi-Ricci curvature  $\Lambda_\alpha(p) = \min_{(e,f) \text{ orthonormal in } T_p M} \text{BRic}_\alpha(e, f)$

Notice that for  $\alpha = 1$ ,  $\text{BRic}_1$  is the classical bi-Ricci curvature as defined in [25].

If  $\Sigma \looparrowright M$  is a hypersurface with unit normal  $\nu$ . We use the following conventions:

- the second fundamental form of  $\Sigma$  is  $A_\Sigma(X, Y) = (\nabla_X \nu, Y) = -(\nabla_X Y, \nu)$  and
- the mean curvature of  $\Sigma$  is  $H = \text{tr } A_\Sigma$ .

If  $\Omega$  is a subset of  $M$ , we denote by  $\mathcal{N}_\rho(\Omega)$  the  $\rho$ -tubular neighborhood of  $\Omega$ : the set of points at distance less than  $\rho$  from  $\Omega$ .

We finish by a simple remark that we use in Section 3.

*Remark 1.* Let  $A \in \mathcal{M}_n(\mathbb{R})$  be a positive definite symmetric matrix and  $B \in \mathbb{R}^n$ . Then the function  $f : X \in \mathbb{R}^n \mapsto X^\top A X + B^\top X \in \mathbb{R}$  is lower bounded and its minimum is given by  $-\frac{1}{4} B^\top A^{-1} B$ .

## 3. SPECTRAL LOWER BOUND FOR THE WEIGHTED BI-RICCI CURVATURE

Let  $F : M^n \looparrowright \mathbb{R}^{n+1}$  be a complete two-sided minimal hypersurface and  $g$  its induced metric. We consider the Gulliver-Lawson conformal metric  $\tilde{g} = r^{-2}g$  where  $r$  is the Euclidean distance function to 0. Notice that if  $F(p) = 0$ ,  $\tilde{g}$  is not defined. So we consider  $N = M \setminus F^{-1}(0)$ . As it was observed by Gulliver and Lawson [18], the metric  $(N, \tilde{g})$  is complete.

The first step of the proof of Theorem 1.1 consists in proving that the stability assumption can be translated in a spectral lower bound for the  $\alpha$ -bi-Ricci curvature of the metric  $\tilde{g}$ . Actually we have the following result.

**Theorem 3.1.** *Let  $M^n \looparrowright \mathbb{R}^{n+1}$  be a two-sided stable minimal hypersurface. Suppose  $n = 5$ , then, for  $a = \frac{11}{10}$ ,  $\alpha = \frac{40}{43}$  and  $\delta = \frac{3}{10}$ , there is a smooth function  $V$  such that*

$$V \geq \delta - \tilde{\Lambda}_\alpha$$

and

$$(1) \quad \int_N |\nabla \varphi|_{\tilde{g}}^2 dv_{\tilde{g}} \geq \int_N \frac{1}{a} V \varphi^2 dv_{\tilde{g}}$$

for any  $\varphi \in C_c^1(N)$ .

**3.1. Recalling some computations.** We first recall some computations and results of [12].

We denote by  $\nu$  the unit normal to  $M$  and by  $|dr|$  the norm of the differential of  $r$  along  $M$  with respect to the  $g$ -metric.

Let  $(e_i)_{1 \leq i \leq n}$  be an orthonormal basis for the metric  $g$  then, for the conformal metric  $\tilde{g}$ , an orthonormal basis is given by  $\tilde{e}_i = re_i$ . The sectional curvatures of  $g$  and  $\tilde{g}$  are related by

$$(2) \quad \tilde{R}_{ijji} = r^2 R_{ijji} + 2 - |dr|^2 - dr(e_i)^2 - dr(e_j)^2 - (p, \nu)(A_{ii} + A_{jj})$$

(see [12, Proposition 3.5]).

The second result that we want to recall is the writing of the stability inequality in the conformal metric  $\tilde{g}$ . We have

$$(3) \quad \int_N |\nabla \varphi|^2 dv_{\tilde{g}} \geq \int_N \left( r^2 |A|^2 - \frac{n(n-2)}{2} + \frac{n^2-4}{4} |dr|^2 \right) \varphi^2 dv_{\tilde{g}}$$

for any  $\varphi \in C_c^1(N)$  (see [12, Proposition 3.10]).

**3.2. Estimating the curvature terms.** In this subsection, we want to relate the curvature term in the stability inequality (3) to the  $\alpha$ -bi-Ricci curvature. We use the notations of the preceding subsection.

**Proposition 3.1.**

$$\begin{aligned} \widetilde{\text{BRic}}_\alpha(\tilde{e}_1, \tilde{e}_2) &= r^2 \text{BRic}_\alpha(e_1, e_2) + 2(n-1 + \alpha(n-2)) - (n + \alpha(n-1))|dr|^2 \\ &\quad - ((n-2 - \alpha)dr(e_1)^2 + \alpha(n-3)dr(e_2)^2) \\ &\quad - (p, \nu)((n-2 - \alpha)A_{11} + \alpha(n-3)A_{22}) \end{aligned}$$

*Proof.* Summing (2) and using  $\text{tr } A = 0$ , we have

$$\begin{aligned} \widetilde{\text{BRic}}_\alpha(\tilde{e}_1, \tilde{e}_2) &= \sum_{i=2}^n \tilde{R}_{1ii1} + \alpha \sum_{j=3}^n \tilde{R}_{2jj2} \\ &= r^2 \text{BRic}_\alpha(e_1, e_2) + 2(n-1) - (n-1)|dr|^2 - (n-1)dr(e_1)^2 \\ &\quad - (|dr|^2 - dr(e_1)^2) - (p, \nu)((n-1)A_{11} - A_{11}) + 2\alpha(n-2) \\ &\quad - \alpha(n-2)|dr|^2 - \alpha(n-2)dr(e_2)^2 - \alpha(|dr|^2 - dr(e_1)^2 - dr(e_2)^2) \\ &\quad - \alpha(p, \nu)((n-2)A_{22} - A_{11} - A_{22}) \\ &= r^2 \text{BRic}_\alpha(e_1, e_2) + 2(n-1 + \alpha(n-2)) - (n + \alpha(n-1))|dr|^2 \\ &\quad - ((n-2 - \alpha)dr(e_1)^2 + \alpha(n-3)dr(e_2)^2) \\ &\quad - (p, \nu)((n-2 - \alpha)A_{11} + \alpha(n-3)A_{22}) \end{aligned}$$

□

**Proposition 3.2.**

$$\text{BRic}_\alpha(e_1, e_2) = - \sum_{i=1}^n A_{1i}^2 - \alpha \sum_{j=2}^n A_{2j}^2 - \alpha A_{11} A_{22}$$

*Proof.* Applying Gauss formula we have

$$\begin{aligned} \text{BRic}_\alpha(e_1, e_2) &= \sum_{i=2}^n (A_{11} A_{ii} - A_{1i}^2) + \alpha \sum_{j=3}^n (A_{22} A_{jj} - A_{2j}^2) \\ &= - \sum_{i=1}^n A_{1i}^2 + \alpha (-A_{22} (A_{11} + A_{22}) - \sum_{j=3}^n A_{2j}^2) \\ &= - \sum_{i=1}^n A_{1i}^2 - \alpha \sum_{j=2}^n A_{2j}^2 - \alpha A_{11} A_{22} \end{aligned}$$

□

Using the above computation, we obtain the following estimate of the curvature term. This estimate introduces some constraints on  $\alpha$  and a second parameter  $a$ .

**Proposition 3.3.** *Let  $a, \alpha > 0$  such that  $a > \frac{1}{2}$ ,  $2a \geq \alpha$  and*

$$W = (a - \frac{1}{2}) (a - \frac{n-2}{2n} (1 + 2\alpha)) - \frac{n-2}{4n} (1 - \alpha)^2 > 0$$

*Let us define*

$$f = \frac{(n-2)^2}{8W} \left( (a - \frac{1}{2}) \frac{n-2}{n} (1 + \alpha \frac{n-4}{n-2})^2 + (1 - \alpha)^2 (a + \frac{n-2}{2n} - \frac{2}{n} \alpha) \right)$$

*Then*

$$ar^2 |A|^2 + f(1 - |dr|^2) \geq -r^2 \text{BRic}_\alpha(e_1, e_2) + (p, \nu) ((n-2-\alpha)A_{11} + \alpha(n-3)A_{22})$$

*Proof.* By Proposition 3.2, the right-hand side of the expected inequality satisfies to

$$\begin{aligned} &-r^2 \text{BRic}_\alpha(e_1, e_2) + (p, \nu) ((n-2-\alpha)A_{11} + \alpha(n-3)A_{22}) \\ &= r^2 \left( \sum_{i=1}^n A_{1i}^2 + \alpha \sum_{j=2}^n A_{2j}^2 + \alpha A_{11} A_{22} \right. \\ &\quad \left. + (\frac{p}{r^2}, \nu) ((n-2-\alpha)A_{11} + \alpha(n-3)A_{22}) \right) \\ (4) \quad &= r^2 \left( A_{11}^2 + \alpha A_{22}^2 + \alpha A_{11} A_{22} + \sum_{i=2}^n A_{1i}^2 + \alpha \sum_{j=3}^n A_{2j}^2 \right. \\ &\quad \left. + (\frac{p}{r^2}, \nu) ((n-2-\alpha)A_{11} + \alpha(n-3)A_{22}) \right) \end{aligned}$$

The vector  $A_\Delta = (A_{11}, \dots, A_{nn})$  belongs to the sub-space  $F_n = \{X \in \mathbb{R}^n \mid x_1 + \dots + x_n = 0\}$ . We write a decomposition in an orthonormal basis of  $F_n$  as

$$\begin{pmatrix} A_{11} \\ \vdots \\ A_{nn} \end{pmatrix} = \sum_{i=1}^{n-3} \begin{pmatrix} 0 \\ 0 \\ E_i \end{pmatrix} x_i + \frac{1}{\sqrt{2n(n-2)}} \begin{pmatrix} n-2 \\ n-2 \\ -2 \\ \vdots \\ -2 \end{pmatrix} z_1 + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} z_2$$

where  $(E_i)_{1 \leq i \leq n-3}$  is an orthonormal basis of  $F_{n-2}$ . So we have

$$\begin{aligned} & A_{11}^2 + \alpha A_{22}^2 + \alpha A_{11} A_{22} + \left(\frac{p}{r^2}, \nu\right) ((n-2-\alpha)A_{11} + \alpha(n-3)A_{22}) \\ &= \left(\frac{\sqrt{n-2}}{\sqrt{2n}} z_1 + \frac{z_2}{\sqrt{2}}\right)^2 + \alpha \left(\frac{\sqrt{n-2}}{\sqrt{2n}} z_1 - \frac{z_2}{\sqrt{2}}\right)^2 + \alpha \left(\frac{n-2}{2n} z_1^2 - \frac{1}{2} z_2^2\right) \\ (5) \quad &+ \left(\frac{p}{r^2}, \nu\right) \left(\frac{\sqrt{n-2}}{\sqrt{2n}} (n-2 + \alpha(n-4)) z_1 + \frac{n-2}{\sqrt{2}} (1-\alpha) z_2\right) \\ &= \frac{n-2}{2n} (1+2\alpha) z_1^2 + \sqrt{\frac{n-2}{n}} (1-\alpha) z_1 z_2 + \frac{1}{2} z_2^2 \\ &+ \left(\frac{p}{r^2}, \nu\right) \frac{n-2}{\sqrt{2}} \left(\sqrt{\frac{n-2}{n}} \left(1 + \alpha \frac{n-4}{n-2}\right) z_1 + (1-\alpha) z_2\right) \end{aligned}$$

For  $a > 0$ , we are interested in the minimum (if it exists) of

$$\begin{aligned} (6) \quad & a(z_1^2 + z_2^2) - \frac{n-2}{2n} (1+2\alpha) z_1^2 - \sqrt{\frac{n-2}{n}} (1-\alpha) z_1 z_2 - \frac{1}{2} z_2^2 \\ & - \left(\frac{p}{r^2}, \nu\right) \frac{n-2}{\sqrt{2}} \left(\sqrt{\frac{n-2}{n}} \left(1 + \alpha \frac{n-4}{n-2}\right) z_1 + (1-\alpha) z_2\right) \end{aligned}$$

The matrix of the quadratic part of the above expression is

$$\begin{pmatrix} a - \frac{n-2}{2n} (1+2\alpha) & -\sqrt{\frac{n-2}{4n}} (1-\alpha) \\ -\sqrt{\frac{n-2}{4n}} (1-\alpha) & a - \frac{1}{2} \end{pmatrix}$$

This matrix is positive definite if  $a > \frac{1}{2}$  and its determinant is positive:

$$W = \left(a - \frac{1}{2}\right) \left(a - \frac{n-2}{2n} (1+2\alpha)\right) - \frac{n-2}{4n} (1-\alpha)^2 > 0$$

If it's the case, by Remark 1 with vector  $B = -(\frac{p}{r^2}, \nu)^{\frac{n-2}{\sqrt{2}}}(\sqrt{\frac{n-2}{n}}(1 + \alpha\frac{n-4}{n-2}), (1 - \alpha))$ , the quantity in (6) is lower bounded by

$$\begin{aligned} & -(\frac{p}{r^2}, \nu)^2 \frac{(n-2)^2}{8W} \left( (a - \frac{1}{2})\frac{n-2}{n} (1 + \alpha\frac{n-4}{n-2})^2 + \frac{n-2}{n} (1 - \alpha)^2 (1 + \alpha\frac{n-4}{n-2}) \right. \\ & \quad \left. + (a - \frac{n-2}{2n} (1 + 2\alpha))(1 - \alpha)^2 \right) \\ &= -(\frac{p}{r^2}, \nu)^2 \frac{(n-2)^2}{8W} \left( (a - \frac{1}{2})\frac{n-2}{n} (1 + \alpha\frac{n-4}{n-2})^2 + (1 - \alpha)^2 (a + \frac{n-2}{2n} - \frac{2}{n}\alpha) \right) \\ &= -(\frac{p}{r^2}, \nu)^2 f \end{aligned}$$

Since  $(\frac{p}{r}, \nu)^2 = (1 - |dr|^2)$ , we have then proved that

$$\begin{aligned} a(z_1^2 + z_2^2) + \frac{f}{r^2}(1 - |dr|^2) &\geq \frac{n-2}{2n}(1 + 2\alpha)z_1^2 + \sqrt{\frac{n-2}{n}}(1 - \alpha)z_1z_2 + \frac{1}{2}z_2^2 \\ &\quad + (\frac{p}{r^2}, \nu) \frac{n-2}{\sqrt{2}} \left( \sqrt{\frac{n-2}{n}}(1 + \alpha\frac{n-4}{n-2})z_1 + (1 - \alpha)z_2 \right) \end{aligned}$$

Combining this with (4) and (5), if  $2a \geq \alpha$ , we have

$$\begin{aligned} a|A|^2 + \frac{f}{r^2}(1 - |dr|^2) &\geq a(|A_\Delta|^2 + \sum_{i \neq j} A_{ij}^2) + \frac{f}{r^2}(1 - |dr|^2) \\ &\geq A_{11}^2 + \alpha A_{22}^2 + \alpha A_{11}A_{22} + (\frac{p}{r^2}, \nu)((n-2 - \alpha)A_{11} + \alpha(n-3)A_{22}) \\ &\quad + a \sum_{i \neq j} A_{ij}^2 \\ &\geq A_{11}^2 + \alpha A_{22}^2 + \alpha A_{11}A_{22} + \sum_{i=2}^n A_{1i}^2 + \alpha \sum_{j=3}^n A_{2j}^2 \\ &\quad + (\frac{p}{r^2}, \nu)((n-2 - \alpha)A_{11} + \alpha(n-3)A_{22}) \\ &\geq -\text{BRic}_\alpha(e_1, e_2) + \frac{(p, \nu)}{r^2}((n-2 - \alpha)A_{11} + \alpha(n-3)A_{22}) \end{aligned}$$

This is the expected estimate □

**3.3. Proof of Theorem 3.1.** Let us assume that the basis is chosen such that  $\tilde{\Lambda}_\alpha = \text{BRic}_\alpha(\tilde{e}_1, \tilde{e}_2)$ . From (3), we are looking for a lower bound for  $r^2|A|^2 - \frac{n(n-2)}{2} + \frac{n^2-4}{4}|dr|^2$ . Under the assumptions of Proposition 3.3,  $\alpha \leq 1$  (such that  $n-2 - \alpha \geq \alpha(n-3)$ ) and using



Proposition 3.1, we have

$$\begin{aligned}
& a\left(r^2|A|^2 - \frac{n(n-2)}{2} + \frac{n^2-4}{4}|dr|^2\right) \\
& \geq -r^2 \text{BRic}_\alpha(e_1, e_2) + (p, \nu)((n-2-\alpha)A_{11} + \alpha(n-3)A_{22}) \\
& \quad - f(1-|dr|^2) - a\frac{n(n-2)}{2} + a\frac{n^2-4}{4}|dr|^2 \\
& \geq -\widetilde{\text{BRic}}_\alpha(\tilde{e}_1, \tilde{e}_2) + 2(n-1+\alpha(n-2)) - (n+\alpha(n-1))|dr|^2 \\
& \quad - ((n-2-\alpha)dr(e_1)^2 + \alpha(n-3)dr(e_2)^2) \\
& \quad - f(1-|dr|^2) - a\frac{n(n-2)}{2} + a\frac{n^2-4}{4}|dr|^2 \\
& \geq C(|dr|^2) - \tilde{\Lambda}_\alpha
\end{aligned}$$

where

$$C(t) = 2(n-1+\alpha(n-2)) - (2n-2+\alpha(n-2))t - f(1-t) - a\frac{n(n-2)}{2} + a\frac{n^2-4}{4}t$$

$C$  is an affine function and  $0 \leq |dr|^2 \leq 1$ , so  $C(|dr|^2) \geq \min(C(0), C(1))$ . We have

$$\begin{aligned}
C(1) &= 2(n-1+\alpha(n-2)) - (2n-2+\alpha(n-2)) - a\frac{n(n-2)}{2} + a\frac{n^2-4}{4} \\
&= \alpha(n-2) - a\frac{(n-2)^2}{4} = (n-2)\left(\alpha - a\frac{n-2}{4}\right)
\end{aligned}$$

and

$$C(0) = 2(n-1+\alpha(n-2)) - f - a\frac{n(n-2)}{2}$$

If we consider  $a = \frac{11}{10}$  and  $\alpha = \frac{40}{43}$ , we have  $a > \frac{1}{2}$ ,  $2a \geq \alpha$ ,  $\alpha \leq 1$  and  $W = \frac{26697}{184900} > 0$ . So the above computations apply. We have

$$C(0) = \frac{731975}{1530628} \simeq 0.47 \quad \text{and} \quad C(1) = \frac{543}{1720} \simeq 0.31$$

So for these values of  $a$  and  $\alpha$ , and with  $\delta = \frac{3}{10} \leq \min(C(0), C(1))$ , we have

$$V = a\left(r^2|A|^2 - \frac{n(n-2)}{2} + \frac{n^2-4}{4}|dr|^2\right) \geq \delta - \tilde{\Lambda}_\alpha$$

By (3), the spectral estimate (1) is true. Theorem 3.1 is proved.

#### 4. THE $\mu$ -BUBBLE CONSTRUCTION

In this section, we produce a warped  $\mu$ -bubble with a spectral Ricci curvature lower bound. So we start with a connected complete non-compact Riemannian manifold  $(N^n, \bar{g})$  with a spectral lower bound on the  $\alpha$ -bi-Ricci curvature: there is a smooth function  $\bar{V}$  on  $N$  such that

$$\bar{V} \geq \delta - \bar{\Lambda}_\alpha$$

and

$$(7) \quad \int_N |\bar{\nabla} \varphi|_{\bar{g}}^2 dv_{\bar{g}} \geq \int_N \frac{1}{a} \bar{V} \varphi^2 dv_{\bar{g}}$$

for any  $\varphi \in C_c^1(N)$

**Theorem 4.1.** *Assume  $(N, \bar{g})$  as above with  $n = 5$ ,  $a = \frac{11}{10}$ ,  $\alpha = \frac{40}{43}$  and  $\delta = \frac{3}{10}$ . Let  $\Omega_+$  be a domain in  $N$  (i.e. an open subset with compact smooth boundary) such that  $N \setminus \bar{\mathcal{N}}_{100\pi}(\Omega_+) \neq \emptyset$ . Then there is a domain  $\Omega_*$  with*

- $\Omega_+ \subset \Omega_* \subset \bar{\mathcal{N}}_{100\pi}(\Omega_+)$  and
- there is a smooth function  $V$  on  $\Sigma = \partial\Omega_*$  such that

$$V \geq \frac{\delta}{2} - \alpha\lambda^\Sigma$$

and

$$(8) \quad \frac{4}{4-a} \int_\Sigma |\nabla\varphi|^2 dv_g \geq \int_\Sigma V\varphi^2 dv_g$$

for any  $\varphi \in C^1(\Sigma)$  where  $g$  is the induced metric on  $\Sigma$ .

**4.1. Construction of the  $\mu$ -bubble.** Because of the spectral control (7) on  $N$ , we know (see [16]) that there is a positive function  $w$  on  $N$  such that

$$(9) \quad -a\bar{\Delta}w = \bar{V}w \geq (\delta - \bar{\Lambda}_\alpha)w$$

Let us recall quickly the construction of the  $\mu$ -bubble. Let  $\Omega_-$  be a domain in  $N$  such that  $\Omega_+ \subset\subset \Omega_- \subset \bar{\mathcal{N}}_{100\pi}(\Omega_+)$ . Let  $h : \Omega_- \setminus \Omega_+ \rightarrow \mathbb{R}$  be a smooth function such that  $\lim_{p \rightarrow \partial\Omega_+} h(p) = +\infty$  and  $\lim_{p \rightarrow \partial\Omega_-} h(p) = -\infty$ . Let  $\underline{\Omega}$  be a domain with  $\Omega_+ \subset\subset \underline{\Omega} \subset\subset \Omega_-$ .

For any sets of finite perimeter  $\Omega$  with  $\Omega_+ \subset\subset \Omega \subset\subset \Omega_-$ , we consider the quantity

$$\mathcal{A}(\Omega) = \int_{\partial^*\Omega} w^a - \int_U (\chi_\Omega - \chi_{\underline{\Omega}})hw^a$$

where  $\partial^*\Omega$  is the reduced boundary of  $\Omega$ . By similar argument to the ones in [11, 28], there is a set of finite perimeter  $\Omega_*$  ( $\Omega_+ \subset\subset \Omega_* \subset\subset \Omega_-$ ) which minimize the functional  $\mathcal{A}$ . Moreover its reduced boundary  $\partial^*\Omega_* = \Sigma$  is non empty ( $N \setminus \bar{\mathcal{N}}_{100\pi}(\Omega_+) \neq \emptyset$ ) and smooth (see for example [21, 27]).

**4.2. Spectral Ricci-curvature bound of the  $\mu$ -bubble.** We denote by  $k = n - 1$  the dimension of  $\Sigma$  and by  $\eta$  the outgoing unit normal to  $\Sigma$ .

As in [12, Proposition 4.2], if  $\varphi$  is a function on  $\Sigma$ , writing the first variation of  $\mathcal{A}$  for a variation  $\{\Omega_t\}$  of  $\Omega_*$  generated by  $\varphi\eta$  gives

$$0 = \frac{d}{dt} \mathcal{A}(\Omega_t)|_{t=0} = \int_\Sigma (Hw^a + aw^{a-1}dw(\eta) - hw^a)\varphi = \int_\Sigma (H + aw^{-1}dw(\eta) - h)w^a\varphi$$

Since this is true for any  $\varphi$ ,

$$(10) \quad H = h - ad \ln w(\eta)$$

Computing the second derivative of  $\mathcal{A}(\Omega_t)$ , we obtain

$$0 \leq \frac{d^2}{dt^2} \mathcal{A}(\Omega_t)|_{t=0} = \int_\Sigma w^a \left( -\varphi\Delta\varphi - (|B|^2 + \bar{\text{Ric}}(\eta, \eta))\varphi^2 - aw^{-2}dw(\eta)^2\varphi^2 \right. \\ \left. + aw^{-1}\bar{\nabla}^2 w(\eta, \eta)\varphi^2 - aw^{-1}(\bar{\nabla}w, \nabla\varphi)\varphi - dh(\eta)\varphi^2 \right)$$

where  $B$  is the second fundamental form of  $\Sigma$ . So

$$0 \leq \int_{\Sigma} -\operatorname{div}(w^a \varphi \nabla \varphi) + w^a \left( |\nabla \varphi|^2 - (|B|^2 + \overline{\operatorname{Ric}}(\eta, \eta)) \varphi^2 - aw^{-2} dw(\eta)^2 \varphi^2 \right. \\ \left. + aw^{-1} \overline{\nabla}^2 w(\eta, \eta) \varphi^2 - dh(\eta) \varphi^2 \right)$$

Using  $\overline{\nabla}^2 w(\eta, \eta) = \overline{\Delta} w - \Delta w - Hdw(\eta)$ , we obtain

$$(11) \quad 0 \leq \int_{\Sigma} w^a \left( |\nabla \varphi|^2 - (|B|^2 + \overline{\operatorname{Ric}}(\eta, \eta)) \varphi^2 - aw^{-2} dw(\eta)^2 \varphi^2 \right. \\ \left. + aw^{-1} (\overline{\Delta} w - \Delta w - Hdw(\eta)) \varphi^2 - dh(\eta) \varphi^2 \right)$$

For  $\varphi = w^{-a/2} \psi$ , we have  $\nabla \varphi = w^{-a/2} \nabla \psi - \frac{a}{2} w^{-a/2-1} \psi \nabla w$ . So we can write

$$\int_{\Sigma} w^a (|\nabla \varphi|^2 - aw^{-1} \Delta w \varphi^2) = \int_{\Sigma} |\nabla \psi|^2 - aw^{-1} \psi (\nabla w, \nabla \psi) + \frac{a^2}{4} \psi^2 w^{-2} |\nabla w|^2 - a \psi^2 w^{-1} \Delta w \\ = \int_{\Sigma} |\nabla \psi|^2 - a \operatorname{div}(\psi^2 w^{-1} \nabla w) + aw^{-1} \psi (\nabla w, \nabla \psi) \\ - \left( a - \frac{a^2}{4} \right) \psi^2 w^{-2} |\nabla w|^2 \\ = \int_{\Sigma} |\nabla \psi|^2 + aw^{-1} \psi (\nabla w, \nabla \psi) - \left( a - \frac{a^2}{4} \right) \psi^2 w^{-2} |\nabla w|^2$$

Using that  $w^{-1} \psi (\nabla w, \nabla \psi) \leq \varepsilon |\nabla \psi|^2 + \frac{1}{4\varepsilon} \psi^2 w^{-2} |\nabla w|^2$  with  $\varepsilon = \frac{1}{4-a}$ , we get

$$\int_{\Sigma} w^a (|\nabla \varphi|^2 - aw^{-1} \Delta w \varphi^2) \leq \frac{4}{4-a} \int_{\Sigma} |\nabla \psi|^2$$

From (11) and (9), we then obtain

$$(12) \quad \frac{4}{4-a} \int_{\Sigma} |\nabla \psi|^2 \geq \int_{\Sigma} \left( |B|^2 + \overline{\operatorname{Ric}}(\eta, \eta) + aw^{-2} dw(\eta)^2 - aw^{-1} \overline{\Delta} w + aHd \ln(\eta) \right) \psi^2 \\ + adh(\eta) \psi^2 \\ \geq \int_{\Sigma} \left( |B|^2 + \overline{\operatorname{Ric}}(\eta, \eta) + \delta - \overline{\Lambda}_{\alpha} + aw^{-2} dw(\eta)^2 + aHd \ln(\eta) \right) \psi^2 \\ + adh(\eta) \psi^2$$

Let  $(e_1, \dots, e_k)$  be an orthonormal basis of  $\Sigma$ . Using Gauss equation, we have

$$\alpha \operatorname{Ric}^{\Sigma}(e_1, e_1) = \alpha \sum_{j=2}^k R_{1jj1}^{\Sigma} = \alpha \sum_{j=2}^k (\overline{R}_{1jj1} + B_{11} B_{jj} - B_{1j}^2) \\ = \overline{\operatorname{BRic}}_{\alpha}(\eta, e_1) - \overline{\operatorname{Ric}}(\eta, \eta) + \alpha \sum_{j=2}^k (B_{11} B_{jj} - B_{1j}^2)$$

So assuming that  $e_1$  is such that  $\text{Ric}^\Sigma(e_1, e_1) = \lambda^\Sigma$ , we have

$$\overline{\text{Ric}}(\eta, \eta) - \overline{\Lambda}_\alpha \geq \overline{\text{Ric}}(\eta, \eta) - \overline{\text{BRic}}_\alpha(\eta, e_1) = -\alpha\lambda^\Sigma + \alpha \sum_{j=2}^k (B_{11}B_{jj} - B_{1j}^2)$$

So the above inequality and using  $\text{tr } B = H$  in (12), we then get the inequality

$$\begin{aligned} \frac{4}{4-a} \int_\Sigma |\nabla \psi|^2 &\geq \int_\Sigma \psi^2 \left( \delta - \alpha\lambda^\Sigma + |B|^2 + \alpha \sum_{j=2}^k (B_{11}B_{jj} - B_{1j}^2) + a(d \ln w(\eta))^2 \right. \\ &\quad \left. + aHd \ln w(\eta) + adh(\eta) \right) \\ &\geq \int_\Sigma \psi^2 \left( \delta - \alpha\lambda^\Sigma + |B|^2 + \alpha HB_{11} - \alpha \sum_{j=1}^k B_{1j}^2 + a(d \ln w(\eta))^2 \right. \\ &\quad \left. + aHd \ln w(\eta) + adh(\eta) \right) \end{aligned}$$

Using (10), we have

$$\begin{aligned} K := |B|^2 + \alpha HB_{11} - \alpha \sum_{j=1}^k B_{1j}^2 + a(d \ln w(\eta))^2 + aHd \ln w(\eta) = \\ |B|^2 + \alpha HB_{11} - \alpha \sum_{j=1}^k B_{1j}^2 + \frac{1}{a}(H-h)^2 + H(h-H) \end{aligned}$$

Let us denote by  $\Phi$  the traceless part of  $B$  and let  $\Phi_\Delta$  denote the vector  $(\Phi_{11}, \dots, \Phi_{kk}) \in F_k$ . Thus, for  $\alpha \leq 2$ , we have

$$K \geq \frac{1}{k}H^2 + |\Phi_\Delta|^2 + \frac{\alpha}{k}H^2 + \alpha H\Phi_{11} - \alpha \left( \frac{1}{k}H + \Phi_{11} \right)^2 + \frac{1}{a}(H-h)^2 + H(h-H)$$

We can write a decomposition of  $\Phi_\Delta$  in an orthonormal basis of  $F_k$

$$\Phi_\Delta = \sum_{i=1}^{k-2} \begin{pmatrix} 0 \\ E_i \end{pmatrix} x_i + \frac{1}{\sqrt{k(k-1)}} \begin{pmatrix} k-1 \\ -1 \\ \vdots \\ -1 \end{pmatrix} z$$

where  $(E_i)_{1 \leq i \leq k-2}$  is an orthonormal basis of  $F_{k-1}$ . We then have

$$\begin{aligned} K &\geq \frac{1}{k}H^2 + z^2 + \frac{\alpha}{k}H^2 + \alpha H \sqrt{\frac{k-1}{k}} z - \alpha \left( \frac{1}{k}H + \sqrt{\frac{k-1}{k}} z \right)^2 + \frac{1}{a}(H-h)^2 + H(h-H) \\ &\geq \left( \frac{1}{k} + \frac{\alpha}{k} - \frac{\alpha}{k^2} + \frac{1}{a} - 1 \right) H^2 + \left( 1 - \alpha \frac{k-1}{k} \right) z^2 + \frac{1}{a} h^2 + \alpha \sqrt{\frac{k-1}{k}} \left( 1 - \frac{2}{k} \right) Hz + \left( 1 - \frac{2}{a} \right) Hh \end{aligned}$$

The above expression is a quadratic form in  $(H, z, h)$  associated to the matrix

$$G = \begin{pmatrix} \frac{1}{k} + \frac{\alpha}{k} - \frac{\alpha}{k^2} + \frac{1}{a} - 1 & \frac{\alpha}{2} \sqrt{\frac{k-1}{k}} \left( 1 - \frac{2}{k} \right) & \frac{1}{2} - \frac{1}{a} \\ \frac{\alpha}{2} \sqrt{\frac{k-1}{k}} \left( 1 - \frac{2}{k} \right) & 1 - \alpha \frac{k-1}{k} & 0 \\ \frac{1}{2} - \frac{1}{a} & 0 & \frac{1}{a} \end{pmatrix}$$

Notice that this matrix is positive definite if  $1 - \alpha \frac{k-1}{k} > 0$  and  $\det(G) > 0$ . Actually for  $k = 4$ ,  $a = \frac{11}{10}$ ,  $\alpha = \frac{40}{43}$ , we have  $1 - \alpha \frac{k-1}{k} = \frac{13}{43} > 0$  and

$$\det \left( G - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{22} \end{pmatrix} \right) = \frac{2599}{1789832} > 0$$

So  $K \geq \frac{1}{22}h^2$ . Finally, for our values of the parameters, we have

$$(13) \quad \frac{4}{4-a} \int_{\Sigma} |\nabla \psi|^2 \geq \int_{\Sigma} \psi^2 \left( \frac{\delta}{2} - \alpha \lambda^{\Sigma} \right) + \psi^2 \left( \frac{\delta}{2} + \frac{1}{22}h^2 + adh(\eta) \right)$$

**4.3. End of the proof.** We need to choose the domain  $\Omega_-$  and the function  $h$ . Let  $\Psi : N \setminus \Omega_+ \rightarrow \mathbb{R}_+$  be a smoothing of the distance function  $d_{\tilde{g}}(\cdot, \partial\Omega_+)$  such that  $\frac{1}{2}d_{\tilde{g}}(\cdot, \partial\Omega_+) \leq \Psi \leq 2d_{\tilde{g}}(\cdot, \partial\Omega_+)$  and  $|\bar{\nabla} \Psi|_{\tilde{g}} \leq 2$ . Let  $\varepsilon > 0$  small such that  $(1+\varepsilon)11\pi\sqrt{\frac{88}{15}}$  is a regular value of  $\Psi$ . Let us define  $\Omega_- = \Omega_+ \cup \{\Phi \leq (1+\varepsilon)11\pi\sqrt{\frac{88}{15}}\}$ . On  $\Omega_-$ ,  $d_{\tilde{g}}(\cdot, \partial\Omega_+) \leq 2(1+\varepsilon)11\pi\sqrt{\frac{88}{15}} \leq 100\pi$ , so  $\Omega_- \subset \bar{\mathcal{N}}_{100\pi}(\Omega_+)$ .

On  $\{0 < \Psi < (1+\varepsilon)11\pi\sqrt{\frac{88}{15}}\}$ , we consider the function  $h$  defined by  $h = k \circ \frac{\Psi}{1+\varepsilon}$  where

$$k(t) = -\sqrt{\frac{33}{10}} \tan\left(\frac{1}{11}\sqrt{\frac{15}{88}}t - \frac{\pi}{2}\right)$$

for  $t \in (0, 11\pi\sqrt{\frac{88}{15}})$ . We have  $\lim_{p \rightarrow \partial\Omega_{\pm}} h(p) = \pm\infty$ . Notice that  $k$  solves  $-k' = \frac{3}{44} + \frac{5}{242}k^2$  so

$$|adh(\eta)| = a|k' \left( \frac{\Psi(p)}{1+\varepsilon} \right)| \frac{|\Psi'(p)|}{1+\varepsilon} \leq \frac{2a}{1+\varepsilon} \left( \frac{3}{44} + \frac{5}{242}h^2 \right) \leq \frac{3}{20} + \frac{1}{22}h^2 = \frac{\delta}{2} + \frac{1}{22}h^2$$

Hence, the above construction applies and, for our choices of parameters, (13) becomes

$$\frac{4}{4-a} \int_{\Sigma} |\nabla \psi|^2 \geq \int_{\Sigma} \psi^2 \left( \frac{\delta}{2} - \alpha \lambda^{\Sigma} \right)$$

This ends the proof of Theorem 4.1.

## 5. STABLE BERNSTEIN PROBLEM

In this section we prove Theorem 1.1. This is a consequence of the following volume growth estimate.

**Proposition 5.1.** *Let  $F : M^5 \looparrowright \mathbb{R}^6$  be a complete, immersed, two-sided, simply-connected stable minimal hypersurface. Let  $\mathcal{B}_{\rho}$  denote the geodesic ball of radius  $\rho > 0$  centered at some point  $p_0$  in  $M$  (for the induced metric  $g$ ). Then*

$$\text{Vol}(\mathcal{B}_{\rho}) \leq \text{Vol}(\mathbb{B}^5) \left( \frac{800}{43} \right)^{5/2} (2 \exp(100\pi))^5 \rho^5$$

*Proof.* First, up to a translation, we may assume that  $F(p_0) = 0$ . Let  $\Omega_+$  be a smooth compact domain in  $M$  such that  $\mathcal{B}_{\rho} \subset \Omega_+ \subset \mathcal{B}_{2\rho}$  and such that  $0 \notin F(\partial\Omega_+)$ . We consider the Gulliver-Lawson conformal metric  $\tilde{g} = r^{-2}g$ . By Theorem 3.1 and Theorem 4.1, there is  $\Omega_*$  a domain in  $M$  such that  $\Omega_+ \subset \Omega_* \subset \tilde{\mathcal{N}}_{100\pi}(\Omega_+)$  and  $\partial\Omega_*$  satisfies the spectral Ricci lower bound (8) for the metric induced by  $\tilde{g}$ .

By [7, Theorem 1],  $M$  has one end. We consider  $\Omega_{**}$  the connected component of  $\Omega_*$  that contains  $\mathcal{B}_\rho$ . We assume  $M$  is simply connected so the unbounded component of  $M \setminus \Omega_{**}$  has only boundary component  $\Sigma_0$ . Let  $\Omega'$  be the bounded component of  $M \setminus \Sigma_0$ . We have  $\mathcal{B}_\rho \subset \Omega'$  and  $\partial\Omega' \subset \tilde{\mathcal{N}}_{100\pi}(\mathcal{B}_{2\rho})$ .

On  $\partial\mathcal{B}_{2\rho}$ , the Euclidean distance function  $r$  is bounded by  $2\rho$ . So, by [10, Lemma 6.2], on  $\tilde{\mathcal{N}}_{100\pi}(\mathcal{B}_{2\rho})$ , the Euclidean distance function  $r$  is bounded by  $2\rho \exp(100\pi)$ .

Now, because of the spectral Ricci lower bound (8) and since  $\frac{4}{(4-a)\alpha} = \frac{43}{29} < \frac{3}{2} = \frac{k-1}{k-2}$ , we can apply the volume estimate of Antonelli and Xu [2, Theorem 1] for the metric  $\tilde{g}$  and obtain

$$\text{Vol}_{\tilde{g}}(\Sigma_0) \leq \left(\frac{\delta}{6\alpha}\right)^{-2} \text{Vol}(\mathbb{S}^4) = \left(\frac{800}{43}\right)^2 \text{Vol}(\mathbb{S}^4)$$

So scaling back to the Euclidean metric

$$\text{Vol}(\Sigma_0) \leq \left(\frac{800}{43}\right)^2 \text{Vol}(\mathbb{S}^4) (2 \exp(100\pi))^4 \rho^4$$

Finally we can apply the isoperimetric inequality for minimal hypersurfaces in  $\mathbb{R}^{n+1}$  [6, 20] to obtain

$$\text{Vol}_g(\mathcal{B}_\rho) \leq \text{Vol}_g(\Omega') \leq \text{Vol}(\mathbb{B}^5) \left(\frac{800}{43}\right)^{5/2} (2 \exp(100\pi))^5 \rho^5$$

□

*Proof of Theorem 1.1.* Let  $M \looparrowright \mathbb{R}^6$  be an immersed, connected, complete, two-sided, stable minimal hypersurface. The stability assumption lifts to the universal cover, so we can assume  $M$  to be simply connected. By Proposition 5.1,  $M$  has Euclidean volume growth. So by [23] (see also [3]), we obtain that  $M$  is a flat hyperplane. □

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