Stable minimal hypersurface in R^6

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STABLE MINIMAL HYPERSURFACES IN $\mathbb{R}^6$

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Abstract. Following the strategy developed by Chodosh, Li, Minter and Stryker, and using the volume estimate of Antonelli and Xu, we prove that, in $\mathbb{R}^6$, a complete, two-sided, stable minimal hypersurfaces is flat.

1. introduction

A minimal hypersurface M^n of $\mathbb{R}^{n+1}$ is a critical point of the n-volume functional. It is characterized by its vanishing mean curvature. If a unit normal vectorfield ν is defined along M and φ is a function with compact support on M, one can consider a deformation of M with initial speed $\varphi\nu$. The computation of the second derivative of the n-volume along this deformation at initial time gives

$$\int_M |\nabla\varphi|^2 - |A_M|^2\varphi^2$$

where A_M is the second fundamental form of M. So asking that this quantity is non negative for any φ means that M is a minimum at order 2 of the n-volume. Such a minimal hypersurface is called stable.

The stable Bernstein problem asks wether a complete stable minimal hypersurface is a flat affine hyperplane. We give a positive answer in the case $n = 5$.

Theorem 1.1. *Let $M^5 \looparrowright \mathbb{R}^6$ be an immersed, complete, connected, two-sided, stable minimal hypersurface. Then M is a Euclidean hyperplane.*

A particular class of stable minimal hypersurface is given by minimal graphs over $\mathbb{R}^n$. In [4], Bernstein proved that a minimal graph over $\mathbb{R}^2$ has to be a plane. In the sixties, the same question for higher dimensions was studied in a series of paper by Fleming [17], De Giorgi [14], Almgren [1] and Simons [26]. They proved that minimal graphs over $\mathbb{R}^n$ are planes if $n \leq 7$. For $n \geq 8$, Bombieri, De Giorgi and Giusti [5] were able to construct counter-examples and gave also in $\mathbb{R}^8$ an example of a stable minimal hypersurfaces that is not a hyperplane.

Concerning the stable Bernstein problem, the question was solved positively in $\mathbb{R}^3$ by Do Carmo and Peng [15], Fischer-Colbrie and Schoen [16] and Porogelov [22] in the early eighties. In higher dimension, Schoen, Simon and Yau [23, 24] were able to settle the stable Bernstein in $\mathbb{R}^{n+1}$, $n \leq 6$, under a Euclidean volume growth assumption (see also the recent work of Bellettini [3]).

Recently Chodosh and Li [9] were able to answer positively the stable Bernstein problem in $\mathbb{R}^4$. Later two alternative proofs came out: one by Catino, Mastrolia and Roncoroni [8] and one by Chodosh and Li [10]. Actually in [10], Chodosh and Li develop a second strategy to prove the result. Then, in a joint work Minter and Stryker [12], they were able to apply this strategy in the case of $\mathbb{R}^5$ to solved the stable Bernstein problem in this dimension as well.

As in [13, 12], it is well known that Theorem 1.1 comes with corollaries like curvature estimates for stable minimal immersions in 6-dimensional manifolds and characterization of finite Morse index minimal hypersurfaces in $\mathbb{R}^6$. For example, we have

Corollary 1.1. *Let (X^6, g) be a complete Riemannian manifold whose sectional curvature satisfies $|sec_g| \leq K$. Then any compact, two-sided, stable minimal immersion $M^5 \looparrowright X$ satisfies*

$$|A_M|(q) \min(1, d_M(q, \partial M)) \leq C(K)$$

for $q \in M$.

The basic idea to prove Theorem 1.1 is to obtain a Euclidean growth estimate for the volume of M and then apply the work of Schoen, Simon and Yau. The strategy of Chodosh and Li is a way towards this estimate. We refer to [10, 12] for a good presentation of their ideas. Let us give some elements. Let M be a stable minimal hypersurface in $\mathbb{R}^{n+1}$ with induced metric g. Inspired by the work of Gulliver and Lawson [18], they consider the conformal metric $\tilde{g} = r^{-2}g$ where r is the Euclidean distance to 0 in $\mathbb{R}^{n+1}$. If M was a hyperplane passing through the origin $(M \setminus \{0\}, \tilde{g})$ would be isometric to the Euclidean product $\mathbb{S}^{n-1} \times \mathbb{R}$. In the general case, the idea of Chodosh and Li is that the stability assumption implies that the geometry of $(M \setminus \{0\}, \tilde{g})$ should look like $\mathbb{S}^{n-1} \times \mathbb{R}$. In [12], the authors consider the bi-Ricci curvature which is a certain combination of sectional curvatures. It was introduced by Shen and Ye in [25], already to study minimal surfaces (see precise definition in Section 2). Notice that on $\mathbb{S}^{n-1} \times \mathbb{R}$, the bi-Ricci curvature is lower bounded by $n-2$. In [12], the authors prove that the stability of M implies a positive spectral lower bound for the bi-Ricci curvature of $(M \setminus \{0\}, \tilde{g})$. More precisely they prove that, on $(M \setminus \{0\}, \tilde{g})$, the operator $-\widetilde{\Delta} + (\widetilde{\mathrm{BRic}}_- - 1)$ is non-negative where $\widetilde{\mathrm{BRic}}_-$ is the punctual minimum of the bi-Ricci curvature of $\tilde{g}$. This should be understood as a weak form of $\widetilde{\mathrm{BRic}} \geq 1$.

The second step of the strategy consists in the construction of a μ-bubble in $(M \setminus \{0\}, \tilde{g})$ with a spectral lower bound for its Ricci curvature. In some sense, they identify in any sufficiently large part of $(M \setminus \{0\}, \tilde{g})$ a hypersurface that play the role of $\mathbb{S}^{n-1} \times \{t\}$ in $\mathbb{S}^{n-1} \times \mathbb{R}$.

The last step in [12] is to obtain a Bishop-Gromov volume estimate for the μ-bubble under the spectral lower bound on the Ricci curvature. In their paper, the proof of this volume estimate was specific to dimension 3. Recently, Antonelli and Xu [2] have proved such a Bishop-Gromov estimate in any dimension.

Once the $\tilde{g}$-volume of the μ-bubble is controlled, this gives an estimate of its volume in the original metric g and then control the growth of the volume of M tanks to an isoperimetric inequality due to Michael and Simon [20] and Brendle [6].

In the present paper, we also follow the above strategy of [12]. Here we consider a weighted bi-Ricci curvature BRic_α where the parameter α does not give the same weight to all sectional curvature in the combination (a similar idea appear in the recent article by Hong and Yan [19]). We prove a spectral lower bound for the weighted bi-Ricci curvature: the operator $-a\widetilde{\Delta} + (\widetilde{\mathrm{BRic}}_{\alpha-} - \delta)$ is non-negative where $a, \delta \in \mathbb{R}$. At that step, a, α are two parameters that should be chosen such that $\delta > 0$.

By imposing some new constraints on a and α, we are then able to construct the μ-bubble with a spectral lower bound on the Ricci curvature. At the last step, we apply the Bishop-Gromov estimate of Antonelli and Xu [2]. In order to do so, this imposes some new

constraints on the parameters a and α. Nevertheless, the choice $a = \frac{11}{10}$ and $\alpha = \frac{40}{43}$ fits all the constraints. The end of the proof then follows the line of [12].

Organization. In Section 2, we fix some notations that we use all along the paper. Section 3 is devoted to the proof the spectral lower bound for $\widetilde{\mathrm{BRic}}_\alpha$ for the Gulliver-Lawson metric on a stable minimal hypersurface. In Section 4, we construct the μ-bubble with a spectral lower Ricci bound. We end the proof of Theorem 1.1 in Section 5. Along the paper, we specify the value of n, a and α only when it is necessary, we hope this allows to understand where the constraints come from.

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2. Preliminaries

Let (M^n, g) be a Riemannian manifold and $(e_i)_{1\leq i\leq n}$ be an orthonormal basis of T_pM. For $\alpha \in \mathbb{R}$, we recall or define

- the Ricci curvature $\mathrm{Ric}(e_1, e_1) = \sum_{i=2}^n R(e_1, e_i, e_i, e_1)$,
- the punctual minimum of the Ricci curvature $\lambda(p) = \min_{v\in T_pM, |e|=1} \mathrm{Ric}(e, e)$,
- the weighted bi-Ricci or α-bi-Ricci curvature

$$\mathrm{BRic}_\alpha(e_1, e_2) = \sum_{i=2}^n R(e_1, e_i, e_i, e_1) + \alpha \sum_{j=3}^n R(e_2, e_j, e_j, e_2)$$

- the minimum of the α-bi-Ricci curvature $\Lambda_\alpha(p) = \min_{(e,f) \text{ orthonormal in } T_pM} \mathrm{BRic}_\alpha(e, f)$

Notice that for $\alpha = 1$, BRic_1 is the classical bi-Ricci curvature as defined in [25].

If $\Sigma \looparrowright M$ is a hypersurface with unit normal ν. We use the following conventions:

- the second fundamental form of Σ is $A_\Sigma(X, Y) = (\nabla_X \nu, Y) = -(\nabla_X Y, \nu)$ and
- the mean curvature of Σ is $H = \mathrm{tr}\, A_\Sigma$.

If Ω is a subset of M, we denote by $\mathcal{N}_\rho(\Omega)$ the ρ-tubular neighborhood of Ω: the set of points at distance less than ρ from Ω.

We finish by a simple remark that we use in Section 3.

Remark 1. Let $A \in \mathcal{M}_n(\mathbb{R})$ be a positive definite symmetric matrix and $B \in \mathbb{R}^n$. Then the function $f : X \in \mathbb{R}^n \mapsto X^\top AX + B^\top X \in \mathbb{R}$ is lower bounded and its minimum is given by $-\frac{1}{4}B^\top A^{-1}B$.

3. Spectral lower bound for the weighted bi-Ricci curvature

Let $F : M^n \looparrowright \mathbb{R}^{n+1}$ be a complete two-sided minimal hypersurface and g its induced metric. We consider the Gulliver-Lawson conformal metric $\tilde{g} = r^{-2}g$ where r is the Euclidean distance function to 0. Notice that if $F(p) = 0$, $\tilde{g}$ is not defined. So we consider $N = M \setminus F^{-1}(0)$. As it was observed by Gulliver and Lawson [18], the metric $(N, \tilde{g})$ is complete.

The first step of the proof of Theorem 1.1 consists in proving that the stability assumption can be translated in a spectral lower bound for the α-bi-Ricci curvature of the metric $\tilde{g}$. Actually we have the following result.

Theorem 3.1. *Let $M^n \looparrowright \mathbb{R}^{n+1}$ be a two-sided stable minimal hypersurface. Suppose $n = 5$, then, for $a = \frac{11}{10}$, $\alpha = \frac{40}{43}$ and $\delta = \frac{3}{10}$, there is a smooth function V such that*

$$V \geq \delta - \widetilde{\Lambda}_\alpha$$

and

$$\int_N |\nabla\varphi|^2_{\tilde{g}} dv_{\tilde{g}} \geq \int_N \frac{1}{a} V\varphi^2 dv_{\tilde{g}} \tag{1}$$

for any $\varphi \in C^1_c(N)$.

3.1. **Recalling some computations.** We first recall some computations and results of [12].

We denote by ν the unit normal to M and by $|dr|$ the norm of the differential of r along M with respect to the g-metric.

Let $(e_i)_{1\leq i\leq n}$ be an orthonormal basis for the metric g then, for the conformal metric $\tilde{g}$, an orthonormal basis is given by $\tilde{e}_i = re_i$. The sectional curvatures of g and $\tilde{g}$ are related by

$$\tilde{R}_{ijji} = r^2 R_{ijji} + 2 - |dr|^2 - dr(e_i)^2 - dr(e_j)^2 - (p,\nu)(A_{ii} + A_{jj}) \tag{2}$$

(see [12, Proposition 3.5]).

The second result that we want to recall is the writing of the stability inequality in the conformal metric $\tilde{g}$. We have

$$\int_N |\nabla\varphi|^2 dv_{\tilde{g}} \geq \int_N \left(r^2|A|^2 - \frac{n(n-2)}{2} + \frac{n^2-4}{4}|dr|^2 \right) \psi^2 dv_{\tilde{g}} \tag{3}$$

for any $\varphi \in C^1_c(N)$ (see [12, Proposition 3.10]).

3.2. **Estimating the curvature terms.** In this subsection, we want to relate the curvature term in the stability inequality (3) to the α-bi-Ricci curvature. We use the notations of the preceding subsection.

Proposition 3.1.

$$\begin{aligned}\widetilde{\mathrm{BRic}}_\alpha(\tilde{e}_1,\tilde{e}_2) =& r^2\,\mathrm{BRic}_\alpha(e_1,e_2) + 2(n-1+\alpha(n-2)) - (n+\alpha(n-1))|dr|^2\\ &- ((n-2-\alpha)dr(e_1)^2 + \alpha(n-3)dr(e_2)^2)\\ &- (p,\nu)((n-2-\alpha)A_{11} + \alpha(n-3)A_{22})\end{aligned}$$

Proof. Summing (2) and using $\operatorname{tr} A = 0$, we have

$$\begin{aligned}\widetilde{\mathrm{BRic}}_\alpha(\tilde{e}_1,\tilde{e}_2) =& \sum_{i=2}^n \tilde{R}_{1ii1} + \alpha\sum_{j=3}^n \tilde{R}_{2jj2}\\ =& r^2\,\mathrm{BRic}_\alpha(e_1,e_2) + 2(n-1) - (n-1)|dr|^2 - (n-1)dr(e_1)^2\\ &- (|dr|^2 - dr(e_1)^2) - (p,\nu)((n-1)A_{11} - A_{11}) + 2\alpha(n-2)\\ &- \alpha(n-2)|dr|^2 - \alpha(n-2)dr(e_2)^2 - \alpha(|dr|^2 - dr(e_1)^2 - dr(e_2)^2)\\ &- \alpha(p,\nu)((n-2)A_{22} - A_{11} - A_{22})\\ =& r^2\,\mathrm{BRic}_\alpha(e_1,e_2) + 2(n-1+\alpha(n-2)) - (n+\alpha(n-1))|dr|^2\\ &- ((n-2-\alpha)dr(e_1)^2 + \alpha(n-3)dr(e_2)^2)\\ &- (p,\nu)((n-2-\alpha)A_{11} + \alpha(n-3)A_{22})\end{aligned}$$

□

Proposition 3.2.

$$\mathrm{BRic}_\alpha(e_1,e_2) = -\sum_{i=1}^n A_{1i}^2 - \alpha\sum_{j=2}^n A_{2j}^2 - \alpha A_{11}A_{22}$$

Proof. Applying Gauss formula we have

$$\begin{aligned}\mathrm{BRic}_\alpha(e_1,e_2) &= \sum_{i=2}^n (A_{11}A_{ii} - A_{1i}^2) + \alpha\sum_{j=3}^n (A_{22}A_{jj} - A_{2j}^2)\\ &= -\sum_{i=1}^n A_{1i}^2 + \alpha(-A_{22}(A_{11}+A_{22}) - \sum_{j=3}^n A_{2j}^2)\\ &= -\sum_{i=1}^n A_{1i}^2 - \alpha\sum_{j=2}^n A_{2j}^2 - \alpha A_{11}A_{22}\end{aligned}$$

□

Using the above computation, we obtain the following estimate of the curvature term. This estimate introduces some constraints on α and a second parameter a.

Proposition 3.3. *Let $a, \alpha > 0$ such that $a > \frac{1}{2}$, $2a \geq \alpha$ and*

$$W = (a-\frac{1}{2})\big(a - \frac{n-2}{2n}(1+2\alpha)\big) - \frac{n-2}{4n}(1-\alpha)^2 > 0$$

Let us define

$$f = \frac{(n-2)^2}{8W}\Big((a-\frac{1}{2})\frac{n-2}{n}(1+\alpha\frac{n-4}{n-2})^2 + (1-\alpha)^2(a+\frac{n-2}{2n}-\frac{2}{n}\alpha)\Big)$$

Then

$$ar^2|A|^2 + f(1-|dr|^2) \geq -r^2\,\mathrm{BRic}_\alpha(e_1,e_2) + (p,\nu)((n-2-\alpha)A_{11} + \alpha(n-3)A_{22})$$

Proof. By Proposition 3.2, the right-hand side of the expected inequality satisfies to

$$\begin{aligned}-r^2\,\mathrm{BRic}_\alpha(e_1,e_2) &+ (p,\nu)((n-2-\alpha)A_{11} + \alpha(n-3)A_{22})\\ &= r^2\Big(\sum_{i=1}^n A_{1i}^2 + \alpha\sum_{j=2}^n A_{2j}^2 + \alpha A_{11}A_{22}\\ &\qquad + (\frac{p}{r^2},\nu)((n-2-\alpha)A_{11} + \alpha(n-3)A_{22})\Big)\\ &= r^2\Big(A_{11}^2 + \alpha A_{22}^2 + \alpha A_{11}A_{22} + \sum_{i=2}^n A_{1i}^2 + \alpha\sum_{j=3}^n A_{2j}^2\\ &\qquad + (\frac{p}{r^2},\nu)((n-2-\alpha)A_{11} + \alpha(n-3)A_{22})\Big)\end{aligned} \tag{4}$$

The vector $A_\Delta = (A_{11}, \cdots, A_{nn})$ belongs to the sub-space $F_n = \{X \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\}$. We write a decomposition in an orthonormal basis of F_n as

$$\begin{pmatrix} A_{11} \\ \vdots \\ A_{nn} \end{pmatrix} = \sum_{i=1}^{n-3} \begin{pmatrix} 0 \\ 0 \\ E_i \end{pmatrix} x_i + \frac{1}{\sqrt{2n(n-2)}} \begin{pmatrix} n-2 \\ n-2 \\ -2 \\ \vdots \\ -2 \end{pmatrix} z_1 + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} z_2$$

where $(E_i)_{1 \leq i \leq n-3}$ is an orthonormal basis of F_{n-2}. So we have

$$\begin{aligned} A_{11}^2 + \alpha A_{22}^2 + \alpha A_{11} A_{22} + (\frac{p}{r^2}, \nu)((n-2-\alpha)A_{11} + \alpha(n-3)A_{22}) \\ = (\frac{\sqrt{n-2}}{\sqrt{2n}} z_1 + \frac{z_2}{\sqrt{2}})^2 + \alpha(\frac{\sqrt{n-2}}{\sqrt{2n}} z_1 - \frac{z_2}{\sqrt{2}})^2 + \alpha(\frac{n-2}{2n} z_1^2 - \frac{1}{2} z_2^2) \\ + (\frac{p}{r^2}, \nu)\big(\frac{\sqrt{n-2}}{\sqrt{2n}}(n-2+\alpha(n-4))z_1 + \frac{n-2}{\sqrt{2}}(1-\alpha)z_2\big) \\ = \frac{n-2}{2n}(1+2\alpha)z_1^2 + \sqrt{\frac{n-2}{n}}(1-\alpha)z_1 z_2 + \frac{1}{2} z_2^2 \\ + (\frac{p}{r^2}, \nu)\frac{n-2}{\sqrt{2}}\big(\sqrt{\frac{n-2}{n}}(1+\alpha\frac{n-4}{n-2})z_1 + (1-\alpha)z_2\big) \end{aligned} \tag{5}$$

For $a > 0$, we are interested in the minimum (if it exists) of

$$\begin{aligned} a(z_1^2 + z_2^2) - \frac{n-2}{2n}(1+2\alpha)z_1^2 - \sqrt{\frac{n-2}{n}}(1-\alpha)z_1 z_2 - \frac{1}{2} z_2^2 \\ - (\frac{p}{r^2}, \nu)\frac{n-2}{\sqrt{2}}(\sqrt{\frac{n-2}{n}}(1+\alpha\frac{n-4}{n-2})z_1 + (1-\alpha)z_2) \end{aligned} \tag{6}$$

The matrix of the quadratic part of the above expression is

$$\begin{pmatrix} a - \frac{n-2}{2n}(1+2\alpha) & -\sqrt{\frac{n-2}{4n}}(1-\alpha) \\ -\sqrt{\frac{n-2}{4n}}(1-\alpha) & a - \frac{1}{2} \end{pmatrix}$$

This matrix is positive definite if $a > \frac{1}{2}$ and its determinant is positive:

$$W = (a - \frac{1}{2})\big(a - \frac{n-2}{2n}(1+2\alpha)\big) - \frac{n-2}{4n}(1-\alpha)^2 > 0$$

If it's the case, by Remark 1 with vector $B = -(\frac{p}{r^2}, \nu)\frac{n-2}{\sqrt{2}}\Big(\sqrt{\frac{n-2}{n}}(1+\alpha\frac{n-4}{n-2}), (1-\alpha)\Big)$, the quantity in (6) is lower bounded by

$$\begin{aligned}
&-(\frac{p}{r^2}, \nu)^2\frac{(n-2)^2}{8W}\Big((a-\frac{1}{2})\frac{n-2}{n}(1+\alpha\frac{n-4}{n-2})^2 + \frac{n-2}{n}(1-\alpha)^2(1+\alpha\frac{n-4}{n-2})\\
&\qquad\qquad + (a - \frac{n-2}{2n}(1+2\alpha))(1-\alpha)^2\Big)\\
=&-(\frac{p}{r^2}, \nu)^2\frac{(n-2)^2}{8W}\Big((a-\frac{1}{2})\frac{n-2}{n}(1+\alpha\frac{n-4}{n-2})^2 + (1-\alpha)^2(a+\frac{n-2}{2n}-\frac{2}{n}\alpha)\Big)\\
=&-(\frac{p}{r^2}, \nu)^2 f
\end{aligned}$$

Since $(\frac{p}{r}, \nu)^2 = (1-|dr|^2)$, we have then proved that

$$\begin{aligned}
a(z_1^2+z_2^2) + \frac{f}{r^2}(1-|dr|^2) \geq&\frac{n-2}{2n}(1+2\alpha)z_1^2 + \sqrt{\frac{n-2}{n}}(1-\alpha)z_1z_2 + \frac{1}{2}z_2^2\\
&+ (\frac{p}{r^2}, \nu)\frac{n-2}{\sqrt{2}}(\sqrt{\frac{n-2}{n}}(1+\alpha\frac{n-4}{n-2})z_1 + (1-\alpha)z_2)
\end{aligned}$$

Combining this with (4) and (5), if $2a \geq \alpha$, we have

$$\begin{aligned}
a|A|^2 + \frac{f}{r^2}(1-|dr|^2) \geq&a(|A_\Delta|^2 + \sum_{i\neq j}A_{ij}^2) + \frac{f}{r^2}(1-|dr|^2)\\
\geq&A_{11}^2 + \alpha A_{22}^2 + \alpha A_{11}A_{22} + (\frac{p}{r^2}, \nu)((n-2-\alpha)A_{11} + \alpha(n-3)A_{22})\\
&+ a\sum_{i\neq j}A_{ij}^2\\
\geq&A_{11}^2 + \alpha A_{22}^2 + \alpha A_{11}A_{22} + \sum_{i=2}^n A_{1i}^2 + \alpha\sum_{j=3}^n A_{2j}^2\\
&+ (\frac{p}{r^2}, \nu)((n-2-\alpha)A_{11} + \alpha(n-3)A_{22})\\
\geq&-\mathrm{BRic}_\alpha(e_1, e_2) + \frac{(p,\nu)}{r^2}((n-2-\alpha)A_{11} + \alpha(n-3)A_{22})
\end{aligned}$$

This is the expected estimate □

3.3. **Proof of Theorem 3.1.** Let us assume that the basis is chosen such that $\widetilde{\Lambda}_\alpha = \widetilde{\mathrm{BRic}}_\alpha(\tilde{e}_1, \tilde{e}_2)$. From (3), we are looking for a lower bound for $r^2|A|^2 - \frac{n(n-2)}{2} + \frac{n^2-4}{4}|dr|^2$. Under the assumptions of Proposition 3.3, $\alpha \leq 1$ (such that $n-2-\alpha \geq \alpha(n-3)$) and using

Proposition 3.1, we have

$$\begin{aligned}
a\Big(r^2|A|^2 - \frac{n(n-2)}{2} + \frac{n^2-4}{4}|dr|^2\Big)&\\
&\geq -r^2\,\mathrm{BRic}_\alpha(e_1,e_2) + (p,\nu)((n-2-\alpha)A_{11} + \alpha(n-3)A_{22})\\
&\quad - f(1-|dr|^2) - a\frac{n(n-2)}{2} + a\frac{n^2-4}{4}|dr|^2\\
&\geq -\widetilde{\mathrm{BRic}}_\alpha(\tilde e_1,\tilde e_2) + 2(n-1+\alpha(n-2)) - (n+\alpha(n-1))|dr|^2\\
&\quad - \big((n-2-\alpha)dr(e_1)^2 + \alpha(n-3)dr(e_2)^2\big)\\
&\quad - f(1-|dr|^2) - a\frac{n(n-2)}{2} + a\frac{n^2-4}{4}|dr|^2\\
&\geq C(|dr|^2) - \widetilde{\Lambda}_\alpha
\end{aligned}$$

where

$$C(t) = 2(n-1+\alpha(n-2)) - (2n-2+\alpha(n-2))t - f(1-t) - a\frac{n(n-2)}{2} + a\frac{n^2-4}{4}t$$

C is an affine function and $0 \leq |dr|^2 \leq 1$, so $C(|dr|^2) \geq \min(C(0), C(1))$. We have

$$\begin{aligned}
C(1) &= 2(n-1+\alpha(n-2)) - (2n-2+\alpha(n-2)) - a\frac{n(n-2)}{2} + a\frac{n^2-4}{4}\\
&= \alpha(n-2) - a\frac{(n-2)^2}{4} = (n-2)(\alpha - a\frac{n-2}{4})
\end{aligned}$$

and

$$C(0) = 2(n-1+\alpha(n-2)) - f - a\frac{n(n-2)}{2}$$

If we consider $a = \frac{11}{10}$ and $\alpha = \frac{40}{43}$, we have $a > \frac{1}{2}$, $2a \geq \alpha$, $\alpha \leq 1$ and $W = \frac{26697}{184900} > 0$. So the above computations apply. We have

$$C(0) = \frac{731975}{1530628} \simeq 0.47 \quad \text{and} \quad C(1) = \frac{543}{1720} \simeq 0.31$$

So for these values of a and α, and with $\delta = \frac{3}{10} \leq \min(C(0), C(1))$, we have

$$V = a\left(r^2|A|^2 - \frac{n(n-2)}{2} + \frac{n^2-4}{4}|dr|^2\right) \geq \delta - \widetilde{\Lambda}_\alpha$$

By (3), the spectral estimate (1) is true. Theorem 3.1 is proved.

4. The μ-bubble construction

In this section, we produce a warped μ-bubble with a spectral Ricci curvature lower bound. So we start with a complete non-compact Riemannian manifold $(N^n, \bar g)$ with a spectral lower bound on the α-bi-Ricci curvature: there is a smooth function $\overline{V}$ on N such that

$$\overline{V} \geq \delta - \overline{\Lambda}_\alpha$$

and

$$\int_N |\overline{\nabla}\varphi|^2_{\bar g} dv_{\bar g} \geq \int_N \frac{1}{a}\overline{V}\varphi^2 dv_{\bar g} \tag{7}$$

for any $\varphi \in C^1_c(N)$

Theorem 4.1. *Assume $(N, \bar{g})$ as above with $n = 5$, $a = \frac{11}{10}$, $\alpha = \frac{40}{43}$ and $\delta = \frac{3}{10}$. Let Ω_+ be a domain in N. Then there is a domain Ω_* with*

- *$\Omega_+ \subset \Omega_* \subset \overline{\mathcal{N}}_{100\pi}(\Omega_+)$ and*
- *there is a smooth function V on $\Sigma = \partial\Omega_*$ such that*

$$V \geq \frac{\delta}{2} - \alpha\lambda^{\Sigma}$$

and

$$\frac{4}{4-a}\int_{\Sigma} |\nabla\varphi|^2 dv_g \geq \int_{\Sigma} V\varphi^2 dv_g \tag{8}$$

for any $\varphi \in C^1(\Sigma)$ where g is the induced metric on Σ.

4.1. **Construction of the μ-bubble.** Because of the spectral control (7) on N, we know (see [16]) that there is a positive function w on N such that

$$-a\overline{\Delta}w = \overline{V}w \geq (\delta - \overline{\Lambda}_\alpha)w \tag{9}$$

Let us recall quickly the construction of the μ-bubble. Let Ω_- be a domain in N such that $\Omega_+ \subset\subset \Omega_- \subset \overline{\mathcal{N}}_{100\pi}(\Omega_+)$. Let $h : \Omega_- \setminus \Omega_+ \to \mathbb{R}$ be a smooth function such that $\lim_{p\to\partial\Omega_+} h(p) = +\infty$ and $\lim_{p\to\partial\Omega_-} h(p) = -\infty$. Let $\underline{\Omega}$ be a domain with $\Omega_+ \subset\subset \underline{\Omega} \subset\subset \Omega_-$.

For any sets of finite perimeter Ω with $\Omega_+ \subset\subset \Omega \subset\subset \Omega_-$, we consider the quantity

$$\mathcal{A}(\Omega) = \int_{\partial^*\Omega} w^a - \int_U (\chi_\Omega - \chi_{\underline{\Omega}}) h w^a$$

where $\partial^*\Omega$ is the reduced boundary of Ω. By similar argument to the ones in [11, 28], there there is a set of finite perimeter Ω_* ($\Omega_+ \subset\subset \Omega_* \subset\subset \Omega_-$) which minimize the functional $\mathcal{A}$. Moreover its reduced boundary $\partial^*\Omega_* = \Sigma$ is smooth (see for example [21, 27]).

4.2. **Spectral Ricci-curvature bound of the μ-bubble.** We denote by $k = n - 1$ the dimension of Σ and by η the outgoing unit normal to Σ.

As in [12, Proposition 4.2], if φ is a function on Σ, writing the first variation of $\mathcal{A}$ for a variation $\{\Omega_t\}$ of Ω_* generated by $\varphi\eta$ gives

$$0 = \frac{d}{dt}\mathcal{A}(\Omega_t)_{|t=0} = \int_\Sigma (Hw^a + aw^{a-1}dw(\eta) - hw^a)\varphi = \int_\Sigma (H + aw^{-1}dw(\eta) - h)w^a\varphi$$

Since this is true for any φ,

$$H = h - a\,d\ln w(\eta) \tag{10}$$

Computing the second derivative of $\mathcal{A}(\Omega_t)$, we obtain

$$\begin{aligned} 0 \leq \frac{d^2}{dt^2}\mathcal{A}(\Omega_t)_{|t=0} = \int_\Sigma w^a\Big(&- \varphi\Delta\varphi - (|B|^2 + \overline{\mathrm{Ric}}(\eta,\eta))\varphi^2 - aw^{-2}dw(\eta)^2\varphi^2 \\ &+ aw^{-1}\overline{\nabla}^2 w(\eta,\eta)\varphi^2 - aw^{-1}(\overline{\nabla}w, \nabla\varphi)\varphi - dh(\eta)\varphi^2\Big) \end{aligned}$$

where B is the second fundamental form of Σ. So

$$\begin{aligned} 0 \leq \int_\Sigma -\operatorname{div}(w^a\varphi\nabla\varphi) + w^a\Big(&|\nabla\varphi|^2 - (|B|^2 + \overline{\mathrm{Ric}}(\eta,\eta))\varphi^2 - aw^{-2}dw(\eta)^2\varphi^2 \\ &+ aw^{-1}\overline{\nabla}^2 w(\eta,\eta)\varphi^2 - dh(\eta)\varphi^2\Big) \end{aligned}$$

Using $\overline{\nabla}^2 w(\eta,\eta) = \overline{\Delta} w - \Delta w - H dw(\eta)$, we obtain

$$\begin{aligned} 0 \leq \int_\Sigma w^a \Big(&|\nabla\varphi|^2 - (|B|^2 + \overline{\mathrm{Ric}}(\eta,\eta))\varphi^2 - a w^{-2} dw(\eta)^2 \varphi^2 \\ &+ a w^{-1} (\overline{\Delta} w - \Delta w - H dw(\eta))\varphi^2 - dh(\eta)\varphi^2 \Big) \end{aligned} \tag{11}$$

For $\varphi = w^{-a/2}\psi$, we have $\nabla\varphi = w^{-a/2}\nabla\psi - \frac{a}{2} w^{-a/2-1}\psi\nabla w$. So we can write

$$\begin{aligned} \int_\Sigma w^a (|\nabla\varphi|^2 - a w^{-1}\Delta w \varphi^2) &= \int_\Sigma |\nabla\psi|^2 - a w^{-1}\psi(\nabla w, \nabla\psi) + \frac{a^2}{4}\psi^2 w^{-2}|\nabla w|^2 - a\psi^2 w^{-1}\Delta w \\ &= \int_\Sigma |\nabla\psi|^2 - a\,\mathrm{div}(\psi^2 w^{-1}\nabla w) + a w^{-1}\psi(\nabla w, \nabla\psi) \\ &\qquad - (a - \frac{a^2}{4})\psi^2 w^{-2}|\nabla w|^2 \\ &= \int_\Sigma |\nabla\psi|^2 + a w^{-1}\psi(\nabla w, \nabla\psi) - (a - \frac{a^2}{4})\psi^2 w^{-2}|\nabla w|^2 \end{aligned}$$

Using that $w^{-1}\psi(\nabla w, \nabla\psi) \leq \varepsilon|\nabla\psi|^2 + \frac{1}{4\varepsilon}\psi^2 w^{-2}|\nabla w|^2$ with $\varepsilon = \frac{1}{4-a}$, we get

$$\int_\Sigma w^a (|\nabla\varphi|^2 - a w^{-1}\Delta w\varphi^2) \leq \frac{4}{4-a}\int_\Sigma |\nabla\psi|^2$$

From (11) and (9), we then obtain

$$\begin{aligned} \frac{4}{4-a}\int_\Sigma |\nabla\psi|^2 &\geq \int_\Sigma \Big(|B|^2 + \overline{\mathrm{Ric}}(\eta,\eta) + a w^{-2} dw(\eta)^2 - a w^{-1}\overline{\Delta} w + aH d\ln(\eta) \Big)\psi^2 \\ &\qquad + a dh(\eta)\psi^2 \\ &\geq \int_\Sigma \Big(|B|^2 + \overline{\mathrm{Ric}}(\eta,\eta) + \delta - \overline{\Lambda}_\alpha + a w^{-2} dw(\eta)^2 + aH d\ln(\eta) \Big)\psi^2 \\ &\qquad + a dh(\eta)\psi^2 \end{aligned} \tag{12}$$

Let $(e_1, \dots, e_k)$ be an orthonormal basis of Σ. Using Gauss equation, we have

$$\begin{aligned} \alpha\,\mathrm{Ric}^\Sigma(e_1,e_1) = \alpha\sum_{j=2}^k R^\Sigma_{1jj1} &= \alpha\sum_{j=2}^k (\overline{R}_{1jj1} + B_{11}B_{jj} - B_{1j}^2) \\ &= \overline{\mathrm{BRic}}_\alpha(\eta,e_1) - \overline{\mathrm{Ric}}(\eta,\eta) + \alpha\sum_{j=2}^k (B_{11}B_{jj} - B_{1j}^2) \end{aligned}$$

So assuming that e_1 is such that $\mathrm{Ric}^\Sigma(e_1,e_1) = \lambda^\Sigma$, we have

$$\overline{\mathrm{Ric}}(\eta,\eta) - \overline{\Lambda}_\alpha \geq \overline{\mathrm{Ric}}(\eta,\eta) - \overline{\mathrm{BRic}}_\alpha(\eta,e_1) = -\alpha\lambda^\Sigma + \alpha\sum_{j=2}^k (B_{11}B_{jj} - B_{1j}^2)$$

So the above inequality and using $\operatorname{tr} B = H$ in (12), we then get the inequality

$$\begin{aligned}\frac{4}{4-a}\int_\Sigma |\nabla\psi|^2 \geq \int_\Sigma \psi^2\Big(&\delta - \alpha\lambda^\Sigma + |B|^2 + \alpha\sum_{j=2}^{k}(B_{11}B_{jj} - B_{1j}^2) + a(d\ln w(\eta))^2 \\ &+ aHd\ln w(\eta) + adh(\eta)\Big) \\ \geq \int_\Sigma \psi^2\Big(&\delta - \alpha\lambda^\Sigma + |B|^2 + \alpha HB_{11} - \alpha\sum_{j=1}^{k} B_{1j}^2 + a(d\ln w(\eta))^2 \\ &+ aHd\ln w(\eta) + adh(\eta)\Big)\end{aligned}$$

Using (10), we have

$$\begin{aligned}K := |B|^2 + \alpha HB_{11} - \alpha\sum_{j=1}^{k} B_{1j}^2 + a(d\ln w(\eta))^2 + aHd\ln w(\eta) = \\ |B|^2 + \alpha HB_{11} - \alpha\sum_{j=1}^{k} B_{1j}^2 + \frac{1}{a}(H-h)^2 + H(h-H)\end{aligned}$$

Let us denote by Φ the traceless part of B and let Φ_Δ denote the vector $(\Phi_{11},\dots,\Phi_{kk}) \in F_k$. Thus, for $\alpha \leq 2$, we have

$$K \geq \frac{1}{k}H^2 + |\Phi_\Delta|^2 + \frac{\alpha}{k}H^2 + \alpha H\Phi_{11} - \alpha(\frac{1}{k}H + \Phi_{11})^2 + \frac{1}{a}(H-h)^2 + H(h-H)$$

We can write a decomposition of Φ_Δ in an orthonormal basis of F_k

$$\Phi_\Delta = \sum_{i=1}^{k-2}\begin{pmatrix}0\\E_i\end{pmatrix}x_i + \frac{1}{\sqrt{k(k-1)}}\begin{pmatrix}k-1\\-1\\\vdots\\-1\end{pmatrix}z$$

where $(E_i)_{1\leq i\leq k-2}$ is an orthonormal basis of F_{k-1}. We then have

$$\begin{aligned}K \geq&\frac{1}{k}H^2 + z^2 + \frac{\alpha}{k}H^2 + \alpha H\sqrt{\frac{k-1}{k}}z - \alpha(\frac{1}{k}H + \sqrt{\frac{k-1}{k}}z)^2 + \frac{1}{a}(H-h)^2 + H(h-H) \\ \geq&(\frac{1}{k} + \frac{\alpha}{k} - \frac{\alpha}{k^2} + \frac{1}{a} - 1)H^2 + (1 - \alpha\frac{k-1}{k})z^2 + \frac{1}{a}h^2 + \alpha\sqrt{\frac{k-1}{k}}(1-\frac{2}{k})Hz + (1-\frac{2}{a})Hh\end{aligned}$$

The above expression is a quadratic form in (H,z,h) associated to the matrix

$$G = \begin{pmatrix}\frac{1}{k} + \frac{\alpha}{k} - \frac{\alpha}{k^2} + \frac{1}{a} - 1 & \frac{\alpha}{2}\sqrt{\frac{k-1}{k}}(1-\frac{2}{k}) & \frac{1}{2} - \frac{1}{a} \\ \frac{\alpha}{2}\sqrt{\frac{k-1}{k}}(1-\frac{2}{k}) & 1 - \alpha\frac{k-1}{k} & 0 \\ \frac{1}{2} - \frac{1}{a} & 0 & \frac{1}{a}\end{pmatrix}$$

Notice that this matrix is positive definite if $1-\alpha\frac{k-1}{k}>0$ and $\det(G)>0$. Actually for $k=4$, $a=\frac{11}{10}$, $\alpha=\frac{40}{43}$, we have $1-\alpha\frac{k-1}{k}=\frac{13}{43}>0$ and

$$\det\Big(G-\begin{pmatrix}0&0&0\\0&0&0\\0&0&\frac{1}{22}\end{pmatrix}\Big)=\frac{2599}{1789832}>0$$

So $K\geq\frac{1}{22}h^2$. Finally, for our values of the parameters, we have

$$\frac{4}{4-a}\int_\Sigma|\nabla\psi|^2\geq\int_\Sigma\psi^2\big(\frac{\delta}{2}-\alpha\lambda^\Sigma\big)+\psi^2\big(\frac{\delta}{2}+\frac{1}{22}h^2+adh(\eta)\big)\tag{13}$$

4.3. **End of the proof.** We need to choose the domain Ω_- and the function h. Let $\Psi: N\backslash\Omega_+\to\mathbb{R}_+$ be a smoothing of the distance function $d_{\bar g}(\cdot,\partial\Omega_+)$ such that $\frac{1}{2}d_{\bar g}(\cdot,\partial\Omega_+)\leq\Psi\leq 2d_{\bar g}(\cdot,\partial\Omega_+)$ and $|\overline{\nabla}\Psi|_{\bar g}\leq 2$. Let $\varepsilon>0$ small such that $(1+\varepsilon)11\pi\sqrt{\frac{88}{15}}$ is a regular value of Ψ. Let us define $\Omega_-=\Omega_+\cup\{\Phi\leq(1+\varepsilon)11\pi\sqrt{\frac{88}{15}}\}$. On Ω_-, $d_{\bar g}(\cdot,\partial\Omega_+)\leq 2(1+\varepsilon)11\pi\sqrt{\frac{88}{15}}\leq 100\pi$, so $\Omega_-\subset\overline{\mathcal{N}}_{100\pi}(\Omega_+)$.

On $\{0<\Psi<(1+\varepsilon)11\pi\sqrt{\frac{88}{15}}\}$, we consider the function h defined by $h=k\circ\frac{\Psi}{1+\varepsilon}$ where

$$k(t)=-\sqrt{\frac{33}{10}}\tan(\frac{1}{11}\sqrt{\frac{15}{88}}t-\frac{\pi}{2})$$

for $t\in(0,11\pi\sqrt{\frac{88}{15}})$. We have $\lim_{p\to\partial\Omega_\pm}h(p)=\pm\infty$. Notice that k solves $-k'=\frac{3}{44}+\frac{5}{242}k^2$ so

$$|adh(\eta)|=a|k'(\frac{\Psi(p)}{1+\varepsilon})|\frac{|\Psi'(p)|}{1+\varepsilon}\leq\frac{2a}{1+\varepsilon}(\frac{3}{44}+\frac{5}{242}h^2)\leq\frac{3}{20}+\frac{1}{22}h^2=\frac{\delta}{2}+\frac{1}{22}h^2$$

Hence, the above construction applies and, for our choices of parameters, (13) becomes

$$\frac{4}{4-a}\int_\Sigma|\nabla\psi|^2\geq\int_\Sigma\psi^2\big(\frac{\delta}{2}-\alpha\lambda^\Sigma\big)$$

This ends the proof of Theorem 4.1.

5. Stable Bernstein problem

In this section we prove Theorem 1.1. This a consequence of the following volume growth estimate.

Proposition 5.1. *Let $F:M^5\looparrowright\mathbb{R}^6$ be a complete, immersed, two-sided, simply-connected stable minimal hypersurface. Let $\mathcal{B}_\rho$ denote the geodesic ball of radius $\rho>0$ centered at some point p_0 in M (for the induced metric g). Then*

$$\mathrm{Vol}(\mathcal{B}_\rho)\leq\mathrm{Vol}(\mathbb{B}^5)(\frac{800}{43})^{5/2}\big(2\exp(100\pi)\big)^5\rho^5$$

Proof. First, up to a translation, we may assume that $F(p_0)=0$. Let Ω_+ be a smooth compact domain in M such that $\mathcal{B}_\rho\subset\Omega_+\subset\mathcal{B}_{2\rho}$ and such that $0\notin F(\partial\Omega_+)$. We consider the Gulliver-Lawson conformal metric $\tilde g=r^{-2}g$. By Theorem 3.1 and Theorem 4.1, there is Ω_* a domain in M such that $\Omega_+\subset\Omega_*\subset\widetilde{\mathcal{N}}_{100\pi}(\Omega_+)$ and $\partial\Omega_*$ satisfies the spectral Ricci lower bound (8) for the metric induced by $\tilde g$.

By [7, Theorem 1], M has one end. We consider Ω_{**} the connected component of Ω_* that contains $\mathcal{B}_\rho$. We assume M is simply connected so the unbounded component of $M \setminus \Omega_{**}$ has only boundary component Σ_0. Let Ω' be the bounded component of $M \setminus \Sigma_0$. We have $\mathcal{B}_\rho \subset \Omega'$ and $\partial\Omega' \subset \widetilde{\mathcal{N}}_{100\pi}(\mathcal{B}_{2\rho})$.

On $\partial\mathcal{B}_{2\rho}$, the Euclidean distance function r is bounded by 2ρ. So, by [10, Lemma 6.2], on $\widetilde{\mathcal{N}}_{100\pi}(\mathcal{B}_{2\rho})$, the Euclidean distance function r is bounded by $2\rho\exp(100\pi)$.

Now, because of the spectral Ricci lower bound (8) and since $\frac{4}{(4-a)\alpha} = \frac{43}{29} < \frac{3}{2} = \frac{k-1}{k-2}$, we can apply the volume estimate of Antonelli and Xu [2, Theorem 1] for the metric $\tilde{g}$ and obtain

$$\mathrm{Vol}_{\tilde{g}}(\Sigma_0) \leq (\frac{\delta}{6\alpha})^{-2}\,\mathrm{Vol}(\mathbb{S}^4) = (\frac{800}{43})^2\,\mathrm{Vol}(\mathbb{S}^4)$$

So scaling back to the Euclidean metric

$$\mathrm{Vol}(\Sigma_0) \leq (\frac{800}{43})^2\,\mathrm{Vol}(\mathbb{S}^4)\big(2\exp(100\pi)\big)^4\rho^4$$

Finally we can apply the isoperimetric inequality for minimal hypersurfaces in $\mathbb{R}^{n+1}$ [6, 20] to obtain

$$\mathrm{Vol}_g(\mathcal{B}_\rho) \leq \mathrm{Vol}_g(\Omega') \leq \mathrm{Vol}(\mathbb{B}^5)(\frac{800}{43})^{5/2}\big(2\exp(100\pi)\big)^5\rho^5$$

□

Proof of Theorem 1.1. Let $M \looparrowright \mathbb{R}^6$ be an immersed, connected,complete, two-sided, stable minimal hypersurface. The stability assumption lifts to the universal cover, so we can assume M to be simply connected. By Proposition 5.1, M has Euclidean volume growth. So by [23] (see also [3]), we obtain that M is a flat hyperplane. □

References

[1] F. J. Almgren, Jr. Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem. *Ann. of Math. (2)*, 84:277–292, 1966.

[2] Gioacchino Antonelli and Kai Xu. New spectral Bishop-Gromov and Bonnet-Myers theorems and application to isoperimetry. preprint, arXiv:2405.08918.

[3] Costante Bellettini. Extensions of Schoen–Simon–Yau and Schoen–Simon theorems via iteration à la De Giorgi. preprint, arXiv:2310.01340.

[4] Serge Bernstein. Über ein geometrisches Theorem und seine Anwendung auf die partiellen Differentialgleichungen vom elliptischen Typus. *Math. Z.*, 26:551–558, 1927.

[5] E. Bombieri, E. De Giorgi, and E. Giusti. Minimal cones and the Bernstein problem. *Invent. Math.*, 7:243–268, 1969.

[6] Simon Brendle. The isoperimetric inequality for a minimal submanifold in Euclidean space. *J. Amer. Math. Soc.*, 34:595–603, 2021.

[7] Huai-Dong Cao, Ying Shen, and Shunhui Zhu. The structure of stable minimal hypersurfaces in $\mathbf{R}^{n+1}$. *Math. Res. Lett.*, 4:637–644, 1997.

[8] Giovanni Catino, Paolo Mastrolia, and Alberto Roncoroni. Two rigidity results for stable minimal hypersurfaces. *Geom. Funct. Anal.*, 34:1–18, 2024.

[9] Otis Chodosh and Chao Li. Stable minimal hypersurfaces in $\mathbf{R}^4$. preprint, arXiv:2108.11462.

[10] Otis Chodosh and Chao Li. Stable anisotropic minimal hypersurfaces in $\mathbf{R}^4$. *Forum Math. Pi*, 11:Paper No. e3, 22, 2023.

[11] Otis Chodosh and Chao Li. Generalized soap bubbles and the topology of manifolds with positive scalar curvature. *Ann. of Math. (2)*, 199:707–740, 2024.

[12] Otis Chodosh, Chao Li, Paul Minter, and Douglas Stryker. Stable minimal hypersurfaces in $\mathbf{R}^5$. preprint, arXiv:2401.01492.

[13] Otis Chodosh, Chao Li, and Douglas Stryker. Complete stable minimal hypersurfaces in positively curved 4-manifolds. preprint, arXiv:2202.07708.
[14] Ennio De Giorgi. Una estensione del teorema di Bernstein. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 19:79–85, 1965.
[15] M. do Carmo and C. K. Peng. Stable complete minimal surfaces in $\mathbf{R}^3$ are planes. *Bull. Amer. Math. Soc. (N.S.)*, 1:903–906, 1979.
[16] Doris Fischer-Colbrie and Richard Schoen. The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature. *Comm. Pure Appl. Math.*, 33:199–211, 1980.
[17] Wendell H. Fleming. On the oriented Plateau problem. *Rend. Circ. Mat. Palermo (2)*, 11:69–90, 1962.
[18] Robert Gulliver and H. Blaine Lawson, Jr. The structure of stable minimal hypersurfaces near a singularity. In *Geometric measure theory and the calculus of variations (Arcata, Calif., 1984)*, volume 44 of *Proc. Sympos. Pure Math.*, pages 213–237. Amer. Math. Soc., Providence, RI, 1986.
[19] Han Hong and Zeitan Yan. Rigidity and nonexistence of CMC hypersurfaces in 5-manifolds. preprint, arXiv:2405.06867.
[20] J. H. Michael and L. M. Simon. Sobolev and mean-value inequalities on generalized submanifolds of R^n. *Comm. Pure Appl. Math.*, 26:361–379, 1973.
[21] Frank Morgan. Regularity of isoperimetric hypersurfaces in riemannian manifolds. *Trans. Amer Math. Soc.*, 355:5041–5052, 2003.
[22] A. V. Pogorelov. On the stability of minimal surfaces. *Dokl. Akad. Nauk SSSR*, 260:293–295, 1981.
[23] R. Schoen, L. Simon, and S. T. Yau. Curvature estimates for minimal hypersurfaces. *Acta Math.*, 134:275–288, 1975.
[24] Richard Schoen and Leon Simon. Regularity of stable minimal hypersurfaces. *Comm. Pure Appl. Math.*, 34:741–797, 1981.
[25] Ying Shen and Rugang Ye. On the geometry and topology of manifolds of positive bi-ricci curvature. preprint, arXiv:9708014.
[26] James Simons. Minimal varieties in riemannian manifolds. *Ann. of Math. (2)*, 88:62–105, 1968.
[27] Italo Tamanini. Regularity results for almost minimal oriented hypersurfaces in R^N. *Quaderni del Dipartimento di Matematica dell' Università di Lecce*, 1, 1984.
[28] Jintian Zhu. Width estimate and doubly warped product. *Trans. Amer. Math. Soc.*, 374:1497–1511, 2021.

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